Algebraic and combinatorial structures on Baxter permutations

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1. Motivation and goals

- Many combinatorial objects like permutations, binary trees, integer compositions and partitions are endowed with a combinatorial Hopf algebra structure.
- An approach to construct most of these structures in a unified way relies on the definition of a plactic-like congruence on words satisfying some structure conditions.
- Every congruence on words leads to the definition of a monoid of combinatorial objects, and, in addition to the construction of combinatorial Hopf algebras, this construction often comes with partial orders, combinatorial algorithms and Robinson-Schensted-like algorithms.
- The goal of this work is to construct similar structures on Baxter permutations.

2. Baxter permutations, pairs of twin binary trees

- A Baxter permutation is a permutation avoiding the generalized permutation patterns $2-41-3$ and $3-14-2$. For example, 436975128 is a Baxter permutation but 421738 56 is not.
- A pair (T_L, T_R) of binary trees with the same number of nodes is a pair of twin binary trees if the canopies (*i.e.* the orientation of internal leaves in binary trees) of T_L and T_R are complementary.
- There is a bijection between Baxter permutations and pairs of twin binary trees.
- First numbers of Baxter permutations by size are 1, 1, 2, 6, 22, 92, 422, 2074, 10754, 58202, 326240.

3. The Baxter monoid

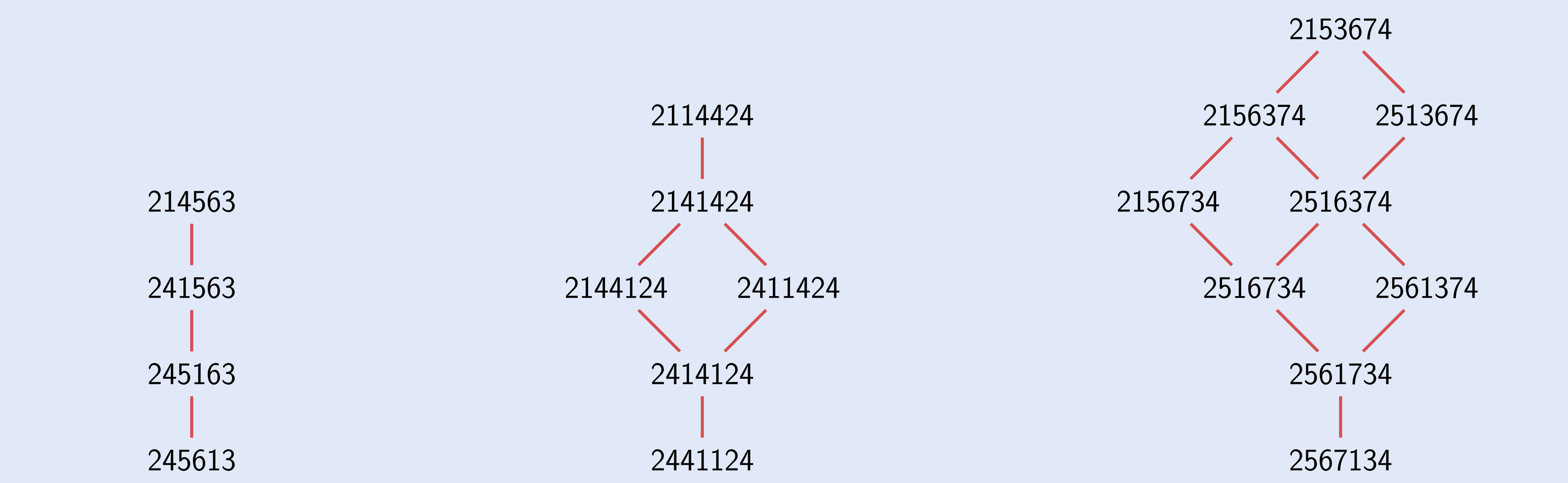
- Let $A := \{a_1 < a_2 < \dots\}$ be a totally ordered infinite alphabet and let A^* be the free monoid spanned by A.

Definition

The Baxter monoid is the quotient of A^* by the congruence $\equiv_B$, that is the transitive closure of the adjacency relations $\leftrightharpoons_B$ and $\rightleftharpoons_B$ defined for $u, v \in A^*$ and $a, b, c, d \in A$ by:

$$c\ u\ ad\ v\ b \leftrightharpoons_B c\ u\ da\ v\ b \quad \text{where} \quad a \leq b < c \leq d,$$
$$b\ u\ da\ v\ c \rightleftharpoons_B b\ u\ ad\ v\ c \quad \text{where} \quad a < b \leq c < d.$$

- Examples:



(a) The $\equiv_B$-equivalence class of the permutation 214563. (b) The $\equiv_B$-equivalence class of the word 2114424. (c) The $\equiv_B$-equivalence class of the permutation 2153674.

Figure: Some $\equiv_B$-equivalence classes. Edges represent adjacency relations.

- The Baxter monoid satisfies some structure properties:

Proposition

The Baxter monoid is compatible with the destandardization process, *i.e.*, for all $u, v \in A^*$, $u \equiv_B v$ iff $\mathrm{std}(u) \equiv_B \mathrm{std}(v)$ and $\mathrm{eval}(u) = \mathrm{eval}(v)$.

Proposition

The Baxter monoid is compatible with the restriction of alphabet intervals, *i.e.*, for all interval I of A and for all $u, v \in A^*$, $u \equiv_B v$ implies $u_{|I} \equiv_B v_{|I}$.

Proposition

The Baxter monoid is compatible with the Schützenberger involution, *i.e.*, for all $u, v \in A^*$, $u \equiv_B v$ implies $u^\# \equiv_B v^\#$.

4. A Robinson-Schensted-like algorithm

- Given an A-labeled pair of twin binary trees (T_L, T_R), one can insert a letter $a \in A$ into (T_L, T_R) with the following algorithm:

Algorithm: Insertion((T_L, T_R), a)

1. Make a leaf insertion of a into the binary tree T_L.
2. Make a root insertion of a into the binary tree T_R.

- The $\mathbb{P}$-symbol of a word $u \in A^*$ is the A-labeled pair of twin binary trees obtained by iteratively inserting, from left to right, the letters of u into the empty pair of twin binary trees $(\perp, \perp)$.
- The $\mathbb{P}$-symbol algorithm allows to decide if two words are $\equiv_B$-equivalent:

Proposition

Let $u, v \in A^*$. Then, $u \equiv_B v$ iff $\mathbb{P}(u) = \mathbb{P}(v)$.

- Example:

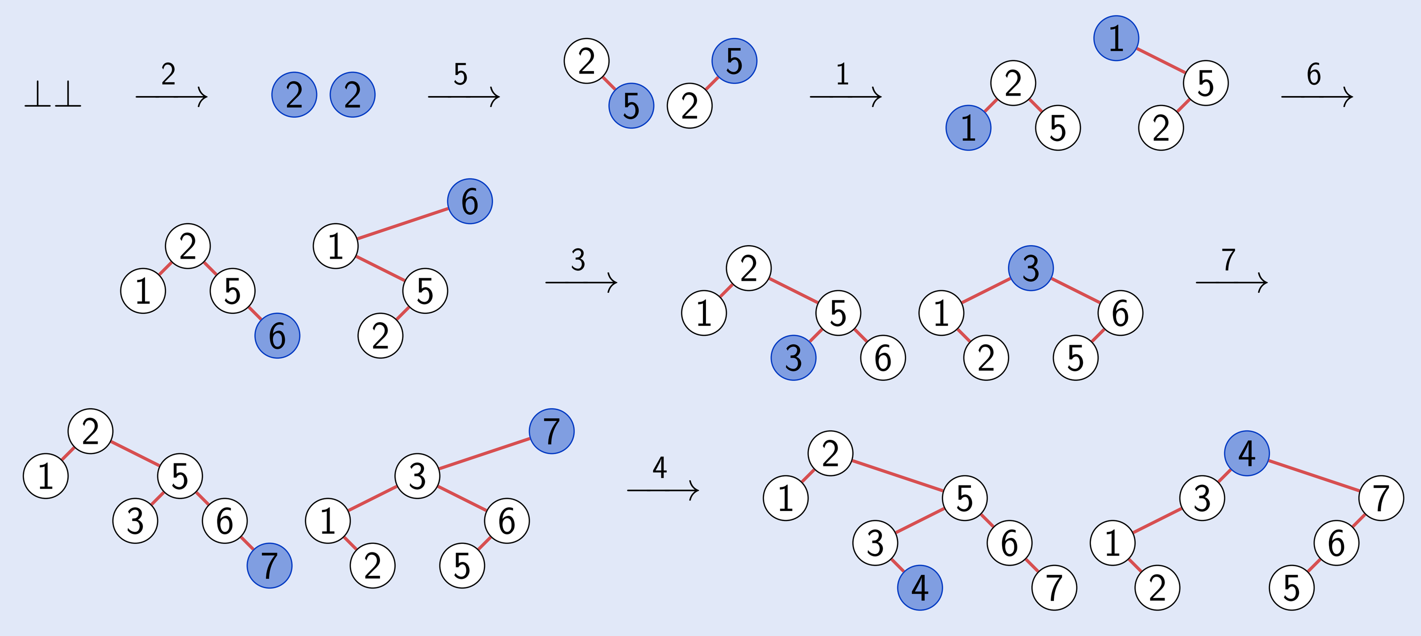


Figure: Steps of the computation of the $\mathbb{P}$-symbol of the word 2516374.

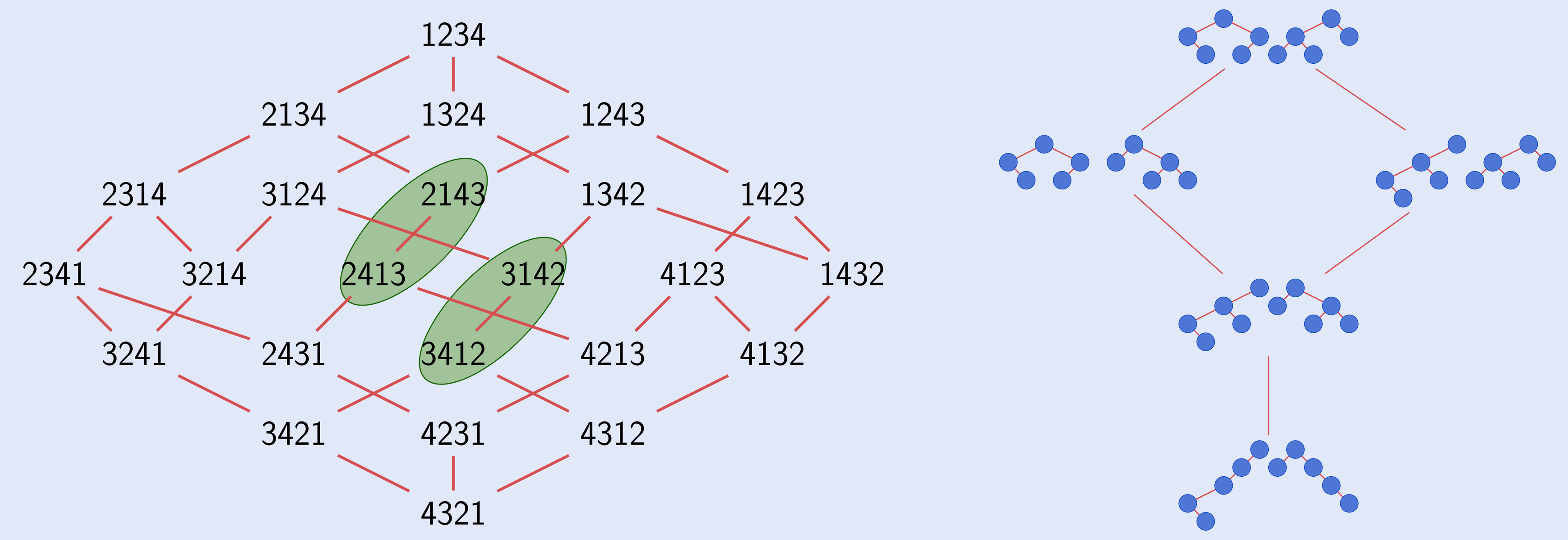
5. The Baxter lattice

- The quotient of the permutohedron by the Baxter equivalence relation defines a lattice over the set of pairs of twin binary trees of a given size.

Proposition

The Baxter equivalence relation is a lattice congruence of the permutohedron, *i.e.*, every $\equiv_B$-equivalence class of permutations is an interval of the permutohedron, and for all permutations σ, ν such that $\sigma \leq_P \nu$, the minimal (resp. maximal) elements σ' and ν' of the $\equiv_B$-equivalence classes of σ and ν satisfy $\sigma' \leq_P \nu'$.

- Covering relations are similar to those of the Tamari lattice and can be described using left and right binary tree rotations: We have $(T_L, T_R) \leq_B (T'_L, T'_R)$ if T'_L (resp. T'_R) can be obtained from T_L (resp. T_R) by performing left (resp. right) binary tree rotations.



(a) The permutohedron of order 4 and the two non-singleton $\equiv_B$-equivalence classes. (b) An interval of the lattice of pairs of twin binary trees of order 5.

Figure: Some pictures about the Baxter lattice.

- Baxter permutations and $\equiv_B$-equivalence classes of permutations are equinumerous:

Theorem

For all $n \geq 0$, each equivalence class of $\mathfrak{S}_n/_{\equiv_B}$ contains exactly one Baxter permutation.

6. The Hopf algebra Baxter

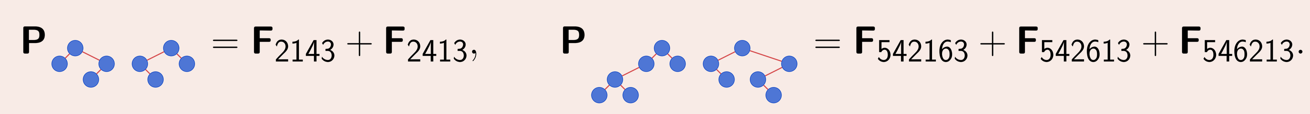
- The family $\{\mathbf{F}_\sigma\}_{\sigma \in \mathfrak{S}}$ forms the fundamental basis of **FQSym**, the combinatorial Hopf algebra of permutations. Its product and its coproduct are defined as follows:

$$\mathbf{F}_\sigma \cdot \mathbf{F}_\nu := \sum_{\pi \in \sigma \,\overline{\sqcup\!\sqcup}\, \nu} \mathbf{F}_\pi, \qquad \Delta(\mathbf{F}_\sigma) := \sum_{u.v = \sigma} \mathbf{F}_{\mathrm{std}(u)} \otimes \mathbf{F}_{\mathrm{std}(v)}.$$

- Let us define the following elements of **FQSym**, indexed by pairs of twin binary trees:

$$\mathbf{P}_J := \sum_{\mathbb{P}(\sigma) = J} \mathbf{F}_\sigma.$$

- Examples:



- Since $\equiv_B$ is a congruence compatible with the destandardization process and also with the restriction of alphabet intervals, we have the following theorem:

Theorem

The family $\{\mathbf{P}_J\}_{J \in \mathcal{TBT}}$ spans a Hopf subalgebra of **FQSym**, namely the Hopf algebra **Baxter**.

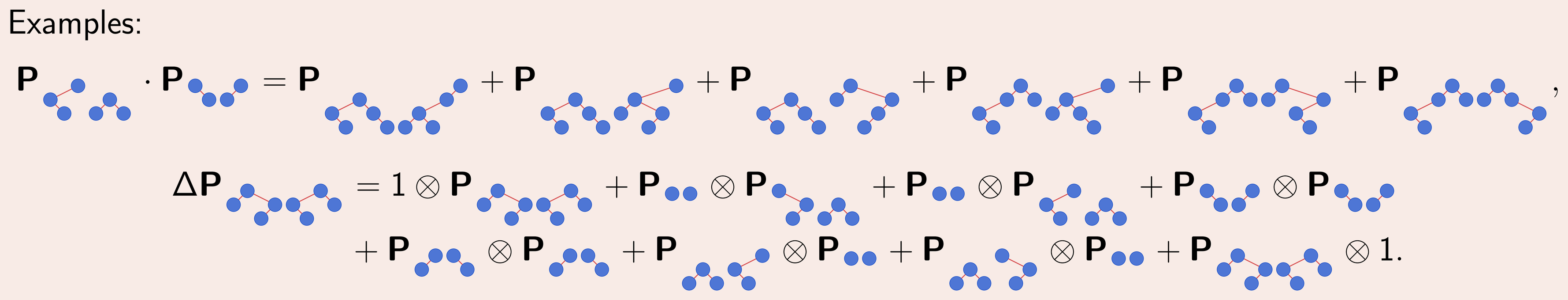
- The product of **Baxter** is deduced from the product of **FQSym** and is expressed as follows:

$$\mathbf{P}_{J_0} \cdot \mathbf{P}_{J_1} = \sum_{\substack{\mathbb{P}(\sigma) = J_0, \mathbb{P}(\nu) = J_1 \\ \pi \in (\sigma \,\overline{\sqcup\!\sqcup}\, \nu) \cap \mathfrak{S}^B}} \mathbf{P}_{\mathbb{P}(\pi)}.$$

- In the same way, the coproduct of **Baxter** is expressed as follows:

$$\Delta(\mathbf{P}_J) = \sum_{\substack{\mathbb{P}(u.v) = J \\ \sigma := \mathrm{std}(u), \nu := \mathrm{std}(v) \in \mathfrak{S}^B}} \mathbf{P}_{\mathbb{P}(\sigma)} \otimes \mathbf{P}_{\mathbb{P}(\nu)}.$$

- Examples:



- We can use the above order relation over pairs of twin binary trees to build multiplicative bases of **Baxter**:

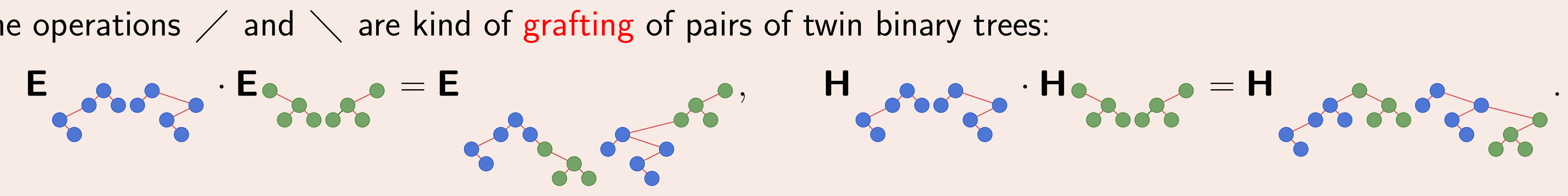
$$\mathbf{E}_J := \sum_{J \leq_B J'} \mathbf{P}_{J'} \quad \text{and} \quad \mathbf{H}_J := \sum_{J' \leq_B J} \mathbf{P}_{J'}.$$

Proposition

The bases $\{\mathbf{E}_J\}_{J \in \mathcal{TBT}}$ and $\{\mathbf{H}_J\}_{J \in \mathcal{TBT}}$ are multiplicative. Indeed, for all $J_0, J_1 \in \mathcal{TBT}$, we have

$$\mathbf{E}_{J_0} \cdot \mathbf{E}_{J_1} = \mathbf{E}_{J_0 \diagup J_1} \quad \text{and} \quad \mathbf{H}_{J_0} \cdot \mathbf{H}_{J_1} = \mathbf{H}_{J_0 \diagdown J_1}.$$

- The operations $\diagup$ and $\diagdown$ are kind of grafting of pairs of twin binary trees:



- Main consequence is

Proposition

The Hopf algebra **Baxter** is free on the elements $\mathbf{E}_J$ where J is a pair of twin binary trees such that all permutations σ satisfying $\mathbb{P}(\sigma) = J$ are connected.

- First dimensions of algebraic generators of **Baxter** are 1, 1, 1, 3, 11, 47, 221, 1113, 5903, 32607, 186143.

References

Samuele Giraudo, *Algebraic and combinatorial structures on Baxter permutations*, FPSAC, 2011.