Unshuffling permutations: Trivial bijections and compositions*

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Abstract. Given permutations π, σ_1 and σ_2, the permutation π (viewed as a string) is said to be a *shuffle* of σ_1 and σ_2, in symbols $\pi \in \sigma_1 \sqcup\!\sqcup \sigma_2$, if π can be formed by interleaving the letters of two strings p_1 and p_2 that are order-isomorphic to σ_1 and σ_2, respectively. Given a permutation $\pi \in S_{2n}$ and a bijective mapping $f : S_n \to S_n$, the f-Unshuffle-Permutation problem is to decide whether there exists a permutation $\sigma \in S_n$ such that $\pi \in \sigma \sqcup\!\sqcup f(\sigma)$. We consider here this problem for the following bijective mappings: inversion, reverse, complementation, and all their possible compositions. In particular, we present combinatorial results about the permutations accepted by this problem. As main results, we obtain that this problem is NP-complete when f is the reverse, the complementation, or the composition of the reverse with the complementation.

Keywords: Permutation · Shuffle product · Computational complexity.

1 Introduction

Given permutations π, σ_1 and σ_2, π is said to be a *shuffle* of σ_1 and σ_2, in symbols $\pi \in \sigma_1 \sqcup\!\sqcup \sigma_2$, if π is the disjoint union of two patterns p_1 and p_2 (*i.e.,* if, viewed as a string, it can be formed by interleaving the letters of p_1 and p_2) that are order-isomorphic to σ_1 and σ_2, respectively. This shuffling operation $\sqcup\!\sqcup$ was introduced by Vargas [12] in an algebraic context and was called *supershuffle*. In case $\sigma = \sigma_1 = \sigma_2$, the permutation π is said to be a *square w.r.t. the shuffle product* (or simply a *square* when clear from the context), and σ is said to be a *square root* of π *w.r.t. the shuffle product* (or simply a *square root* of π when clear from the context). Note that a permutation may have several square roots. For example, $\pi = 18346752 \in 1234 \sqcup\!\sqcup 4321$ since π is a shuffle of the patterns $p_1 = 1347$ and $p_2 = 8652$ that are order-isomorphic to $\sigma_1 = 1234$ and

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$\sigma_2 = 4321$, respectively (as shown in ${}^{1}{}_{8}{}^{34}{}_{6}{}^{7}{}_{52}$). However, π is not a square. Besides, $\pi' = 24317856$ is a square as it is a shuffle of the patterns 2175 and 4386 that are both order-isomorphic to 2143 (as shown in ${}^{2}{}_{43}{}^{17}{}_{8}{}^{5}{}_{6}$). Note that 2143 is not the unique square root of π' since π' is also a shuffle of patterns 2156 and 4378 that are both order-isomorphic to 2134 (as shown in ${}^{2}{}_{43}{}^{1}{}_{78}{}^{56}$).

The above definitions are of course intended to be natural counterparts to the ordinary shuffle of words and languages. Given words u, v_1 and v_2, u is said to be a *shuffle* of v_1 and v_2 (denoted $u \in v_1 \sqcup\!\sqcup\, v_2$), if u can be formed by interleaving the letters of v_1 and v_2 in a way that maintains the left-to-right ordering of the letter from each word. For example, $u = abccbabacbb \in abcab \sqcup\!\sqcup\, cbabcb$ (as shown in ${}^{ab}{}_{c}{}^{c}{}_{bab}{}^{a}{}_{c}{}^{b}{}_{b}$). Similarly, in case $v = v_1 = v_2$, the word u is said to be a *square w.r.t. the shuffle product*, and v is said to be a *square root of u w.r.t. the shuffle product*. For example, $abaaabaabb$ is a square w.r.t. the shuffle product since it belongs to the shuffle of $abaab$ with itself (as shown in ${}^{ab}{}_{a}{}^{aa}{}_{baa}{}^{b}{}_{b}$). Given words u, v_1 and v_2, deciding whether $u \in v_1 \sqcup\!\sqcup\, v_2$ is $O(|u|^2/\log(|u|))$ time solvable [6]. To the best of our knowledge, the first $O(|u|^2)$ time algorithm for this problem appeared in [7]. However, deciding whether a given word u is a square w.r.t. the shuffle product is NP-complete [9, 2]. Finally, for a given word u, deciding whether there exists a word v such that u is in the shuffle of v with its reverse v^{R} (*i.e.,* $u \in v \sqcup\!\sqcup\, v^{\mathrm{R}}$) is NP-complete as well [9]. However, the problem is polynomial-time solvable for words built from a binary alphabet.

Coming back to permutations, by using a pattern avoidance criterion on directed perfect matchings, it is proved in [4] that recognizing permutations that are squares is NP-complete, and a bijection between $(213, 231)$-avoiding square permutations and square binary words is presented. Given $(3142, 2413)$-avoiding (*a.k.a. separable*) permutations π, σ_1 and σ_2, it can be decided in polynomial time whether $\pi \in \sigma_1 \sqcup\!\sqcup\, \sigma_2$ [8].

We finally observe that deciding whether a permutation is a shuffle of two monotone permutations is in P:

(i) *merge permutations* are the union of two increasing subsequences and are characterized by the fact that they contain no decreasing subsequence of length 3 [5], whereas (ii) *skew-merged permutations* are the union of an increasing subsequence with a decreasing subsequence and are characterized by the fact that they contain no subsequence ordered in the same way as 2143 or 3412 [11].

In this paper, we are interested in a generalization of the problem consisting in recognizing square permutations. We call this new problem the f-Unshuffle-Permutation problem, which is defined as follows. Given a permutation $\pi \in S_{2n}$ and a bijective mapping $f : S_n \to S_n$, the f-Unshuffle-Permutation problem asks whether there exists a permutation $\sigma \in S_n$ such that $\pi \in \sigma \sqcup\!\sqcup\, f(\sigma)$. We say in this case that π is a *generalized square permutation*. In case $f = \mathrm{id}$, the id-Unshuffle-Permutation problem reduces to recognizing square permutations, and hence is NP-complete. This paper is devoted to studying the f-Unshuffle-Permutation problem in case f is either a trivial bijection (identity, complement, reverse or inverse) or obtained by composing trivial bijections. These bijections act on permutations by performing a transformation on their permutation matrices and can hence be seen as elements of the Dihedral group D_4, as discussed in the next section. The paper is organized as follows. In Section 2, we provide the needed definitions. Section 3 is devoted to presenting generalized square permutations and some associated enumerative properties. Finally, in Section 4, we show hardness of recognizing some generalized square permutations.

2 Definitions

For any nonnegative integer n, $[n]$ is the set $\{1,\ldots,n\}$. We follow the usual terminology on words [3]. Let us recall here the most important ones. Let u be a word. The length of u is denoted by $|u|$. The *empty word*, the only word of null length, is denoted by ϵ. For any $i \in [|u|]$, the i-th letter of u is denoted by $u(i)$. If I is a subset of $[|u|]$, $u_{|I}$ is the subword of u consisting in the letters of u at the positions specified by the elements of I. A *permutation* of size n is a word π of length n on the alphabet $[n]$ such that each letter admits exactly one occurrence. The set of all permutations of size n is denoted by S_n.

Definition 1 (Reduced form). *If λ is a list of distinct integers, the* reduced form *of λ, denoted* $\mathrm{reduce}(\lambda)$*, is the permutation obtained from λ by replacing its i-th smallest entry with i. For instance,* $\mathrm{reduce}(31845) = 21534$*.*

Definition 2 (Order-isomorphism). *Let λ_1 and λ_2 be two words of distinct integers such that* $\mathrm{reduce}(\lambda_1) = \mathrm{reduce}(\lambda_2)$*. We say that λ_1 and λ_2 are* order-isomorphic *and we denote it by $\lambda_1 \simeq \lambda_2$.*

Definition 3 (Pattern containment). *A permutation σ is said to be* contained *in, or to be a* subpermutation *of, another permutation π, written $\sigma \preceq \pi$, if π has a (not necessarily contiguous) subsequence whose terms are order-isomorphic to σ. We also say that π* admits an occurrence *of the* pattern *σ.*

Thus, $\sigma \preceq \pi$ if there is a set I of positions in π such that $\mathrm{reduce}(\pi_{|I}) = \sigma$. For example, $1423 \preceq 149362785$ since $\mathrm{reduce}\left(149362785_{|\{2,3,5,8\}}\right) = \mathrm{reduce}\,(4968) = 1423$.

Definition 4 (Trivial bijections). *Let $\pi = \pi(1)\pi(2)\ldots\pi(n)$ be a permutation of size n. The* reverse *of π is the permutation $\mathrm{r}(\pi) = \pi(n)\pi(n-1)\ldots\pi(1)$. The* complement *of π is the permutation $\mathrm{c}(\pi) = \pi'(1)\pi'(2)\cdots\pi'(n)$, where $\pi'(i) = n - \pi(i) + 1$. The* inverse *is the regular group theoretical inverse on permutations, that is the $\pi(i)$-th position of the inverse $\mathrm{i}(\pi)$ is occupied by i. The reverse, complement and inverse are called the* trivial bijections *from S_n to itself.*

Abusing notations, for any $S \subseteq S_n$, we let $\mathrm{c}(S)$, $\mathrm{i}(S)$ and $\mathrm{r}(S)$ stand respectively for $\{\mathrm{c}(\pi) : \pi \in S\}$, $\{\mathrm{i}(\pi) : \pi \in S\}$ and $\{\mathrm{r}(\pi) : \pi \in S\}$.

Definition 5 (Shuffle). *Let $\sigma_1 \in S_{k_1}$ and $\sigma_2 \in S_{k_2}$ be two permutations. A permutation $\pi = \pi(1)\pi(2)\ldots\pi(k_1+k_2)$ is a* shuffle *of σ_1 and σ_2 if there is a subset I of $[n]$ such that $\pi_{|I} \simeq \sigma_1$ and $\pi_{|\bar{I}} \simeq \sigma_2$, where $\bar{I} = [n] \setminus I$ and $n = k_1 + k_2$.*

In other terms, π is obtained by interleaving the letters of two words respectively order-isomorphic with σ_1 and σ_2. We denote by $\sigma_1 \sqcup\!\sqcup \sigma_2$ the set of all shuffles of σ_1 and σ_2.

For example,

$$\begin{aligned} 12 \sqcup\!\sqcup 21 = \{&{}^{12}43, {}^{1}32^{4}, {}^{13}42, {}^{1}4^{2}3, {}^{14}32, 21^{34}, 2^{3}1^{4}, 2^{34}1, {}^{2}41^{3}, 2^{43}1, \\ &3^{1}2^{4}, 3^{14}2, {}^{3}21^{4}, {}^{3}2^{4}1, {}^{34}21, 4^{12}3, 4^{13}2, 42^{13}, 4^{23}1, 43^{12}\} \,. \end{aligned}$$

We are now ready to define the f-Unshuffle-Permutation problem we focus on in this paper.

Definition 6 (f-Unshuffle-Permutation). *Given a permutation $\pi \in S_{2n}$ and a bijective mapping $f : S_n \to S_n$, the f-Unshuffle-Permutation problem is to decide whether there exists a permutation $\sigma \in S_n$ such that $\pi \in \sigma \sqcup\!\sqcup f(\sigma)$.*

3 Generalized square permutations and enumerative properties

The bijections c, i, and r are involutions and can be seen as particular elements of the dihedral group D_4 encoding all the symmetries of the square. Before explaining why, let us recall a definition for D_4.

The group D_4 is generated by two elements `a` and `b` subjected exactly to the nontrivial relations

$$\mathtt{aa} = \epsilon, \quad \mathtt{bbbb} = \epsilon, \quad \mathtt{abab} = \epsilon, \tag{1}$$

where ϵ is the unit of D_4. One can think of element `a` (resp. `b`) acting on a square by performing a symmetry through the vertical axis (resp. a 90°clockwise rotation). Now, we can regard, for any $n \in \mathbb{N}$, each element of D_4 as a map $\phi : S_n \to S_n$ such that $\phi(\pi)$, $\pi \in S_n$, is the permutation obtained by performing on its permutation matrix the transformations specified by ϕ. The maps c, i, and r can thus be expressed as

$$\mathrm{c} = \mathtt{bba}, \quad \mathrm{i} = \mathtt{ba}, \quad \mathrm{r} = \mathtt{a}. \tag{2}$$

Figure 1 shows the Cayley graph of this group. The main interest of seeing the three trivial bijections on S_n as elements of D_4 lies in the fact that all the other bijections obtained by composing the trivial ones can be expressed by compositions of the two elements `a` and `b`. This will alow us to gain concision in some proofs presented in the sequel.

We now turn to defining *identity squares*, *complement squares*, *reverse squares* and *inverse squares* that are natural generalizations of square permutations to trivial bijections from S_n to itself.

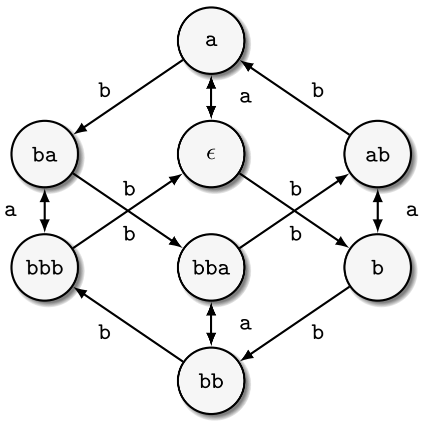


Fig. 1. The Cayley graph of the group D_4.

Definition 7 (Square permutations for trivial bijections). *For $n \in \mathbb{N}$, define*

$$\begin{aligned}
\mathcal{S}^{\epsilon}_{2n} &= \{\pi \in S_{2n} : \exists\, \sigma \in S_n, \textit{ such that } \pi \in \sigma \sqcup\!\sqcup \sigma\},\\
\mathcal{S}^{\mathrm{c}}_{2n} &= \{\pi \in S_{2n} : \exists\, \sigma \in S_n \textit{ such that } \pi \in \sigma \sqcup\!\sqcup \mathrm{c}(\sigma)\},\\
\mathcal{S}^{\mathrm{r}}_{2n} &= \{\pi \in S_{2n} : \exists\, \sigma \in S_n \textit{ such that } \pi \in \sigma \sqcup\!\sqcup \mathrm{r}(\sigma)\},\\
\mathcal{S}^{\mathrm{i}}_{2n} &= \{\pi \in S_{2n} : \exists\, \sigma \in S_n \textit{ such that } \pi \in \sigma \sqcup\!\sqcup \mathrm{i}(\sigma)\}.
\end{aligned}$$

We begin by proving that applying any trivial bijection is in some sense compatible with the shuffle operator.

Lemma 1 (Trivial bijections and shuffle). *Let π, σ_1 and σ_2 be permutations such that $\pi \in \sigma_1 \sqcup\!\sqcup \sigma_2$. Then*

- $\mathrm{c}(\pi) \in \mathrm{c}(\sigma_1) \sqcup\!\sqcup \mathrm{c}(\sigma_2)$*;*
- $\mathrm{r}(\pi) \in \mathrm{r}(\sigma_1) \sqcup\!\sqcup \mathrm{r}(\sigma_2)$*;*
- $\mathrm{i}(\pi) \in \mathrm{i}(\sigma_1) \sqcup\!\sqcup \mathrm{i}(\sigma_2)$.

Proof. Let n be the size of π. Since π is a shuffle of σ_1 and σ_2, there is a subset $I = \{i_1 < \cdots < i_k\}$ of $[n]$ such that $\pi_{|I} \simeq \sigma_1$ and $\pi_{|\bar{I}} \simeq \sigma_2$.

By setting $J = \{n - i_k + 1 < \cdots < n - i_1 + 1\}$, we have

$$\mathrm{r}(\pi)_{|J} = (\pi(n) \ldots \pi(1))_{|J} = \pi(i_k) \ldots \pi(i_1) \simeq \mathrm{r}(\sigma_1). \tag{3}$$

For the same reason, we have also $\mathrm{r}(\pi)_{|\bar{J}} \simeq \mathrm{r}(\sigma_2)$. Hence, $\mathrm{r}(\pi)$ is a shuffle of $\mathrm{r}(\sigma_1)$ and $\mathrm{r}(\sigma_2)$.

Now, set $J = \{\pi(i) : i \in I\} = \{\pi(\ell_1) < \cdots < \pi(\ell_k)\}$ where $\ell_1, \ldots, \ell_k$ are elements of I. We have

$$\mathrm{i}(\pi)_{|J} = \left(\pi^{-1}(1) \ldots \pi^{-1}(n)\right)_{|J} = \pi^{-1}(\pi(\ell_1)) \ldots \pi^{-1}(\pi(\ell_k)) = \ell_1 \ldots \ell_k \simeq \mathrm{i}(\sigma_1). \tag{4}$$

For the same reason, we have also $\mathrm{i}(\pi)_{|\bar{J}} \simeq \mathrm{i}(\sigma_2)$. Hence, $\mathrm{i}(\pi)$ is also shuffle of $\mathrm{i}(\sigma_1)$ and $\mathrm{i}(\sigma_2)$.

Finally, since $\mathtt{b} = \mathrm{i} \circ \mathrm{r}$ in D_4, the map c can be expressed as compositions involving r and i. Hence, $\mathrm{c}(\pi)$ is a shuffle of $\mathrm{c}(\pi_1)$ and $\mathrm{c}(\pi_2)$. ∎

The following lemma is well-known and is a consequence of interpreting c, r, and i as compositions involving $\mathtt{a}$ and $\mathtt{b}$ according to (2) and the relations (1) between $\mathtt{a}$ and $\mathtt{b}$.

Lemma 2 (Compositions of trivial bijections). *For any permutation* π*,*

- $(\mathrm{c} \circ \mathrm{r})(\pi) = (\mathrm{r} \circ \mathrm{c})(\pi)$*;*
- $(\mathrm{i} \circ \mathrm{c})(\pi) = (\mathrm{r} \circ \mathrm{i})(\pi)$*;*
- $(\mathrm{i} \circ \mathrm{r})(\pi) = (\mathrm{c} \circ \mathrm{i})(\pi)$*.*

According to Lemma 2 (and the fact that the group D_4 has exactly eight elements), we thus only need to consider the following compositions in the rest of the paper.

Definition 8 (Square permutations for compositions of trivial bijections). *For* $n \in \mathbb{N}$*, define*

$$\begin{aligned}
\mathcal{S}^{\mathrm{rc}}_{2n} &= \{\pi \in S_{2n} : \exists\, \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup\!\sqcup (\mathrm{r} \circ \mathrm{c})(\sigma)\},\\
\mathcal{S}^{\mathrm{ci}}_{2n} &= \{\pi \in S_{2n} : \exists\, \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup\!\sqcup (\mathrm{c} \circ \mathrm{i})(\sigma)\},\\
\mathcal{S}^{\mathrm{ic}}_{2n} &= \{\pi \in S_{2n} : \exists\, \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup\!\sqcup (\mathrm{i} \circ \mathrm{c})(\sigma)\},\\
\mathcal{S}^{\mathrm{irc}}_{2n} &= \{\pi \in S_{2n} : \exists\, \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup\!\sqcup (\mathrm{i} \circ \mathrm{r} \circ \mathrm{c})(\sigma)\}.
\end{aligned}$$

We hence obtain the eight sets $\mathcal{S}^{\epsilon}_{2n}$, $\mathcal{S}^{\mathrm{c}}_{2n}$, $\mathcal{S}^{\mathrm{r}}_{2n}$, $\mathcal{S}^{\mathrm{i}}_{2n}$, $\mathcal{S}^{\mathrm{rc}}_{2n}$, $\mathcal{S}^{\mathrm{ci}}_{2n}$, $\mathcal{S}^{\mathrm{ic}}_{2n}$, and $\mathcal{S}^{\mathrm{irc}}_{2n}$ of square permutations provided by Definitions 7 and 8. Let us now investigate some easy bijective and enumerative properties satisfied by these sets.

Proposition 1 (Bijection between $\mathcal{S}^{\mathrm{c}}_{2n}$ and $\mathcal{S}^{\mathrm{r}}_{2n}$). *For any* $n \in \mathbb{N}$*,* $\mathrm{i}\left(\mathcal{S}^{\mathrm{c}}_{2n}\right) = \mathcal{S}^{\mathrm{r}}_{2n}$*.*

Proof. Let $\pi \in S_{2n}$. We have

$$\begin{aligned}
\pi \in \mathcal{S}^{\mathrm{c}}_{2n} &\Leftrightarrow \exists \sigma \in S_n,\ \pi \in \sigma \sqcup\!\sqcup \mathrm{c}(\sigma)\\
&\Leftrightarrow \exists \sigma \in S_n,\ \mathrm{i}(\pi) \in \mathrm{i}(\sigma) \sqcup\!\sqcup (\mathrm{i} \circ \mathrm{c})(\sigma) && \text{(Lemma 1)}\\
&\Leftrightarrow \exists \sigma \in S_n,\ \mathrm{i}(\pi) \in \mathrm{i}(\sigma) \sqcup\!\sqcup (\mathrm{r} \circ \mathrm{i})(\sigma) && \text{(Lemma 2)}\\
&\Leftrightarrow \exists \sigma \in S_n,\ \mathrm{i}(\pi) \in \sigma \sqcup\!\sqcup \mathrm{r}(\sigma)\\
&\Leftrightarrow \mathrm{i}(\pi) \in \mathcal{S}^{\mathrm{r}}_{2n}.
\end{aligned}$$
∎

Proposition 2 (Bijection between $\mathcal{S}^{\mathrm{ic}}_{2n}$ and $\mathcal{S}^{\mathrm{ci}}_{2n}$). *For any* $n \in \mathbb{N}$*,* $\mathrm{c}\left(\mathcal{S}^{\mathrm{ic}}_{2n}\right) = \mathcal{S}^{\mathrm{ci}}_{2n}$*,* $\mathrm{i}\left(\mathcal{S}^{\mathrm{ic}}_{2n}\right) = \mathcal{S}^{\mathrm{ci}}_{2n}$*, and* $\mathrm{r}\left(\mathcal{S}^{\mathrm{ic}}_{2n}\right) = \mathcal{S}^{\mathrm{ci}}_{2n}$*.*

Proof. Let $\pi \in S_{2n}$. We have

$$\begin{aligned}
\pi \in \mathcal{S}^{\mathrm{ic}}_{2n} &\Leftrightarrow \exists \sigma \in S_n,\ \pi \in \sigma \sqcup (\mathrm{i} \circ \mathrm{c})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n,\ \mathrm{c}(\pi) \in \mathrm{c}(\sigma) \sqcup (\mathrm{c} \circ \mathrm{i} \circ \mathrm{c})(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n,\ \mathrm{c}(\pi) \in \sigma \sqcup (\mathrm{c} \circ \mathrm{i})(\sigma) \\
&\Leftrightarrow \mathrm{c}(\pi) \in \mathcal{S}^{\mathrm{ci}}_{2n}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\pi \in \mathcal{S}^{\mathrm{ic}}_{2n} &\Leftrightarrow \exists \sigma \in S_n,\ \pi \in \sigma \sqcup (\mathrm{i} \circ \mathrm{c})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n,\ \mathrm{i}(\pi) \in \mathrm{i}(\sigma) \sqcup (\mathrm{i} \circ \mathrm{i} \circ \mathrm{c})(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n,\ \mathrm{i}(\pi) \in \mathrm{i}(\sigma) \sqcup (\mathrm{i} \circ \mathrm{r} \circ \mathrm{i})(\sigma) && \text{(Lemma 2)} \\
&\Leftrightarrow \exists \sigma \in S_n,\ \mathrm{i}(\pi) \in \sigma \sqcup (\mathrm{i} \circ \mathrm{r})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n,\ \mathrm{i}(\pi) \in \sigma \sqcup (\mathrm{c} \circ \mathrm{i})(\sigma) && \text{(Lemma 2)} \\
&\Leftrightarrow \mathrm{i}(\pi) \in \mathcal{S}^{\mathrm{ci}}_{2n}.
\end{aligned}$$

Finally, the fact that $\mathrm{i} \circ \mathrm{c} \circ \mathrm{i} = \mathrm{r}$ implies that r is also a bijection between $\mathcal{S}^{\mathrm{ic}}_{2n}$ and $\mathcal{S}^{\mathrm{ci}}_{2n}$. ∎

Proposition 3 (Bijection between $\mathcal{S}^{\mathrm{i}}_{2n}$ and $\mathcal{S}^{\mathrm{irc}}_{2n}$). *For any $n \in \mathbb{N}$, $(\mathrm{i} \circ \mathrm{c})\left(\mathcal{S}^{\mathrm{irc}}_{2n}\right) = \mathcal{S}^{\mathrm{i}}_{2n}$.*

Proof. Let $\pi \in S_{2n}$. We have

$$\begin{aligned}
\pi \in \mathcal{S}^{\mathrm{irc}}_{2n} &\Leftrightarrow \exists \sigma \in S_n,\ \pi \in \sigma \sqcup (\mathrm{i} \circ \mathrm{r} \circ \mathrm{c})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n,\ \mathrm{i}(\pi) \in \mathrm{i}(\sigma) \sqcup (\mathrm{r} \circ \mathrm{c})(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n,\ (\mathrm{r} \circ \mathrm{i})(\pi) \in (\mathrm{r} \circ \mathrm{i})(\sigma) \sqcup \mathrm{c}(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n,\ (\mathrm{i} \circ \mathrm{c})(\pi) \in (\mathrm{i} \circ \mathrm{c})(\sigma) \sqcup \mathrm{c}(\sigma) && \text{(Lemma 2)} \\
&\Leftrightarrow \exists \sigma \in S_n,\ (\mathrm{i} \circ \mathrm{c})(\pi) \in \mathrm{i}(\sigma) \sqcup \sigma \\
&\Leftrightarrow (\mathrm{i} \circ \mathrm{c})(\pi) \in \mathcal{S}^{\mathrm{i}}_{2n}.
\end{aligned}$$

∎

To simplify notations, we write s^{ϵ}_{2n}, s^{c}_{2n}, s^{r}_{2n}, s^{i}_{2n}, s^{rc}_{2n}, s^{ci}_{2n}, s^{ic}_{2n}, and s^{irc}_{2n} for the cardinalities of the sets $\mathcal{S}^{\epsilon}_{2n}$, $\mathcal{S}^{\mathrm{c}}_{2n}$, $\mathcal{S}^{\mathrm{r}}_{2n}$, $\mathcal{S}^{\mathrm{i}}_{2n}$, $\mathcal{S}^{\mathrm{rc}}_{2n}$, $\mathcal{S}^{\mathrm{ci}}_{2n}$, $\mathcal{S}^{\mathrm{ic}}_{2n}$, and $\mathcal{S}^{\mathrm{irc}}_{2n}$, respectively. Table 1 shows the first cardinalities. We note that, at this time, none of the corresponding integer sequence does appear in OEIS [10].

Lemma 3 (Upper bound for the number of generalized squares). *For any $n \in \mathbb{N}$,*

$$|Q_{2n}| \leq \binom{2n-1}{n-1}\binom{2n}{n} n!, \tag{5}$$

where Q_{2n} is any of the sets $\mathcal{S}^{\epsilon}_{2n}$, $\mathcal{S}^{\mathrm{c}}_{2n}$, $\mathcal{S}^{\mathrm{r}}_{2n}$, $\mathcal{S}^{\mathrm{i}}_{2n}$, $\mathcal{S}^{\mathrm{rc}}_{2n}$, $\mathcal{S}^{\mathrm{ci}}_{2n}$, $\mathcal{S}^{\mathrm{ic}}_{2n}$, or $\mathcal{S}^{\mathrm{irc}}_{2n}$.

n	0	1	2	3	4	5
s^{rc}_{2n}	1	2	20	472	18 988	1 112 688
$s^{\mathrm{c}}_{2n} = s^{\mathrm{r}}_{2n}$	1	2	20	480	19 744	1 185 264
$s^{\mathrm{i}}_{2n} = s^{\mathrm{irc}}_{2n}$	1	2	20	488	20 250	1 229 858
s^{ϵ}_{2n}	1	2	20	504	21 032	1 293 418
$s^{\mathrm{ci}}_{2n} = s^{\mathrm{ic}}_{2n}$	1	2	20	586	27 990	2 044 596
$(2n)!$	1	2	24	720	40 320	3 628 800

Table 1. Cardinalities of the sets of square permutations of sizes $0 \leq 2n \leq 10$.

Proof. This is a consequence of the fact that to construct a permutation π of Q_{2n}, we can proceed by first choosing a set of n letters among the alphabet $[2n]$ ($\binom{2n}{n}$ choices), then by specifying an order from left to right for these letters ($n!$ choices), and finally by deploying the chosen letters onto a set I of n positions among the set of all possible positions $[2n]$ ($\binom{2n}{n}$ choices). The empty positions are filled by the unique authorized completion such that $\pi_{|I} \simeq \phi\left(\pi_{|\bar{I}}\right)$ where ϕ is the considered bijection on S_n. For this reason, there is no more than $\binom{2n}{n}^2 n!$ elements in Q_{2n}. The tighter bound (5) is the consequence of the fact that we can assume that the first position can always be fixed. ∎

Let us denote by $\mathcal{S}^\bullet{}_{2n}$ the union of all the sets $\mathcal{S}^{\epsilon}_{2n}$, $\mathcal{S}^{\mathrm{c}}_{2n}$, $\mathcal{S}^{\mathrm{r}}_{2n}$, $\mathcal{S}^{\mathrm{i}}_{2n}$, $\mathcal{S}^{\mathrm{rc}}_{2n}$, $\mathcal{S}^{\mathrm{ci}}_{2n}$, $\mathcal{S}^{\mathrm{ic}}_{2n}$, and $\mathcal{S}^{\mathrm{irc}}_{2n}$. We show that S_{2n} and $\mathcal{S}^\bullet{}_{2n}$ do not coincide.

Proposition 4 (Existence of non-square permutations). *For any* $n \in \mathbb{N}, n \geq 9, S_{2n} \neq \mathcal{S}^\bullet{}_{2n}$.

Proof. By Lemma 3 (and using its notations), we have

$$|\,\mathcal{S}^\bullet{}_{2n}\,| \leq 8\,|Q_{2n}| \leq 8\binom{2n-1}{n-1}\binom{2n}{n}n!. \tag{6}$$

The proportion of the elements of $\mathcal{S}^\bullet{}_{2n}$ among all the permutations of S_{2n} admits as upper bound, when $n \geq 1$,

$$\frac{|\,\mathcal{S}^\bullet{}_{2n}\,|}{|S_{2n}|} \leq \frac{8\binom{2n-1}{n-1}\binom{2n}{n}n!}{(2n)!} = 4\binom{2n}{n}\frac{1}{n!}. \tag{7}$$

But $4\binom{2n}{n}\frac{1}{n!} < 1$ for $n \geq 9$, thereby proving the result. ∎

Proposition 5 (Square permutations in some intersections). *Let* $n \in \mathbb{N}$ *such that* $2n \equiv 0 \pmod 4$. *Then,*

the permutation $\pi = 12\ldots(n-1)(n)\ \ (2n)(2n-1)\ldots(n+2)(n+1)$
belongs to $\mathcal{S}^{\epsilon}_{2n} \cap \mathcal{S}^{\mathrm{r}}_{2n} \cap \mathcal{S}^{\mathrm{c}}_{2n} \cap \mathcal{S}^{\mathrm{i}}_{2n} \cap \mathcal{S}^{\mathrm{ci}}_{2n} \cap \mathcal{S}^{\mathrm{ic}}_{2n}$.

Proof. Let $I = \{1, 3, 5, \ldots, 2n-1\}$ and $\bar{I} = [2n] \setminus I$. Then, since

$$\pi_{|I} \simeq 12\ldots\frac{n}{2}\ \ n(n-1)\ldots\left(\frac{n}{2}+1\right) \simeq \pi_{|\bar{I}}, \tag{8}$$

we have $\pi_{|I} \simeq \pi_{|\bar{I}} \simeq \mathrm{i}(\pi_{|\bar{I}})$. This shows that π belongs to $\mathcal{S}^{\epsilon}_{2n} \cap \mathcal{S}^{\mathrm{i}}_{2n}$.

Now, let $J = \{1, 2, \ldots, n\}$ and $\bar{J} = [2n] \setminus J$. Then, since

$$\pi_{|J} \simeq 12 \ldots n \quad \text{and} \quad \pi_{|\bar{J}} \simeq n \ldots 21, \tag{9}$$

we have $\pi_{|J} \simeq \mathrm{r}(\pi_{|\bar{J}}) \simeq \mathrm{c}(\pi_{|\bar{J}}) \simeq (\mathrm{i} \circ \mathrm{c})(\pi_{|\bar{J}}) \simeq (\mathrm{c} \circ \mathrm{i})(\pi_{|\bar{J}})$. This shows that π belongs to $\mathcal{S}^{\mathrm{r}}_{2n} \cap \mathcal{S}^{\mathrm{c}}_{2n} \cap \mathcal{S}^{\mathrm{ic}}_{2n} \cap \mathcal{S}^{\mathrm{ci}}_{2n}$. ∎

4 Recognizing generalized square permutations

This section is devoted to proving that deciding membership to $\mathcal{S}^{\mathrm{r}}_{2n}$, $\mathcal{S}^{\mathrm{c}}_{2n}$ and $\mathcal{S}^{\mathrm{rc}}_{2n}$ is NP-complete. The approach relies on constraint directed matchings on permutations.

Definition 9 (Directed matching on permutations). *Let $\pi \in S_n$. A* directed matching *on π is a set $\mathcal{M}$ of pairwise disjoint arcs (i, j), $1 \leq i, j \leq n$ and $i \neq j$, that connect pairs of elements of π. The directed matching $\mathcal{M}$ is* perfect *if every element of π is the source or the sink of an arc of $\mathcal{M}$.*

Definition 10 (Two-arcs pattern occurrences). *A* two-arc pattern *is a perfect directed matching on the set* [4]*. A perfect directed matching $\mathcal{M}$ on a permutation π of size $2n$ admits an occurrence of a two-arc pattern $\mathcal{A}$ if there is an increasing map $\phi : [4] \to [2n]$ (*i.e.*, $i < i'$ implies $\phi(i) < \phi(i')$) such that, if (i, i') is an arc of $\mathcal{A}$, then $(\phi(i), \phi(i'))$ is an arc of $\mathcal{M}$. We say moreover that $\mathcal{M}$* avoids *$\mathcal{A}$ if $\mathcal{A}$ does not admits any occurrence of $\mathcal{A}$.*

We shall draw two-arcs patterns by unlabeled graphs wherein vertices are implicitly indexed from 1 to 4 from left to right. Figure 4 shows an example of a directed perfect matching on a permutation. The directed matching does contain the patterns , , , and and does avoid the patterns , , , , , and .

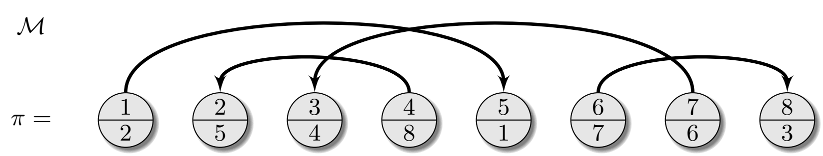


Fig. 2. A directed perfect matching $\mathcal{M} = \{(1, 5), (4, 2), (7, 3), (6, 8)\}$ on the permutation $\pi = 25481763$. Recall that arcs refer to positions in π.

In case π is written as a concatenation of contiguous patterns $\pi = \pi_1\ \pi_2\ \cdots\ \pi_k$, we write $\mathcal{M}_{\pi_i \to \pi_j}$ for the subset of arcs of $\mathcal{M}$ with source index in π_i and sink index in π_j. Hence,

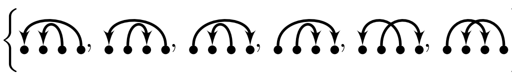
$$\mathcal{M} = \bigsqcup_{\substack{i \in [k] \\ j \in [k]}} \mathcal{M}_{\pi_i \to \pi_j}. \tag{10}$$

The following properties will prove useful for defining equivalences between squares and restricted directed perfect matchings.

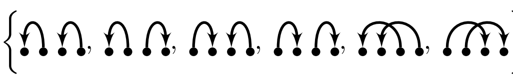
Definition 11 (Property $\mathbf{P_1}$ — Order isomorphism). *Let π be a permutation. A directed perfect matching $\mathcal{M}$ on π is said to have property $\mathbf{P_1}$ if, for any two distinct arcs (i, i') and (j, j') of $\mathcal{M}$, we have $\pi(i) < \pi(j)$ if and only if $\pi(i') < \pi(j')$.*

Definition 12 (Property $\mathbf{P_2}$ — Order anti-isomorphism). *Let π be a permutation. A directed perfect matching $\mathcal{M}$ on π is said to have property $\mathbf{P_2}$ if, for any two distinct arcs (i, i') and (j, j') of $\mathcal{M}$, we have $\pi(i) < \pi(j)$ if and only if $\pi(i') > \pi(j')$.*

Definition 13 (Property $\mathbf{Q_1}$ — First start first terminal). *Let π be a permutation. A directed perfect matching $\mathcal{M}$ on π is said to have property $\mathbf{Q_1}$ if it avoids the following set of two-arcs patterns:*

$\mathcal{Q}_1 = \{$  $\}$.

Definition 14 (Property $\mathbf{Q_2}$ — First start last terminal). *Let π be a permutation. A directed perfect matching $\mathcal{M}$ on π is said to have property $\mathbf{Q_2}$ if it avoids the following set of two-arcs patterns:*

$\mathcal{Q}_2 = \{$  $\}$.

Observe that $\mathcal{Q}_1 \sqcup \mathcal{Q}_2$ is the set of all two-arcs patterns.

Theorem 1 ([4]). *Let $\pi \in S_{2n}$. The two following statements are equivalent.*

1. *$\pi \in \mathcal{S}_{2n}^{\epsilon}$.*
2. *There exists a directed perfect matching $\mathcal{M}$ on π that satisfies properties $\mathbf{P_1}$ and $\mathbf{Q_1}$.*

The following two definitions will intervene in the next constructions.

Definition 15 (Lifting). *Let $\pi = \pi(1)\,\pi(2)\,\ldots\,\pi(n)$ be a permutation of size n and k be a positive integer. We call k-*lifting *of π, denoted $\pi[k]$, the permutation $(k+\pi(1))\,(k+\pi(2))\,\ldots\,(k+\pi(n))$ on the alphabet $\{k+1,\ldots,k+n\}$.*

Definition 16 (Monotone). *For any positive integer k, we let $\nearrow_k$ stand for the increasing permutation $1\,2\,\ldots\,k$ and $\searrow_k$ stand for the decreasing permutation $k\,(k-1)\,\ldots\,1$.*

Lemma 4 (One shot lemma). *Let $\pi \in S_{2n}$ and $\mathcal{M}$ be a perfect directed matching on π that satisfies properties $\mathbf{P}_i$ and $\mathbf{Q}_j$, $i \in \{1,2\}$ and $j \in \{1,2\}$. Then, the perfect directed matching $\mathcal{M}'$ obtained by reversing each arc of $\mathcal{M}$ satisfies properties $\mathbf{P}_i$ and $\mathbf{Q}_j$.*

Lemma 5. *Let* $I^1 = \{i^1_j : j \in [n]\}$*,* $i^1_1 < i^1_2 < \cdots < i^1_n$*, and* $I^2 = \{i^2_j : j \in [n]\}$*,* $i^2_1 < i^2_2 < \cdots < i^2_n$*, be such that* $I^1 \cap I^2 = \emptyset$*. The directed perfect matching* $\mathcal{M} = ([2n], E)$ *defined by* $E = \left\{\left(i^1_j, i^2_j\right) : j \in [n]\right\}$ *avoids the patterns* , , , , *and* .

Proof. Suppose, aiming at a contradiction, that $\mathcal{M}$ does contain the pattern , , , , , or , say for arcs $\left(i^1_j, i^2_j\right)$ and $\left(i^1_k, i^2_k\right)$. Wlog, assume $i^1_j < i^1_k$ (see Figure 3). This is a contradiction since $i^1_j < i^1_k$ implies $j < k$, and hence, $i^2_j < i^2_k$. ■

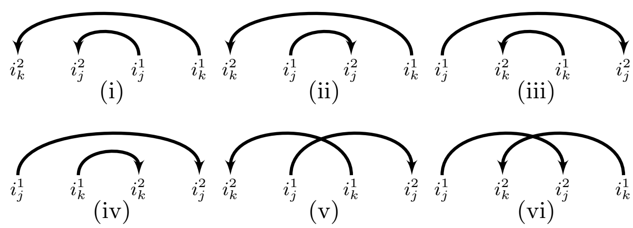


Fig. 3. Arcs (i^1_j, i^2_j) and (i^1_k, i^2_k) with $i^1_j < i^2_k$.

Lemma 6. *Let* $I^1 = \{i^1_j : j \in [n]\}$*,* $i^1_1 < i^1_2 < \cdots < i^1_n$*, and* $I^2 = \{i^2_j : j \in [n]\}$*,* $i^2_1 < i^2_2 < \cdots < i^2_n$*, be such that* $I^1 \cap I^2 = \emptyset$*. The directed perfect matching* $\mathcal{M} = ([2n], E)$ *defined by* $E = \left\{\left(i^1_j, i^2_{n-j+1}\right) : j \in [n]\right\}$ *avoids the patterns* , , , , , *and* .

Proof. Suppose, aiming at a contradiction, that $\mathcal{M}$ does contain the pattern , , , , , or , say for arcs $\left(i^1_j, i^2_{n-j+1}\right)$ and $\left(i^1_k, i^2_{n-k+1}\right)$. Wlog, assume $i^1_j < i^1_k$ (see Figure 4). This is a contradiction since $i^1_j < i^1_k$ implies $j < k$, and hence, $i^2_{n-j+1} > i^2_{n-k+1}$. ■

Proposition 6. *Let* $\pi \in S_{2n}$*. The two following statements are equivalent.*

1. $\pi \in \mathcal{S}^{\mathrm{r}}_{2n}$.
2. *There exists a directed perfect matching* $\mathcal{M}$ *on* π *that satisfies properties* $\mathbf{P_1}$ *and* $\mathbf{Q_2}$.

Proof. ($\mathbf{1} \Rightarrow \mathbf{2}$) Let $\pi \in \mathcal{S}^{\mathrm{r}}_{2n}$. Then, there exists $\sigma \in S_n$ such that $\pi \in \sigma \sqcup \mathrm{r}(\sigma)$, and hence there exist two disjoint set of indexes $I^1 = \{i^1_j : j \in [n]\}$, $i^1_1 < i^1_2 < \cdots < i^1_n$, and $I^2 = \{i^2_j : j \in [n]\}$, $i^2_1 < i^2_2 < \cdots < i^2_n$, such that

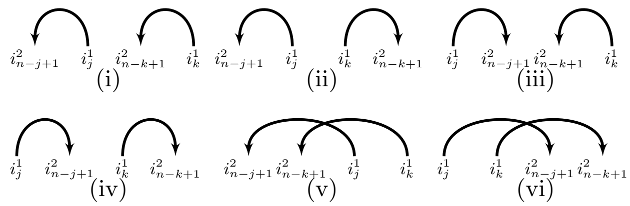


Fig. 4. Arcs (i^1_j, i^2_{n-j+1}) and (i^1_k, i^2_{n-k+1}) with $i^1_j < i^2_k$.

$\pi(i^1_1)\pi(i^1_2)\cdots\pi(i^1_n)$ is order-isomorphic to σ and $\pi(i^2_1)\pi(i^2_2)\cdots\pi(i^2_n)$ is order-isomorphic to $\mathrm{r}(\sigma)$. Let $\mathcal{M} = (V, E)$ be the directed graph defined by $V = [2n]$ and $E = \left\{\left(i^1_j, i^2_{n-j+1}\right) : j \in [n]\right\}$. Clearly, $\mathcal{M}$ is a directed perfect matching since $I^1 \cap I^2 = \emptyset$ and $I^1 \cup I^2 = [2n]$. According to Lemma 6, $\mathcal{M}$ does avoid the patterns , , , , , and . Finally, for any two distinct arcs $\left(i^1_j, i^2_{n-j+1}\right)$ and $\left(i^1_k, i^2_{n-k+1}\right)$ of $\mathcal{M}$, we have $\pi\left(i^1_j\right) < \pi\left(i^1_k\right)$ if and only if $\pi\left(i^2_{n-j+1}\right) < \pi\left(i^2_{n-k+1}\right)$ since $\pi(i^1_1)\pi(i^1_2)\cdots\pi(i^1_n)$ is order-isomorphic to σ and $\pi(i^2_1)\pi(i^2_2)\cdots\pi(i^2_n)$ is order-isomorphic to $\mathrm{r}(\sigma)$. Therefore, $\mathcal{M}$ satisfies properties $\mathbf{P_1}$ and $\mathbf{Q_2}$.

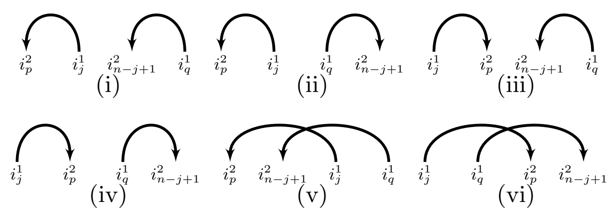


Fig. 5. All possible configurations considered in Proof of Proposition 6 ($\mathbf{2} \Rightarrow \mathbf{1}$).

($\mathbf{2} \Rightarrow \mathbf{1}$) Let $\mathcal{M}$ be a directed perfect matching that satisfies properties $\mathbf{P_1}$ and $\mathbf{Q_2}$. Let $I^1 = \{i^1_j : j \in [n]\}$, $i^1_1 < i^1_2 < \cdots < i^1_n$, be the set of the sources of the arcs of $\mathcal{M}$ and let $I^2 = \{i^2_j : j \in [n]\}$, $i^2_1 < i^2_2 < \cdots < i^2_n$, be the set of the sinks of the arcs of $\mathcal{M}$. We first show that, for every $j \in [n]$, $\left(i^1_j, i^2_{n-j+1}\right)$ is an arc of $\mathcal{M}$. The proof is by induction on $j = 1, 2, \ldots, n$.

- Base. Suppose, aiming at a contradiction that $\left(i^1_1, i^2_n\right)$ is not an arc of $\mathcal{M}$. Then, there exist $1 \leq p < n$ and $1 < q \leq n$ such that $\left(i^1_1, i^2_p\right)$ and $\left(i^1_q, i^2_n\right)$ are two arcs of $\mathcal{M}$. But $i^1_1 < i^1_q$ and $i^2_p < i^2_n$, and hence one of the configurations of Figure 5 (with $j = 1$ and $k = n$) does occur in $\mathcal{M}$. This is a contradiction since $\mathcal{M}$ satisfies Property $\mathbf{Q_2}$, and hence $\left(i^1_1, i^2_n\right)$ is an arc of $\mathcal{M}$.
- Induction step. Assume that $\left(i^1_k, i^2_{n-k+1}\right)$ is an arc of $\mathcal{M}$ for $1 \leq k < j$, and suppose, aiming at a contradiction, that $\left(i^1_j, i^2_{n-j+1}\right)$ is not an arc of $\mathcal{M}$.

Then, there exist $1 \leq p < n-j+1$ and $j < q \leq n$ such that $\left(i_j^1, i_p^2\right)$ and $\left(i_q^1, i_{n-j+1}^2\right)$ are two arcs of $\mathcal{M}$. But $i_j^1 < i_q^1$ and $i_p^2 < i_{n-j+1}^2$, and hence one of the configurations of Figure 5 does occur in $\mathcal{M}$. This is a contradiction since $\mathcal{M}$ satisfies Property $\mathbf{Q_2}$, and hence $\left(i_j^1, i_{n-j+1}^2\right)$ is an arc of $\mathcal{M}$.

Now, let σ_1 be the pattern of π induced by the sources of $\mathcal{M}$ and σ_2 be the pattern of π induced by the sinks of $\mathcal{M}$ (*i.e.*, $\sigma_1 = \pi\left(i_1^1\right)\pi\left(i_2^1\right)\dots\pi\left(i_n^1\right)$ and $\sigma_2 = \pi\left(j_1^2\right)\pi\left(j_2^2\right)\dots\pi\left(i_n^2\right)$). Clearly σ_1 and σ_2 are disjoint in π (since $\mathcal{M}$ is a matching) and cover π (since $\mathcal{M}$ is perfect). Moreover, the fact that $\mathcal{M}$ satisfies $\mathbf{P_1}$ implies immediately that reduce(σ_1) is order-isomorphic to $(\text{reduce} \circ \text{r})(\sigma_2)$, and hence this shows that $\pi \in \mathcal{S}_{2n}^{\text{r}}$. ■

Lemma 7. *Let $\pi \in S_{2n}$ and $\mathcal{M}$ be a perfect directed matching on π that avoids the patterns , , , and . Then, for any arc $(i,j) \in \mathcal{M}$, either $1 \leq i \leq n < j \leq 2n$ or $1 \leq j \leq n < i \leq 2n$.*

Proof. We only prove the case $1 \leq i \leq n < j \leq 2n$ since the proof for $1 \leq j \leq n < i \leq 2n$ is similar.

Suppose, aiming at a contradiction, that there exists an arc $(i,j) \in \mathcal{M}$ with $1 \leq i < j \leq n$. Since $\mathcal{M}$ avoids the patterns , , and , there is no arc $(k,\ell) \in \mathcal{M}$ with $j < k \leq 2n$ and $j < \ell \leq 2n$, $k \neq \ell$. Then it follows that there exist $2n - j \geq n$ distinct arcs $(k,\ell) \in \mathcal{M}$ with $1 \leq \min\{k,\ell\} \leq j-1$ and $j+1 \leq \max\{k,\ell\} \leq 2n$, $k \neq \ell$. This is a contradiction since $(i,j) \in \mathcal{M}$ with $1 \leq i < j \leq n$. ■

The following technical lemma is needed to simplify the proof of upcoming Proposition 7.

Lemma 8. *Let $k, \ell_1, \ell_2, \ell_3$ and ℓ_4 be positive integers and $\pi \in \mathcal{S}^{\text{r}}_{2k+\ell_1+\ell_2+\ell_3+\ell_4}$ be the permutation defined by $\pi = X_1\ L\ X_2\ X_3\ R\ X_4$, where $X_1 = \mu_1[k+\ell_2]$, $L = \nearrow_k [\ell_2]$, $X_2 = \mu_2$, $X_3 = \mu_3[k+\ell_1+\ell_2]$, $R = \searrow_k[k+\ell_1+\ell_2+\ell_3]$ and $X_4 = \mu_4[2k+\ell_1+\ell_2+\ell_3]$ for some permutations $\mu_i \in S_{\ell_i}$, $1 \leq i \leq 4$ (see Figure 6). Let $\mathcal{M}$ be a directed perfect matching on π that satisfies properties $\mathbf{P_1}$ and $\mathbf{Q_2}$. If $\mathcal{M}_{L\to R} \neq \emptyset$ or $\mathcal{M}_{R\to L} \neq \emptyset$, then either $|\mathcal{M}_{L\to R}| = k$ or $|\mathcal{M}_{R\to L}| = k$.*

Proof. Suppose first, aiming at a contradiction, that $\mathcal{M}_{L\to R} \neq \emptyset$ and $\mathcal{M}_{R\to L} \neq \emptyset$. Let $(i,j) \in \mathcal{M}_{L\to R}$ and $(i',j') \in \mathcal{M}_{R\to L}$. By construction, we have $\pi(i) < \pi(i')$ and $\pi(j) > \pi(j')$ thereby contradicting Property $\mathbf{P_1}$. Then it follows that either $\mathcal{M}_{L\to R} \neq \emptyset$ or $\mathcal{M}_{R\to L} \neq \emptyset$, but not both.

Suppose now that $\mathcal{M}_{L\to R} \neq \emptyset$ and $\mathcal{M}_{R\to L} = \emptyset$ (the case $\mathcal{M}_{L\to R} = \emptyset$ and $\mathcal{M}_{R\to L} \neq \emptyset$ can be proved with the same arguments). Suppose, aiming at a contradiction, that $|\mathcal{M}_{L\to R}| \neq k$ and let $(i,j) \in \mathcal{M}_{L\to R}$. According to Lemma 7, we are left with considering the four following cases.

- There exists $(i',j') \in \mathcal{M}_{L\to\mu_3}$. Since $\mathcal{M}$ avoids , we have $i < i'$, and hence $\pi(i) < \pi(i')$ and $\pi(j) > \pi(j')$. This is a contradiction since $\mathcal{M}$ satisfies Property $\mathbf{P_1}$.

– There exists $(i', j') \in \mathcal{M}_{\mu_3 \to L}$ and hence $\pi(i) < \pi(i')$ and $\pi(j) > \pi(j')$. This is a contradiction since $\mathcal{M}$ satisfies Property $\mathbf{P_1}$.
– There exists $(i', j') \in \mathcal{M}_{L \to \mu_4}$. Since $\mathcal{M}$ avoids , we have $i' < i$, and hence $\pi(i) > \pi(i')$ and $\pi(j) < \pi(j')$. This is a contradiction since $\mathcal{M}$ satisfies Property $\mathbf{P_1}$.
– There exists $(i', j') \in \mathcal{M}_{\mu_4 \to L}$ and hence $\pi(i) < \pi(i')$ and $\pi(j) > \pi(j')$. This is a contradiction since $\mathcal{M}$ satisfies Property $\mathbf{P_1}$.

Therefore, $|\mathcal{M}_{L \to R}| = k$. ■

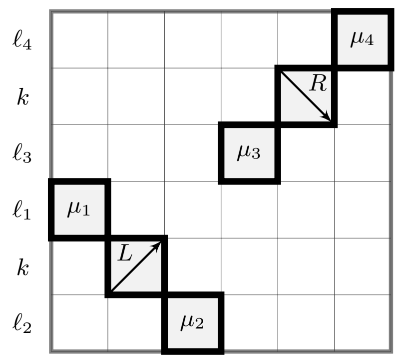


Fig. 6. Ilustration of Lemma 8, where $\pi = X_1 \;\; L \;\; X_2 \;\; X_3 \;\; R \;\; X_4$.

Proposition 7. r-Unshuffle-Permutation *is* NP*-complete.*

Proof. We reduce from Permutation Pattern which, given two permutations $\pi \in S_n$ and $\sigma \in S_k$ with $k \leq n$, asks whether σ is a pattern of π. Permutation Pattern is known to be NP-complete [1]. Let $\pi \in S_n$ and $k \in S_k$ be an instance of Permutation Pattern. Set $N_2 = n + k + 1$ and $N_1 = 2N_2$. Define a target permutation $\mu \in S_{2n+2k+2N_1+2N_2}$ by $\mu = \mu_\pi \;\; \mu_2 \;\; \mu_1 \;\; \mu_\sigma \;\; \mu'_2 \;\; \mu'_\pi \;\; \mu'_1 \;\; \mu'_\sigma$, where

$$\mu_\sigma = \sigma, \qquad \mu_\pi = \pi[k + N_1 + N_2],$$
$$\mu'_\pi = \mathrm{r}(\pi)[2n + k + 2N_1 + 2N_2] \qquad \mu'_\sigma = \mathrm{r}(\sigma)[n + k + N_1 + 2N_2],$$
$$\mu_1 = \nearrow_{N_1}[k], \qquad \mu'_1 = \searrow_{N_1}[2n + k + N_1 + 2N_2],$$
$$\mu_2 = \nearrow_{N_2}[k + N_1], \qquad \mu'_2 = \searrow_{N_2}[n + k + N_1 + N_2].$$

Clearly, the construction can be carried out in polynomial time. We claim that σ is a pattern of π if and only if $\mu \in \mathcal{S}^{\mathrm{r}}_{2n+2k+2N_1+2N_2}$.

($\Rightarrow$) Suppose that σ is a pattern of π. Then there exist indices $P = \{p_1, p_2, \ldots, p_k\}$, $1 \leq p_1 < p_2 < \cdots < p_k \leq n$, such that σ and $\pi(p_1)\,\pi(p_2)\cdots\pi(p_k)$ are order-isomorphic. For the sake of simplification, let $Q = [n] \setminus P$ and write

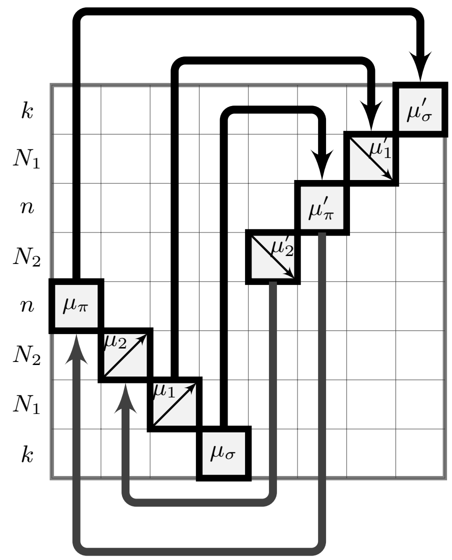


Fig. 7. Schematic representation of the construction used in Proposition 7.

$Q = \{q_1, q_2, \cdots, q_{n-k}\}$, $1 \leq q_1 < q_2 < \cdots < q_{n-k} \leq n$. Define a directed matching

$$\mathcal{M} = \mathcal{M}_{\mu_\sigma \to \mu'_\pi} \uplus \mathcal{M}_{\mu'_\pi \to \mu_\pi} \uplus \mathcal{M}_{\mu_\pi \to \mu'_\sigma} \uplus \mathcal{M}_{\mu_1 \to \mu'_1} \uplus \mathcal{M}_{\mu_2 \to \mu'_2}$$

on μ as follows.

$$\begin{aligned}
\mathcal{M}_{\mu_\sigma \to \mu'_\pi} &= \{(n + N_1 + N_2 + i, n + k + N_1 + 2N_2 + p_i) : 1 \leq i \leq k\} \\
\mathcal{M}_{\mu'_\pi \to \mu_\pi} &= \{(2n + k + N_1 + 2N_2 - q_i + 1, q_i) : 1 \leq i \leq n - k\} \\
\mathcal{M}_{\mu_\pi \to \mu'_\sigma} &= \{(p_i, 2n + 2k + 2N_1 + 2N_2 - i) : 1 \leq i \leq k\} \\
\mathcal{M}_{\mu_1 \to \mu'_1} &= \{(n + N_2 + i, 2n + k + 2N_1 + 2N_2 - i + 1) : 1 \leq i \leq N_1\} \\
\mathcal{M}_{\mu'_2 \to \mu_2} &= \{(n + k + N_1 + 2N_2 - i + 1, n + i) : 1 \leq i \leq N_2\}.
\end{aligned}$$

Informally,

- $\mathcal{M}_{\mu_\sigma \to \mu'_\pi}$ is the directed (left-to-right) matching describing the (reversed) occurrence of σ in $\mathrm{r}(\pi)$,
- $\mathcal{M}_{\mu'_\pi \to \mu_\pi}$ is the directed (right-to-left) matching connecting the elements of $\mathrm{r}(\pi)$ and π that are not part of the occurrence of σ in π,
- $\mathcal{M}_{\mu_\pi \to \mu'_\sigma}$ is the directed (left-to-right) matching describing the (reversed) occurrence of $\mathrm{r}(\sigma)$ in π,
- $\mathcal{M}_{\mu_1 \to \mu'_1}$ is a directed (left-to-right) matching that fully connects μ_1 to μ'_1 and
- $\mathcal{M}_{\mu'_2 \to \mu_2}$ is a directed (right-to-left) matching that fully connects μ'_2 to μ_2.

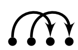
It is now a simple matter to check that $\mathcal{M}$ is a directed matching on μ that satisfies properties $\mathbf{P_1}$ and $\mathbf{Q_2}$. Therefore, according to Proposition 6, $\mu \in \mathcal{S}^{\mathrm{r}}_{2n+2k+2N_1+2N_2}$.

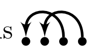
($\Leftarrow$) Suppose that $\mu \in \mathcal{S}^{\mathrm{r}}_{2n}$. Therefore, according to Proposition 6, there exists a directed perfect matching $\mathcal{M}$ on μ that satisfies properties $\mathbf{P_1}$ and $\mathbf{Q_2}$. We begin with a sequence of claims that help defining the general structure of $\mathcal{M}$.

Claim. We may assume that $|\mathcal{M}_{\mu_1 \to \mu'_1}| = N_1$.

Proof. Combining Lemma 7 and $N_1 = 2N_2 > n + k + N_2$, we conclude that $\mathcal{M}_{\mu_1 \to \mu'_1} \neq \emptyset$ or $\mathcal{M}_{\mu'_1 \to \mu_1} \neq \emptyset$. Thus, applying Lemma 8, we obtain $|\mathcal{M}_{\mu_1 \to \mu'_1}| = N_1$ or $|\mathcal{M}_{\mu'_1 \to \mu_1}| = N_1$. Therefore, by Lemma 4 (One shot lemma), we may assume that $\mathcal{M}_{\mu_1 \to \mu'_1} \neq \emptyset$. ■

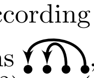
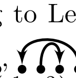
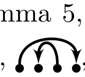
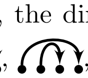
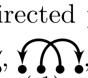
Claim. Assuming $|\mathcal{M}_{\mu_1 \to \mu'_1}| = N_1$, we have $|\mathcal{M}_{\mu'_2 \to \mu_2}| = N_2$.

Proof. Combining Claim 4, Lemma 7 and $N_2 > n + k$, we conclude that $\mathcal{M}_{\mu_2 \to \mu'_2} \neq \emptyset$ or $\mathcal{M}_{\mu'_2 \to \mu_2} \neq \emptyset$. Applying Lemma 8, we obtain $|\mathcal{M}_{\mu_2 \to \mu'_2}| = N_2$ or $|\mathcal{M}_{\mu'_2 \to \mu_2}| = N_2$. The claim now follows from $\mathcal{M}_{\mu_1 \to \mu'_1} \neq \emptyset$ (Claim 4) and the fact that $\mathcal{M}$ avoids  (Property $\mathbf{Q_2}$). ■

From the above two claims and the fact that $\mathcal{M}$ avoids  (Property $\mathbf{Q_2}$), we conclude that $|\mathcal{M}_{\mu_\sigma \to \mu'_\pi}| = k$. But $\mathcal{M}$ satisfies Property $\mathbf{P_1}$ and hence there exists an occurrence of $\mathrm{r}(\sigma)$ in $\mathrm{r}(\pi)$. It follows that σ is a pattern of π. ■

Proposition 8. *Let $\pi \in S_{2n}$. The two following statements are equivalent.*

1. $\pi \in \mathcal{S}^{\mathrm{c}}_{2n}$.
2. *There exists a directed perfect matching $\mathcal{M}$ that satisfies properties $\mathbf{P_2}$ and $\mathbf{Q_1}$.*

Proof. ($\mathbf{1} \Rightarrow \mathbf{2}$) Let $\pi \in \mathcal{S}^{\mathrm{r}}_{2n}$. Then, there exists $\sigma \in S_n$ such that $\pi \in \sigma \sqcup \mathrm{c}(\sigma)$, and hence there exist two disjoint set of indices $I^1 = \{i^1_j : j \in [n]\}$, $i^1_1 < i^1_2 < \cdots < i^1_n$, and $I^2 = \{i^2_j : j \in [n]\}$, $i^2_1 < i^2_2 < \cdots < i^2_n$, such that $\pi(i^1_1)\pi(i^1_2)\cdots\pi(i^1_n)$ is order-isomorphic to σ and $\pi(i^2_1)\pi(i^2_2)\cdots\pi(i^2_n)$ is order-isomorphic to $\mathrm{c}(\sigma)$. Let $\mathcal{M} = (V, E)$ be the directed graph defined by $V = [2n]$ and $E = \left\{\left(i^1_j, i^2_j\right) : j \in [n]\right\}$. Clearly, $\mathcal{M}$ is a directed perfect matching since $I^1 \cap I^2 = \emptyset$ and $I^1 \cup I^2 = [2n]$. According to Lemma 5, the directed perfect matching $\mathcal{M}$ does avoid the patterns , , , , , . Finally, for any two distinct arcs $\left(i^1_j, i^2_j\right)$ and $\left(i^1_k, i^2_k\right)$ of $\mathcal{M}$, we have $\pi\left(i^1_j\right) < \pi\left(i^1_k\right)$ if and only if $\pi\left(i^2_j\right) > \pi\left(i^2_k\right)$ since $\pi(i^1_1)\pi(i^1_2)\cdots\pi(i^1_n)$ is order-isomorphic to σ and $\pi(i^2_1)\pi(i^2_2)\cdots\pi(i^2_n)$ is order-isomorphic to $\mathrm{c}(\sigma)$ (and hence $\pi(i^2_j) = n - \pi(i^1_j)$ and $\pi(i^2_k) = n - \pi(i^1_k)$). Therefore, $\mathcal{M}$ satisfies properties $\mathbf{P_2}$ and $\mathbf{Q_1}$.

($\mathbf{2} \Rightarrow \mathbf{1}$) Let $\mathcal{M}$ be a directed perfect matching that satisfies properties $\mathbf{P_2}$ and $\mathbf{Q_1}$. Let $I^1 = \{i^1_j : j \in [n]\}$, $i^1_1 < i^1_2 < \cdots < i^1_n$, be the set the sources of the arcs of $\mathcal{M}$ and let $I^2 = \{i^2_j : j \in [n]\}$, $i^2_1 < i^2_2 < \cdots < i^2_n$, be the set of the sinks of the arcs of $\mathcal{M}$. We first show that, for every $j \in [n]$, $\left(i^1_j, i^2_j\right)$ is an arc of $\mathcal{M}$. The proof is by induction on $j = 1, 2, \cdots, n$.

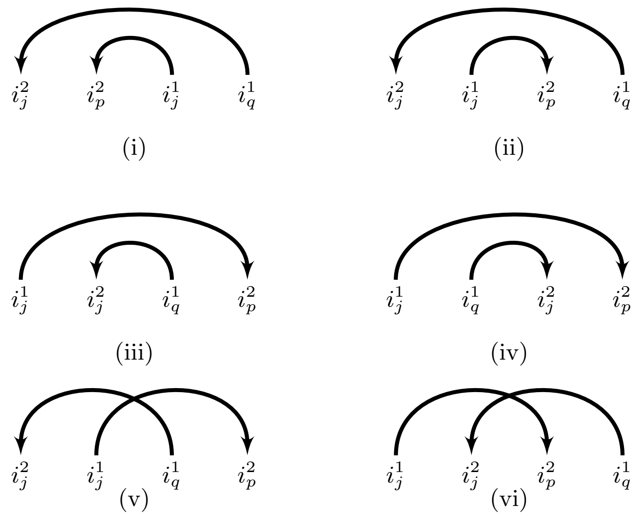


Fig. 8. All possible configurations considered in Proof of Proposition 8 (**2** ⇒ **1**).

- Base. Suppose, aiming at a contradiction that $\left(i_1^1, i_1^2\right)$ is not an arc of $\mathcal{M}$. Then, there exist $1 < p \leq n$ and $1 < q \leq n$ such that $\left(i_1^1, i_p^2\right)$ and $\left(i_q^1, i_1^2\right)$ are two arcs of $\mathcal{M}$. But $i_1^1 < i_q^1$ and $i_1^2 < i_p^2$, and hence one of the configurations of Figure 8 (with $j = 1$) does occur in $\mathcal{M}$. This is a contradiction since $\mathcal{M}$ satisfies Property $\mathbf{Q_1}$, and hence $\left(i_1^1, i_1^2\right)$ is an arc of $\mathcal{M}$.
- Induction step. Assume that $\left(i_k^1, i_{n-k+1}^2\right)$ is an arc of $\mathcal{M}$ for $1 \leq k < j$, and suppose, aiming at a contradiction, that $\left(i_j^1, i_j^2\right)$ is not an arc of $\mathcal{M}$. Then, there exist $j < p \leq n$ and $j < q \leq n$ such that $\left(i_j^1, i_p^2\right)$ and $\left(i_q^1, i_{n-j+1}^2\right)$ are two arcs of $\mathcal{M}$. But $i_j^1 < i_q^1$ and $i_j^2 < i_p^2$, and hence one of the configurations of Figure 8 does occur in $\mathcal{M}$. This is a contradiction since $\mathcal{M}$ satisfies Property $\mathbf{Q_1}$, and hence $\left(i_j^1, i_j^2\right)$ is an arc of $\mathcal{M}$.

Now, let σ_1 be the pattern of π induced by the sources of $\mathcal{M}$ and σ_2 be the pattern of π induced by the sinks of $\mathcal{M}$ (*i.e.*, $\sigma_1 = \pi\left(i_1^1\right)\pi\left(i_2^1\right)\ldots\pi\left(i_n^1\right)$ and $\sigma_2 = \pi\left(j_1^2\right)\pi\left(j_2^2\right)\ldots\pi\left(i_n^2\right)$). Clearly σ_1 and σ_2 are disjoint in π (since $\mathcal{M}$ is a matching) and cover π (since $\mathcal{M}$ is perfect). Moreover, the fact that $\mathcal{M}$ satisfies $\mathbf{P_2}$ implies immediately that reduce(σ_1) is order-isomorphic to $(\text{reduce} \circ \text{c})(\sigma_2)$, and hence this shows that $\pi \in \mathcal{S}_{2n}^{\text{c}}$. ■

Corollary 1. c-Unshuffle-Permutation *is* NP*-complete.*

Proof. Combine Proposition 7 with Proposition 1. ■

Proposition 9. *Let* $\pi \in S_{2n}$. *The two following statements are equivalent.*

1. $\pi \in \mathcal{S}_{2n}^{\text{rc}}$.

2. There exists a directed perfect matching $\mathcal{M}$ that satisfies properties $\mathbf{P_2}$ *and* $\mathbf{Q_2}$*.*

Proof. ($\mathbf{1} \Rightarrow \mathbf{2}$) Let $\pi \in \mathcal{S}^{\mathrm{rc}}_{2n}$. Then, there exists $\sigma \in S_n$ such that $\pi \in \sigma \sqcup (\mathrm{r} \circ \mathrm{c})(\sigma)$, and hence there exist two disjoint set of indices $I^1 = \{i^1_j : j \in [n]\}$, $i^1_1 < i^1_2 < \cdots < i^1_n$, and $I^2 = \{i^2_j : j \in [n]\}$, $i^2_1 < i^2_2 < \cdots < i^2_n$, such that $\pi(i^1_1)\pi(i^1_2)\cdots\pi(i^1_n)$ is order-isomorphic to σ and $\pi(i^2_1)\pi(i^2_2)\cdots\pi(i^2_n)$ is order-isomorphic to $(\mathrm{r} \circ \mathrm{c})(\sigma)$. Let $\mathcal{M} = (V, E)$ be the directed graph defined by $V = [2n]$ and $E = \left\{\left(i^1_j, i^2_{n-j+1}\right) : j \in [n]\right\}$. Clearly, $\mathcal{M}$ is a directed perfect matching since $I^1 \cap I^2 = \emptyset$ and $I^1 \cup I^2 = [2n]$. According to Lemma 6, the directed perfect matching $\mathcal{M}$ does avoid the patterns , , , , and . Finally, for any two distinct arcs $\left(i^1_j, i^2_{n-j+1}\right)$ and $\left(i^1_k, i^2_{n-k+1}\right)$ of $\mathcal{M}$, we have $\pi\left(i^1_j\right) < \pi\left(i^1_k\right)$ if and only if $\pi\left(i^2_{n-j+1}\right) > \pi\left(i^2_{n-k+1}\right)$ since $\pi(i^1_1)\pi(i^1_2)\cdots\pi(i^1_n)$ is order-isomorphic to σ and $\pi(i^2_1)\pi(i^2_2)\cdots\pi(i^2_n)$ is order-isomorphic to $(\mathrm{r} \circ \mathrm{c})(\sigma)$. Therefore, $\mathcal{M}$ satisfies properties $\mathbf{P_1}$ and $\mathbf{Q_2}$.

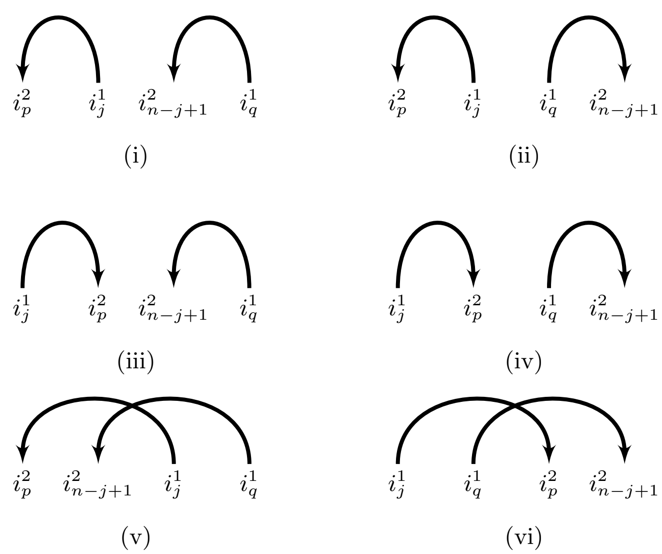


Fig. 9. All possible configurations considered in Proof of Proposition 9 ($\mathbf{2} \Rightarrow \mathbf{1}$).

($\mathbf{2} \Rightarrow \mathbf{1}$) Let $\mathcal{M}$ be a directed perfect matching that satisfies properties $\mathbf{P_1}$ and $\mathbf{Q_2}$. Let $I^1 = \{i^1_j : j \in [n]\}$, $i^1_1 < i^1_2 < \cdots < i^1_n$, be the set the sources of the arcs of $\mathcal{M}$ and let $I^2 = \{i^2_j : j \in [n]\}$, $i^2_1 < i^2_2 < \cdots < i^2_n$, be the set of the sinks of the arcs of $\mathcal{M}$. We first show that, for every $j \in [n]$, $\left(i^1_j, i^2_{n-j+1}\right)$ is an arc of $\mathcal{M}$. The proof is by induction on $j = 1, 2, \cdots, n$.

– Base. Suppose, aiming at a contradiction that $\left(i_1^1, i_n^2\right)$ is not an arc of $\mathcal{M}$. Then, there exist $1 \leq p < n$ and $1 < q \leq n$ such that $\left(i_1^1, i_p^2\right)$ and $\left(i_q^1, i_n^2\right)$ are two arcs of $\mathcal{M}$. But $i_1^1 < i_q^1$ and $i_p^2 < i_n^2$, and hence one of the configurations of Figure 9 (with $j = 1$ and $k = n$) does occur in $\mathcal{M}$. This is a contradiction since $\mathcal{M}$ satisfies Property $\mathbf{Q_2}$, and hence $\left(i_1^1, i_n^2\right)$ is an arc of $\mathcal{M}$.
– Induction step. Assume that $\left(i_k^1, i_{n-k+1}^2\right)$ is an arc of $\mathcal{M}$ for $1 \leq k < j$, and suppose, aiming at a contradiction, that $\left(i_j^1, i_{n-j+1}^2\right)$ is not an arc of $\mathcal{M}$. Then, there exist $1 \leq p < n - j + 1$ and $j < q \leq n$ such that $\left(i_j^1, i_p^2\right)$ and $\left(i_q^1, i_{n-j+1}^2\right)$ are two arcs of $\mathcal{M}$. But $i_j^1 < i_q^1$ and $i_p^2 < i_{n-j+1}^2$, and hence one of the configurations of Figure 9 does occur in $\mathcal{M}$. This is a contradiction since $\mathcal{M}$ satisfies Property $\mathbf{Q_2}$, and hence $\left(i_j^1, i_{n-j+1}^2\right)$ is an arc of $\mathcal{M}$.

Now, let σ_1 be the pattern of π induced by the sources of $\mathcal{M}$ and σ_2 be the pattern of π induced by the sinks of $\mathcal{M}$ (*i.e.*, $\sigma_1 = \pi\left(i_1^1\right)\pi\left(i_2^1\right)\ldots\pi\left(i_n^1\right)$ and $\sigma_2 = \pi\left(j_1^2\right)\pi\left(j_2^2\right)\ldots\pi\left(i_n^2\right)$). Clearly σ_1 and σ_2 are disjoint in π (since $\mathcal{M}$ is a matching) and cover π (since $\mathcal{M}$ is perfect). Moreover, the fact that $\mathcal{M}$ satisfies $\mathbf{P_2}$ implies immediately that reduce(σ_1) is order-isomorphic to $(\text{reduce} \circ \text{r} \circ \text{c})(\sigma_2)$, and hence this shows that $\pi \in \mathcal{S}_{2n}^{\text{rc}}$. ■

Proposition 10. $(\text{r} \circ \text{c})$-Unshuffle-Permutation *is* NP-*complete.*

Proof (sketch). The proof is similar to Proposition 7, replacing μ'_1 by $\nearrow_{N_1}$ $[2n+k+N_1+2N_2]$, μ'_2 by $\nearrow_{N_2}$ $[n+k+N_1+N_2]$, μ'_π by $(\text{r} \circ \text{c})(\pi)[2n+k+2N_1+2N_2]$ and μ'_σ by $(\text{r} \circ \text{c})(\sigma)[n + k + N_1 + 2N_2]$. ■

5 Conclusion and perspectives

In this paper we have proposed to investigate the problem of recognizing those permutations π for which there exists a permutation σ such that $\pi \in \sigma \sqcup f(\sigma)$ for some bijection $f : S_n \to S_n$. Quite a number of problems are left open by the results presented here. For instance, can we efficiently decide membership to $\mathcal{S}_{2n}^{\text{i}}$ (*i.e.,* the permutations π for witch there exists a permutation σ such that $\pi \in \sigma \sqcup \text{i}(\sigma)$)? To conclude, we wish to mention two conjectures.

Conjecture 1 (Enumeration). For any $n \in \mathbb{N}$, $s_{2n}^{\text{rc}} \leq s_{2n}^{\text{c}} = s_{2n}^{\text{r}} \leq s_{2n}^{\text{irc}} = s_{2n}^{\text{i}} \leq s_{2n}^{\epsilon} \leq s_{2n}^{\text{ci}} = s_{2n}^{\text{ic}}$.

Conjecture 2 (Ubiquitous generalized squares). For any $n \in \mathbb{N}$, $\mathcal{S}_{2n}^{\epsilon} \cap \mathcal{S}_{2n}^{\text{r}} \cap \mathcal{S}_{2n}^{\text{c}} \cap \mathcal{S}_{2n}^{\text{i}} \cap \mathcal{S}_{2n}^{\text{ci}} \cap \mathcal{S}_{2n}^{\text{ic}} \cap \mathcal{S}_{2n}^{\text{rc}} \cap \mathcal{S}_{2n}^{\text{irc}} \neq \emptyset$.

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