Operads of decorated cliques II: Noncrossing cliques

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Abstract. A complete study of an operad $\mathrm{NC}\mathcal{M}$ of noncrossing configurations of chords introduced in previous work of the author is performed. This operad is defined on the linear span of all noncrossing $\mathcal{M}$-cliques. These are noncrossing configurations of chords with arcs labeled by a unitary magma $\mathcal{M}$. The magmatic product of $\mathcal{M}$ intervenes for the computation of the operadic composition of $\mathcal{M}$-cliques. We show that this operad is binary, quadratic, and Koszul by considering techniques coming from rewrite systems on trees. We also compute a presentation for its Koszul dual. Finally, we explain how $\mathrm{NC}\mathcal{M}$ allows one to obtain alternative constructions of already known operads like operads of formal fractions and the operad of bicolored noncrossing configurations.

Contents

Date: 2024-02-02.
2020 *Mathematics Subject Classification.* 05C05, 05C76, 18D99, 05E99.
Key words and phrases. Tree; Graph; Noncrossing configuration; Rewrite rule; Operad; Koszul duality.
This research has been partially supported by the projects CARPLO (ANR-20-CE40-0007) and ALCOHOL (ANR-19-CE40-0006) of the Agence nationale de la recherche.

1 Introduction

Noncrossing configurations of chords on regular polygons are combinatorial objects appearing in various contexts (see for instance [FN99; CP92; DRS10; PR14]). A simple generalization of these consists in allowing several colors for the arcs of the configurations. This has been considered in previous work [Gir20] of the author, where the arcs are labeled by elements of a unitary magma $\mathcal{M}$. More precisely, given a unitary magma $\mathcal{M}$, $\mathcal{C}_{\mathcal{M}}$ is the set of regular polygons $\mathfrak{p}$ where all possible arcs of $\mathfrak{p}$ are labeled by elements of $\mathcal{M}$. When an arc is labeled by the unit $\mathbb{1}_{\mathcal{M}}$ of $\mathcal{M}$, it is considered as missing, so that the notion of noncrossing configurations makes sense. More precisely, noncrossing configurations have no diagonals labeled by elements different from $\mathbb{1}_{\mathcal{M}}$ crossing another such diagonal. These objects have been named noncrossing $\mathcal{M}$-cliques and are the main combinatorial objects studied in the present article.

The linear span $\mathrm{C}\mathcal{M}$ of all $\mathcal{M}$-cliques (by relaxing the noncrossing condition) over any field of characteristic zero forms an operad with rich algebraic and combinatorial properties, introduced and studied in [Gir20]. The operad structure added on these objects allows one to compose two $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ by gluing a special arc of $\mathfrak{q}$ (called the base) onto a selected edge of $\mathfrak{p}$. The magmatic product of $\mathcal{M}$ encodes how to relabel some edges of the resulting $\mathcal{M}$-clique. We showed in this previous work that the subspace $\mathrm{NC}\mathcal{M}$ of $\mathrm{C}\mathcal{M}$ generated by all noncrossing $\mathcal{M}$-cliques forms a suboperad of $\mathrm{C}\mathcal{M}$. The purpose of the present paper is to perform a complete study of this operad. A first particularity of $\mathrm{NC}\mathcal{M}$ motivating this objective is that it has a special status among all the suboperads of $\mathrm{C}\mathcal{M}$. Indeed, if $\mathcal{M}$ is nontrivial, then $\mathrm{C}\mathcal{M}$ is not a binary operad, and $\mathrm{NC}\mathcal{M}$ is precisely the biggest binary suboperad of $\mathrm{C}\mathcal{M}$.

Let us now give an overview of the main properties of $\mathrm{NC}\mathcal{M}$ and the main results contained in this article. First, by considering the dual trees of noncrossing $\mathcal{M}$-cliques, we can view each noncrossing $\mathcal{M}$-clique of $\mathrm{NC}\mathcal{M}$ as a Schröder tree (an ordered tree where all internal nodes have two or more children) with edges labeled by $\mathcal{M}$ satisfying some conditions. Under this point of view, $\mathrm{NC}\mathcal{M}$ is an operad of such Schröder trees endowed with a composition operation which is essentially a grafting operation with a relabeling of edges or a contraction of edges. As a consequence of this alternative combinatorial realization of $\mathrm{NC}\mathcal{M}$, we obtain a formula for its dimensions involving Narayana numbers [Nar55]. To complete the study of $\mathrm{NC}\mathcal{M}$, a natural and important question is to exhibit one of its presentations by generators and relations. In order to compute the space of relations of $\mathrm{NC}\mathcal{M}$, we use techniques of rewrite systems of trees [BN98]. Thus, we define a convergent rewrite rule $\to$ and show that the space induced by $\to$ is the space of relations of $\mathrm{NC}\mathcal{M}$, leading to a presentation by generators and relations of $\mathrm{NC}\mathcal{M}$. As an important consequence, this proves that $\mathrm{NC}\mathcal{M}$ is always quadratic, regardless of $\mathcal{M}$. The existence of such a convergent orientation of the space of relations of $\mathrm{NC}\mathcal{M}$ implies also by [Hof10] that this operad is Koszul.

We also study some structures related with $\mathrm{NC}\mathcal{M}$. This includes the suboperads of $\mathrm{NC}\mathcal{M}$ generated by some finite families of bubbles (the latter being noncrossing $\mathcal{M}$-cliques with no diagonals). Under some conditions on the considered sets of bubbles, we can describe the Hilbert series of these suboperads of $\mathrm{NC}\mathcal{M}$ by a system of algebraic equations. We give two examples of suboperads of $\mathrm{NC}\mathcal{M}$ generated by some subsets of triangles, including one which is a suboperad of $\mathrm{NC}\mathbb{D}_0$ ($\mathbb{D}_0$ is the multiplicative monoid on $\{0, 1\}$) isomorphic to the operad Motz of Motzkin paths defined in [Gir15]. Moreover, since $\mathrm{NC}\mathcal{M}$ is a binary and quadratic operad, its Koszul dual $\mathrm{NC}\mathcal{M}^!$ is well-defined [GK94]. We compute its presentation, present an algebraic equation for its Hilbert series, give a formula for its dimensions, and establish a combinatorial realization of $\mathrm{NC}\mathcal{M}^!$ as a graded space involving dual $\mathcal{M}$-cliques, which are $\mathcal{M}^2$-cliques with some constraints for the labels of their arcs.

Furthermore, by selecting appropriate unitary magmas $\mathcal{M}$, it is possible to provide alternative constructions of already known operads as suboperads of $\mathrm{NC}\mathcal{M}$. We hence construct the operad NCT of based noncrossing trees [Cha07; Ler11], the suboperad $\mathcal{F}\mathcal{F}_4$ of the operad of formal fractions $\mathcal{F}\mathcal{F}$ [CHN16], and the operad of bicolored noncrossing configurations BNC [CG14]. As a consequence of this last construction, all the suboperads of BNC can be obtained from the construction NC. This includes for example the operad of noncrossing plants [Cha07], the dipterous operad [LR03; Zin12], and the 2-associative operad [LR06; Zin12].

This text is organized as follows. Section 2 sets our notations about trees, syntax trees, rewrite rules on trees, free operads, and Koszul duality of operads. In Section 3, we perform the aforementioned study of the operad $\mathrm{NC}\mathcal{M}$ and in Section 4, of $\mathrm{NC}\mathcal{M}^!$. Finally, in Section 5, we use the construction NC to provide alternative definitions of some known operads.

This paper is an extended version of [Gir17], containing the proofs of the presented results. It is also a sequel of [Gir20].

General notations and conventions. All the algebraic structures of this article have a field of characteristic zero $\mathbb{K}$ as ground field. For a set S, $\mathbb{K}\langle S\rangle$ denotes the linear span of the elements of S. For integers a and c, $[a,c]$ denotes the set $\{b \in \mathbb{N} : a \leqslant b \leqslant c\}$ and $[n]$ is short for the set $[1,n]$. The cardinality of a finite set S is denoted by $\#S$. For a set A, A^* denotes the set of finite sequences, called words, of elements of A. For an integer $n \geqslant 0$, A^n (resp. $A^{\geqslant n}$) is the set of words on A of length n (resp. at least n). The word of length 0 is the empty word denoted by ϵ. If u is a word, its letters are indexed from left to right from 1 to its length $|u|$. For $i \in [|u|]$, u_i is the letter of u at position i. If a is a letter and n is a nonnegative integer, a^n denotes the word consisting in n occurrences of a. For a letter a, $|u|_a$ denotes the number of occurrences of a in u.

2 Elementary definitions and tools

The main purposes of this section are to provide tools to compute presentations and to prove Koszulity of operads. For this, it is important to have precise definitions about free operads, trees, and rewrite rules on trees at hand.

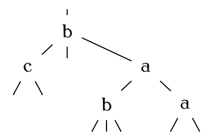
2.1 Trees and rewrite rules

Unless otherwise specified, we use in the sequel the standard terminology (*i.e.*, *node*, *edge*, *root*, *child*, *etc.*) about ordered trees [Knu97]. For the sake of completeness, we recall the most important definitions and set our notations.

2.1.1 Trees.

Let $\mathfrak{t}$ be an ordered tree. The *arity* of a node of $\mathfrak{t}$ is its number of children. An *internal node* (resp. a *leaf*) of $\mathfrak{t}$ is a node with a nonzero (resp. null) arity. Internal nodes can be *labeled*, that is, each internal node of a tree is associated with an element of a certain set. Given an internal node x of $\mathfrak{t}$, the children of x are by definition totally ordered from left to right and are thus indexed from 1 to the arity ℓ of x. For $i \in [\ell]$, the *ith subtree* of $\mathfrak{t}$ is the tree rooted at the ith child of $\mathfrak{t}$. Similarly, the leaves of $\mathfrak{t}$ are totally ordered from left to right and thus are indexed from 1 to the number of its leaves. In our graphical representations, each ordered tree is depicted so that its root is the uppermost node. Since we consider in the sequel only ordered rooted trees, we shall call these simply *trees*.

2.1.2 Syntax trees. Let $G := \bigsqcup_{n \geqslant 1} G(n)$ be a graded set. The *arity* of an element x of G is n provided that $x \in G(n)$. A *syntax tree* on G is a tree such that its internal nodes of arity n are labeled by elements of arity n of G. The *degree* (resp. *arity*) of a syntax tree is its number of internal nodes (resp. leaves). For instance, if $G := G(2) \sqcup G(3)$ with $G(2) := \{\mathtt{a}, \mathtt{c}\}$ and $G(3) := \{\mathtt{b}\}$,

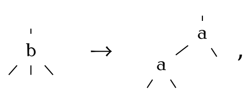

(2.1.1)

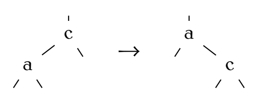
is a syntax tree on G of degree 5 and arity 8. Its root is labeled by $\mathtt{b}$ and has arity 3.

A syntax tree $\mathfrak{s}$ is a *subtree* of a syntax tree $\mathfrak{t}$ if it is possible to fit $\mathfrak{s}$ at a certain place of $\mathfrak{t}$, by possibly superimposing leaves of $\mathfrak{s}$ and internal nodes of $\mathfrak{t}$. In this case, we say that $\mathfrak{t}$ *admits an occurrence* of (the *pattern*) $\mathfrak{s}$. Conversely, we say that $\mathfrak{t}$ *avoids* $\mathfrak{s}$ if there is no occurrence of $\mathfrak{s}$ in $\mathfrak{t}$.

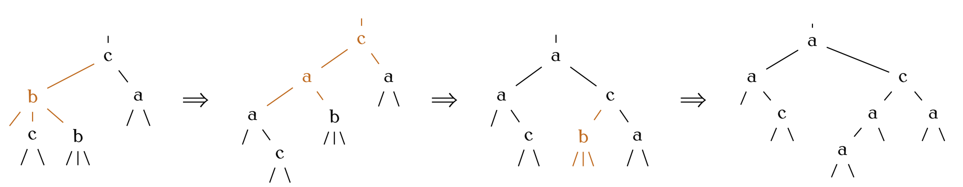
2.1.3 Rewrite rules. Let S be a set of trees. A *rewrite rule* on S is a binary relation $\to$ on S which has the property that, if $\mathfrak{s} \to \mathfrak{s}'$ for two trees $\mathfrak{s}$ and $\mathfrak{s}'$, then $\mathfrak{s}$ and $\mathfrak{s}'$ have the same number of leaves. We say that a tree $\mathfrak{t}$ is *rewritable in one step* into $\mathfrak{t}'$ by $\to$ if there exist two trees $\mathfrak{s}$ and $\mathfrak{s}'$ satisfying $\mathfrak{s} \to \mathfrak{s}'$ and $\mathfrak{t}$ has a subtree $\mathfrak{s}$ such that, by replacing $\mathfrak{s}$ by $\mathfrak{s}'$ in $\mathfrak{t}$, we obtain $\mathfrak{t}'$. We denote this property by $\mathfrak{t} \Rightarrow \mathfrak{t}'$, so that $\Rightarrow$ is a binary relation on S. If $\mathfrak{t} = \mathfrak{t}'$ or if there exists a sequence of trees $(\mathfrak{t}_1, \dots, \mathfrak{t}_{k-1})$ with $k \geqslant 1$ such that $\mathfrak{t} \Rightarrow \mathfrak{t}_1 \Rightarrow \cdots \Rightarrow \mathfrak{t}_{k-1} \Rightarrow \mathfrak{t}'$, we say that $\mathfrak{t}$ is *rewritable* by $\Rightarrow$ into $\mathfrak{t}'$, and we denote this property by $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{t}'$. In other words, $\overset{*}{\Rightarrow}$ is the reflexive and transitive closure of $\Rightarrow$. We write $\overset{*}{\to}$ for the reflexive and transitive closure of $\to$, and we write $\overset{*}{\leftrightarrow}$ (resp. $\overset{*}{\Leftrightarrow}$) for the reflexive, transitive, and symmetric closure of $\to$ (resp. $\Rightarrow$). The *vector space induced* by $\to$ is the subspace of the linear span $\mathbb{K}\langle S \rangle$ of all trees of S generated by the family of all $\mathfrak{t} - \mathfrak{t}'$ such that $\mathfrak{t} \overset{*}{\leftrightarrow} \mathfrak{t}'$.

For instance, let S be the set of trees where internal nodes are labeled by $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and consider the rewrite rule $\to$ on S satisfying


(2.1.2a)


(2.1.2b)

We then have the following steps of rewritings by $\to$:

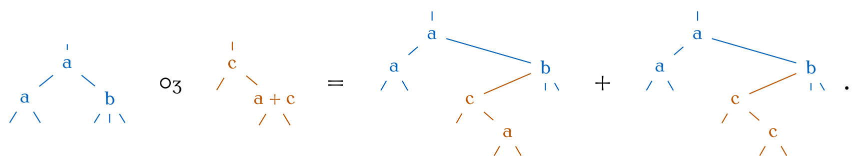

(2.1.3)

We shall use the standard terminology (*terminating*, *normal form*, *confluent*, *convergent*, *etc.*) about rewrite rules [BN98]. Let us recall now the most important definitions. Let $\to$ be a rewrite rule on a set S of trees. We say that $\to$ is *terminating* if there is no infinite chain $\mathfrak{t} \Rightarrow \mathfrak{t}_1 \Rightarrow \mathfrak{t}_2 \Rightarrow \cdots$. In this case, any tree $\mathfrak{t}$ of S that cannot be rewritten by $\to$ is a *normal form* for $\to$. We say that $\to$ is *confluent* if for any trees $\mathfrak{t}$, $\mathfrak{r}_1$, and $\mathfrak{r}_2$ such that $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{r}_1$ and $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{r}_2$, there exists a tree $\mathfrak{t}'$ such that $\mathfrak{r}_1 \overset{*}{\Rightarrow} \mathfrak{t}'$ and $\mathfrak{r}_2 \overset{*}{\Rightarrow} \mathfrak{t}'$. If $\to$ is both terminating and confluent, then $\to$ is *convergent*.

2.2 Free operads and Koszul duality

All notations and conventions about operads come from Section 1.1 of [Gir20].

2.2.1 Free operads. Let $\mathfrak{G} := \bigoplus_{n \geqslant 1} \mathfrak{G}(n)$ be a graded vector space. In particular, $\mathfrak{G}$ is a graded set so that we can consider syntax trees on $\mathfrak{G}$. The *free operad* over $\mathfrak{G}$ is the operad $\mathrm{Free}(\mathfrak{G})$ wherein, for $n \geqslant 1$, $\mathrm{Free}(\mathfrak{G})(n)$ is the linear span of the syntax trees on $\mathfrak{G}$ of arity n. The labeling of the internal nodes of the trees of $\mathrm{Free}(\mathfrak{G})$ is linear in the sense that if $\mathfrak{t}$ is a syntax tree on $\mathfrak{G}$ having an internal node labeled by $x + \lambda y \in \mathfrak{G}$, $\lambda \in \mathbb{K}$, then, in $\mathrm{Free}(\mathfrak{G})$, we have $\mathfrak{t} = \mathfrak{t}_x + \lambda \mathfrak{t}_y$, where $\mathfrak{t}_x$ (resp. $\mathfrak{t}_y$) is the tree obtained by labeling by x (resp. y) the considered node labeled by $x + \lambda y$ in $\mathfrak{t}$. The partial composition $\mathfrak{s} \circ_i \mathfrak{t}$ of $\mathrm{Free}(\mathfrak{G})$ of two syntax trees $\mathfrak{s}$ and $\mathfrak{t}$ on $\mathfrak{G}$ consists in grafting the root of $\mathfrak{t}$ on the ith leaf of $\mathfrak{s}$. The unit $\perp$ of $\mathrm{Free}(\mathfrak{G})$ is the tree consisting in one leaf. For instance, by setting $\mathfrak{G} := \mathbb{K}\langle G \rangle$ where G is the graded set defined in the previous example, in $\mathrm{Free}(\mathfrak{G})$ we have



(2.2.1)

We denote by $\mathrm{c} : \mathfrak{G} \to \mathrm{Free}(\mathfrak{G})$ the inclusion map, sending any x of $\mathfrak{G}$ to the *corolla* labeled by x, that is, the syntax tree consisting in a single internal node labeled by x attached to a required number of leaves. In the sequel, if required by the context, we shall implicitly view an element x of $\mathfrak{G}$ as the corolla $\mathrm{c}(x)$ of $\mathrm{Free}(\mathfrak{G})$. For instance, given two elements x and y of $\mathfrak{G}$, we shall denote the syntax tree $\mathrm{c}(x) \circ_i \mathrm{c}(y)$ simply by $x \circ_i y$ for all valid integers i.

Free operads satisfy a universality property. Indeed, $\mathrm{Free}(\mathfrak{G})$ is the unique operad (up to isomorphism) such that for an operad $\mathcal{O}$ and a linear map $f : \mathfrak{G} \to \mathcal{O}$ respecting the arities, there exists a unique operad morphism $\phi : \mathrm{Free}(\mathfrak{G}) \to \mathcal{O}$ such that $f = \phi \circ \mathrm{c}$.

2.2.2 Evaluations and treelike expressions. Let us first fix a notation. If $\mathcal{O}$ is an operad, the *complete composition map* of $\mathcal{O}$ is the linear map

$$\circ : \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n), \tag{2.2.2}$$

defined, for $x \in \mathcal{O}(n)$ and $y_1, \ldots, y_n \in \mathcal{O}$, by

$$x \circ [y_1, \ldots, y_n] := (\ldots ((x \circ_n y_n) \circ_{n-1} y_{n-1}) \ldots) \circ_1 y_1. \tag{2.2.3}$$

For an operad $\mathcal{O}$, by viewing $\mathcal{O}$ as a graded vector space, $\mathrm{Free}(\mathcal{O})$ is by definition the free operad on $\mathcal{O}$. The *evaluation map* of $\mathcal{O}$ is the linear map

$$\mathrm{ev} : \mathrm{Free}(\mathcal{O}) \to \mathcal{O}, \tag{2.2.4}$$

defined recursively, for any syntax tree $\mathfrak{t}$ on $\mathcal{O}$, by

$$\mathrm{ev}(\mathfrak{t}) := \begin{cases} \mathbb{1} \in \mathcal{O}, & \text{if } \mathfrak{t} = \perp, \\ x \circ [\mathrm{ev}(\mathfrak{t}_1), \ldots, \mathrm{ev}(\mathfrak{t}_k)], & \text{otherwise,} \end{cases} \tag{2.2.5}$$

where x is the label of the root of $\mathfrak{t}$ and $\mathfrak{t}_1, \dots, \mathfrak{t}_k$ are, from left to right, the subtrees of $\mathfrak{t}$. This map is the unique surjective operad morphism from $\mathrm{Free}(\mathcal{O})$ to $\mathcal{O}$ satisfying $\mathrm{ev}(\mathfrak{c}(x)) = x$ for all $x \in \mathcal{O}$. If S is a subspace of $\mathcal{O}$, a *treelike expression* on S of $x \in \mathcal{O}$ is a tree $\mathfrak{t}$ of $\mathrm{Free}(\mathcal{O})$ such that $\mathrm{ev}(\mathfrak{t}) = x$ and all internal nodes of $\mathfrak{t}$ are labeled by S.

2.2.3 Presentations by generators and relations. Let $G := \bigsqcup_{n \geqslant 1} G(n)$ be a graded set. Setting $\mathfrak{G} := \mathbb{K}\langle G \rangle$, we denote the operad ideal of $\mathrm{Free}(\mathfrak{G})$ generated by the subspace $\mathfrak{R}$ of $\mathrm{Free}(\mathfrak{G})$ by $\langle \mathfrak{R} \rangle$. Given an operad $\mathcal{O}$, the pair $(G, \mathfrak{R})$ is a *presentation* of $\mathcal{O}$ if $\mathcal{O}$ is isomorphic to $\mathrm{Free}(\mathfrak{G})/_{\langle \mathfrak{R} \rangle}$. In this case, we call $\mathfrak{G}$ the *space of generators* and $\mathfrak{R}$ the *space of relations* of $\mathcal{O}$. We say that $\mathcal{O}$ is *quadratic* if there is a presentation $(G, \mathfrak{R})$ of $\mathcal{O}$ such that $\mathfrak{R}$ is a homogeneous subspace of $\mathrm{Free}(\mathfrak{G})$ consisting in syntax trees of degree 2. Furthermore, we say that $\mathcal{O}$ is *binary* if there is a presentation $(G, \mathfrak{R})$ of $\mathcal{O}$ such that $\mathfrak{G}$ is concentrated in arity 2. Furthermore, if $\mathcal{O}$ admits a presentation $(G, \mathfrak{R})$ and $\to$ is a rewrite rule on $\mathrm{Free}(\mathfrak{G})$ such that the space induced by $\to$ is $\mathfrak{R}$, we say that $\to$ is an *orientation* of $\mathfrak{R}$.

2.2.4 Koszul duality and Koszulity. In [GK94], Ginzburg and Kapranov extended the notion of Koszul duality of quadratic associative algebras to quadratic operads. Starting with an operad $\mathcal{O}$ admitting a binary and quadratic presentation $(G, \mathfrak{R})$ where G is finite, the *Koszul dual* of $\mathcal{O}$ is the operad $\mathcal{O}^!$, isomorphic to the operad admitting the presentation $\left(G, \mathfrak{R}^\perp\right)$ where $\mathfrak{R}^\perp$ is the annihilator of $\mathfrak{R}$ in $\mathrm{Free}(\mathfrak{G})$, $\mathfrak{G}$ being the space $\mathbb{K}\langle G \rangle$, with respect to the bilinear map

$$\langle -, - \rangle : \mathrm{Free}(\mathfrak{G})(3) \otimes \mathrm{Free}(\mathfrak{G})(3) \to \mathbb{K} \tag{2.2.6}$$

defined, for all $x, x', y, y' \in \mathfrak{G}(2)$, by

$$\langle x \circ_i y, x' \circ_{i'} y' \rangle := \begin{cases} 1, & \text{if } x = x', y = y', \text{ and } i = i' = 1, \\ -1, & \text{if } x = x', y = y', \text{ and } i = i' = 2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.7}$$

Then, with knowledge of a presentation of $\mathcal{O}$, one can compute a presentation of $\mathcal{O}^!$.

Recall that a quadratic operad $\mathcal{O}$ is *Koszul* if its Koszul complex is acyclic [GK94; LV12]. Furthermore, if $\mathcal{O}$ is Koszul and admits an Hilbert series, then the Hilbert series of $\mathcal{O}$ and of its Koszul dual $\mathcal{O}^!$ are related [GK94] by

$$\mathcal{H}_{\mathcal{O}}\left(-\mathcal{H}_{\mathcal{O}^!}(-t)\right) = t. \tag{2.2.8}$$

Relation (2.2.8) can be either used to prove that an operad is not Koszul (this is the case when the coefficients of the hypothetical Hilbert series of the Koszul dual admit coefficients that are not nonnegative integers) or to compute the Hilbert series of the Koszul dual of a Koszul operad.

Here, to prove the Koszulity of an operad $\mathcal{O}$, we shall make use of a tool introduced by Dotsenko and Khoroshkin [DK10] in the context of Gröbner bases for operads, which in our context can be reformulated in the following way by using rewrite rules.

▶ **Lemma 2.2.1** — *Let $\mathcal{O}$ be an operad admitting a quadratic presentation $(G, \mathfrak{R})$. If there exists an orientation $\to$ of $\mathfrak{R}$ such that $\to$ is a convergent rewrite rule, then $\mathcal{O}$ is Koszul.*

If $\to$ satisfies the conditions contained in the statement of Lemma 2.2.1, then the set of normal forms of $\to$ forms a basis of $\mathcal{O}$, called *Poincaré–Birkhoff–Witt basis*. These bases arise from the work of Hoffbeck [Hof10] (see also [LV12]).

2.2.5 Algebras over operads. An operad $\mathcal{O}$ encodes a category of algebras whose objects are called *$\mathcal{O}$-algebras*. An $\mathcal{O}$-algebra $\mathcal{A}_\mathcal{O}$ is a vector space endowed with a linear left action

$$\cdot : \mathcal{O}(n) \otimes \mathcal{A}_\mathcal{O}^{\otimes n} \to \mathcal{A}_\mathcal{O}, \qquad n \geqslant 1, \tag{2.2.9}$$

satisfying the relations imposed by the structure of $\mathcal{O}$, which are

$$(x \circ_i y) \cdot (a_1 \otimes \dots \otimes a_{n+m-1}) \\ = x \cdot (a_1 \otimes \dots \otimes a_{i-1} \otimes y \cdot (a_i \otimes \dots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \dots \otimes a_{n+m-1}), \tag{2.2.10}$$

for all $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, $i \in [n]$, and $a_1 \otimes \dots \otimes a_{n+m-1} \in \mathcal{A}_\mathcal{O}^{\otimes n+m-1}$.

Notice that, by (2.2.10), if G is a generating set of $\mathcal{O}$, it is enough to define the action of each $x \in G$ on $\mathcal{A}_\mathcal{O}^{\otimes |x|}$ to wholly define $\cdot$. In other words, any element x of $\mathcal{O}$ of arity n plays the role of a linear operation

$$x : \mathcal{A}_\mathcal{O}^{\otimes n} \to \mathcal{A}_\mathcal{O}, \tag{2.2.11}$$

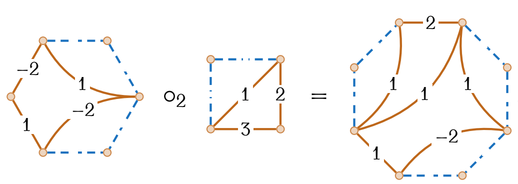
taking n elements of $\mathcal{A}_\mathcal{O}$ as inputs and computing an element of $\mathcal{A}_\mathcal{O}$. By a slight but convenient abuse of notation, for $x \in \mathcal{O}(n)$, we shall write $x(a_1, \dots, a_n)$, or $a_1 \, x \, a_2$ if x has arity 2, for the element $x \cdot (a_1 \otimes \dots \otimes a_n)$ of $\mathcal{A}_\mathcal{O}$, for any $a_1 \otimes \dots \otimes a_n \in \mathcal{A}_\mathcal{O}^{\otimes n}$. Observe that by (2.2.10) an associative element of $\mathcal{O}$ gives rise to an associative operation on $\mathcal{A}_\mathcal{O}$.

3 Operads of noncrossing decorated cliques

We use all the notations and definitions of Sections 1.2 and 2 of [Gir20] about decorated $\mathcal{M}$-cliques and the $\mathcal{M}$-clique operad $\mathrm{C}\mathcal{M}$. We perform here a complete study of the suboperad $\mathrm{Cro}_0\mathcal{M}$ of noncrossing $\mathcal{M}$-cliques defined in Section 3.1.3 of the aforementioned paper. For simplicity, this operad is denoted in the sequel as $\mathrm{NC}\mathcal{M}$ and called the *noncrossing $\mathcal{M}$-clique operad*. The process which produces from a unitary magma $\mathcal{M}$ the operad $\mathrm{NC}\mathcal{M}$ is called the *noncrossing clique construction*.

3.1 General properties

As shown in [Gir20], $\mathrm{NC}\mathcal{M}$ is an operad defined on the linear span of all noncrossing $\mathcal{M}$-cliques and can be seen as a suboperad of $\mathrm{C}\mathcal{M}$ restrained on $\mathcal{M}$-cliques with 0 as crossing number. By definition of $\mathrm{NC}\mathcal{M}$, the partial composition $\mathfrak{p} \circ_i \mathfrak{q}$ of two noncrossing $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathrm{NC}\mathcal{M}$ is equal to the partial composition $\mathfrak{p} \circ_i \mathfrak{q}$ in $\mathrm{C}\mathcal{M}$. Recall that the partial composition $\mathfrak{p} \circ_i \mathfrak{q}$ is the noncrossing $\mathcal{M}$-clique obtained by gluing the base of $\mathfrak{q}$ onto the ith edge of $\mathfrak{p}$ and by relabeling the common arcs between $\mathfrak{p}$ and $\mathfrak{q}$, respectively the arcs $(i, i+1)$ and $(1, m+1)$, by $\mathfrak{p}_i \star \mathfrak{q}_0$, where $\star$ is the magmatic product of $\mathcal{M}$. For instance, in $\mathrm{C}\mathbb{Z}$, we have



(3.1.1)

We call *fundamental basis* of $\mathrm{NC}\mathcal{M}$ the fundamental basis of $\mathrm{C}\mathcal{M}$ restricted to noncrossing $\mathcal{M}$-cliques. Observe that the fundamental basis of $\mathrm{NC}\mathcal{M}$ is a set-operad basis.

To study $\mathrm{NC}\mathcal{M}$, we begin by establishing the fact that $\mathrm{NC}\mathcal{M}$ inherits some properties of $\mathrm{C}\mathcal{M}$. Then we shall describe a realization of $\mathrm{NC}\mathcal{M}$ in terms of decorated Schröder trees, compute a minimal generating set of $\mathrm{NC}\mathcal{M}$, and compute its dimensions.

3.1.1 First properties.

▶ **Proposition 3.1.1** — *Let $\mathcal{M}$ be a unitary magma. Then,*

(i) the associative elements of $\mathrm{NC}\mathcal{M}$ *are the ones of* $\mathrm{C}\mathcal{M}$*;*

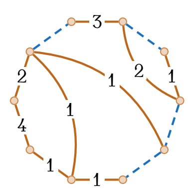
(ii) the group of symmetries of $\mathrm{NC}\mathcal{M}$ *contains the map* ref *(defined by* (2.2.14) *in [Gir20]) and all the maps* $\mathrm{C}\theta$ *where* θ *are unitary magma automorphisms of* $\mathcal{M}$*;*

(iii) the fundamental basis of $\mathrm{NC}\mathcal{M}$ *is a basic set-operad basis if and only if* $\mathcal{M}$ *is right cancelable;*

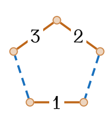
(iv) the map ρ *(defined by* (2.2.16) *in [Gir20]) is a rotation map of* $\mathrm{NC}\mathcal{M}$ *endowing it with a cyclic operad structure.*

◀ **Proof** — First, since $\mathrm{NC}\mathcal{M}$ is a suboperad of $\mathrm{C}\mathcal{M}$, each associative element of $\mathrm{NC}\mathcal{M}$ is an associative element of $\mathrm{C}\mathcal{M}$. Moreover, since all $\mathcal{M}$-bubbles are in $\mathrm{NC}\mathcal{M}$ and, as shown in [Gir20], all associative elements of $\mathrm{C}\mathcal{M}$ are linear combinations of $\mathcal{M}$-bubbles, each associative element of $\mathrm{C}\mathcal{M}$ belongs to $\mathrm{NC}\mathcal{M}$. This shows *(i)*. Moreover, since for any noncrossing $\mathcal{M}$-clique $\mathfrak{p}$, $\mathrm{ref}(\mathfrak{p})$ (resp. $\rho(\mathfrak{p})$) is still noncrossing and ref belongs to the group of symmetries of $\mathrm{C}\mathcal{M}$ (resp. ρ is a rotation map of $\mathrm{C}\mathcal{M}$), *(ii)* (resp. *(iv)*) holds. Finally, again since $\mathrm{NC}\mathcal{M}$ is a suboperad of $\mathrm{C}\mathcal{M}$, since the fundamental basis of $\mathrm{C}\mathcal{M}$ is a basic set-operad basis, and since $\mathrm{NC}\mathcal{M}(2) = \mathrm{C}\mathcal{M}(2)$, *(iii)* holds. ■

3.1.2 Treelike expressions on bubbles. Let $\mathfrak{p}$ be a noncrossing $\mathcal{M}$-clique of arity $n \geqslant 2$, and (x, y) be a diagonal or the base of $\mathfrak{p}$. Let $\{z_1, \dots, z_k\}$ be the set of vertices of $\mathfrak{p}$ such that $x = z_1 < \dots < z_k = y$ and for any $i \in [k-1]$, z_{i+1} is the greatest vertex of $\mathfrak{p}$ such that (z_i, z_{i+1}) is a solid diagonal or a (not necessarily solid) edge of $\mathfrak{p}$. The *area* of $\mathfrak{p}$ adjacent to (x, y) is the $\mathcal{M}$-bubble $\mathfrak{q}$ of arity k whose base is labeled by $\mathfrak{p}(x, y)$ and $\mathfrak{q}_i = \mathfrak{p}(z_i, z_{i+1})$ for all $i \in [k]$. From a geometric point of view, $\mathfrak{q}$ is the unique maximal component of $\mathfrak{p}$ adjacent to the arc (x, y), without solid diagonals, and bounded by solid diagonals or edges of $\mathfrak{p}$. For instance, for the noncrossing $\mathbb{Z}$-clique

$\mathfrak{p} :=$  , (3.1.2)

the path associated with the diagonal $(4, 9)$ of $\mathfrak{p}$ is $(4, 5, 6, 8, 9)$. For this reason, the area of $\mathfrak{p}$ adjacent to $(4, 9)$ is the $\mathbb{Z}$-bubble

 . (3.1.3)

▶ **Proposition 3.1.2** — *Let $\mathcal{M}$ be a unitary magma and $\mathfrak{p}$ be a noncrossing $\mathcal{M}$-clique of arity greater than 1. Then there is a unique $\mathcal{M}$-bubble $\mathfrak{q}$ with a maximal arity $k \geqslant 2$ such that $\mathfrak{p} = \mathfrak{q} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_k]$, where each $\mathfrak{r}_i$, $i \in [k]$, is a noncrossing $\mathcal{M}$-clique with a base labeled by $\mathbb{1}_{\mathcal{M}}$.*

◀ **Proof** — Let $\mathfrak{q}'$ be the area of $\mathfrak{p}$ adjacent to its base and k' be the arity of $\mathfrak{q}'$. By definition of the partial composition of $\mathrm{NC}\mathcal{M}$, for all $\mathcal{M}$-cliques $\mathfrak{u}$, $\mathfrak{u}'$, $\mathfrak{u}_1$, and $\mathfrak{u}_2$, if $\mathfrak{u} = \mathfrak{u}' \circ_i \mathfrak{u}_1 = \mathfrak{u}' \circ_i \mathfrak{u}_2$ and $\mathfrak{u}_1$ and $\mathfrak{u}_2$ have bases labeled by $\mathbb{1}_{\mathcal{M}}$, then $\mathfrak{u}_1 = \mathfrak{u}_2$. This implies in particular that there are unique noncrossing $\mathcal{M}$-cliques $\mathfrak{r}'_i$, $i \in [k']$, with bases labeled by $\mathbb{1}_{\mathcal{M}}$ such that $\mathfrak{p} = \mathfrak{q}' \circ [\mathfrak{r}'_1, \dots, \mathfrak{r}'_{k'}]$. Finally, the fact that $\mathfrak{q}'$ is the area of $\mathfrak{p}$ adjacent to its base implies the maximality for the arity of $\mathfrak{q}'$. The statement of the proposition follows. ■

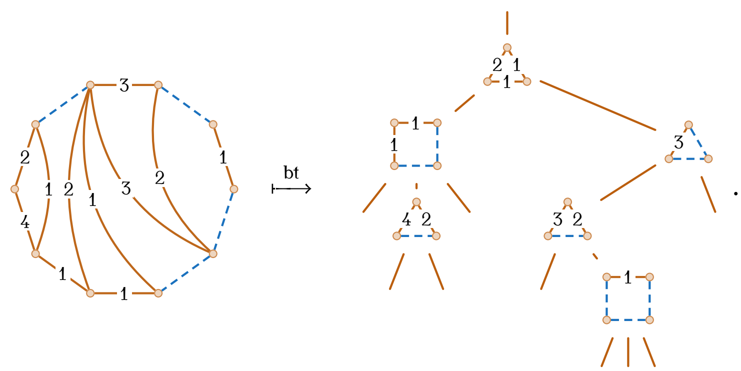
Consider the linear map

$$\mathrm{bt} : \mathrm{NC}\mathcal{M} \to \mathrm{Free}\left(\mathbb{K}\langle \mathcal{B}_{\mathcal{M}} \rangle\right) \tag{3.1.4}$$

defined recursively by $\mathrm{bt}(\circ\!-\!-\!\circ) := \perp$ and, for a noncrossing $\mathcal{M}$-clique $\mathfrak{p}$ of arity greater than 1, by

$$\mathrm{bt}(\mathfrak{p}) := \mathrm{c}(\mathfrak{q}) \circ [\mathrm{bt}(\mathfrak{r}_1), \dots, \mathrm{bt}(\mathfrak{r}_k)], \tag{3.1.5}$$

where $\mathfrak{p} = \mathfrak{q} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_k]$ is the unique decomposition of $\mathfrak{p}$ stated in Proposition 3.1.2. We call $\mathrm{bt}(\mathfrak{p})$ the *bubble tree* of $\mathfrak{p}$. For instance, in $\mathrm{NC}\mathbb{Z}$,



(3.1.6)

▶ Lemma 3.1.3 — *Let $\mathcal{M}$ be a unitary magma. For a noncrossing $\mathcal{M}$-clique $\mathfrak{p}$, $\mathrm{bt}(\mathfrak{p})$ is a treelike expression on $\mathcal{B}_{\mathcal{M}}$ of $\mathfrak{p}$.*

◀ Proof — We proceed by induction on the arity n of $\mathfrak{p}$. If $n = 1$, since $\mathfrak{p} = \circ\!-\!-\!\circ$ and $\mathrm{bt}(\circ\!-\!-\!\circ) = \perp$, the statement of the lemma immediately follows. Otherwise, we have $\mathrm{bt}(\mathfrak{p}) = \mathrm{c}(\mathfrak{q}) \circ [\mathrm{bt}(\mathfrak{r}_1), \dots, \mathrm{bt}(\mathfrak{r}_k)]$ where $\mathfrak{p}$ uniquely decomposes as $\mathfrak{p} = \mathfrak{q} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_k]$ under the conditions stated by Proposition 3.1.2. By definition of area and of the map bt, $\mathfrak{q}$ is an $\mathcal{M}$-bubble. Moreover, by induction hypothesis, any $\mathrm{bt}(\mathfrak{r}_i)$, $i \in [k]$, is a treelike expression on $\mathcal{B}_{\mathcal{M}}$ of $\mathfrak{r}_i$. Hence, $\mathrm{bt}(\mathfrak{p})$ is a treelike expression on $\mathcal{B}_{\mathcal{M}}$ of $\mathfrak{p}$. ■

▶ Proposition 3.1.4 — *Let $\mathcal{M}$ be a unitary magma. Then the map bt is injective and the image of bt is the linear span of all syntax trees $\mathfrak{t}$ on $\mathcal{B}_{\mathcal{M}}$ such that*

- *(i) the root of $\mathfrak{t}$ is labeled by an $\mathcal{M}$-bubble;*
- *(ii) the internal nodes of $\mathfrak{t}$ different from the root are labeled by $\mathcal{M}$-bubbles whose bases are labeled by $\mathbb{1}_{\mathcal{M}}$;*
- *(iii) if x and y are two internal nodes of $\mathfrak{t}$ such that y is the ith child of x, the ith edge of the bubble labeling x is solid.*

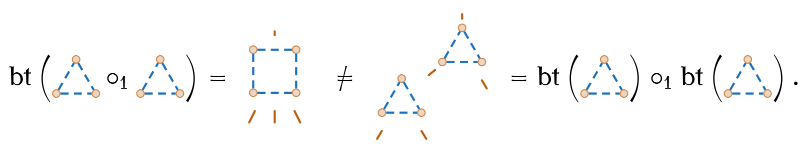
◀ Proof — First of all, since by definition bt sends a basis element of $\mathrm{NC}\mathcal{M}$ to a basis element of $\mathrm{Free}\left(\mathbb{K}\langle \mathcal{B}_{\mathcal{M}} \rangle\right)$, it is sufficient to show that bt is injective as a map from $\mathcal{C}_{\mathcal{M}}$ to the set of syntax trees on $\mathcal{B}_{\mathcal{M}}$ to establish that it is an injective linear map. For this, we proceed by induction on the arity n. If $n = 1$, since $\mathrm{bt}(\circ\!-\!-\!\circ) = \perp$ and $\mathrm{NC}\mathcal{M}(1)$ is of dimension 1, bt is injective. Assume now that $\mathfrak{p}$ and $\mathfrak{p}'$ are two noncrossing $\mathcal{M}$-cliques of arity n such that $\mathrm{bt}(\mathfrak{p}) = \mathrm{bt}(\mathfrak{p}')$. Hence, $\mathfrak{p}$ (resp. $\mathfrak{p}'$) uniquely decomposes as $\mathfrak{p} = \mathfrak{q} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_k]$ (resp. $\mathfrak{p}' = \mathfrak{q}' \circ [\mathfrak{r}'_1, \dots, \mathfrak{r}'_k]$) as stated by Proposition 3.1.2 and

$$\mathrm{bt}(\mathfrak{p}) = \mathrm{c}(\mathfrak{q}) \circ [\mathrm{bt}(\mathfrak{r}_1), \dots, \mathrm{bt}(\mathfrak{r}_k)] = \mathrm{c}(\mathfrak{q}') \circ \left[\mathrm{bt}(\mathfrak{r}'_1), \dots, \mathrm{bt}(\mathfrak{r}'_k)\right] = \mathrm{bt}(\mathfrak{p}'). \tag{3.1.7}$$

Now, because by definition of area, all bases of the $\mathfrak{r}_i$ and $\mathfrak{r}'_i$, $i \in [k]$, are labeled by $\mathbb{1}_{\mathcal{M}}$, this implies that $\mathfrak{q} = \mathfrak{q}'$. Therefore, we have $\mathrm{bt}(\mathfrak{r}_i) = \mathrm{bt}(\mathfrak{r}'_i)$ for all $i \in [k]$, so that, by induction hypothesis, $\mathfrak{r}_i = \mathfrak{r}'_i$ for all $i \in [k]$. Hence, bt is injective.

The definition of bt together with Proposition 3.1.2 leads to the fact that, for a noncrossing $\mathcal{M}$-clique $\mathfrak{p}$, the syntax tree $\mathrm{bt}(\mathfrak{p})$ satisfies *(i)*, *(ii)*, and *(iii)*. Conversely, let $\mathfrak{t}$ be a syntax tree satisfying *(i)*, *(ii)*, and *(iii)*. We show by structural induction on $\mathfrak{t}$ that there is a noncrossing $\mathcal{M}$-clique $\mathfrak{p}$ such that $\mathrm{bt}(\mathfrak{p}) = \mathfrak{t}$. If $\mathfrak{t} = \perp$, the property holds because bt(∘- -∘) $= \perp$. Otherwise, we have $\mathfrak{t} = \mathfrak{s} \circ [\mathfrak{u}_1, \dots, \mathfrak{u}_k]$ where $\mathfrak{s}$ is a syntax tree of degree 1 and the $\mathfrak{u}_i$, $i \in [k]$, are syntax trees. Since $\mathfrak{t}$ satisfies *(i)*, *(ii)*, and *(iii)*, the trees $\mathfrak{s}$ and $\mathfrak{u}_i$, $i \in [k]$, satisfy the same three properties. Therefore, by induction hypothesis, there are noncrossing $\mathcal{M}$-cliques $\mathfrak{q}$ and $\mathfrak{r}_i$, $i \in [k]$, such that $\mathrm{bt}(\mathfrak{q}) = \mathfrak{s}$ and $\mathrm{bt}(\mathfrak{r}_i) = \mathfrak{u}_i$. Now define $\mathfrak{p}$ as the noncrossing $\mathcal{M}$-clique $\mathfrak{q} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_k]$. By definition of the map bt and the unique decomposition stated in Proposition 3.1.2 for $\mathfrak{p}$, one obtains that $\mathrm{bt}(\mathfrak{p}) = \mathfrak{t}$. ■

Observe that bt is not an operad morphism. Indeed,

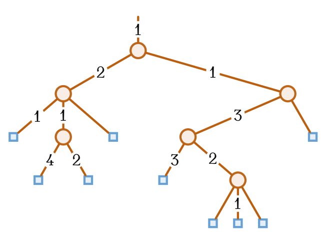


(3.1.8)

Observe that (3.1.8) holds for all unitary magmas $\mathcal{M}$ since $\mathbb{1}_{\mathcal{M}}$ is always idempotent.

3.1.3 Realization in terms of decorated Schröder trees. Recall that a *Schröder tree* is a tree such that all internal nodes have at least two children. An *$\mathcal{M}$-Schröder tree* $\mathfrak{t}$ is a Schröder tree such that each edge connecting two internal nodes is labeled by $\bar{\mathcal{M}}$, each edge connecting an internal node and a leaf is labeled by $\mathcal{M}$, and the outgoing edge from the root of $\mathfrak{t}$ is labeled by $\mathcal{M}$ (see (3.1.9) for an example of a $\mathbb{Z}$-Schröder tree).

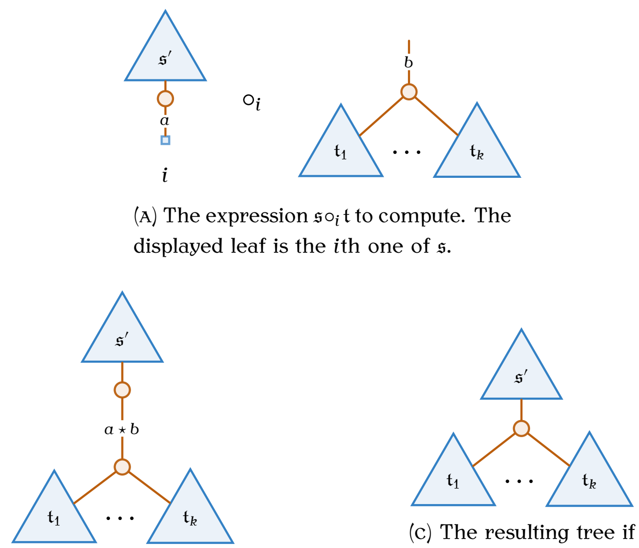
From the description of the image of the map bt provided by Proposition 3.1.4, any bubble tree $\mathfrak{t}$ of a noncrossing $\mathcal{M}$-clique $\mathfrak{p}$ of arity n can be encoded by an $\mathcal{M}$-Schröder tree $\mathfrak{s}$ with n leaves. Indeed, this $\mathcal{M}$-Schröder tree is obtained by considering each internal node x of $\mathfrak{t}$ and by labeling the edge connecting x and its ith child by the label of the ith edge of the $\mathcal{M}$-bubble labeling x. The outgoing edge from the root of $\mathfrak{s}$ is labeled by the label of the base of the $\mathcal{M}$-bubble labeling the root of $\mathfrak{t}$. For instance, the bubble tree of (3.1.6) is encoded by the $\mathbb{Z}$-Schröder tree



, (3.1.9)

where the labels of the edges are placed in their centers and where unlabeled edges are implicitly labeled by $\mathbb{1}_{\mathcal{M}}$. We shall use these drawing conventions in the sequel. As a side remark, observe that the $\mathcal{M}$-Schröder tree encoding a noncrossing $\mathcal{M}$-clique $\mathfrak{p}$ and the dual tree of $\mathfrak{p}$ (in the usual meaning) have the same underlying unlabeled tree.

This encoding of noncrossing $\mathcal{M}$-cliques by bubble trees is reversible, and hence one can interpret $\mathrm{NC}\mathcal{M}$ as an operad on the linear span of all $\mathcal{M}$-Schröder trees. Hence, through this interpretation, if $\mathfrak{s}$ and $\mathfrak{t}$ are two $\mathscr{M}$-Schröder trees and i is a valid integer, the tree $\mathfrak{s} \circ_i \mathfrak{t}$ is computed by grafting the root of $\mathfrak{t}$ to the ith leaf of $\mathfrak{s}$. Then, by writing b for the label of the edge adjacent to the root of $\mathfrak{t}$ and a for the label of the edge adjacent to the ith leaf of $\mathfrak{s}$, we have two cases to consider, depending on the value of $c := a \star b$. If $c \neq \mathbb{1}_{\mathscr{M}}$, we label the edge connecting $\mathfrak{s}$ and $\mathfrak{t}$ by c. Otherwise, if $c = \mathbb{1}_{\mathscr{M}}$, we contract the edge connecting $\mathfrak{s}$ and $\mathfrak{t}$ by merging the root of $\mathfrak{t}$ and the direct ancestor of the ith leaf of $\mathfrak{s}$ (see Figure 1). For instance, in $\mathrm{NC}\mathbb{N}_3$, we have the



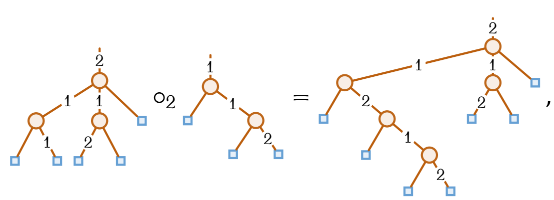
(A) The expression $\mathfrak{s} \circ_i \mathfrak{t}$ to compute. The displayed leaf is the ith one of $\mathfrak{s}$.

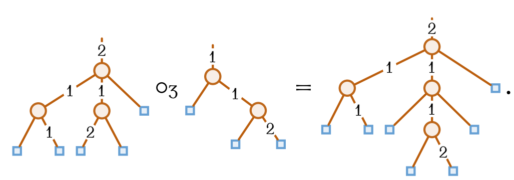
(B) The resulting tree if $a \star b \neq \mathbb{1}_{\mathscr{M}}$.

(C) The resulting tree if $a \star b = \mathbb{1}_{\mathscr{M}}$.

FIGURE 1. The partial composition of $\mathrm{NC}\mathscr{M}$ realized on $\mathscr{M}$-Schröder trees. Here, the two cases (b) and (c) for the computation of $\mathfrak{s} \circ_i \mathfrak{t}$ are shown, where $\mathfrak{s}$ and $\mathfrak{t}$ are two $\mathscr{M}$-Schröder trees. In these drawings, the triangles denote subtrees.

two partial compositions

 (3.1.10a)

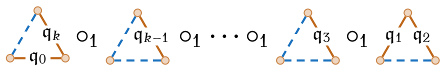
 (3.1.10b)

In the sequel, we shall indifferently view $\mathrm{NC}\mathscr{M}$ as an operad on noncrossing $\mathscr{M}$-cliques or on $\mathscr{M}$-Schröder trees.

3.1.4 Minimal generating set.

▶ **Proposition 3.1.5 —** *Let $\mathcal{M}$ be a unitary magma. The set $\mathfrak{T}_{\mathcal{M}}$ of all $\mathcal{M}$-triangles is a minimal generating set of* $\mathrm{NC}\mathcal{M}$*.*

◀ **Proof —** We start by showing by induction on the arity that the suboperad $(\mathrm{NC}\mathcal{M})^{\mathfrak{T}_{\mathcal{M}}}$ of $\mathrm{NC}\mathcal{M}$ generated by $\mathfrak{T}_{\mathcal{M}}$ is $\mathrm{NC}\mathcal{M}$. This is immediately true in arity 1. Let $\mathfrak{p}$ be a noncrossing $\mathcal{M}$-clique of arity $n \geqslant 2$. Proposition 3.1.2 says in particular that we can express $\mathfrak{p}$ as $\mathfrak{p} = \mathfrak{q} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_k]$ where $\mathfrak{q}$ is an $\mathcal{M}$-bubble of arity $k \geqslant 2$ and the $\mathfrak{r}_i$, $i \in [k]$, are noncrossing $\mathcal{M}$-cliques. Since $\mathfrak{q}$ is an $\mathcal{M}$-bubble, it can be expressed as

$\mathfrak{q} =$ 

(3.1.11)

Observe that, in (3.1.11), brackets are not necessary since $\circ_1$ is associative. Since $k \geqslant 2$, the arities of each $\mathfrak{r}_i$, $i \in [k]$, are smaller than the one of $\mathfrak{p}$. For this reason, by induction hypothesis, each $\mathfrak{r}_i$ belongs to $(\mathrm{NC}\mathcal{M})^{\mathfrak{T}_{\mathcal{M}}}$. Moreover, since (3.1.11) shows an expression of $\mathfrak{q}$ by partial compositions of $\mathcal{M}$-triangles, $\mathfrak{q}$ also belongs to $(\mathrm{NC}\mathcal{M})^{\mathfrak{T}_{\mathcal{M}}}$. This implies that this is also the case for $\mathfrak{p}$. Hence, $\mathrm{NC}\mathcal{M}$ is generated by $\mathfrak{T}_{\mathcal{M}}$.

Finally, due to the fact that the partial composition of two $\mathcal{M}$-triangles is an $\mathcal{M}$-clique of arity 3, if $\mathfrak{p}$ is an $\mathcal{M}$-triangle, $\mathfrak{p}$ cannot be expressed as a partial composition of $\mathcal{M}$-triangles. Moreover, since the space $\mathrm{NC}\mathcal{M}(1)$ is trivial, these arguments imply that $\mathfrak{T}_{\mathcal{M}}$ is a minimal generating set of $\mathrm{NC}\mathcal{M}$. ■

Proposition 3.1.5 also says that $\mathrm{NC}\mathcal{M}$ is the smallest suboperad of $\mathrm{C}\mathcal{M}$ that contains all $\mathcal{M}$-triangles and that $\mathrm{NC}\mathcal{M}$ is the biggest binary suboperad of $\mathrm{C}\mathcal{M}$.

3.1.5 DIMENSIONS. We now use the notion of bubble trees introduced in Section 3.1.2 to compute the dimensions of $\mathrm{NC}\mathcal{M}$.

▶ **Proposition 3.1.6 —** *Let $\mathcal{M}$ be a finite unitary magma. The Hilbert series $\mathcal{H}_{\mathrm{NC}\mathcal{M}}(t)$ of* $\mathrm{NC}\mathcal{M}$ *satisfies*

$$t + \left(m^3 - 2m^2 + 2m - 1\right) t^2 + \left(2m^2 t - 3mt + 2t - 1\right) \mathcal{H}_{\mathrm{NC}\mathcal{M}}(t) + (m-1)\, \mathcal{H}_{\mathrm{NC}\mathcal{M}}(t)^2 = 0, \quad (3.1.12)$$

where $m := \#\mathcal{M}$.

◀ **Proof —** By Proposition 3.1.4, the set of noncrossing $\mathcal{M}$-cliques is in one-to-one correspondence with the set of syntax trees on $\mathcal{B}_{\mathcal{M}}$ that satisfy *(i)*, *(ii)*, and *(iii)*. We call $T(t)$ the generating series of these trees and $S(t)$ the generating series of these trees with the extra condition that the roots are labeled by $\mathcal{M}$-bubbles whose bases are labeled by $\mathbb{1}_{\mathcal{M}}$. Immediately from its description, $S(t)$ satisfies

$$S(t) = t + \sum_{n \geqslant 2} ((m-1)S(t) + t)^n \,, \quad (3.1.13)$$

and $T(t)$ satisfies

$$T(t) = t + m(S(t) - t). \quad (3.1.14)$$

As the set of noncrossing $\mathcal{M}$-cliques forms the fundamental basis of $\mathrm{NC}\mathcal{M}$, we have $\mathcal{H}_{\mathrm{NC}\mathcal{M}}(t) = T(t)$. We eventually obtain (3.1.12) from (3.1.13) and (3.1.14) by a direct computation. ■

From Proposition 3.1.6, we deduce that the Hilbert series of $\mathrm{NC}\mathcal{M}$ satisfies

$$\mathcal{H}_{\mathrm{NC}\mathcal{M}}(t) = \frac{1 - (2m^2 - 3m + 2)t - \sqrt{1 - 2(2m^2 - m)t + m^2 t^2}}{2(m-1)}, \quad (3.1.15)$$

where $m := \#\mathcal{M} \neq 1$.

By using Narayana numbers, whose definition is recalled in Section 3.1.6 of [Gir20], we can state the following result.

▶ Proposition 3.1.7 — *Let $\mathcal{M}$ be a finite unitary magma. For all $n \geqslant 2$,*

$$\dim \mathrm{NC}\mathcal{M}(n) = \sum_{0 \leqslant k \leqslant n-2} m^{n+k+1} (m-1)^{n-k-2} \operatorname{nar}(n, k), \tag{3.1.16}$$

where $m := \#\mathcal{M}$.

◀ Proof — As shown by Proposition 3.1.4, each noncrossing $\mathcal{M}$-clique $\mathfrak{p}$ of $\mathrm{NC}\mathcal{M}(n)$ can be encoded by a unique syntax tree $\mathrm{bt}(\mathfrak{p})$ on $\mathfrak{B}_{\mathcal{M}}$ satisfying some conditions. Moreover, Proposition 3.1.5 shows that a noncrossing $\mathcal{M}$-clique can be expressed (not necessarily in a unique way) as partial compositions of several $\mathcal{M}$-triangles. By combining these two results, we obtain that a noncrossing $\mathcal{M}$-clique $\mathfrak{p}$ can be encoded by a syntax tree on $\mathcal{T}_{\mathcal{M}}$ obtained from $\mathrm{bt}(\mathfrak{p})$ by replacing each of its nodes $\mathfrak{s}$ of arity $\ell \geqslant 3$ by left comb binary syntax trees $\mathfrak{s}'$ on $\mathcal{T}_{\mathcal{M}}$ satisfying

$$\mathfrak{s}' := \mathrm{c}\left(\mathfrak{q}^1\right) \circ_1 \mathrm{c}\left(\mathfrak{q}^2\right) \circ_1 \cdots \circ_1 \mathrm{c}\left(\mathfrak{q}^{\ell-1}\right), \tag{3.1.17}$$

where the $\mathfrak{q}^i$, $i \in [\ell - 1]$, are the unique $\mathcal{M}$-triangles such that, for every $i \in [2, \ell-1]$, the base of $\mathfrak{q}^i$ is labeled by $\mathbb{1}_{\mathcal{M}}$, for every $j \in [\ell - 2]$, the first edge of $\mathfrak{q}^j$ is labeled by $\mathbb{1}_{\mathcal{M}}$, and $\mathrm{ev}(\mathfrak{s}') = \mathrm{ev}(\mathfrak{s})$. Observe that, in (3.1.17), brackets are not necessary since $\circ_1$ is associative. Therefore, $\mathfrak{p}$ can be encoded in a unique way as a binary syntax tree $\mathfrak{t}$ on $\mathcal{T}_{\mathcal{M}}$ satisfying the following restrictions:

(i) the $\mathcal{M}$-triangles labeling the internal nodes of $\mathfrak{t}$ which are not the root have bases labeled by $\mathbb{1}_{\mathcal{M}}$;

(ii) if x and y are two internal nodes of $\mathfrak{t}$ such that y is the right child of x, the second edge of the bubble labeling x is solid.

To establish (3.1.16), since the set of noncrossing $\mathcal{M}$-cliques forms the fundamental basis of $\mathrm{NC}\mathcal{M}$, we now have to count these binary trees. Consider a binary tree $\mathfrak{t}$ of arity $n \geqslant 2$ with exactly $k \in [0, n-2]$ internal nodes having an internal node as a left child. There are m ways to label the base of the $\mathcal{M}$-triangle labeling the root of $\mathfrak{t}$, m^k ways to label the first edges of the $\mathcal{M}$-triangles labeling the internal nodes of $\mathfrak{t}$ that have an internal node as left child, m^n ways to label the first (resp. second) edges of the $\mathcal{M}$-triangles labeling the internal nodes of $\mathfrak{t}$ having a leaf as left (resp. right) child, and, since there are exactly $n - k - 2$ internal nodes of $\mathfrak{t}$ having an internal node as a right child, there are $(m-1)^{n-k-2}$ ways to label the second edges of the $\mathcal{M}$-triangles labeling these internal nodes. Now, since $\operatorname{nar}(n, k)$ counts the binary trees with n leaves and exactly k internal nodes having an internal node as a left child, and a binary tree with n leaves can have at most $n - 2$ internal nodes having an internal node as left child, (3.1.16) follows. ■

We can use Proposition 3.1.7 to compute the first dimensions of $\mathrm{NC}\mathcal{M}$. For instance, depending on $m := \#\mathcal{M}$, we have the following sequences of dimensions:

$$1, 1, 1, 1, 1, 1, 1, 1, \qquad m = 1, \tag{3.1.18a}$$

$$1, 8, 48, 352, 2880, 25216, 231168, 2190848, \qquad m = 2, \tag{3.1.18b}$$

$$1, 27, 405, 7533, 156735, 349263, 81520425, 1967414265, \qquad m = 3, \tag{3.1.18c}$$

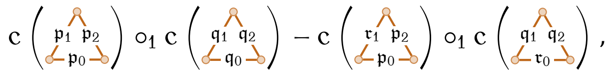
$$1, 64, 1792, 62464, 2437120, 101859328, 4459528192, 201889939456. \qquad m = 4, \tag{3.1.18d}$$

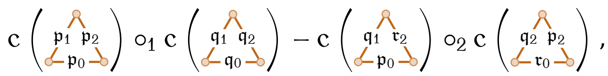
The second one forms, except for the first terms, Sequence **A054726** of [Slo]. The last two sequences are not listed in [Slo] at this time.

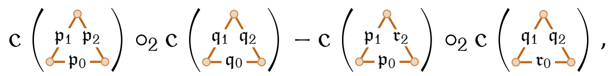
3.2 Presentation and Koszulity

The aim of this section is to establish a presentation by generators and relations of $\mathrm{NC}\mathcal{M}$. For this, we will define an adequate rewrite rule on the set of syntax trees on $\mathcal{T}_{\mathcal{M}}$ and prove that it admits the required properties.

3.2.1 Space of relations. Let $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$ be the subspace of $\mathrm{Free}(\mathbb{K}\langle\mathcal{T}_{\mathcal{M}}\rangle)(3)$ generated by the elements

 if $\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{r}_1 \star \mathfrak{r}_0 \neq \mathbb{1}_{\mathcal{M}}$, (3.2.1a)

 if $\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{r}_2 \star \mathfrak{r}_0 = \mathbb{1}_{\mathcal{M}}$, (3.2.1b)

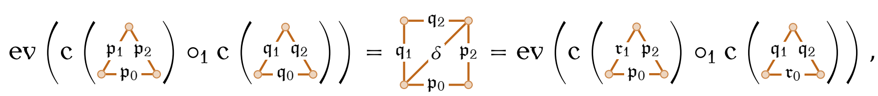
 if $\mathfrak{p}_2 \star \mathfrak{q}_0 = \mathfrak{r}_2 \star \mathfrak{r}_0 \neq \mathbb{1}_{\mathcal{M}}$, (3.2.1c)

where $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{r}$ are $\mathcal{M}$-triangles.

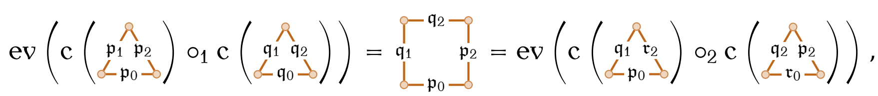
▶ **Lemma 3.2.1** — *Let $\mathcal{M}$ be a unitary magma, and $\mathfrak{s}$ and $\mathfrak{t}$ be two syntax trees of arity 3 on $\mathcal{T}_{\mathcal{M}}$. Then $\mathfrak{s} - \mathfrak{t}$ belongs to $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$ if and only if $\mathrm{ev}(\mathfrak{s}) = \mathrm{ev}(\mathfrak{t})$.*

◀ **Proof** — Assume first that $\mathfrak{s} - \mathfrak{t}$ belongs to $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$. Then $\mathfrak{s} - \mathfrak{t}$ is a linear combination of elements of the form (3.2.1a), (3.2.1b), and (3.2.1c). Now, observe that, if $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{r}$ are three $\mathcal{M}$-triangles,

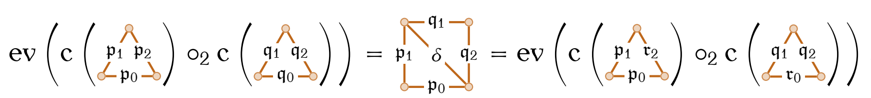
(a) if $\delta := \mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{r}_1 \star \mathfrak{r}_0 \neq \mathbb{1}_{\mathcal{M}}$, we have

 (3.2.2)

(b) if $\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{r}_2 \star \mathfrak{r}_0 = \mathbb{1}_{\mathcal{M}}$, we have

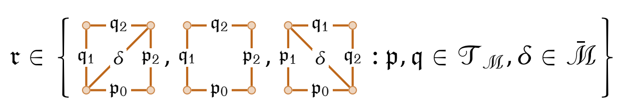
 (3.2.3)

(c) if $\delta := \mathfrak{p}_2 \star \mathfrak{q}_0 = \mathfrak{r}_2 \star \mathfrak{r}_0 \neq \mathbb{1}_{\mathcal{M}}$, we have

 (3.2.4)

This shows that all evaluations in $\mathrm{NC}\mathcal{M}$ of (3.2.1a), (3.2.1b), and (3.2.1c) are equal to zero. Therefore, $\mathrm{ev}(\mathfrak{s} - \mathfrak{t}) = 0$, and hence we have $\mathrm{ev}(\mathfrak{s}) - \mathrm{ev}(\mathfrak{t}) = 0$ and, as expected, $\mathrm{ev}(\mathfrak{s}) = \mathrm{ev}(\mathfrak{t})$.

We now assume that $\mathrm{ev}(\mathfrak{s}) = \mathrm{ev}(\mathfrak{t})$ and let $\mathfrak{r} := \mathrm{ev}(\mathfrak{s})$. As $\mathfrak{s}$ is of arity 3, $\mathfrak{r}$ also is of arity 3 and thus,

 (3.2.5)

Now, by definition of the partial composition of $\mathrm{NC}\mathcal{M}$, if $\mathfrak{r}$ has the form of the first (resp. second, third) noncrossing $\mathcal{M}$-clique appearing in (3.2.5), $\mathfrak{s}$ and $\mathfrak{t}$ are of the form of the first and second syntax trees of (3.2.1a) (resp. (3.2.1b), (3.2.1c)). Hence, in all cases, $\mathfrak{s} - \mathfrak{t}$ is in $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$. ■

▶ **Proposition 3.2.2** — *Let $\mathcal{M}$ be a finite unitary magma. Then the dimension of the space $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$ satisfies*

$$\dim \mathfrak{R}_{\mathrm{NC}\mathcal{M}} = 2m^6 - 2m^5 + m^4, \tag{3.2.6}$$

where $m := \#\mathcal{M}$.

◀ **Proof —** For $x \in \mathcal{M}$, let $f(x)$ be the number of ordered pairs $(y, z) \in \mathcal{M}^2$ such that $x = y \star z$. Since $\mathcal{M}$ is finite, $f : \mathcal{M} \to \mathbb{N}$ is a well-defined map.

Let $\equiv$ be the equivalence relation on the set of syntax trees on $\mathcal{T}_{\mathcal{M}}$ of arity 3 satisfying $\mathfrak{s} \equiv \mathfrak{t}$ if $\mathfrak{s}$ and $\mathfrak{t}$ are two such syntax trees satisfying $\mathrm{ev}(\mathfrak{s}) = \mathrm{ev}(\mathfrak{t})$. Let also C be the set of noncrossing $\mathcal{M}$-cliques of arity 3. For $\mathfrak{r} \in C$, we denote the set of syntax trees $\mathfrak{s}$ satisfying $\mathrm{ev}(\mathfrak{s}) = \mathfrak{r}$ by $[\mathfrak{r}]_\equiv$. Proposition 3.1.5 says in particular that any $\mathfrak{r} \in C$ can be obtained by a partial composition of two $\mathcal{M}$-triangles, and hence all $[\mathfrak{r}]_\equiv$ are nonempty sets and thus, are $\equiv$-equivalence classes.

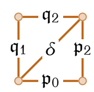
Moreover, by Lemma 3.2.1, for syntax trees $\mathfrak{s}$ and $\mathfrak{t}$, we have $\mathfrak{s} \equiv \mathfrak{t}$ if and only if $\mathfrak{s} - \mathfrak{t}$ is in $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$. For this reason, the dimension of $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$ is linked with the cardinalities of all $\equiv$-equivalence classes by

$$\dim \mathfrak{R}_{\mathrm{NC}\mathcal{M}} = \sum_{\mathfrak{r} \in C} \left(\#[\mathfrak{r}]_\equiv - 1\right). \tag{3.2.7}$$

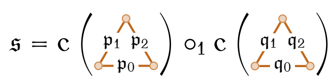
We now compute (3.2.7) by enumerating each $\equiv$-equivalence class $[\mathfrak{r}]_\equiv$.

Observe that, since $\mathfrak{r}$ is of arity 3, it can be of three different forms according to the presence of a solid diagonal.

(a) If

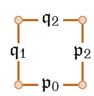
$\mathfrak{r} =$  (3.2.8)

for some $\mathfrak{p}_0, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathcal{M}$ and $\delta \in \bar{\mathcal{M}}$, to have $\mathfrak{s} \in [\mathfrak{r}]_\equiv$, we necessarily have

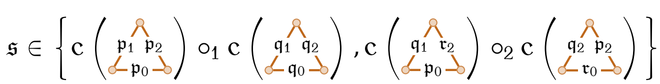
 (3.2.9)

where $\mathfrak{p}_1, \mathfrak{q}_0 \in \mathcal{M}$ and $\mathfrak{p}_1 \star \mathfrak{q}_0 = \delta$. Hence, $\#[\mathfrak{r}]_\equiv = f(\delta)$.

(b) If

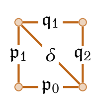
$\mathfrak{r} =$  (3.2.10)

for some $\mathfrak{p}_0, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathcal{M}$, to have $\mathfrak{s} \in [\mathfrak{r}]_\equiv$, we necessarily have

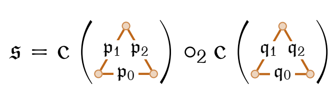
 (3.2.11)

where $\mathfrak{p}_1, \mathfrak{q}_0, \mathfrak{r}_0, \mathfrak{r}_2 \in \mathcal{M}$, $\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}$, and $\mathfrak{r}_2 \star \mathfrak{r}_0 = \mathbb{1}_{\mathcal{M}}$. Hence, $\#[\mathfrak{r}]_\equiv = 2f(\mathbb{1}_{\mathcal{M}})$.

(c) Otherwise,

$\mathfrak{r} =$  (3.2.12)

for some $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathcal{M}$ and $\delta \in \bar{\mathcal{M}}$, and to have $\mathfrak{s} \in [\mathfrak{r}]_\equiv$, we necessarily have

 (3.2.13)

where $\mathfrak{p}_2, \mathfrak{q}_0 \in \mathcal{M}$ and $\mathfrak{p}_2 \star \mathfrak{q}_0 = \delta$. Hence, $\#[\mathfrak{r}]_\equiv = f(\delta)$.

Therefore, by using the fact that

$$\sum_{\delta \in \mathcal{M}} f(\delta) = m^2, \tag{3.2.14}$$

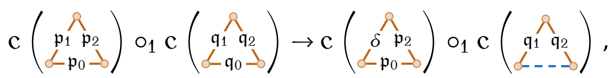
from (3.2.7) we obtain

$$\begin{aligned}\dim \mathfrak{R}_{\mathrm{NC}\mathcal{M}} &= \sum_{\substack{\mathfrak{p}_0,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2\in\mathcal{M}\\ \delta\in\bar{\mathcal{M}}}} (f(\delta)-1) + \sum_{\mathfrak{p}_0,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2\in\mathcal{M}} (2f(\mathbb{1}_{\mathcal{M}})-1) + \sum_{\substack{\mathfrak{p}_0,\mathfrak{p}_1,\mathfrak{q}_1,\mathfrak{q}_2\in\mathcal{M}\\ \delta\in\bar{\mathcal{M}}}} (f(\delta)-1)\\ &= m^4\left(2\sum_{\delta\in\bar{\mathcal{M}}}(f(\delta)-1) + 2f(\mathbb{1}_{\mathcal{M}})-1\right)\\ &= 2m^6-2m^5+m^4,\end{aligned} \tag{3.2.15}$$

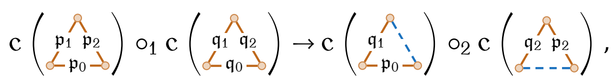
establishing the statement of the proposition. ■

Observe that, by Proposition 3.2.2, the dimension of $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$ only depends on the cardinality of $\mathcal{M}$ and not on its operation $\star$.

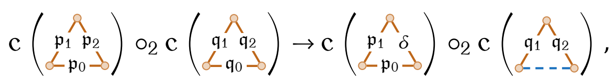
3.2.2 Rewrite rule. Let $\to$ be the rewrite rule on the set of syntax trees on $\mathcal{T}_{\mathcal{M}}$ satisfying



if $\mathfrak{q}_0 \neq \mathbb{1}_{\mathcal{M}}$, where $\delta := \mathfrak{p}_1 \star \mathfrak{q}_0$, (3.2.16a)



if $\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}$, (3.2.16b)



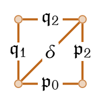
if $\mathfrak{q}_0 \neq \mathbb{1}_{\mathcal{M}}$, where $\delta := \mathfrak{p}_2 \star \mathfrak{q}_0$, (3.2.16c)

where $\mathfrak{p}$ and $\mathfrak{q}$ are $\mathcal{M}$-triangles.

▶ Lemma 3.2.3 — *Let $\mathcal{M}$ be a unitary magma. Then the vector space induced by the rewrite rule $\to$ is $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$.*

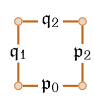
◀ Proof — Let $\mathfrak{s}$ and $\mathfrak{t}$ be two syntax trees on $\mathcal{T}_{\mathcal{M}}$ such that $\mathfrak{s} \to \mathfrak{t}$. We have three cases to consider depending on the form of $\mathfrak{s}$ and $\mathfrak{t}$.

(a) if $\mathfrak{s}$ (resp. $\mathfrak{t}$) is of the form described by the left (resp. right) side of (3.2.16a), we have

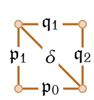
$\mathrm{ev}(\mathfrak{s}) =$  $= \mathrm{ev}(\mathfrak{t}),$ (3.2.17)

where $\delta := \mathfrak{p}_1 \star \mathfrak{q}_0$.

(b) If $\mathfrak{s}$ (resp. $\mathfrak{t}$) is of the form described by the left (resp. right) side of (3.2.16b), we have

$\mathrm{ev}(\mathfrak{s}) =$  $= \mathrm{ev}(\mathfrak{t}).$ (3.2.18)

(c) Otherwise, $\mathfrak{s}$ (resp. $\mathfrak{t}$) is of the form described by the left (resp. right) side of (3.2.16c). We have

$\mathrm{ev}(\mathfrak{s}) =$  $= \mathrm{ev}(\mathfrak{t}),$ (3.2.19)

where $\delta := \mathfrak{p}_2 \star \mathfrak{q}_0$.

Therefore, by Lemma 3.2.1 we have $\mathfrak{s} - \mathfrak{t} \in \mathfrak{R}_{\mathrm{NC}\mathcal{M}}$ for each case. This leads to the fact that $\mathfrak{s} \overset{*}{\leftrightarrow} \mathfrak{t}$ implies $\mathfrak{s} - \mathfrak{t} \in \mathfrak{R}_{\mathrm{NC}\mathcal{M}}$, and shows that the space induced by $\to$ is a subspace of $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$.

We now assume that $\mathfrak{s}$ and $\mathfrak{t}$ are two syntax trees on $\mathcal{T}_{\mathcal{M}}$ such that $\mathfrak{s} - \mathfrak{t}$ is a generator of $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$ among (3.2.1a), (3.2.1b), and (3.2.1c).

(a) If $\mathfrak{s}$ (resp. $\mathfrak{t}$) is of the form described by the left (resp. right) side of (3.2.1a), we have by (3.2.16a),

$$\mathfrak{s} \xrightarrow{*} \mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\delta\ \mathfrak{p}_2}\right) \circ_1 \mathrm{c}\left(\triangle_{- - -}^{\mathfrak{q}_1\ \mathfrak{q}_2}\right) \quad \text{and} \quad \mathfrak{t} \xrightarrow{*} \mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\delta'\ \mathfrak{p}_2}\right) \circ_1 \mathrm{c}\left(\triangle_{- - -}^{\mathfrak{q}_1\ \mathfrak{q}_2}\right), \tag{3.2.20}$$

where $\delta := \mathfrak{p}_1 \star \mathfrak{q}_0$ and $\delta' := \mathfrak{r}_1 \star \mathfrak{r}_0$. Since by (3.2.1a), we have $\delta = \delta'$, we obtain that $\mathfrak{s} \overset{*}{\leftrightarrow} \mathfrak{t}$.

(b) If $\mathfrak{s}$ (resp. $\mathfrak{t}$) is of the form described by the left (resp. right) side of (3.2.1b), we have by (3.2.16b) and by (3.2.16c),

$$\mathfrak{s} \to \mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\mathfrak{q}_1\ -}\right) \circ_2 \mathrm{c}\left(\triangle_{- - -}^{\mathfrak{q}_2\ \mathfrak{p}_2}\right) \quad \text{and} \quad \mathfrak{t} \xrightarrow{*} \mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\mathfrak{q}_1\ -}\right) \circ_2 \mathrm{c}\left(\triangle_{- - -}^{\mathfrak{q}_2\ \mathfrak{p}_2}\right). \tag{3.2.21}$$

We obtain that $\mathfrak{s} \overset{*}{\leftrightarrow} \mathfrak{t}$.

(c) Otherwise, $\mathfrak{s}$ (resp. $\mathfrak{t}$) is of the form described by the left (resp. right) side of (3.2.1c). We have by (3.2.16c),

$$\mathfrak{s} \xrightarrow{*} \mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\ \delta}\right) \circ_2 \mathrm{c}\left(\triangle_{- - -}^{\mathfrak{q}_1\ \mathfrak{q}_2}\right) \quad \text{and} \quad \mathfrak{t} \xrightarrow{*} \mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\ \delta'}\right) \circ_2 \mathrm{c}\left(\triangle_{- - -}^{\mathfrak{q}_1\ \mathfrak{q}_2}\right), \tag{3.2.22}$$

where $\delta := \mathfrak{p}_2 \star \mathfrak{q}_0$ and $\delta' := \mathfrak{r}_2 \star \mathfrak{r}_0$. Since by (3.2.1c), $\delta = \delta'$, we obtain that $\mathfrak{s} \overset{*}{\leftrightarrow} \mathfrak{t}$.

Hence, for each case, we have $\mathfrak{s} \overset{*}{\leftrightarrow} \mathfrak{t}$. This shows that $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$ is a subspace of the space induced by $\to$. The statement of the lemma follows. ■

▶ **Lemma 3.2.4 —** *For a unitary magma $\mathcal{M}$, the rewrite rule $\to$ is terminating.*

◀ **Proof —** Writing T_n for the set of syntax trees on $\mathcal{T}_{\mathcal{M}}$ of arity n, let $\phi : T_n \to \mathbb{N}^2$ be the map defined in the following way. For a syntax tree $\mathfrak{t}$ of T_n, $\phi(\mathfrak{t}) := (\alpha, \beta)$, where α is the sum of the number of internal nodes in the left subtree of x taken over all internal nodes x of $\mathfrak{t}$, and β is the number of internal nodes of $\mathfrak{t}$ labeled by an $\mathcal{M}$-triangle whose base is not labeled by $\mathbb{1}_{\mathcal{M}}$. Let $\mathfrak{s}$ and $\mathfrak{t}$ be two syntax trees of T_3 such that $\mathfrak{s} \to \mathfrak{t}$. Due to the definition of $\to$, we have three configurations to explore. In what follows, $\eta : \mathcal{M} \to \mathbb{N}$ is the map satisfying $\eta(a) := 0$ if $a = \mathbb{1}_{\mathcal{M}}$ and $\eta(a) := 1$ otherwise.

(a) If $\mathfrak{s}$ (resp. $\mathfrak{t}$) is of the form described by the left (resp. right) side of (3.2.16a), writing $\leqslant$ for the lexicographic order on $\mathbb{N}^2$, we have

$$\begin{aligned} \phi\left(\mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\ \mathfrak{p}_2}\right) \circ_1 \mathrm{c}\left(\triangle_{\mathfrak{q}_0}^{\mathfrak{q}_1\ \mathfrak{q}_2}\right)\right) &= (1, \eta(\mathfrak{p}_0) + 1) \\ &> (1, \eta(\mathfrak{p}_0)) = \phi\left(\mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\delta\ \mathfrak{p}_2}\right) \circ_1 \mathrm{c}\left(\triangle_{- - -}^{\mathfrak{q}_1\ \mathfrak{q}_2}\right)\right), \end{aligned} \tag{3.2.23}$$

where $\delta := \mathfrak{p}_1 \star \mathfrak{q}_0$.

(b) If $\mathfrak{s}$ (resp. $\mathfrak{t}$) is of the form described by the left (resp. right) side of (3.2.16b), we have

$$\begin{aligned} \phi\left(\mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\ \mathfrak{p}_2}\right) \circ_1 \mathrm{c}\left(\triangle_{\mathfrak{q}_0}^{\mathfrak{q}_1\ \mathfrak{q}_2}\right)\right) &= (1, \eta(\mathfrak{p}_0) + \eta(\mathfrak{q}_0)) \\ &> (0, \eta(\mathfrak{p}_0)) = \phi\left(\mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\mathfrak{q}_1\ -}\right) \circ_2 \mathrm{c}\left(\triangle_{- - -}^{\mathfrak{q}_2\ \mathfrak{p}_2}\right)\right). \end{aligned} \tag{3.2.24}$$

(c) Otherwise, $\mathfrak{s}$ (resp. $\mathfrak{t}$) is of the form described by the left (resp. right) side of (3.2.16c). We have

$$\begin{aligned} \phi\left(\mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\ \mathfrak{p}_2}\right) \circ_2 \mathrm{c}\left(\triangle_{\mathfrak{q}_0}^{\mathfrak{q}_1\ \mathfrak{q}_2}\right)\right) &= (0, \eta(\mathfrak{p}_0) + 1) \\ &> (0, \eta(\mathfrak{p}_0)) = \phi\left(\mathrm{c}\left(\triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\ \delta}\right) \circ_2 \mathrm{c}\left(\triangle_{- - -}^{\mathfrak{q}_1\ \mathfrak{q}_2}\right)\right), \end{aligned} \tag{3.2.25}$$

where $\delta := \mathfrak{p}_2 \star \mathfrak{q}_0$.

Therefore, for all syntax trees $\mathfrak{s}$ and $\mathfrak{t}$ such that $\mathfrak{s} \to \mathfrak{t}$, $\phi(\mathfrak{s}) > \phi(\mathfrak{t})$. This implies that, for all syntax trees $\mathfrak{s}$ and $\mathfrak{t}$ such that $\mathfrak{s} \neq \mathfrak{t}$ and $\mathfrak{s} \xrightarrow{*} \mathfrak{t}$, we have $\phi(\mathfrak{s}) > \phi(\mathfrak{t})$. Since $(0,0)$ is the smallest element of $\mathbb{N}^2$ with respect to the lexicographic order $\leqslant$, the statement of the lemma follows. ■

▶ Lemma 3.2.5 — *Let $\mathcal{M}$ be a unitary magma. The set of normal forms of the rewrite rule $\to$ is the set of syntax trees $\mathfrak{t}$ on $\mathcal{T}_{\mathcal{M}}$ such that, for any internal nodes x and y of $\mathfrak{t}$ where y is a child of x,*

(i) the base of the $\mathcal{M}$-triangle labeling y is labeled by $\mathbb{1}_{\mathcal{M}}$;

(ii) if y is a left child of x, the first edge of the $\mathcal{M}$-triangle labeling x is not labeled by $\mathbb{1}_{\mathcal{M}}$.

◀ Proof — By Lemma 3.2.4, $\to$ is terminating. Therefore, $\to$ admits normal forms, which are by definition the syntax trees on $\mathcal{T}_{\mathcal{M}}$ that cannot be rewritten by $\to$.

Let $\mathfrak{t}$ be a normal form of $\to$. The fact that $\mathfrak{t}$ satisfies *(i)* is an immediate consequence of the fact that $\mathfrak{t}$ avoids the patterns appearing as left sides of (3.2.16a) and (3.2.16c). Moreover, since $\mathfrak{t}$ avoids the patterns appearing as left sides of (3.2.16b), one cannot have $\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}$, where $\mathfrak{p}$ (resp. $\mathfrak{q}$) is the label of x (resp. y). Since by *(i)*, we have $\mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}$, we necessarily get $\mathfrak{p}_1 \neq \mathbb{1}_{\mathcal{M}}$. Hence, $\mathfrak{t}$ satisfies *(ii)*.

Conversely, if $\mathfrak{t}$ is a syntax tree on $\mathcal{T}_{\mathcal{M}}$ satisfying *(i)* and *(ii)*, a direct inspection shows that one cannot rewrite $\mathfrak{t}$ by $\to$. Therefore, $\mathfrak{t}$ is a normal form of $\to$. ■

▶ Lemma 3.2.6 — *Let $\mathcal{M}$ be a finite unitary magma. The generating series of the normal forms of the rewrite rule $\to$ is the Hilbert series $\mathcal{H}_{\mathrm{NC}\mathcal{M}(t)}$ of $\mathrm{NC}\mathcal{M}$.*

◀ Proof — First, since by Lemma 3.2.4, $\to$ is terminating, and since for $n \geqslant 1$, due to the finiteness of $\mathcal{M}$, there are finitely many syntax trees on $\mathcal{T}_{\mathcal{M}}$ of arity n, the generating series $T(t)$ of the normal forms of $\to$ is well-defined.

Let $S(t)$ be the generating series of the normal forms of $\to$ such that the bases of the $\mathcal{M}$-triangles labeling the roots are labeled by $\mathbb{1}_{\mathcal{M}}$. From the description of the normal forms of $\to$ provided by Lemma 3.2.5, we obtain that $S(t)$ satisfies

$$S(t) = t + mtS(t) + (m-1)mS(t)^2. \tag{3.2.26}$$

Again by Lemma 3.2.5, we have

$$T(t) = t + m(S(t) - t). \tag{3.2.27}$$

A direct computation shows that $T(t)$ satisfies the algebraic equation

$$t + \left(m^3 - 2m^2 + 2m - 1\right) t^2 + \left(2m^2 t - 3mt + 2t - 1\right) T(t) + (m-1)\, T(t)^2 = 0. \tag{3.2.28}$$

Hence, by Proposition 3.1.6, we observe that $T(t) = \mathcal{H}_{\mathrm{NC}\mathcal{M}(t)}$. ■

▶ Lemma 3.2.7 — *For a finite unitary magma $\mathcal{M}$, the rewrite rule $\to$ is confluent.*

◀ Proof — Arguing by contradiction, assume that $\to$ is not confluent. Since by Lemma 3.2.4, $\to$ is terminating, there is an integer $n \geqslant 1$ and two normal forms $\mathfrak{t}$ and $\mathfrak{t}'$ of $\to$ of arity n such that $\mathfrak{t} \neq \mathfrak{t}'$ and $\mathfrak{t} \overset{*}{\Leftrightarrow} \mathfrak{t}'$. Now, Lemma 3.2.1 together with Lemma 3.2.3 implies that $\mathrm{ev}(\mathfrak{t}) = \mathrm{ev}(\mathfrak{t}')$. By Proposition 3.1.5, the map $\mathrm{ev} : \mathrm{Free}\left(\mathbb{K}\langle \mathcal{T}_{\mathcal{M}} \rangle\right) \to \mathrm{NC}\mathcal{M}$ is surjective, leading to the fact that the number of normal forms of $\to$ of arity n is greater than the number of noncrossing $\mathcal{M}$-cliques of arity n. However, by Lemma 3.2.6, there are as many normal forms of $\to$ of arity n as noncrossing $\mathcal{M}$-cliques of arity n. This leads to a contradiction and proves the statement of the lemma. ■

3.2.3 Presentation and Koszulity. The results of Sections 3.2.1 and 3.2.2 are finally used here to provide a presentation of $\mathrm{NC}\mathcal{M}$ and prove that $\mathrm{NC}\mathcal{M}$ is a Koszul operad.

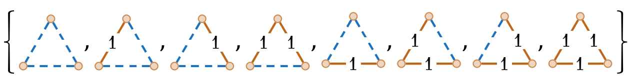
▶ Theorem 3.2.8 — *Let $\mathcal{M}$ be a finite unitary magma. Then $\mathrm{NC}\mathcal{M}$ admits the presentation $(\mathfrak{T}_{\mathcal{M}}, \mathfrak{R}_{\mathrm{NC}\mathcal{M}})$.*

◀ Proof — First, since by Lemmas 3.2.4 and 3.2.7, $\to$ is a convergent rewrite rule, and since by Lemma 3.2.3, the space induced by $\to$ is $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$, we can regard the underlying space of the quotient operad

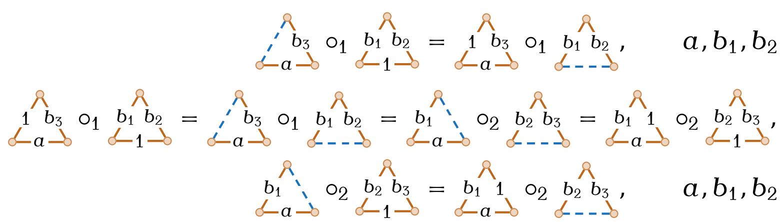
$$\mathcal{O} := \mathrm{Free}\left(\mathbb{K}\langle \mathfrak{T}_{\mathcal{M}}\rangle\right) /_{\langle \mathfrak{R}_{\mathrm{NC}\mathcal{M}}\rangle} \tag{3.2.29}$$

as the linear span of all normal forms of $\to$. Moreover, as a consequence of Lemma 3.2.1, the linear map $\phi : \mathcal{O} \to \mathrm{NC}\mathcal{M}$ defined for any normal form $\mathfrak{t}$ of $\to$ by $\phi(\mathfrak{t}) := \mathrm{ev}(\mathfrak{t})$ is an operad morphism. Now, by Proposition 3.1.5, ϕ is surjective. Moreover, by Lemma 3.2.6, the dimensions of the spaces $\mathcal{O}(n)$, $n \geqslant 1$ are the ones of $\mathrm{NC}\mathcal{M}(n)$. Hence, ϕ is an operad isomorphism and the statement of the theorem follows. ■

We use Theorem 3.2.8 to express the presentations of the operads $\mathrm{NC}\mathbb{N}_2$ and $\mathrm{NC}\mathbb{D}_0$, where $\mathbb{N}_2$ is the cyclic monoid $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{D}_0$ is the multiplicative monoid on $\{0, 1\}$. The operad $\mathrm{NC}\mathbb{N}_2$ is generated by

$\mathfrak{T}_{\mathbb{N}_2} =$  (3.2.30)

and these generators satisfies only the nontrivial relations

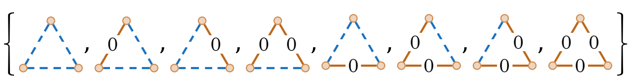


$a, b_1, b_2, b_3 \in \mathbb{N}_2$, (3.2.31a)

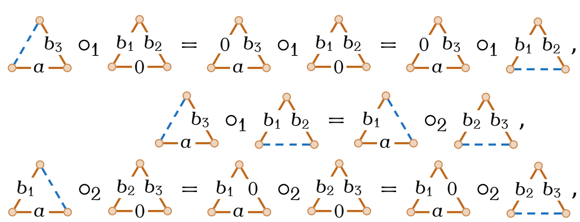
$a, b_1, b_2, b_3 \in \mathbb{N}_2$, (3.2.31b)

$a, b_1, b_2, b_3 \in \mathbb{N}_2$. (3.2.31c)

On the other hand, the operad $\mathrm{NC}\mathbb{D}_0$ is generated by

$\mathfrak{T}_{\mathbb{D}_0} =$  (3.2.32)

and these generators satisfies only the nontrivial relations



$a, b_1, b_2, b_3 \in \mathbb{D}_0$, (3.2.33a)

$a, b_1, b_2, b_3 \in \mathbb{D}_0$, (3.2.33b)

$a, b_1, b_2, b_3 \in \mathbb{D}_0$. (3.2.33c)

▶ Theorem 3.2.9 — *For a finite unitary magma $\mathcal{M}$, $\mathrm{NC}\mathcal{M}$ is Koszul and the set of normal forms of $\to$ forms a Poincaré–Birkhoff–Witt basis of $\mathrm{NC}\mathcal{M}$.*

◀ Proof — By Lemma 3.2.3 and Theorem 3.2.8, the rewrite rule $\to$ is an orientation of the space of relations $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$ of $\mathrm{NC}\mathcal{M}$. Moreover, by Lemmas 3.2.4 and 3.2.7, this rewrite rule is convergent. Therefore, by Lemma 2.2.1, $\mathrm{NC}\mathcal{M}$ is Koszul. Finally, the set of normal forms of $\to$ described by Lemma 3.2.5 is, by definition, a Poincaré–Birkhoff–Witt basis of $\mathrm{NC}\mathcal{M}$. ■

3.3 Suboperads generated by bubbles

In this section, we consider suboperads of $\mathrm{NC}\mathcal{M}$ generated by finite sets of $\mathcal{M}$-bubbles. We assume here that $\mathcal{M}$ is endowed with an arbitrary total order so that $\mathcal{M} = \{x_0, x_1, \dots\}$ with $x_0 = \mathbb{1}_{\mathcal{M}}$.

If $\mathfrak{p}$ is an $\mathcal{M}$-clique, the *border* of $\mathfrak{p}$ is the word $\mathrm{bor}(\mathfrak{p})$ of length n such that, for $i \in [n]$, we have $\mathrm{bor}(\mathfrak{p})_i = \mathfrak{p}_i$.

3.3.1 Treelike expressions on bubbles. Let B and E be two subsets of $\mathcal{M}$. We write $\mathcal{B}_{\mathcal{M}}^{B,E}$ for the set of $\mathcal{M}$-bubbles $\mathfrak{p}$ such that the bases of $\mathfrak{p}$ are labeled by B and all edges of $\mathfrak{p}$ are labeled by E. Moreover, we say that $\mathcal{M}$ is *(E, B)-quasi-injective* if for all $x, x' \in E$ and $y, y' \in B$, $x \star y = x' \star y' \neq \mathbb{1}_{\mathcal{M}}$ implies $x = x'$ and $y = y'$.

▶ **Lemma 3.3.1** — *Let $\mathcal{M}$ be a unitary magma, and B and E be two subsets of $\mathcal{M}$. If $\mathcal{M}$ is (E, B)-quasi-injective, then any $\mathcal{M}$-clique admits at most one treelike expression on $\mathcal{B}_{\mathcal{M}}^{B,E}$ of minimal degree.*

◀ **Proof** — Assume that $\mathfrak{p}$ is an $\mathcal{M}$-clique admitting a treelike expression on $\mathcal{B}_{\mathcal{M}}^{B,E}$. This implies that the base of $\mathfrak{p}$ is labeled by B, all solid diagonals of $\mathfrak{p}$ are labeled by $B \star E$, and all edges of $\mathfrak{p}$ are labeled by E. By Proposition 3.1.2 and Lemma 3.1.3, the tree $\mathfrak{t} := \mathrm{bt}(\mathfrak{p})$ is a treelike expression of $\mathfrak{p}$ on $\mathcal{B}_{\mathcal{M}}$ of minimal degree. Now, observe that $\mathfrak{t}$ is not necessarily a syntax tree on $\mathcal{B}_{\mathcal{M}}^{B,E}$ as required since some of its internal nodes may be labeled by bubbles that do not belong to $\mathcal{B}_{\mathcal{M}}^{B,E}$. Since $\mathcal{M}$ is (E, B)-quasi-injective, there is a unique way to relabel the internal nodes of $\mathfrak{t}$ by bubbles of $\mathcal{B}_{\mathcal{M}}^{B,E}$ to obtain a syntax tree on $\mathcal{B}_{\mathcal{M}}^{B,E}$ such that $\mathrm{ev}(\mathfrak{t}') = \mathrm{ev}(\mathfrak{t})$. By construction, $\mathfrak{t}'$ satisfies the properties of the statement of the lemma. ■

3.3.2 Dimensions. Let G be a set of $\mathcal{M}$-bubbles and $\Xi := \{\xi_{x_0}, \xi_{x_1}, \dots\}$ be a set of noncommutative variables. Given $x_i \in \mathcal{M}$, let B_{x_i} be the series of $\mathbb{N}\langle\langle \Xi \rangle\rangle$ defined by

$$\mathrm{B}_{x_i}(\xi_{x_0}, \xi_{x_1}, \dots) := \sum_{\substack{\mathfrak{p} \in \mathcal{B}_{\mathcal{M}}^{G} \\ \mathfrak{p} \neq \circ\text{--}\circ}} \prod_{i \in [|\mathfrak{p}|]} \xi_{\mathfrak{p}_i}, \tag{3.3.1}$$

where $\mathcal{B}_{\mathcal{M}}^{G}$ is the set of $\mathcal{M}$-bubbles that can be obtained by partial compositions of elements of G. Observe from (3.3.1) that a noncommutative monomial $u \in \Xi^{\geqslant 2}$ appears in B_{x_i} with 1 as coefficient if and only if there is an $\mathcal{M}$-bubble with a base labeled by x_i and with u as border in the suboperad of $\mathrm{NC}\mathcal{M}$ generated by G.

Moreover, for $x_i \in \mathcal{M}$, let the series F_{x_i} of $\mathbb{N}\langle\langle t \rangle\rangle$ defined by

$$\mathrm{F}_{x_i}(t) := \mathrm{B}_{x_i}\left(t + \bar{\mathrm{F}}_{x_0}(t), t + \bar{\mathrm{F}}_{x_1}(t), \dots\right), \tag{3.3.2}$$

where, for $x_i \in \mathcal{M}$,

$$\bar{\mathrm{F}}_{x_i}(t) := \sum_{\substack{x_j \in \mathcal{M} \\ x_i \star x_j \neq \mathbb{1}_{\mathcal{M}}}} \mathrm{F}_{x_j}(t). \tag{3.3.3}$$

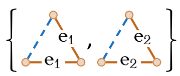
▶ **Proposition 3.3.2** — *Let $\mathcal{M}$ be a unitary magma and G be a finite set of $\mathcal{M}$-bubbles such that, writing B (resp. E) for the set of labels of the bases (resp. edges) of the elements of G, $\mathcal{M}$ is (E, B)-quasi-injective. Then, the Hilbert series $\mathcal{H}_{(\mathrm{NC}\mathcal{M})^G}(t)$ of the suboperad of $\mathrm{NC}\mathcal{M}$ generated by G satisfies*

$$\mathcal{H}_{(\mathrm{NC}\mathcal{M})^G}(t) = t + \sum_{x_i \in \mathcal{M}} \mathrm{F}_{x_i}(t). \tag{3.3.4}$$

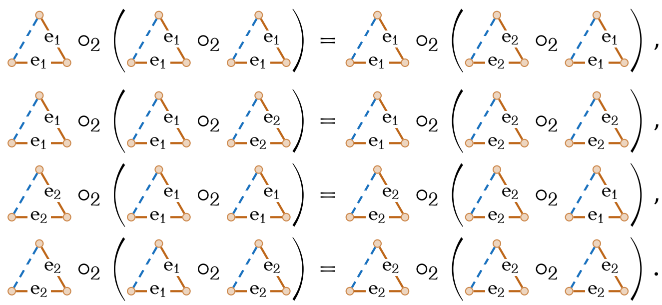
◀ **Proof —** By Lemma 3.3.1, an $\mathcal{M}$-clique of $(\mathrm{NC}\mathcal{M})^G$ admits exactly one treelike expression on $\mathcal{M}$-bubbles of $(\mathrm{NC}\mathcal{M})^G$ of minimal degree. For this reason, and as a consequence of the definition (3.3.3) of the series $\bar{\mathrm{F}}_{x_i}(t)$, $x_i \in \mathcal{M}$, the series $\mathrm{F}_{x_i}(t)$ is the generating series of all $\mathcal{M}$-cliques of $(\mathrm{NC}\mathcal{M})^G$ different from ∘--∘ and with a base labeled by $x_i \in \mathcal{M}$. Therefore, the expression (3.3.4) for the Hilbert series of $(\mathrm{NC}\mathcal{M})^G$ follows. ■

As a side remark, Proposition 3.3.2 can be proved by using the notion of bubble decompositions of operads developed in [CG14]. This result provides a practical method to compute the dimensions of some suboperads $(\mathrm{NC}\mathcal{M})^G$ of $\mathrm{NC}\mathcal{M}$ by describing the series (3.3.1) of the bubbles of $\mathcal{B}^G_{\mathcal{M}}$. If G satisfies the requirement of Proposition 3.3.2, this result also implies that the Hilbert series of $(\mathrm{NC}\mathcal{M})^G$ is algebraic.

3.3.3 First example : a cubic suboperad. Consider the suboperad of $\mathrm{NC}\mathbb{E}_2$ generated by

$G :=$  . (3.3.5)

Computer experiments show that the generators of $(\mathrm{NC}\mathbb{E}_2)^G$ do not satisfy any nontrivial quadratic relation and that they satisfy the only four nontrivial cubic relations



(3.3.6a)
(3.3.6b)
(3.3.6c)
(3.3.6d)

Hence, $(\mathrm{NC}\mathbb{E}_2)^G$ is not a quadratic operad. Moreover, it is possible to prove that this operad does not admit any other nontrivial relations between its generators. This can be performed by defining a rewrite rule on the syntax trees on G, consisting in rewriting the left patterns of (3.3.6a), (3.3.6b), (3.3.6c), and (3.3.6d) into their respective right patterns, and by checking that this rewrite rule admits the required properties (like the ones establishing the presentation of $\mathrm{NC}\mathcal{M}$ by Theorem 3.2.8). The existence of this nonquadratic operad shows that $\mathrm{NC}\mathcal{M}$ contains nonquadratic suboperads even if it is quadratic.

One can prove by induction on the arity that the set of bubbles of $(\mathrm{NC}\mathbb{E}_2)^G$ is the set $B_1 \sqcup B_2$ where B_1 (resp. B_2) is the set of bubbles whose bases are labeled by e_1 (resp. e_2) and the border is $\mathbb{1}\mathrm{e}_1$ (resp. $\mathbb{1}\mathrm{e}_2$), or $\mathbb{1}\mathbb{1}\mathbb{1}^*\mathrm{e}_1$, or $\mathbb{1}\mathbb{1}\mathbb{1}^*\mathrm{e}_2$. Hence, we obtain

$$\mathrm{B}_{\mathbb{1}}\left(\xi_{\mathbb{1}}, \xi_{\mathrm{e}_1}, \xi_{\mathrm{e}_2}\right) = 0, \tag{3.3.7a}$$

$$\mathrm{B}_{\mathrm{e}_1}\left(\xi_{\mathbb{1}}, \xi_{\mathrm{e}_1}, \xi_{\mathrm{e}_2}\right) = \frac{\xi_{\mathbb{1}}}{1 - \xi_{\mathbb{1}}} \left(\xi_{\mathrm{e}_1} + \xi_{\mathbb{1}} \xi_{\mathrm{e}_2}\right) = \mathrm{B}_{\mathrm{e}_2}\left(\xi_{\mathbb{1}}, \xi_{\mathrm{e}_2}, \xi_{\mathrm{e}_1}\right), \tag{3.3.7b}$$

Moreover, one can check that G satisfies the conditions required by Proposition 3.3.2. We hence have

$$\bar{\mathrm{F}}_{\mathbb{1}}(t) = \mathrm{F}_{\mathrm{e}_1}(t) + \mathrm{F}_{\mathrm{e}_2}(t), \tag{3.3.8a}$$

$$\bar{\mathrm{F}}_{\mathrm{e}_1}(t) = \mathrm{F}_{\mathbb{1}}(t) = \bar{\mathrm{F}}_{\mathrm{e}_2}(t), \tag{3.3.8b}$$

and

$$\mathrm{F}_{\mathbb{1}}(t) = 0, \tag{3.3.9a}$$

$$\mathrm{F}_{\mathrm{e}_1}(t) = \mathrm{B}_{\mathrm{e}_1}(t + \mathrm{F}_{\mathrm{e}_1}(t) + \mathrm{F}_{\mathrm{e}_2}(t), t, t) = \mathrm{B}_{\mathrm{e}_2}(t + \mathrm{F}_{\mathrm{e}_1}(t) + \mathrm{F}_{\mathrm{e}_2}(t), t, t) = \mathrm{F}_{\mathrm{e}_2}(t), \tag{3.3.9b}$$

By Proposition 3.3.2, the Hilbert series of $(\mathrm{NC}\mathbb{E}_2)^G$ satisfies

$$\mathcal{H}_{(\mathrm{NC}\mathbb{E}_2)^G}(t) = t + \mathrm{F}_{\mathbb{1}}(t) + \mathrm{F}_{\mathrm{e}_1}(t) + \mathrm{F}_{\mathrm{e}_2}(t) = t + 2\mathrm{F}_{\mathrm{e}_1}(t), \tag{3.3.10}$$

and, by a straightforward computation, we obtain that this series satisfies the algebraic equation

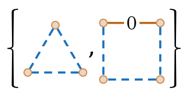
$$t + (t-1)\mathcal{H}_{(\mathrm{NC}\mathbb{E}_2)^G}(t) + (2t+1)\mathcal{H}_{(\mathrm{NC}\mathbb{E}_2)^G}(t)^2 = 0. \tag{3.3.11}$$

The first dimensions of $(\mathrm{NC}\mathbb{E}_2)^G$ are

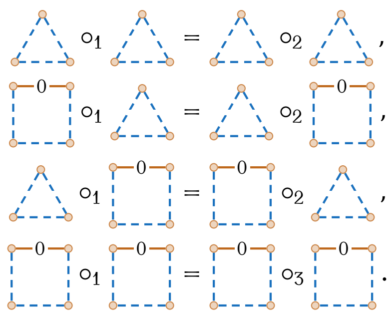
$$1, 2, 8, 36, 180, 956, 5300, 30316, \tag{3.3.12}$$

and form Sequence **A129148** of [Slo].

3.3.4 Second example: a suboperad of Motzkin paths. Consider the suboperad of $\mathrm{NC}\mathbb{D}_0$ generated by

$G :=$  . (3.3.13)

Computer experiments show that the generators of $(\mathrm{NC}\mathbb{D}_0)^G$ satisfy the only four nontrivial quadratic relations



(3.3.14a)
(3.3.14b)
(3.3.14c)
(3.3.14d)

It is possible to prove that this operad does not admit any other nontrivial relations between its generators. This can be performed by defining a rewrite rule on the syntax trees on G, consisting in rewriting the left patterns of (3.3.14a), (3.3.14b), (3.3.14c), and (3.3.14d) into their respective right patterns, and by checking that this rewrite rule admits the required properties (like the ones establishing the presentation of $\mathrm{NC}\mathcal{M}$ by Theorem 3.2.8).

One can prove by induction on the arity that the set of bubbles of $(\mathrm{NC}\mathbb{D}_0)^G$ is the set of bubbles whose bases are labeled by $\mathbb{1}$ and borders are words of $\{\mathbb{1}, 0\}^{\geqslant 2}$ such that each occurrence of 0 has a $\mathbb{1}$ immediately to its left and a $\mathbb{1}$ immediately to its right. Hence, we obtain

$$\mathrm{B}_{\mathbb{1}}(\xi_{\mathbb{1}}, \xi_0) = \frac{1}{1 - \xi_{\mathbb{1}} - \xi_{\mathbb{1}}\xi_0}\xi_{\mathbb{1}} - \xi_{\mathbb{1}}, \tag{3.3.15a}$$

$$\mathrm{B}_0(\xi_{\mathbb{1}}, \xi_0) = 0. \tag{3.3.15b}$$

Moreover, one can check that G satisfies the conditions required by Proposition 3.3.2. We hence have

$$\bar{\mathrm{F}}_{\mathbb{1}}(t) = \mathrm{F}_0(t), \tag{3.3.16a}$$

$$\bar{\mathrm{F}}_0(t) = \mathrm{F}_{\mathbb{1}}(t) + \mathrm{F}_0(t), \tag{3.3.16b}$$

and

$$\mathrm{F}_{\mathbb{1}}(t) = \mathrm{B}_{\mathbb{1}}(t, t + \mathrm{F}_{\mathbb{1}}(t)), \tag{3.3.17a}$$

$$\mathrm{F}_0(t) = 0. \tag{3.3.17b}$$

By Proposition 3.3.2, the Hilbert series of $(\mathrm{NC}\mathbb{D}_0)^G$ satisfies

$$\mathcal{H}_{(\mathrm{NC}\mathbb{D}_0)^G}(t) = t + \mathrm{F}_{\mathbb{1}}(t), \tag{3.3.18}$$

and, by a straightforward computation, we obtain that this series satisfies the algebraic equation

$$t + (t-1)\mathcal{H}_{(\mathrm{NC}\mathbb{D}_0)^G}(t) + t\mathcal{H}_{(\mathrm{NC}\mathbb{D}_0)^G}(t)^2 = 0. \tag{3.3.19}$$

The first dimensions of $(\mathrm{NC}\mathbb{D}_0)^G$ are

$$1, 1, 2, 4, 9, 21, 51, 127, \tag{3.3.20}$$

and form Sequence **A001006** of [Slo]. The operad $(\mathrm{NC}\mathbb{D}_0)^G$ has the same presentation by generators and relations (and thus, the same Hilbert series) as the operad Motz defined in [Gir15], involving Motzkin paths. Hence, $(\mathrm{NC}\mathbb{D}_0)^G$ and Motz are two isomorphic operads. Note in passing that these two operads are not isomorphic to the operad $\mathrm{Mot}\mathbb{D}_0$ constructed in Section 3.2.4 of [Gir20] and involving Motzkin configurations. Indeed, the sequence of the dimensions of this last operad is a shifted version of the one of $(\mathrm{NC}\mathbb{D}_0)^G$ and Motz.

3.4 Algebras over the noncrossing clique operads

We begin by briefly describing $\mathrm{NC}\mathcal{M}$-algebras in terms of relations between their operations and the free $\mathrm{NC}\mathcal{M}$-algebras over one generator. We continue this section by providing two ways to construct (not necessarily free) $\mathrm{NC}\mathcal{M}$-algebras. The first one takes as input an associative algebra endowed with endofunctions satisfying some conditions, and the second one takes as input a monoid.

3.4.1 Relations.

From the presentation of $\mathrm{NC}\mathcal{M}$ established by Theorem 3.2.8, any $\mathrm{NC}\mathcal{M}$-algebra is a vector space $\mathcal{A}$ endowed with binary linear operations

$$\triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\,\mathfrak{p}_2} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \qquad \mathfrak{p} \in \mathcal{T}_{\mathcal{M}}, \tag{3.4.1}$$

satisfying, for all $a_1, a_2, a_3 \in \mathcal{A}$, the relations

$$\left(a_1 \triangle_{\mathfrak{q}_0}^{\mathfrak{q}_1\,\mathfrak{q}_2} a_2\right) \triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\,\mathfrak{p}_2} a_3 = \left(a_1 \triangle_{\mathfrak{r}_0}^{\mathfrak{q}_1\,\mathfrak{q}_2} a_2\right) \triangle_{\mathfrak{p}_0}^{\mathfrak{r}_1\,\mathfrak{p}_2} a_3, \qquad \text{if } \mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{r}_1 \star \mathfrak{r}_0 \neq \mathbb{1}_{\mathcal{M}}, \tag{3.4.2a}$$

$$\left(a_1 \triangle_{\mathfrak{q}_0}^{\mathfrak{q}_1\,\mathfrak{q}_2} a_2\right) \triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\,\mathfrak{p}_2} a_3 = a_1 \triangle_{\mathfrak{p}_0}^{\mathfrak{q}_1\,\mathfrak{r}_2} \left(a_2 \triangle_{\mathfrak{r}_0}^{\mathfrak{q}_2\,\mathfrak{p}_2} a_3\right), \qquad \text{if } \mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{r}_2 \star \mathfrak{r}_0 = \mathbb{1}_{\mathcal{M}}, \tag{3.4.2b}$$

$$a_1 \triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\,\mathfrak{p}_2} \left(a_2 \triangle_{\mathfrak{q}_0}^{\mathfrak{q}_1\,\mathfrak{q}_2} a_3\right) = a_1 \triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\,\mathfrak{r}_2} \left(a_2 \triangle_{\mathfrak{r}_0}^{\mathfrak{q}_1\,\mathfrak{q}_2} a_3\right), \qquad \text{if } \mathfrak{p}_2 \star \mathfrak{q}_0 = \mathfrak{r}_2 \star \mathfrak{r}_0 \neq \mathbb{1}_{\mathcal{M}}, \tag{3.4.2c}$$

where $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{r}$ are $\mathcal{M}$-triangles. Observe that $\mathcal{M}$ has to be finite because Theorem 3.2.8 requires this property as premise.

3.4.2 Free algebras over one generator.

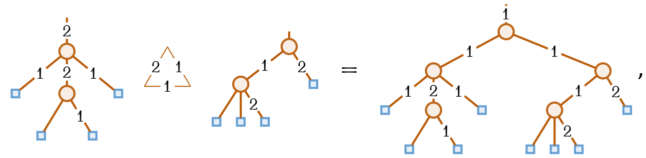
From the realization of $\mathrm{NC}\mathcal{M}$ coming from its definition as a suboperad of $\mathrm{C}\mathcal{M}$, the free $\mathrm{NC}\mathcal{M}$-algebra over one generator is the linear span $\mathrm{NC}\mathcal{M}$ of all noncrossing $\mathcal{M}$-cliques endowed with the linear operations

$$\triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\,\mathfrak{p}_2} : \mathrm{NC}\mathcal{M}(n) \otimes \mathrm{NC}\mathcal{M}(m) \to \mathrm{NC}\mathcal{M}(n+m), \qquad \mathfrak{p} \in \mathcal{T}_{\mathcal{M}}, n, m \geqslant 1, \tag{3.4.3}$$

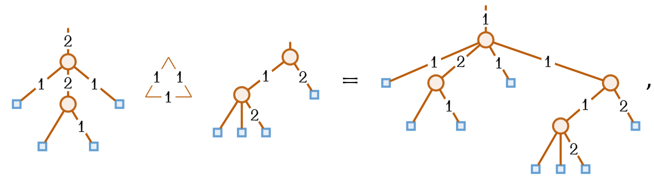
defined, for noncrossing $\mathcal{M}$-cliques $\mathfrak{q}$ and $\mathfrak{r}$, by

$$\mathfrak{q} \triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\,\mathfrak{p}_2} \mathfrak{r} := \left(\triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1\,\mathfrak{p}_2} \circ_2 \mathfrak{r}\right) \circ_1 \mathfrak{q}. \tag{3.4.4}$$

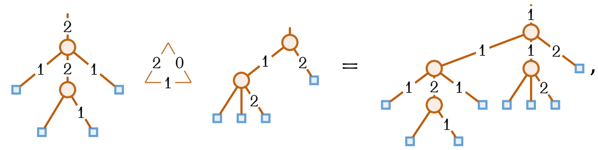
In terms of $\mathcal{M}$-Schröder trees (see Section 3.1.3), (3.4.4) is the $\mathcal{M}$-Schröder tree obtained by grafting the $\mathcal{M}$-Schröder trees of $\mathfrak{q}$ and $\mathfrak{r}$ respectively as left and right children of a binary corolla having its edge adjacent to the root labeled by $\mathfrak{p}_0$, its first edge labeled by $\mathfrak{p}_1 \star \mathfrak{q}_0$, and second edge labeled by $\mathfrak{p}_2 \star \mathfrak{r}_0$, and by contracting each of these two edges when labeled by $\mathbb{1}_{\mathcal{M}}$. For instance, in the free $\mathrm{NC}\mathbb{N}_3$-algebra, we have



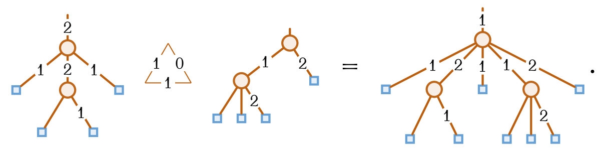
(3.4.5a)



(3.4.5b)



(3.4.5c)



(3.4.5d)

3.4.3 NC$\mathcal{M}$-algebras from associative algebras. Let $\mathcal{A}$ be an associative algebra with associative product denoted by $\odot$, and

$$\omega_x : \mathcal{A} \to \mathcal{A}, \qquad x \in \mathcal{M}, \tag{3.4.6}$$

be a family of linear maps, not necessarily associative algebra morphisms, indexed by the elements of $\mathcal{M}$. We say that $\mathcal{A}$ together with this family (3.4.6) of maps is *$\mathcal{M}$-compatible* if

$$\omega_{\mathbb{1}_{\mathcal{M}}} = \mathrm{Id}_{\mathcal{A}} \tag{3.4.7}$$

where $\mathrm{Id}_{\mathcal{A}}$ is the identity map on $\mathcal{A}$, and

$$\omega_x \circ \omega_y = \omega_{x \star y}, \tag{3.4.8}$$

for all $x, y \in \mathcal{M}$. We now use $\mathcal{M}$-compatible associative algebras to construct NC$\mathcal{M}$-algebras.

▶ **Theorem 3.4.1** — *Let $\mathcal{M}$ be a finite unitary magma and $\mathcal{A}$ be an $\mathcal{M}$-compatible associative algebra. The vector space $\mathcal{A}$ endowed with the binary linear operations*

$$\overset{\mathfrak{p}_1\ \mathfrak{p}_2}{\underset{\mathfrak{p}_0}{\triangle}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \qquad \mathfrak{p} \in \mathcal{T}_{\mathcal{M}}, \tag{3.4.9}$$

defined for each $\mathcal{M}$-triangle $\mathfrak{p}$ and $a_1, a_2 \in \mathcal{A}$ by

$$a_1 \overset{\mathfrak{p}_1\ \mathfrak{p}_2}{\underset{\mathfrak{p}_0}{\triangle}} a_2 := \omega_{\mathfrak{p}_0}\left(\omega_{\mathfrak{p}_1}(a_1) \odot \omega_{\mathfrak{p}_2}(a_2)\right), \tag{3.4.10}$$

is an NC$\mathcal{M}$-algebra.

◀ **Proof** — We prove that the operations (3.4.9) satisfy Relations (3.4.2a), (3.4.2b), and (3.4.2c) of NC$\mathcal{M}$-algebras. Since $\mathcal{M}$ is finite, this amounts to showing that these operations endow $\mathcal{A}$ with an NC$\mathcal{M}$-algebra structure. For this, let a_1, a_2, and a_3 be three elements of $\mathcal{A}$, and $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{r}$ be three $\mathcal{M}$-triangles.

(a) If $\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{r}_1 \star \mathfrak{r}_0 \neq \mathbb{1}_{\mathcal{M}}$, then, since by (3.4.8) we have $\omega_{\mathfrak{p}_1} \circ \omega_{\mathfrak{q}_0} = \omega_{\mathfrak{r}_1} \circ \omega_{\mathfrak{r}_0}$, we obtain

$$\begin{aligned}\left(a_1 \underset{\mathfrak{q}_0}{\overset{\mathfrak{q}_1\,\mathfrak{q}_2}{\triangle}} a_2\right) \underset{\mathfrak{p}_0}{\overset{\mathfrak{p}_1\,\mathfrak{p}_2}{\triangle}} a_3 &= \omega_{\mathfrak{q}_0}(\omega_{\mathfrak{q}_1}(a_1) \odot \omega_{\mathfrak{q}_2}(a_2)) \underset{\mathfrak{p}_0}{\overset{\mathfrak{p}_1\,\mathfrak{p}_2}{\triangle}} a_3 \\ &= \omega_{\mathfrak{p}_0}(\omega_{\mathfrak{p}_1}(\omega_{\mathfrak{q}_0}(\omega_{\mathfrak{q}_1}(a_1) \odot \omega_{\mathfrak{q}_2}(a_2))) \odot \omega_{\mathfrak{p}_2}(a_3)) \\ &= \omega_{\mathfrak{p}_0}(\omega_{\mathfrak{r}_1}(\omega_{\mathfrak{r}_0}(\omega_{\mathfrak{q}_1}(a_1) \odot \omega_{\mathfrak{q}_2}(a_2))) \odot \omega_{\mathfrak{p}_2}(a_3)) \\ &= \left(a_1 \underset{\mathfrak{r}_0}{\overset{\mathfrak{q}_1\,\mathfrak{q}_2}{\triangle}} a_2\right) \underset{\mathfrak{p}_0}{\overset{\mathfrak{r}_1\,\mathfrak{p}_2}{\triangle}} a_3.\end{aligned} \tag{3.4.11}$$

Hence (3.4.2a) holds.

(b) If $\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{r}_2 \star \mathfrak{r}_0 = \mathbb{1}_{\mathcal{M}}$, then, since by (3.4.7) we have $\omega_{\mathfrak{p}_1} \circ \omega_{\mathfrak{q}_0} = \omega_{\mathfrak{r}_2} \circ \omega_{\mathfrak{r}_0} = \mathrm{Id}_{\mathcal{A}}$ and since $\odot$ is associative, we get

$$\begin{aligned}\left(a_1 \underset{\mathfrak{q}_0}{\overset{\mathfrak{q}_1\,\mathfrak{q}_2}{\triangle}} a_2\right) \underset{\mathfrak{p}_0}{\overset{\mathfrak{p}_1\,\mathfrak{p}_2}{\triangle}} a_3 &= \omega_{\mathfrak{q}_0}(\omega_{\mathfrak{q}_1}(a_1) \odot \omega_{\mathfrak{q}_2}(a_2)) \underset{\mathfrak{p}_0}{\overset{\mathfrak{p}_1\,\mathfrak{p}_2}{\triangle}} a_3 \\ &= \omega_{\mathfrak{p}_0}((\omega_{\mathfrak{p}_1}(\omega_{\mathfrak{q}_0}(\omega_{\mathfrak{q}_1}(a_1) \odot \omega_{\mathfrak{q}_2}(a_2)))) \odot \omega_{\mathfrak{p}_2}(a_3)) \\ &= \omega_{\mathfrak{p}_0}(\omega_{\mathfrak{q}_1}(a_1) \odot \omega_{\mathfrak{q}_2}(a_2) \odot \omega_{\mathfrak{p}_2}(a_3)) \\ &= \omega_{\mathfrak{p}_0}(\omega_{\mathfrak{q}_1}(a_1) \odot (\omega_{\mathfrak{q}_2}(a_2) \odot \omega_{\mathfrak{p}_2}(a_3))) \\ &= \omega_{\mathfrak{p}_0}(\omega_{\mathfrak{q}_1}(a_1) \odot \omega_{\mathfrak{r}_2}(\omega_{\mathfrak{r}_0}(\omega_{\mathfrak{q}_2}(a_2) \odot \omega_{\mathfrak{p}_2}(a_3)))) \\ &= a_1 \underset{\mathfrak{p}_0}{\overset{\mathfrak{q}_1\,\mathfrak{r}_2}{\triangle}} \left(a_2 \underset{\mathfrak{r}_0}{\overset{\mathfrak{q}_2\,\mathfrak{p}_2}{\triangle}} a_3\right).\end{aligned} \tag{3.4.12}$$

Hence (3.4.2b) holds.

(c) If $\mathfrak{p}_2 \star \mathfrak{q}_0 = \mathfrak{r}_2 \star \mathfrak{r}_0 \neq \mathbb{1}_{\mathcal{M}}$, then, since by (3.4.8) we have $\omega_{\mathfrak{p}_2} \circ \omega_{\mathfrak{q}_0} = \omega_{\mathfrak{r}_2} \circ \omega_{\mathfrak{r}_0}$, we obtain

$$\begin{aligned}a_1 \underset{\mathfrak{p}_0}{\overset{\mathfrak{p}_1\,\mathfrak{p}_2}{\triangle}} \left(a_2 \underset{\mathfrak{q}_0}{\overset{\mathfrak{q}_1\,\mathfrak{q}_2}{\triangle}} a_3\right) &= a_1 \underset{\mathfrak{p}_0}{\overset{\mathfrak{p}_1\,\mathfrak{p}_2}{\triangle}} \omega_{\mathfrak{q}_0}(\omega_{\mathfrak{q}_1}(a_2) \odot \omega_{\mathfrak{q}_2}(a_3)) \\ &= \omega_{\mathfrak{p}_0}(\omega_{\mathfrak{p}_1}(a_1) \odot \omega_{\mathfrak{p}_2}(\omega_{\mathfrak{q}_0}(\omega_{\mathfrak{q}_1}(a_2) \odot \omega_{\mathfrak{q}_2}(a_3)))) \\ &= \omega_{\mathfrak{p}_0}(\omega_{\mathfrak{p}_1}(a_1) \odot \omega_{\mathfrak{r}_2}(\omega_{\mathfrak{r}_0}(\omega_{\mathfrak{q}_1}(a_2) \odot \omega_{\mathfrak{q}_2}(a_3)))) \\ &= a_1 \underset{\mathfrak{p}_0}{\overset{\mathfrak{p}_1\,\mathfrak{r}_2}{\triangle}} \left(a_2 \underset{\mathfrak{r}_0}{\overset{\mathfrak{q}_1\,\mathfrak{q}_2}{\triangle}} a_3\right).\end{aligned} \tag{3.4.13}$$

Hence (3.4.2c) holds.

Consequently, $\mathcal{A}$ is an NC$\mathcal{M}$-algebra. ∎

By Theorem 3.4.1, $\mathcal{A}$ has the structure of an NC$\mathcal{M}$-algebra. Hence, there is a left action $\cdot$ of the operad NC$\mathcal{M}$ on the tensor algebra of $\mathcal{A}$ of the form

$$\cdot : \mathrm{NC}\mathcal{M}(n) \otimes \mathcal{A}^{\otimes n} \to \mathcal{A}, \qquad n \geqslant 1, \tag{3.4.14}$$

whose definition comes from the ones of the operations (3.4.9) and Relation (2.2.10). We describe here an algorithm to compute the action of an element of NC$\mathcal{M}$ of arity n on tensors $a_1 \otimes \cdots \otimes a_n$ of $\mathcal{A}^{\otimes n}$. First, if $\mathfrak{b}$ is an $\mathcal{M}$-bubble of arity n,

$$\mathfrak{b} \cdot (a_1 \otimes \cdots \otimes a_n) = \omega_{\mathfrak{b}_0}\left(\prod_{i \in [n]} \omega_{\mathfrak{b}_i}(a_i)\right), \tag{3.4.15}$$

where the product of (3.4.15) denotes the iterated version of the associative product $\odot$ of $\mathcal{A}$. If $\mathfrak{p}$ is a noncrossing $\mathcal{M}$-clique of arity n, then $\mathfrak{p}$ acts recursively on $a_1 \otimes \cdots \otimes a_n$ as follows. We have

$$\mathfrak{p} \cdot a_1 = a_1 \tag{3.4.16}$$

if $\mathfrak{p} =$ ∘--∘, and

$$\mathfrak{p} \cdot (a_1 \otimes \cdots \otimes a_n) = \mathfrak{b} \cdot \left(\left(\mathfrak{r}_1 \cdot \left(a_1 \otimes \cdots \otimes a_{|\mathfrak{r}_1|}\right)\right) \otimes \cdots \otimes \left(\mathfrak{r}_k \cdot \left(a_{|\mathfrak{r}_1| + \cdots + |\mathfrak{r}_{k-1}| + 1} \otimes \cdots \otimes a_n\right)\right)\right), \tag{3.4.17}$$

where, by setting $\mathfrak{t}$ as the bubble tree $\mathrm{bt}(\mathfrak{p})$ of $\mathfrak{p}$ (see Section 3.1.2), $\mathfrak{b}$ and $\mathfrak{r}_1, \dots, \mathfrak{r}_k$ are the unique $\mathscr{M}$-bubble and noncrossing $\mathscr{M}$-cliques such that $\mathfrak{t} = \mathrm{c}(\mathfrak{b}) \circ [\mathrm{bt}(\mathfrak{r}_1), \dots, \mathrm{bt}(\mathfrak{r}_k)]$.

Here are a few examples of the construction provided by Theorem 3.4.1.

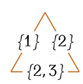
Noncommutative polynomials and selected concatenation: Let us consider the unitary magma $\mathbb{S}_\ell$ of the subsets of $[\ell]$ with the union as product. Let $A := \{a_j : j \in [\ell]\}$ be an alphabet of noncommutative letters. On the associative algebra $\mathbb{K}\langle A\rangle$ of polynomials on A, we define the linear maps

$$\omega_S : \mathbb{K}\langle A\rangle \to \mathbb{K}\langle A\rangle, \qquad S \in \mathbb{S}_\ell, \tag{3.4.18}$$

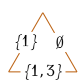
as follows. For $u \in A^*$ and $S \in \mathbb{S}_\ell$, we set

$$\omega_S(u) := \begin{cases} u, & \text{if } |u|_{a_j} \geqslant 1 \text{ for all } j \in S, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.19}$$

Since, for all $u \in A^*$, $\omega_\emptyset(u) = u$ and $(\omega_S \circ \omega_{S'})(u) = \omega_{S \cup S'}(u)$, and $\emptyset$ is the unit of $\mathbb{S}_\ell$, we obtain from Theorem 3.4.1 that the operations (3.4.9) endow $\mathbb{K}\langle A\rangle$ with an $\mathrm{NC}\mathbb{S}_\ell$-algebra structure. For instance, with $\ell := 3$ we have

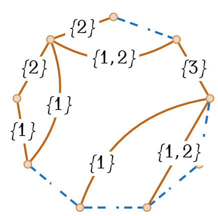
$$(a_1 + a_1 a_3 + a_2 a_2)$$ 

$$(1 + a_3 + a_2 a_1) = a_1 a_3 a_2 a_1, \tag{3.4.20a}$$

$$(a_1 + a_1 a_3 + a_2 a_2)$$ 

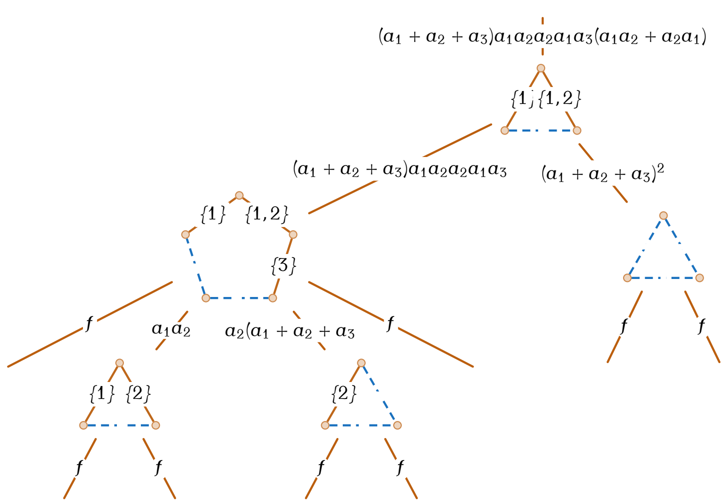
$$(1 + a_3 + a_2 a_1) = 2\, a_1 a_3 + a_1 a_3 a_3 + a_1 a_3 a_2 a_1. \tag{3.4.20b}$$

Moreover, to compute the action



$$\cdot\, (f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f), \tag{3.4.21}$$

where $f := a_1 + a_2 + a_3$, we use the above algorithm and (3.4.15) and (3.4.17). By presenting the computation on the bubble tree of the noncrossing $\mathbb{S}_3$-clique of (3.4.21), we obtain



$$, \tag{3.4.22}$$

so that (3.4.21) is equal to the polynomial $(a_1 + a_2 + a_3)a_1 a_2 a_2 a_1 a_3(a_1 a_2 + a_2 a_1)$.

Noncommutative polynomials and constant term product: Let us now consider the unitary magma $\mathbb{D}_0$. Let $A := \{a_1, a_2, \dots\}$ be an infinite alphabet of noncommutative letters. On the associative algebra $\mathbb{K}\langle A\rangle$ of polynomials on A, we define the linear maps

$$\omega_{\mathbb{1}}, \omega_0 : \mathbb{K}\langle A\rangle \to \mathbb{K}\langle A\rangle, \tag{3.4.23}$$

as follows. For $u \in A^*$, we set $\omega_{\mathbb{1}}(u) := u$, and

$$\omega_0(u) := \begin{cases} 1, & \text{if } u = \epsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.24}$$

In other terms, $\omega_0(f)$ is the constant term, denoted by $f(0)$, of the polynomial $f \in \mathbb{K}\langle A\rangle$. Since $\omega_{\mathbb{1}}$ is the identity map on $\mathbb{K}\langle A\rangle$ and, for all $u \in A^*$,

$$(\omega_0 \circ \omega_0)(f) = (f(0))(0) = f(0) = \omega_0(f), \tag{3.4.25}$$

we obtain from Theorem 3.4.1 that the operations (3.4.9) endow $\mathbb{K}\langle A\rangle$ with an $\mathrm{NC}\mathbb{D}_0$-algebra structure. For instance, for all polynomials f_1 and f_2 of $\mathbb{K}\langle A\rangle$, we have

$$f_1 \, \triangle^{\mathbb{1}\,\mathbb{1}}_{\mathbb{1}} \, f_2 = f_1 f_2, \tag{3.4.26a}$$

$$f_1 \, \triangle^{\mathbb{1}\,\mathbb{1}}_{0} \, f_2 = (f_1 f_2)(0) = f_1(0)\, f_2(0), \tag{3.4.26b}$$

$$f_1 \, \triangle^{0\,\mathbb{1}}_{\mathbb{1}} \, f_2 = f_1(0)\, f_2, \tag{3.4.26c}$$

$$f_1 \, \triangle^{\mathbb{1}\,0}_{\mathbb{1}} \, f_2 = f_1\, (f_2(0)). \tag{3.4.26d}$$

If $f_1(0) = 1 = f_2(0)$, we obtain from (3.4.26c) and (3.4.26d) that

$$f_1 \left(\triangle^{0\,\mathbb{1}}_{\mathbb{1}} + \triangle^{\mathbb{1}\,0}_{\mathbb{1}} \right) f_2 = f_1(0)\, f_2 + f_1\, (f_2(0)) = f_1 + f_2. \tag{3.4.27}$$

3.4.4 NC$\mathcal{M}$-algebras from monoids. If $\mathcal{M}$ is a monoid, with binary associative operation $\star$ and unit $\mathbb{1}_{\mathcal{M}}$, we write $\mathbb{K}\langle \mathcal{M}^*\rangle$ for the space of all noncommutative polynomials on $\mathcal{M}$, viewed as an alphabet, with coefficients in $\mathbb{K}$. This space can be endowed with an NC$\mathcal{M}$-algebra structure as follows.

For $x \in \mathcal{M}$ and a word $w = w_1 \dots w_{|w|} \in \mathcal{M}^*$, let

$$x * w := (x \star w_1) \dots (x \star w_{|w|}). \tag{3.4.28}$$

This operation $*$ is linearly extended on the right to $\mathbb{K}\langle \mathcal{M}^*\rangle$.

▶ **Proposition 3.4.2** — *Let $\mathcal{M}$ be a finite monoid. The vector space $\mathbb{K}\langle \mathcal{M}^*\rangle$ endowed with the binary linear operations*

$$\triangle^{\mathfrak{p}_1\,\mathfrak{p}_2}_{\mathfrak{p}_0} : \mathbb{K}\langle \mathcal{M}^*\rangle \otimes \mathbb{K}\langle \mathcal{M}^*\rangle \to \mathbb{K}\langle \mathcal{M}^*\rangle, \qquad \mathfrak{p} \in \mathcal{T}_{\mathcal{M}}, \tag{3.4.29}$$

defined for each $\mathcal{M}$-triangle $\mathfrak{p}$ and $f_1, f_2 \in \mathbb{K}\langle \mathcal{M}^*\rangle$ by

$$f_1 \, \triangle^{\mathfrak{p}_1\,\mathfrak{p}_2}_{\mathfrak{p}_0} \, f_2 := \mathfrak{p}_0 * ((\mathfrak{p}_1 * f_1)\ (\mathfrak{p}_2 * f_2)), \tag{3.4.30}$$

is an NC$\mathcal{M}$-algebra.

◀ **Proof** — This follows from Theorem 3.4.1 as a particular case of the general construction it provides. Indeed, $\mathbb{K}\langle \mathcal{M}^*\rangle$ is an associative algebra for the concatenation product of words. Moreover, by defining linear maps $\omega_x : \mathbb{K}\langle \mathcal{M}^*\rangle \to \mathbb{K}\langle \mathcal{M}^*\rangle$, $x \in \mathcal{M}$, by $\omega_x(u) := x * u$ for a word $u \in \mathcal{M}^*$, we obtain, since $\mathcal{M}$ is a monoid, that this family of maps satisfies (3.4.7) and (3.4.8). Now, since the definition (3.4.30) is the specialization of the definition (3.4.10) in this particular case, the statement of the proposition follows. ■

Here are a few examples of the construction provided by Proposition 3.4.2.

Words and double shifted concatenation: Consider the monoid $\mathbb{N}_\ell := \mathbb{Z}/_{\ell\mathbb{Z}}$ for an $\ell \geqslant 1$. By Proposition 3.4.2, the operations (3.4.29) endow $\mathbb{K}\langle\mathbb{N}_\ell^*\rangle$ with the structure of an $\mathrm{NC}\mathbb{N}_\ell$-algebra. For instance, in $\mathbb{K}\langle\mathbb{N}_4^*\rangle$ we have

$$0211 \ \triangle^{2\;0}_{1}\ 312 = 3100023. \tag{3.4.31}$$

Words and erasing concatenation: Consider here the monoid $\mathbb{D}_\ell$ for an $\ell \geqslant 0$ defined in [Gir20]. By Proposition 3.4.2, the operations (3.4.29) endow $\mathbb{K}\langle\mathbb{D}_\ell^*\rangle$ with the structure of an $\mathrm{NC}\mathbb{D}_\ell$-algebra. For instance, for all words u and v of $\mathbb{D}_\ell^*$, we have

$$u \ \triangle^{\mathbb{1}\;\mathbb{1}}_{\mathbb{1}}\ v = uv, \tag{3.4.32a}$$

$$u \ \triangle^{\mathbb{1}\;\mathbb{1}}_{\mathrm{d}_i}\ v = (uv)_{\mathrm{d}_i}, \tag{3.4.32b}$$

$$u \ \triangle^{\mathbb{1}\;\mathbb{1}}_{0}\ v = 0^{|u|+|v|}, \tag{3.4.32c}$$

$$u \ \triangle^{\mathrm{d}_i\;\mathrm{d}_j}_{\mathbb{1}}\ v = u_{\mathrm{d}_i}\ v_{\mathrm{d}_j}, \tag{3.4.32d}$$

where, for a word w of $\mathbb{D}_\ell^*$ and an element d_j of $\mathbb{D}_\ell$, $j \in [\ell]$, w_{d_j} is the word obtained from w by replacing each occurrence of $\mathbb{1}$ by d_j and each occurrence of d_i, $i \in [\ell]$, by 0.

4 Koszul dual

Since, by Theorem 3.2.8, the operad $\mathrm{NC}\mathcal{M}$ is binary and quadratic, this operad admits a Koszul dual $\mathrm{NC}\mathcal{M}^!$. We end the study of $\mathrm{NC}\mathcal{M}$ by collecting the main properties of $\mathrm{NC}\mathcal{M}^!$.

4.1 Presentation

Let $\mathfrak{R}^!_{\mathrm{NC}\mathcal{M}}$ be the subspace of $\mathrm{Free}(\mathbb{K}\langle\mathcal{T}_\mathcal{M}\rangle)(3)$ generated by the elements

$$\sum_{\substack{\mathfrak{p}_1,\mathfrak{q}_0\in\mathcal{M}\\ \mathfrak{p}_1\star\mathfrak{q}_0=\delta}} \mathrm{c}\left(\triangle^{\mathfrak{p}_1\;\mathfrak{p}_2}_{\mathfrak{p}_0}\right) \circ_1 \mathrm{c}\left(\triangle^{\mathfrak{q}_1\;\mathfrak{q}_2}_{\mathfrak{q}_0}\right), \qquad \mathfrak{p}_0,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2 \in \mathcal{M}, \delta \in \bar{\mathcal{M}}, \tag{4.1.1a}$$

$$\sum_{\substack{\mathfrak{p}_1,\mathfrak{q}_0\in\mathcal{M}\\ \mathfrak{p}_1\star\mathfrak{q}_0=\mathbb{1}_\mathcal{M}}} \left(\mathrm{c}\left(\triangle^{\mathfrak{p}_1\;\mathfrak{p}_2}_{\mathfrak{p}_0}\right) \circ_1 \mathrm{c}\left(\triangle^{\mathfrak{q}_1\;\mathfrak{q}_2}_{\mathfrak{q}_0}\right) - \mathrm{c}\left(\triangle^{\mathfrak{q}_1\;\mathfrak{p}_1}_{\mathfrak{p}_0}\right) \circ_2 \mathrm{c}\left(\triangle^{\mathfrak{q}_2\;\mathfrak{p}_2}_{\mathfrak{q}_0}\right)\right), \qquad \mathfrak{p}_0,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2 \in \mathcal{M}, \tag{4.1.1b}$$

$$\sum_{\substack{\mathfrak{p}_2,\mathfrak{q}_0\in\mathcal{M}\\ \mathfrak{p}_2\star\mathfrak{q}_0=\delta}} \mathrm{c}\left(\triangle^{\mathfrak{p}_1\;\mathfrak{p}_2}_{\mathfrak{p}_0}\right) \circ_2 \mathrm{c}\left(\triangle^{\mathfrak{q}_1\;\mathfrak{q}_2}_{\mathfrak{q}_0}\right), \qquad \mathfrak{p}_0,\mathfrak{p}_1,\mathfrak{q}_1,\mathfrak{q}_2 \in \mathcal{M}, \delta \in \bar{\mathcal{M}}, \tag{4.1.1c}$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are $\mathcal{M}$-triangles.

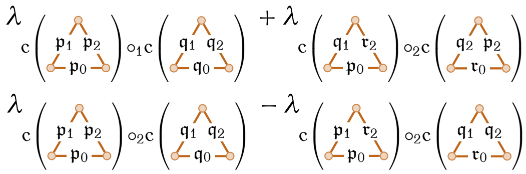
▶ **Proposition 4.1.1** — *Let $\mathcal{M}$ be a finite unitary magma. Then the Koszul dual $\mathrm{NC}\mathcal{M}^!$ of $\mathrm{NC}\mathcal{M}$ admits the presentation $(\mathcal{T}_\mathcal{M}, \mathfrak{R}^!_{\mathrm{NC}\mathcal{M}})$.*

◀ **Proof** — Let

$$f := \sum_{\mathfrak{t}\in T_3} \lambda_\mathfrak{t}\mathfrak{t} \tag{4.1.2}$$

be a generic element of $\mathfrak{R}^!_{\mathrm{NC}\mathcal{M}}$, where T_3 is the set of all syntax trees on $\mathcal{T}_\mathcal{M}$ of arity 3 and the $\lambda_\mathfrak{t}$ are coefficients of $\mathbb{K}$. By definition of Koszul duality of operads, $\langle r, f\rangle = 0$ for all $r \in \mathfrak{R}_{\mathrm{NC}\mathcal{M}}$, where $\langle -, -\rangle$ is the scalar product defined in (2.2.7). Then, since $\mathfrak{R}_{\mathrm{NC}\mathcal{M}}$ is the subspace of $\mathrm{Free}(\mathbb{K}\langle\mathcal{T}_\mathcal{M}\rangle)(3)$ generated by (3.2.1a), (3.2.1b), and (3.2.1c), we have

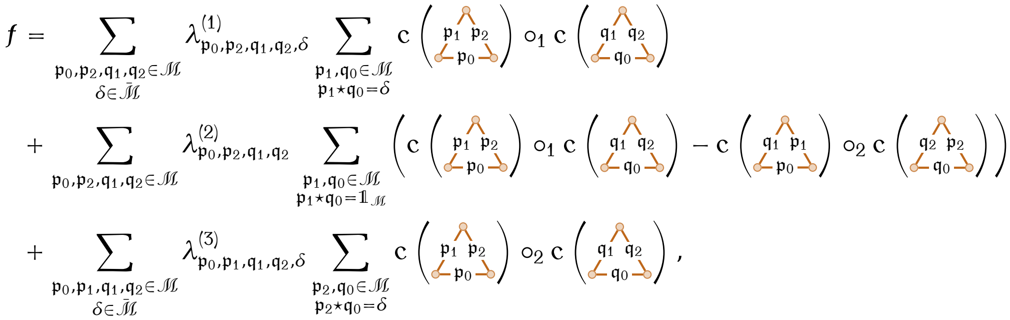
$$\lambda_{\mathrm{c}\left(\triangle^{\mathfrak{p}_1\;\mathfrak{p}_2}_{\mathfrak{p}_0}\right) \circ_1 \mathrm{c}\left(\triangle^{\mathfrak{q}_1\;\mathfrak{q}_2}_{\mathfrak{q}_0}\right)} - \lambda_{\mathrm{c}\left(\triangle^{\mathfrak{r}_1\;\mathfrak{p}_2}_{\mathfrak{p}_0}\right) \circ_1 \mathrm{c}\left(\triangle^{\mathfrak{q}_1\;\mathfrak{q}_2}_{\mathfrak{r}_0}\right)} = 0, \qquad \mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{r}_1 \star \mathfrak{r}_0 \neq \mathbb{1}_\mathcal{M}, \tag{4.1.3a}$$



$\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathfrak{r}_2 \star \mathfrak{r}_0 = \mathbb{1}_{\mathcal{M}}$, (4.1.3b)

$\mathfrak{p}_2 \star \mathfrak{q}_0 = \mathfrak{r}_2 \star \mathfrak{r}_0 \neq \mathbb{1}_{\mathcal{M}}$, (4.1.3c)

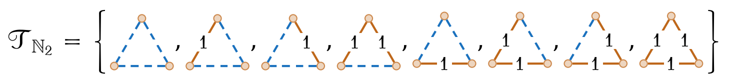
where $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{r}$ are $\mathcal{M}$-triangles. This implies that f is of the form



(4.1.4)

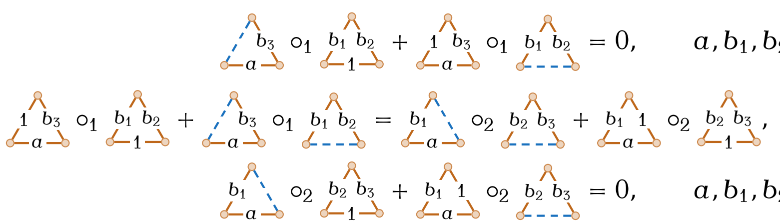
where, for $\mathcal{M}$-triangles $\mathfrak{p}$ and $\mathfrak{q}$ and $\delta \in \bar{\mathcal{M}}$, the $\lambda^{(1)}_{\mathfrak{p}_0,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_0,\delta}$, $\lambda^{(2)}_{\mathfrak{p}_0,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2}$, and $\lambda^{(3)}_{\mathfrak{p}_1,\mathfrak{p}_0,\mathfrak{q}_1,\mathfrak{q}_2,\delta}$ are coefficients of $\mathbb{K}$. Therefore, f belongs to the space generated by (4.1.1a), (4.1.1b), and (4.1.1c). Finally, since the coefficients of each of these relations satisfy (4.1.3a), (4.1.3b), and (4.1.3c), the statement of the proposition follows. ■

We use Proposition 4.1.1 to express the presentations of the operads $\mathrm{NC}\mathbb{N}_2^!$ and $\mathrm{NC}\mathbb{D}_0^!$. The operad $\mathrm{NC}\mathbb{N}_2^!$ is generated by



(4.1.5)

and these generators satisfy only the nontrivial relations

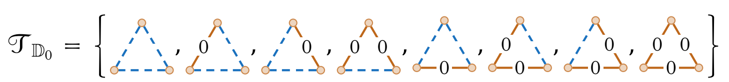


$a, b_1, b_2, b_3 \in \mathbb{N}_2$, (4.1.6a)

$a, b_1, b_2, b_3 \in \mathbb{N}_2$, (4.1.6b)

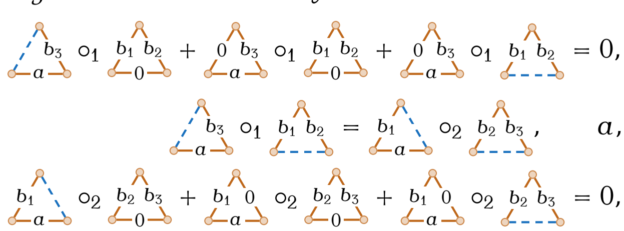
$a, b_1, b_2, b_3 \in \mathbb{N}_2$. (4.1.6c)

On the other hand, the operad $\mathrm{NC}\mathbb{D}_0^!$ is generated by



(4.1.7)

and these generators satisfies only the nontrivial relations



$a, b_1, b_2, b_3 \in \mathbb{D}_0$, (4.1.8a)

$a, b_1, b_2, b_3 \in \mathbb{D}_0$, (4.1.8b)

$a, b_1, b_2, b_3 \in \mathbb{D}_0$. (4.1.8c)

▶ **Proposition 4.1.2** — *Let $\mathcal{M}$ be a finite unitary magma. Then the dimension of the space $\mathfrak{R}^!_{\mathrm{NC}\mathcal{M}}$ is given by*

$$\dim \mathfrak{R}^!_{\mathrm{NC}\mathcal{M}} = 2m^5 - m^4, \tag{4.1.9}$$

where $m := \#\mathcal{M}$.

◀ **Proof —** To compute the dimension of the space of relations $\mathfrak{R}^!_{\mathrm{NC}\mathcal{M}}$ of $\mathrm{NC}\mathcal{M}^!$, we consider the presentation of $\mathrm{NC}\mathcal{M}^!$ provided by Proposition 4.1.1. Consider the space $\mathfrak{R}_1$ generated by the family consisting in the elements (4.1.1a). Since this family is linearly independent and each of its element is totally specified by a tuple $(\mathfrak{p}_0, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2, \delta) \in \mathcal{M}^4 \times \bar{\mathcal{M}}$, we obtain

$$\dim \mathfrak{R}_1 = m^4(m-1). \tag{4.1.10}$$

For the same reason, the dimension of the space $\mathfrak{R}_3$ generated by the elements (4.1.1c) satisfies $\dim \mathfrak{R}_3 = \dim \mathfrak{R}_1$. Now, let $\mathfrak{R}_2$ be the space generated by the elements (4.1.1b). Since this family is linearly independent and each of its elements is totally specified by a tuple $(\mathfrak{p}_0, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2) \in \mathcal{M}^4$, we obtain

$$\dim \mathfrak{R}_2 = m^4. \tag{4.1.11}$$

Therefore, since

$$\mathfrak{R}^!_{\mathrm{NC}\mathcal{M}} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \oplus \mathfrak{R}_3, \tag{4.1.12}$$

we obtain the stated formula (4.1.9) by summing the dimensions of $\mathfrak{R}_1$, $\mathfrak{R}_2$, and $\mathfrak{R}_3$. ■

Observe that, by Propositions 3.2.2 and 4.1.2, we have

$$\begin{aligned}\dim \mathfrak{R}_{\mathrm{NC}\mathcal{M}} + \dim \mathfrak{R}^!_{\mathrm{NC}\mathcal{M}} &= 2m^6 - 2m^5 + m^4 + 2m^5 - m^4 \\ &= 2m^6 \\ &= \dim \mathrm{Free}\left(\mathbb{K}\langle \mathcal{T}_{\mathcal{M}}\rangle\right)(3),\end{aligned} \tag{4.1.13}$$

as expected by Koszul duality, where $m := \#\mathcal{M}$.

4.2 Hilbert series and dimensions

An algebraic equation for the Hilbert series of $\mathrm{NC}\mathcal{M}^!$ is described and a formula involving Narayana numbers to compute its coefficients is provided.

▶ **Proposition 4.2.1 —** *Let $\mathcal{M}$ be a finite unitary magma. The Hilbert series $\mathcal{H}_{\mathrm{NC}\mathcal{M}^!}(t)$ of $\mathrm{NC}\mathcal{M}^!$ satisfies*

$$t + (m-1)t^2 + \left(2m^2t - 3mt + 2t - 1\right)\mathcal{H}_{\mathrm{NC}\mathcal{M}^!}(t) + \left(m^3 - 2m^2 + 2m - 1\right)\mathcal{H}_{\mathrm{NC}\mathcal{M}^!}(t)^2 = 0, \tag{4.2.1}$$

where $m := \#\mathcal{M}$.

◀ **Proof —** Let $G(t)$ be the generating series such that $G(-t)$ satisfies (4.2.1). Therefore, $G(t)$ satisfies

$$-t + (m-1)t^2 + \left(-2m^2t + 3mt - 2t - 1\right)G(t) + \left(m^3 - 2m^2 + 2m - 1\right)G(t)^2 = 0, \tag{4.2.2}$$

and, by solving (4.2.2) as a quadratic equation where t is the unknown, we obtain

$$t = \frac{1 + (2m^2 - 3m + 2)G(t) - \sqrt{1 + 2(2m^2 - m)G(t) + m^2G(t)^2}}{2(m-1)}. \tag{4.2.3}$$

Moreover, by Proposition 3.1.6 and (3.1.15), by setting $F(t) := \mathcal{H}_{\mathrm{NC}\mathcal{M}}(-t)$, we have

$$F(G(t)) = \frac{1 + (2m^2 - 3m + 2)G(t) - \sqrt{1 + 2(2m^2 - m)G(t) + m^2G(t)^2}}{2(m-1)} = t, \tag{4.2.4}$$

showing that $F(t)$ and $G(t)$ are the inverses of each other for series composition.

Now, since by Theorem 3.2.9, $\mathrm{NC}\mathcal{M}$ is a Koszul operad, the Hilbert series of $\mathrm{NC}\mathcal{M}$ and $\mathrm{NC}\mathcal{M}^!$ satisfy (2.2.8). Therefore, (4.2.4) implies that the Hilbert series of $\mathrm{NC}\mathcal{M}^!$ is the series $\mathcal{H}_{\mathrm{NC}\mathcal{M}^!}(t)$, satisfying the stated relation (4.2.1). ■

From Proposition 4.2.1 we deduce that the Hilbert series of $\mathrm{NC}\mathcal{M}^!$ satisfies

$$\mathcal{H}_{\mathrm{NC}\mathcal{M}^!}(t) = \frac{1 - (2m^2 - 3m + 2)t - \sqrt{1 - 2(2m^3 - 2m^2 + m)t + m^2t^2}}{2(m^3 - 2m^2 + 2m - 1)}, \tag{4.2.5}$$

where $m := \#\mathcal{M} \neq 1$.

▶ Proposition 4.2.2 — *Let $\mathcal{M}$ be a finite unitary magma. For all $n \geqslant 2$,*

$$\dim \mathrm{NC}\mathcal{M}^!(n) = \sum_{0 \leqslant k \leqslant n-2} m^{n+1}(m(m-1)+1)^k(m(m-1))^{n-k-2} \operatorname{nar}(n,k). \tag{4.2.6}$$

◀ Proof — The proof consists in enumerating dual $\mathcal{M}$-cliques, introduced in the upcoming Section 4.3. Indeed, by Proposition 4.3.1, $\dim \mathrm{NC}\mathcal{M}^!(n)$ is equal to the number of dual $\mathcal{M}$-cliques of arity n. The expression for $\dim \mathrm{NC}\mathcal{M}^!(n)$ claimed by (4.2.6) can be proved by using similar arguments as the ones intervening in the proof of Proposition 3.1.7 for the expression (3.1.16) of $\dim \mathrm{NC}\mathcal{M}(n)$. ■

We can use Proposition 4.2.2 to compute the first dimensions of $\mathrm{NC}\mathcal{M}^!$. For instance, depending on $m := \#\mathcal{M}$, we have the following sequences of dimensions:

$$1, 1, 1, 1, 1, 1, 1, 1, \qquad m = 1, \tag{4.2.7a}$$

$$1, 8, 80, 992, 13760, 204416, 3180800, 51176960, \qquad m = 2, \tag{4.2.7b}$$

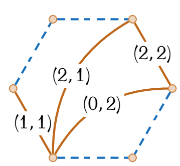
$$1, 27, 1053, 51273, 2795715, 163318599, 9994719033, 632496651597, \qquad m = 3, \tag{4.2.7c}$$

$$1, 64, 6400, 799744, 111923200, 16782082048, 2636161024000, 428208345579520, \qquad m = 4. \tag{4.2.7d}$$

The second one is Sequence **A234596** of [Slo]. The last two sequences are not listed in [Slo] at this time. It is worthwhile to observe that the dimensions of $\mathrm{NC}\mathcal{M}^!$ for $\#\mathcal{M} = 2$ are the ones of the operad BNC of bicolored noncrossing configurations (see Section 5.2).

4.3 Combinatorial basis

To describe a basis of $\mathrm{NC}\mathcal{M}^!$, we introduce the following sort of $\mathcal{M}$-decorated cliques. A *dual $\mathcal{M}$-clique* is an $\mathcal{M}^2$-clique such that its base and its edges are labeled by pairs $(a, a) \in \mathcal{M}^2$, and all solid diagonals are labeled by pairs $(a, b) \in \mathcal{M}^2$ with $a \neq b$. Observe that a non-solid diagonal of a dual $\mathcal{M}$-clique is labeled by $(\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}})$. All definitions about $\mathcal{M}$-cliques given in [Gir20] remain valid for dual $\mathcal{M}$-cliques. For example,



(4.3.1)

is a noncrossing dual $\mathbb{N}_3$-clique.

▶ Proposition 4.3.1 — *Let $\mathcal{M}$ be a finite unitary magma. The underlying graded vector space of $\mathrm{NC}\mathcal{M}^!$ is the linear span of all noncrossing dual $\mathcal{M}$-cliques.*

◀ Proof — The statement of the proposition is equivalent to the fact that the generating series of noncrossing dual $\mathcal{M}$-cliques is the Hilbert series $\mathcal{H}_{\mathrm{NC}\mathcal{M}^!}(t)$ of $\mathrm{NC}\mathcal{M}^!$. From the definition of dual $\mathcal{M}$-cliques, we obtain that the set of dual $\mathcal{M}$-cliques of arity n, $n \geqslant 1$, is in bijection with the set of $\mathcal{M}^2$-Schröder trees of arity n having the outgoing edges from the root and the edges connecting internal nodes with leaves labeled by pairs $(a, a) \in \mathcal{M}^2$, and the edges connecting two internal nodes labeled by pairs $(a, b) \in \mathcal{M}^2$ with $a \neq b$. The map bt defined in Section 3.1.2 (see also Section 3.1.3) realizes such a bijection. Let $T(t)$ be the generating series of these $\mathcal{M}^2$-Schröder trees, and let $S(t)$ be the generating series of the $\mathcal{M}^2$-Schröder trees of arities greater than 1 and such that the outgoing edges from the roots and the edges connecting two internal nodes are labeled by pairs $(a, b) \in \mathcal{M}^2$ with $a \neq b$, and the edges connecting internal nodes with leaves are labeled by pairs $(a, a) \in \mathcal{M}^2$. From the description of these trees, we have

$$S(t) = m(m-1)\frac{(mt + S(t))^2}{1 - mt - S}, \tag{4.3.2}$$

where $m := \#\mathcal{M}$. Moreover, for $m \neq 1$, $T(t)$ satisfies

$$T(t) = t + \frac{S(t)}{m-1}, \tag{4.3.3}$$

and we obtain that $T(t)$ admits (4.2.5) as solution. Then, by Proposition 4.2.1, for $m \neq 1$, this implies the statement of the proposition. If $m = 1$, it follows from Proposition 4.1.1 that $\mathrm{NC}\mathcal{M}^!$ is isomorphic to the associative operad As (see for instance [LV12] for the definition of this operad). Hence, in this case, $\dim \mathrm{NC}\mathcal{M}^!(n) = 1$ for all $n \geqslant 1$. Since there is exactly one dual $\mathcal{M}$-clique of arity n for $n \geqslant 1$, the statement of the proposition is satisfied. ■

Proposition 4.3.1 gives a combinatorial description of the elements of $\mathrm{NC}\mathcal{M}^!$. Nevertheless, for the time being we do not know a partial composition on the linear span of these elements providing a realization of $\mathrm{NC}\mathcal{M}^!$.

5 Concrete constructions

The clique construction provides alternative definitions of known operads. We explore here the cases of the operad NCT of based noncrossing trees, the operad $\mathcal{FF}_4$ of formal fractions, and the operad BNC of bicolored noncrossing configurations.

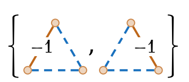
5.1 Rational functions and related operads

We use here the noncrossing clique construction to interpret a few operads related to the operad RatFct of rational functions of Loday [Lod10] (see also Section 2.2.8 of [Gir20]).

5.1.1 Dendriform and based noncrossing tree operads.

The *operad of based noncrossing trees* NCT is an operad introduced in [Cha07]. This operad is generated by two binary elements $\prec$ and $\succ$ satisfying one nontrivial quadratic relation. The algebras over NCT are *L-algebras* and have been studied in [Ler11]. We do not describe NCT in detail here because this is not essential for the sequel. We just explain how to construct NCT through the clique construction and interpret a known link between NCT and the dendriform operad through the rational functions associated with $\mathbb{Z}$-cliques (see Section 2.2.8 of [Gir20]).

Let $\mathcal{O}_{\mathrm{NCT}}$ be the suboperad of $\mathrm{C}\mathbb{Z}$ generated by



(5.1.1)

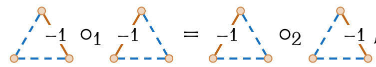
By using Proposition 3.3.2, we find that the Hilbert series $\mathcal{H}_{\mathcal{O}_{\mathrm{NCT}}}(t)$ of $\mathcal{O}_{\mathrm{NCT}}$ satisfies

$$t - \mathcal{H}_{\mathcal{O}_{\mathrm{NCT}}}(t) + 2\mathcal{H}_{\mathcal{O}_{\mathrm{NCT}}}(t)^2 - \mathcal{H}_{\mathcal{O}_{\mathrm{NCT}}}(t)^3 = 0. \tag{5.1.2}$$

The first dimensions of $\mathcal{O}$ are

$$1, 2, 7, 30, 143, 728, 3876, 21318, \tag{5.1.3}$$

and form Sequence **A006013** of [Slo]. Moreover, one can see that

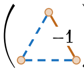


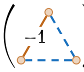
(5.1.4)

is the only nontrivial relation of degree 2 between the generators of $\mathcal{O}_{\mathrm{NCT}}$.

▶ **Proposition 5.1.1** — *The operad* $\mathcal{O}_{\mathrm{NCT}}$ *is isomorphic to the operad* NCT.

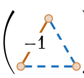
◀ **Proof** — Let $\phi_{\mathrm{NCT}} : \mathcal{O}_{\mathrm{NCT}}(2) \to \mathrm{NCT}(2)$ be the linear map satisfying

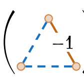
ϕ_{NCT} () $= \prec$, (5.1.5a)

ϕ_{NCT} () $= \succ$, (5.1.5b)

where $\prec$ and $\succ$ are the two binary generators of NCT. In [Cha07], a presentation of NCT is described wherein its generators satisfy one nontrivial relation of degree 2. This relation can be obtained by replacing each $\mathbb{Z}$-clique appearing in (5.1.4) by its image by ϕ_{NCT}. For this reason, ϕ_{NCT} uniquely extends to an operad morphism. Moreover, because the image of ϕ_{NCT} contains all the generators of NCT, this morphism is surjective. Finally, the Hilbert series of NCT satisfies (5.1.2), so that $\mathcal{O}_{\mathrm{NCT}}$ and NCT have the same dimensions. Therefore, ϕ_{NCT} is an operad isomorphism. ■

Loday as shown in [Lod10] that the suboperad of RatFct generated by the rational functions $f_1(u_1, u_2) := u_1^{-1}$ and $f_2(u_1, u_2) := u_2^{-1}$ is isomorphic to the dendriform operad Dendr [Lod01]. This operad is generated by two binary elements $\prec$ and $\succ$ satisfying three nontrivial quadratic relations. An isomorphism between Dendr and the suboperad of RatFct generated by f_1 and f_2 sends $\prec$ to f_2 and $\succ$ to f_1. The map F_{Id} introduced in [Gir20] is an operad morphism from $\mathcal{C}\mathbb{Z}$ to RatFct. Hence, the restriction of F_{Id} on $\mathcal{O}_{\mathrm{NCT}}$ is also an operad morphism from $\mathcal{O}_{\mathrm{NCT}}$ to RatFct. Moreover, since

F_{Id} () $= \dfrac{1}{u_1} = f_1$, (5.1.6a)

F_{Id} () $= \dfrac{1}{u_2} = f_2$, (5.1.6b)

the map F_{Id} is a surjective operad morphism from $\mathcal{O}_{\mathrm{NCT}}$ to Dendr.

5.1.2 Operad of formal fractions. The *operad of formal fractions* $\mathcal{FF}$ is an operad introduced in [CHN16]. Its elements of arity $n \geqslant 1$ are fractions whose numerators and denominators are formal products of subsets of $[n]$. For instance,

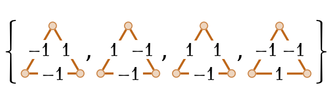
$$\frac{\{1,3,4\}\{2\}\{4,6\}}{\{2,3,5\}\{4\}} \tag{5.1.7}$$

is an element of arity 6 of $\mathcal{FF}$. We do not describe the partial composition of this operad since its knowledge is not essential for the sequel. The operad $\mathcal{FF}$ admits a suboperad $\mathcal{FF}_4$, defined as the binary suboperad of $\mathcal{FF}$ generated by

$$\left\{ \frac{1}{\{1\}\{1,2\}}, \frac{1}{\{2\}\{1,2\}}, \frac{1}{\{1,2\}}, \frac{1}{\{1\}\{2\}} \right\}. \tag{5.1.8}$$

We explain here how to construct $\mathcal{FF}_4$ through the clique construction.

Let $\mathcal{O}_{\mathcal{FF}_4}$ be the suboperad of $\mathcal{C}\mathbb{Z}$ generated by



(5.1.9)

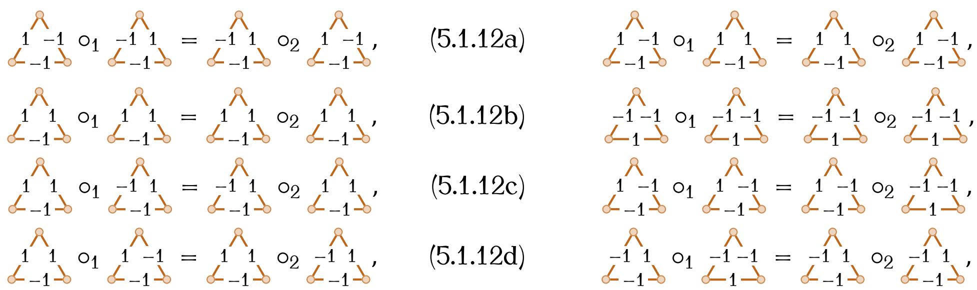
By using Proposition 3.3.2, we find that the Hilbert series $\mathcal{H}_{\mathcal{O}_{\mathcal{FF}_4}}(t)$ of $\mathcal{O}_{\mathcal{FF}_4}$ satisfies

$$t + (2t-1)\mathcal{H}_{\mathcal{O}_{\mathcal{FF}_4}}(t) + 2\mathcal{H}_{\mathcal{O}_{\mathcal{FF}_4}}(t)^2 = 0. \tag{5.1.10}$$

The first dimensions of $\mathcal{O}_{\mathcal{FF}_4}$ are

$$1, 4, 24, 176, 1440, 12608, 115584, 1095424, \tag{5.1.11}$$

and form Sequence **A156017** of [Slo]. Moreover, by computer exploration, we obtain the list

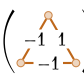


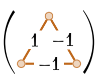
(5.1.12a) (5.1.12b) (5.1.12c) (5.1.12d) (5.1.12e) (5.1.12f) (5.1.12g) (5.1.12h)

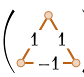
of all nontrivial relations of degree 2 between the generators of $\mathcal{O}_{\mathcal{FF}_4}$.

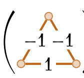
▶ **Proposition 5.1.2 —** *The operad $\mathcal{O}_{\mathcal{FF}_4}$ is isomorphic to the operad $\mathcal{FF}_4$.*

◀ **Proof —** Let $\phi_{\mathcal{FF}_4} : \mathcal{O}_{\mathcal{FF}_4}(2) \to \mathcal{FF}_4(2)$ be the linear map satisfying

$\phi_{\mathcal{FF}_4}$ () $= \dfrac{1}{\{1\}\{1,2\}}$, (5.1.13a)

$\phi_{\mathcal{FF}_4}$ () $= \dfrac{1}{\{2\}\{1,2\}}$, (5.1.13b)

$\phi_{\mathcal{FF}_4}$ () $= \dfrac{1}{\{1,2\}}$, (5.1.13c)

$\phi_{\mathcal{FF}_4}$ () $= \dfrac{1}{\{1\}\{2\}}$. (5.1.13d)

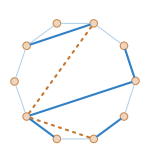
In [CHN16], a presentation of $\mathcal{FF}_4$ is described wherein its generators satisfy eight nontrivial relations of degree 2. These relations can be obtained by replacing each $\mathbb{Z}$-clique appearing in (5.1.12a)–(5.1.12h) by its image by $\phi_{\mathcal{FF}_4}$. For this reason, $\phi_{\mathcal{FF}_4}$ uniquely extends to an operad morphism. Moreover, because the image of $\phi_{\mathcal{FF}_4}$ contains all the generators of $\mathcal{FF}_4$, this morphism is surjective. Finally, again by [CHN16], the Hilbert series of $\mathcal{FF}_4$ satisfies (5.1.10), so that $\mathcal{O}_{\mathcal{FF}_4}$ and $\mathcal{FF}_4$ have the same dimensions. Therefore, $\phi_{\mathcal{FF}_4}$ is an operad isomorphism. ■

Hence, Proposition 5.1.2 shows that the operad $\mathcal{FF}_4$ can be built through the construction C. Observe also that, as a consequence of Proposition 5.1.2, all suboperads of $\mathcal{FF}_4$ defined in [CHN16] that are generated by a subset of (5.1.8) can be constructed by the clique construction.

5.2 Operad of bicolored noncrossing configurations

The *operad of bicolored noncrossing configurations* BNC is an operad defined in [CG14]. Let us describe this operad.

A *bicolored noncrossing configuration* is a noncrossing configuration $\mathfrak{c}$ where each labeled arc is either *thick* (drawn as a thick line) of *dotted* (drawn as a dotted line) and such that all dotted arcs are diagonals. For instance,



(5.2.1)

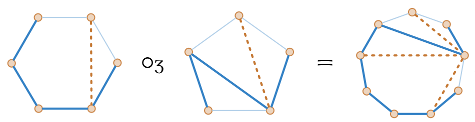
is a bicolored noncrossing configuration of size 9. For $n \geqslant 2$, $\mathrm{BNC}(n)$ is the linear span of all bicolored noncrossing configurations of size n. Moreover, $\mathrm{BNC}(1)$ is the linear span of the singleton containing the only polygon of size 1 where its only arc is unlabeled. The partial composition of BNC is defined graphically as follows. For bicolored noncrossing configurations $\mathfrak{c}$ and $\mathfrak{d}$ of respective arities n and m, and $i \in [n]$, the bicolored noncrossing configuration $\mathfrak{c} \circ_i \mathfrak{d}$ is obtained by gluing the base of $\mathfrak{d}$ onto the ith edge of $\mathfrak{c}$, and then,

(a) if the base of $\mathfrak{d}$ and the ith edge of $\mathfrak{c}$ are both unlabeled, the arc $(i, i+m)$ of $\mathfrak{c} \circ_i \mathfrak{d}$ becomes dotted;

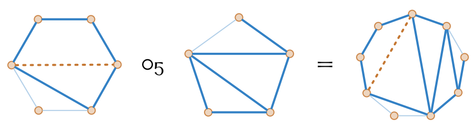
(b) if the base of $\mathfrak{d}$ and the ith edge of $\mathfrak{c}$ are both thick, the arc $(i, i+m)$ of $\mathfrak{c} \circ_i \mathfrak{d}$ becomes thick;

(c) otherwise, the arc $(i, i+m)$ of $\mathfrak{c} \circ_i \mathfrak{d}$ is unlabeled.

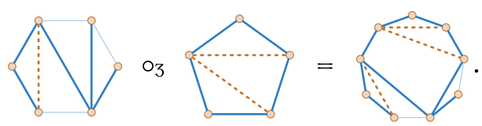
For example,



(5.2.2a)



(5.2.2b)



(5.2.2c)

We now consider the unitary magma $\mathcal{M}_{\mathrm{BNC}} := \{\mathbb{1}, \mathtt{a}, \mathtt{b}\}$ wherein operation $\star$ is defined by the Cayley table

$\star$	$\mathbb{1}$	a	b
$\mathbb{1}$	$\mathbb{1}$	a	b
a	a	a	$\mathbb{1}$
b	b	$\mathbb{1}$	b

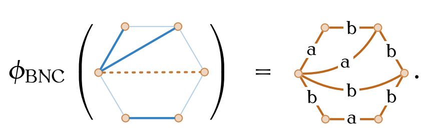
. (5.2.3)

In other words, $\mathcal{M}_{\mathrm{BNC}}$ is the unitary magma wherein a and b are idempotent, and $\mathtt{a} \star \mathtt{b} = \mathbb{1} = \mathtt{b} \star \mathtt{a}$. Observe that $\mathcal{M}_{\mathrm{BNC}}$ is a commutative unitary magma, but, since

$$(\mathtt{b} \star \mathtt{a}) \star \mathtt{a} = \mathbb{1} \star \mathtt{a} = \mathtt{a} \neq \mathtt{b} = \mathtt{b} \star \mathbb{1} = \mathtt{b} \star (\mathtt{a} \star \mathtt{a}), \tag{5.2.4}$$

the operation $\star$ is not associative.

Let $\phi_{\mathrm{BNC}} : \mathrm{BNC} \to \mathrm{NC}\mathcal{M}_{\mathrm{BNC}}$ be the linear map defined in the following way. For a bicolored noncrossing configuration $\mathfrak{c}$, $\phi_{\mathrm{BNC}}(\mathfrak{c})$ is the noncrossing $\mathcal{M}_{\mathrm{BNC}}$-clique of $\mathrm{NC}\mathcal{M}_{\mathrm{BNC}}$ obtained by replacing all thick arcs of $\mathfrak{c}$ by arcs labeled by a, all dotted diagonals of $\mathfrak{c}$ by diagonals labeled by b, all unlabeled edges and bases of $\mathfrak{c}$ by edges labeled by b, and all unlabeled diagonals of $\mathfrak{c}$ by diagonals labeled by $\mathbb{1}$. For instance,



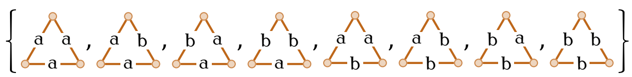
(5.2.5)

▶ **Proposition 5.2.1** — *The linear span of* ⊶ *together with all noncrossing* $\mathcal{M}_{\mathrm{BNC}}$*-cliques without edges nor bases labeled by* $\mathbb{1}$ *forms a suboperad of* $\mathrm{NC}\mathcal{M}_{\mathrm{BNC}}$ *isomorphic to* BNC*. Moreover,* ϕ_{BNC} *is an isomorphism between these two operads.*

◀ **Proof** — We denote the subspace of $\mathrm{NC}\mathcal{M}_{\mathrm{BNC}}$ described in the statement of the proposition by $\mathcal{O}_{\mathrm{BNC}}$. First of all, it follows from the definition of the partial composition of $\mathrm{NC}\mathcal{M}_{\mathrm{BNC}}$ that $\mathcal{O}_{\mathrm{BNC}}$ is closed under the partial composition operation. Hence, and since $\mathcal{O}_{\mathrm{BNC}}$ contains the unit of $\mathrm{NC}\mathcal{M}_{\mathrm{BNC}}$, $\mathcal{O}_{\mathrm{BNC}}$ is an operad. Second, observe that the image of ϕ_{BNC} is the underlying space of $\mathcal{O}_{\mathrm{BNC}}$ and, from the definition of the partial composition of BNC, one can check that ϕ_{BNC} is an operad morphism. Finally, since ϕ_{BNC} is a bijection from BNC to $\mathcal{O}_{\mathrm{BNC}}$, the statement of the proposition follows. ■

Hence, Proposition 5.2.1 shows that the operad BNC can be built through the noncrossing clique construction. Moreover, observe that in [CG14], an automorphism of BNC called *complement* is considered. The complement of a bicolored noncrossing configuration is an involution acting by modifying the labels of some of its arcs. Under our setting, this automorphism translates simply into the map $\mathrm{C}\theta : \mathcal{O}_{\mathrm{BNC}} \to \mathcal{O}_{\mathrm{BNC}}$ where $\mathcal{O}_{\mathrm{BNC}}$ is the operad isomorphic to BNC described in the statement of Proposition 5.2.1 and $\theta : \mathcal{M}_{\mathrm{BNC}} \to \mathcal{M}_{\mathrm{BNC}}$ is the unitary magma automorphism of $\mathcal{M}_{\mathrm{BNC}}$ satisfying $\theta(\mathbb{1}) = \mathbb{1}$, $\theta(\mathrm{a}) = \mathrm{b}$, and $\theta(\mathrm{b}) = \mathrm{a}$.

Moreover, it is shown in [CG14] that the set of bicolored noncrossing configurations of arity 2 is a minimal generating set of BNC. Thus, by Proposition 5.2.1, the set



(5.2.6)

is a minimal generating set of the suboperad $\mathcal{O}_{\mathrm{BNC}}$ of $\mathrm{NC}\mathcal{M}_{\mathrm{BNC}}$ isomorphic to BNC. As a consequence, all the suboperads of BNC defined in [CG14] which are generated by a subset of the set of generators of BNC can be constructed by the noncrossing clique construction. This includes, among others, the magmatic operad, the free operad on two binary generators, the operad of noncrossing plants [Cha07], the dipterous operad [LR03; Zin12], and the 2-associative operad [LR06; Zin12].

6 Conclusion and perspectives

In this article we have completed the study of the clique construction introduced in [Gir20] by focusing on the suboperad $\mathrm{NC}\mathcal{M}$ of $\mathrm{C}\mathcal{M}$ of noncrossing $\mathcal{M}$-cliques. As noticed in the previous sections, $\mathrm{NC}\mathcal{M}$ has a particular status among the suboperads of $\mathrm{C}\mathcal{M}$ because $\mathrm{NC}\mathcal{M}$ is the smallest suboperad of $\mathrm{C}\mathcal{M}$ that contains all elements of arity 2 (the $\mathcal{M}$-triangles) and is the biggest binary suboperad of $\mathrm{C}\mathcal{M}$. This operad is also a Koszul operad when $\mathcal{M}$ is a finite unitary magma.

An open question concerns the Koszul dual $\mathrm{NC}\mathcal{M}^!$ of $\mathrm{NC}\mathcal{M}$. Section 4 contains results about this operad, such as a description of its presentation and a formula for its dimensions. We have also established the fact that, as graded vector space, $\mathrm{NC}\mathcal{M}^!$ is isomorphic to the linear span of all noncrossing dual $\mathcal{M}$-cliques. To obtain a realization of $\mathrm{NC}\mathcal{M}^!$, it is now enough to endow this last space with an adequate partial composition. Finding such a composition is worth to obtain.

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