
Cliff operads: a hierarchy of operads on words

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ABSTRACT. A new hierarchy of operads over the linear spans of δ -cliffs, which are some words of integers, is introduced. These operads are intended to be analogues of the operad of permutations, also known as the associative symmetric operad. We obtain operads whose partial compositions can be described in terms of intervals of the lattice of δ -cliffs. These operads are very peculiar in the world of the combinatorial operads since, despite the relative simplicity for their construction, they are infinitely generated and they have nonquadratic and nonhomogeneous nontrivial relations. We provide a general construction for some of their quotients. We use it to endow the spaces of permutations, m -increasing trees, c -rectangular paths, and m -Dyck paths with operad structures. The operads on c -rectangular paths admit, as Koszul duals, operads generalizing the duplicial and triplicial operads.

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INTRODUCTION

Endowing sets of combinatorial objects with operations has a long-term tradition. The literature abounds of examples of monoids, lattices, pre-Lie algebras, associative algebras, Hopf bialgebras, and operads defined on the linear span of combinatorial sets (see among others the recent works [MR95, LR02, BF03, AL07, Gir12, Gir15, LNPRR20]). Adding such an algebraic dimension offers a new point of view of the objects: we can see these as assemblies, through the offered operations, of generators of the considered structure. This gives among others tools to enumerate combinatorial sets or to establish transformations (and in particular bijections) between two sets of combinatorial objects. All this also maintains connections with partial order theory because the operations in most of these structures can be described as intervals of some partial orders on the underlying objects (as few examples, see [LNPRR20] for some operads on m -Dyck paths, and [LR02] for Hopf bialgebras on permutations and binary trees). In this vein, there is a very rich operad structure on the linear span $\mathbf{As}$ of all permutations for which the operadic partial composition can be described as intervals of the right weak order [AL07].

The primary impetus for this work was the will to introduce a similar operad structure on the linear span of the set of all $\mathbf{1}$ -cliffs, a set of words of integers which is in one-to-one correspondence with the set of permutations. These objects admit some generalizations named δ -cliffs, depending on a parameter δ which is a map from $\mathbb{N} \setminus \{0\}$ to $\mathbb{N}$ (or equivalently, an infinite word of integers). In our context, we search for an analogue of $\mathbf{As}$ involving δ -cliffs instead, and for which the operadic partial composition can be described as intervals of a lattice on δ -cliffs introduced in [CG20, CG22]. As shown in this previous work of the two present authors, the linear span $\mathbf{Cl}_\delta$ of all δ -cliffs when δ is unimodal admits the structure of an associative algebra. In the present work, we show that this space admits (up to a shift in its graduation) also the structure of an operad. Surprisingly, the construction works again only when δ is unimodal. In a remarkable way, in the same way as for $\mathbf{As}$, the partial composition of $\mathbf{Cl}_\delta$ can be described in terms of intervals of the poset of δ -cliffs. More precisely, there is a basis of $\mathbf{Cl}_\delta$ for which the partial composition of two basis elements admits as support the empty set or an interval of the poset of δ -cliffs.

The main results presented here include the construction of three different bases of $\mathbf{Cl}_\delta$, a necessary and sufficient condition on δ for the fact that $\mathbf{Cl}_\delta$ is finitely generated, and a sufficient condition on δ for the fact that the space of nontrivial relations of $\mathbf{Cl}_\delta$ is not finitely generated. We also explore a way to construct quotient operads $\mathbf{Cl}_S$ of $\mathbf{Cl}_\delta$ whose bases are indexed by particular subsets S of δ -cliffs. We show here that when S is a sublattice of the lattice of δ -cliffs, the partial composition of $\mathbf{Cl}_S$ can be described in terms of intervals of S . We finally explore some concrete examples of operads $\mathbf{Cl}_\delta$. These operads appear, unexpectedly, to have a very rich and complex structure. For instance, the space of nontrivial relations of $\mathbf{Cl}_1$ is infinitely generated and has elements which are nonquadratic and nonhomogeneous in terms of degrees. This is quite rare in the panorama of combinatorial operads and seems to brought to the light a very singular new object. The bases of the operads $\mathbf{Cl}_m$, where m is the arithmetic sequence with a common difference of m , are indexed by some labeled planar rooted trees and have also an intricate structure. We also explore the case of the quotients $\mathbf{Cl}_S$ where S is the set of weakly increasing δ -cliffs, called δ -hills in [CG20, CG22]. We obtain here operads whose dimensions are provided by shifted Fuss-Catalan numbers (which are hence different

from the operads of [LNPRR20] whose dimensions are given by Fuss-Catalan numbers) and where the Stanley lattice [Sta75] is the underlying partial order for the description of the partial composition. We also construct a last family of operads whose dimensions are provided by some binomial coefficients and whose bases are indexed by some paths formed by east and north steps in rectangles. They have the interesting property to have, as Koszul duals, operads having dimensions enumerated by (not shifted) Fuss-Catalan numbers which can be thought as generalizations of the duplicial operad [BF03] and the triplicial operad [Ler11].

This paper is organized as follows. Section 1 exposes some definitions about δ -cliffs and some background and notations about operads. In Section 2, we provide the construction of the operads $\mathbf{Cl}_\delta$ and their first properties. Section 3 is devoted to the study of the quotient operads $\mathbf{Cl}_S$ of $\mathbf{Cl}_\delta$. Section 4 presents some particular examples arising from our constructions.

General notations and conventions. For any integers i and j , $[i, j]$ denotes the set $\{i, i + 1, \dots, j\}$. For any integer i , $[i]$ denotes the set $[1, i]$ and $\llbracket i \rrbracket$ denotes the set $[0, i]$. For any set A , A^* is the set of all words on A . If a is a letter and n is a nonnegative integer, a^n is the word consisting in n occurrences of a . In particular, a^0 is the empty word ϵ . For any $w \in A^*$, $\ell(w)$ is the length of w , and for any $i \in [\ell(w)]$, $w(i)$ is the i -th letter of w . For any $i \leq j \in [\ell(w)]$, $w(i, j)$ is the word $w(i)w(i + 1) \dots w(j)$. All algebraic structures considered in this work have a field $\mathbb{K}$ of characteristic zero as ground field.

1. PRELIMINARIES

This first part contains elementary definitions about cliffs and related combinatorial objects. We also provide brief recalls about nonsymmetric operads and finish by describing interstice operads. These operads contain, as suboperads or quotients, the forthcoming operads on cliffs.

1.1. Cliffs and related objects. Cliffs are essentially words of nonnegative integers satisfying some properties [CG20, CG22]. We give here basic definitions about these objects and explain how particular families of cliffs can encode other known combinatorial families (like integer compositions, permutations, m -increasing trees, c -rectangular paths, and m -Dyck paths).

1.1.1. Graded sets. A *graded set* is a set expressed as a disjoint union

$$S := \bigsqcup_{n \in \mathbb{N}} S(n) \tag{1.1.1}$$

such that all $S(n)$, $n \in \mathbb{N}$, are sets. The *size* $|x|$ of an $x \in S$ is the unique integer n such that $x \in S(n)$. If S and S' are two graded sets, a map $\theta : S \rightarrow S'$ is a *graded set morphism* if for any $x \in S$, $|\theta(x)| = |x|$. Besides S' is a *graded subset* of S if for any $n \in \mathbb{N}$, $S'(n) \subseteq S(n)$.

1.1.2. Cliffs. A *range map* is a map $\delta : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$. For any $m \geq 0$, we denote by $\mathbf{m}$ the range map satisfying $\mathbf{m}(i) = (i - 1)m$ for any $i \geq 1$. Moreover, for any $c \geq 0$, we denote by $\underline{c}$ the range map satisfying $\underline{c}(i) = c$ for any $i \geq 1$. We shall specify range maps as infinite words $\delta = \delta(1)\delta(2)\dots$. For this purpose, for any $a \in \mathbb{N}$, we shall denote by a^ω the infinite word having all its letters equal to a . For instance, the notation $\delta := 2113^\omega$ stands the range map δ satisfying $\delta(1) = 2$, $\delta(2) = \delta(3) = 1$, and $\delta(i) = 3$ for all $i \geq 4$. A range map δ is *unimodal* if

for any $1 \leq i_1 < i_2 < i_3$, $\delta(i_1) > \delta(i_2) < \delta(i_3)$ never occurs. Besides, δ is *1-dominated* if there is a $k \geq 1$ such that for all $k' \geq k$, $\delta(1) \geq \delta(k')$.

A word w on $\mathbb{N}$ is a *δ -cliff* if for any $i \in [\ell(w)]$, $w(i) \in \llbracket \delta(i) \rrbracket$. The *size* $|w|$ of a δ -cliff w is $\ell(w) + 1$. The graded set of all δ -cliffs is denoted by Cl_δ . For instance $\text{Cl}_1(4) = \{000, 001, 002, 010, 011, 012\}$. Let also Hi_δ be the graded subset of Cl_δ containing all weakly increasing δ -cliffs. For instance $\text{Hi}_1(4) = \{000, 001, 002, 011, 012\}$. Any element of Hi_δ is a *δ -hill*. The *δ -reduction* of a word w on $\mathbb{N}$ is the δ -cliff $r_\delta(w)$ satisfying

$$(r_\delta(w))(i) = \min\{w(i), \delta(i)\} \quad (1.1.2)$$

for any $i \in [\ell(w)]$. For instance, $r_1(212066) = 012045$ and $r_2(212066) = 012066$.

Let $\preceq$ be the partial order relation on Cl_δ satisfying $u \preceq v$ for any $u, v \in \text{Cl}_\delta$ such that $|u| = |v|$ and for all $i \in [\ell(u)]$, $u_i \leq v_i$. For any $n \geq 1$, the poset $(\text{Cl}_\delta(n), \preceq)$ is the *δ -cliff poset* of order n . This poset is a Cartesian product of total orders. For this reason, it is a distributive lattice. For any $u, v \in \text{Cl}(n)$ such that $u \preceq v$, we denote by $[u, v]_\preceq$ the interval between u and v in this poset.

We now present graded sets which are in one-to-one correspondence with some particular sets of cliffs.

1.1.3. Integer compositions. For any $n \geq 1$, $\text{Cl}_1(n)$ is the set of all binary words of length $n - 1$. For this reason, this set is in one-to-one correspondence with the set of all *integer compositions* of n , which are sequences $(\lambda_1, \dots, \lambda_k)$ of positive integers such that $\lambda_1 + \dots + \lambda_k = n$. A possible bijection sends $w \in \text{Cl}_1(n)$ to the integer composition $\text{comp}(w) := (\lambda_1, \dots, \lambda_k)$ where w has $k - 1$ occurrences of 1 and provided that $w = 0^{\lambda_1-1}10^{\lambda_2-1}1 \dots 10^{\lambda_k-1}$. For instance, in $\text{Cl}_1(8)$,

$$\text{comp}(1100010) = (1, 1, 4, 2). \quad (1.1.3)$$

Observe that the poset $(\text{Cl}_1(n), \preceq)$ is the Boolean lattice of order $n - 1$.

1.1.4. Permutations. For any $n \geq 1$, $\text{Cl}_1(n)$ is the set of all words w of length $n - 1$ such that $w(i) \in \llbracket i - 1 \rrbracket$ for all $i \in [n - 1]$. For this reason, this set is in one-to-one correspondence with the set of all permutations of size $n - 1$. A possible bijection sends $w \in \text{Cl}_1(n)$ to the permutation $\text{perm}(w) := \sigma$ of size $n - 1$ where σ is the permutation such that for any $i \in [n - 1]$, there are in σ exactly $w(i)$ letters on the right of i which are smaller than i . The word w is sometimes called the *Lehmer code* of σ [Leh60], up to a slight variation. For instance, in $\text{Cl}_1(7)$,

$$\text{perm}(002323) = 436512. \quad (1.1.4)$$

The poset $(\text{Cl}_1(n), \preceq)$ is studied in [Den13].

1.1.5. *m*-increasing trees. For any $m \geq 0$ and any $n \geq 1$, $\text{Cl}_m(n)$ is the set of all words w of length $n - 1$ such that $w(i) \in \llbracket (i - 1)m \rrbracket$ for all $i \in [n - 1]$. For any $m \geq 0$ and $n \geq 1$,

$$\#\text{Cl}_m(n) = \prod_{i \in [n-1]} (1 + (i - 1)m). \quad (1.1.5)$$

This set is in one-to-one correspondence with the set of all *m -increasing trees* with $n - 1$ internal nodes, which are planar rooted trees where internal nodes are bijectively labeled from 1 to $n - 1$, have $m + 1$ children, and the sequence of the labels of the nodes of any path starting from the root to the leaves is increasing. A possible bijection [CG22] sends $w \in \text{Cl}_m(n)$

$$\text{tree}_2(0230228) = \begin{array}{c} \text{1} \\ \swarrow \quad \searrow \\ \text{4} \quad \text{2} \\ \swarrow \searrow \quad \swarrow \searrow \quad \swarrow \searrow \\ \text{5} \quad \text{7} \quad \text{3} \\ \swarrow \searrow \quad \swarrow \searrow \quad \swarrow \searrow \\ \text{6} \quad \text{7} \quad \text{3} \end{array} \quad (1.1.6)$$

1.1.6. *c-rectangular paths.* For any $c \geq 0$ and any $n \geq 1$, $\text{Hi}_c(n)$ is the set of all words w of length $n - 1$ such that $w = 0^{\alpha_0} 1^{\alpha_1} 2^{\alpha_2} \dots c^{\alpha_c}$ for a sequence $(\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_c)$ of nonnegative integers such that $\alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_c = n - 1$. It is straightforward to show that

This set is in one-to-one correspondence with the set of all *c-rectangular paths* of size n that are paths from $(0, 0)$ to $(n - 1, c)$ made of east steps $(1, 0)$ and north steps $(0, 1)$. A possible bijection sends $w \in \text{Hi}_{\mathbb{C}}(n)$ to the c -rectangular path $\text{path}_c(w)$ having α_i east steps at ordinate i . For instance, in $\text{Hi}_4(8)$,

[illegible]

1.1.7. *m-Dyck paths.* For any $m \geq 0$ and any $n \geq 1$, $\text{Hi}_m(n)$ is the set of all weakly increasing words w of length $n-1$ such that $w(i) \in [(i-1)m]$ for all $i \in [n-1]$. It is shown in [CG20, CG22] that $\#\text{Hi}_m(n) = \text{cat}_m(n-1)$ where

$$\text{cat}_m(n) := \frac{1}{mn+1} \binom{mn+n}{n} \quad (1.1.9)$$

is the n -th m -Fuss-Catalan number [DM47]. This set is in one-to-one correspondence with the set of all m -Dyck paths of size $n - 1$, that are paths from $(0, 0)$ to $((m + 1)n, 0)$ staying above the x -axis and consisting in steps $(1, m)$ and $(1, -1)$. A possible bijection sends $w \in \text{Hi}_m(n)$ to the m -Dyck path $\text{dyck}_m(w)$ such that for any $i \in [n - 1]$, the i -th step $(1, m)$ of $\text{dyck}_m(w)$ has $w(i)$ steps $(1, -1)$ on its left. For instance, in $\text{Hi}_2(6)$,

[illegible]

1.2. Operads and interstice operads. Let us provide some elementary definitions about non-symmetric operads and interstice operads.

1.2.1. *Graded spaces.* A *graded space* is a set expressed as a direct sum

$$\mathcal{V} := \bigoplus_{n \in \mathbb{N}} \mathcal{V}(n) \quad (1.2.1)$$

such that all $\mathcal{V}(n)$, $n \in \mathbb{N}$, are spaces. Given $f \in \mathcal{V}$, if there is an $n \in \mathbb{N}$ such that $f \in \mathcal{V}(n)$, f is *homogeneous*. In this case, this n is unique and is the *rank* $|f|$ of f . If all $\mathcal{V}(n)$, $n \in \mathbb{N}$, have finite dimensions, the *Hilbert series* of $\mathcal{V}$ is the generating series $\sum_{n \in \mathbb{N}} \dim \mathcal{V}(n) t^n$. If $\mathcal{V}$ and $\mathcal{V}'$ are two graded spaces, a linear map $\phi : \mathcal{V} \rightarrow \mathcal{V}'$ is a *graded space morphism* if for any $n \in \mathbb{N}$ and any $f \in \mathcal{V}(n)$, $\phi(f) \in \mathcal{V}'(n)$. Besides, $\mathcal{V}'$ is a *graded subspace* of $\mathcal{V}$ if for any $n \in \mathbb{N}$, $\mathcal{V}'(n) \subseteq \mathcal{V}(n)$. Given a graded set S , the linear span $\text{Span}(S)$ of S is the graded space defined as the direct sum of the linear spans of each $S(n)$, $n \in \mathbb{N}$. By definition, the bases of $\text{Span}(S)$ are indexed by S . The *elementary basis* (or *E-basis* for short) of $\text{Span}(S)$ is the set $\{E_x : x \in S\}$ where each E_x , $x \in S$ is a formal symbol. We shall consider several bases of a same graded space $\text{Span}(S)$ defined from the E-basis.

1.2.2. *Nonsymmetric operads.* A *nonsymmetric operad in the category of spaces*, or a *nonsymmetric operad* for short, is a graded space $\mathcal{O}$ together with maps

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \rightarrow \mathcal{O}(n + m - 1), \quad 1 \leq i \leq n, \quad 1 \leq m, \quad (1.2.2)$$

called *partial compositions*, and a distinguished element $\mathbb{1} \in \mathcal{O}(1)$, the *unit* of $\mathcal{O}$. These data have to satisfy, for any homogeneous elements f_1, f_2 , and f_3 of $\mathcal{O}$, the three relations

$$(f_1 \circ_i f_2) \circ_{i+j-1} f_3 = f_1 \circ_i (f_2 \circ_j f_3), \quad i \in [|f_1|], \quad j \in [|f_2|], \quad (1.2.3a)$$

$$(f_1 \circ_i f_2) \circ_{j+|f_2|-1} f_3 = (f_1 \circ_j f_3) \circ_i f_2, \quad i, j \in [|f_1|], \quad i < j, \quad (1.2.3b)$$

$$\mathbb{1} \circ_i f = f = f \circ_i \mathbb{1}, \quad i \in [|f|]. \quad (1.2.3c)$$

We use in this work the definitions and conventions about nonsymmetric operads presented in [Gir18, Chapter 5]. Since this work deals only with nonsymmetric operads, we call them simply *operads*. Other usual references about operads are [Mé15] for a combinatorial point of view and [LV12] for an algebraic one. We shall use notions like set-operads, free operads, ideals and quotients, presentations by generators and relations, and Koszul duality.

1.2.3. *Interstice operads.* Given a set A , we see the set A^* of the words on A as a graded set such that the size of $w \in A^*$ is $\ell(w) + 1$. Let us define on the graded set $\mathbf{I}(A) := \text{Span}(A^*)$ the partial composition maps $\circ_i$ defined linearly on the E-basis of $\mathbf{I}(A)$, for any $u, v \in A^*$ and $i \in [|u|]$, by $E_u \circ_i E_v := E_{u \sqcup_i v}$, where

$$u \sqcup_i v := u(1, i - 1) \vee u(i, \ell(u)). \quad (1.2.4)$$

For instance, in $\mathbf{I}(\{a, b, c\})$ we have

$$E_{aabacb} \circ_4 E_{cbaa} = E_{aabcbaaacb}. \quad (1.2.5)$$

It is straightforward to check that this structure is an operad admitting moreover E_ϵ as unit. We call $\mathbf{I}(A)$ the *A-interstice operad*.

This operad is generated by the set $\mathfrak{G} := \{E_a : a \in A\}$ of binary elements. These generators are subjected exactly to the nontrivial relations

$$E_b \circ_1 E_a - E_a \circ_2 E_b \quad (1.2.6)$$

for all $a, b \in A$. In other terms, all elements of A are binary generators, and these elements are all associative with respect to all others. In particular, the algebras over the operad $\mathbf{I}(\{a, b\})$ are known under the name of *duplexes of vertices of cubes* [Pir03, Section 6.3] (see also [AL07, Section 3] and [ZGG20, Definition 3.8]).

2. OPERADS OF CLIFFS

By construction, $\mathbf{I}(\mathbb{N})$ is an operad on the linear span of the set all words of nonnegative integers. Our aim is to build a substructure of $\mathbf{I}(\mathbb{N})$ on the linear span of Cl_δ for the largest possible class of range maps δ . We propose a construction in the case where δ is unimodal. By defining alternative bases of the obtained operads, we shall prove that these operads are set-operads and provide some properties about their generators and their nontrivial relations.

2.1. A quotient of an interstice operad. We detail here the construction of the operads $\mathbf{Cl}_\delta$ when δ are unimodal range maps.

2.1.1. For weakly increasing range maps. First of all, if δ is a weakly increasing range map, it is immediate that the linear span of the set $\{E_w : w \in \text{Cl}_\delta\}$ forms a suboperad of $\mathbf{I}(\mathbb{N})$. Let us denote by $\mathbf{Cl}_\delta$ this operad.

2.1.2. Failure of direct quotients. Observe that, given a (even unimodal) range map δ , the graded subspace $\mathcal{V}_\delta$ of $\mathbf{I}(\mathbb{N})$ defined as the linear span of the set $\{E_w : w \in \mathbb{N}^* \setminus \text{Cl}_\delta\}$ is not always an operad ideal of $\mathbf{I}(\mathbb{N})$. Indeed, for $\delta := 0110^\omega$, one has $E_{11} \in \mathcal{V}_\delta$ and $E_0 \in \mathbf{I}(\mathbb{N})$, but the element $E_{11} \circ_1 E_0 = E_{011}$ is not in $\mathcal{V}_\delta$. As a matter of fact, it is possible to prove that $\mathcal{V}_\delta$ is an operad ideal of $\mathbf{I}(\mathbb{N})$ if and only if δ is weakly increasing.

2.1.3. For unimodal range maps. The key of the construction is not to consider quotients of $\mathbf{I}(\mathbb{N})$ but quotients of some suboperads of $\mathbf{I}(\mathbb{N})$ instead.

For this, given any range map δ , we denote by $\bar{\delta}$ the range map defined by $\bar{\delta}(i) := \max\{\delta(1), \dots, \delta(i)\}$ for any $i \geq 1$. For instance, if $\delta = 10032242^\omega$, then $\bar{\delta} = 11133344^\omega$. Remark that when δ is weakly increasing, $\bar{\delta} = \delta$. By construction, the range map $\bar{\delta}$ is weakly increasing. For this reason, $\mathbf{Cl}_{\bar{\delta}}$ is a well-defined operad. Observe moreover that Cl_δ is a subset of $\text{Cl}_{\bar{\delta}}$. Let $\mathcal{V}_\delta$ be the graded subspace of $\mathbf{Cl}_{\bar{\delta}}$ defined as the linear span of the set $\{E_w : w \in \text{Cl}_{\bar{\delta}} \setminus \text{Cl}_\delta\}$.

Proposition 2.1.1. *For any range map δ , the space $\mathcal{V}_\delta$ is an operad ideal of the operad $\mathbf{Cl}_{\bar{\delta}}$ if and only if δ is unimodal.*

Proof. Assume first that δ is not unimodal. Thus, there are indices $1 \leq \alpha_1 < \alpha_2 < \alpha_3$ such that $\delta(\alpha_1) > \delta(\alpha_2) < \delta(\alpha_3)$. Let $u := 0^{\alpha_2-1}(\delta(\alpha_2) + 1)$. By construction, $u \notin \text{Cl}_\delta$ and, since $\delta(\alpha_1) \geq \delta(\alpha_2) + 1$, $u \in \text{Cl}_{\bar{\delta}}$. Therefore, $E_u \in \mathcal{V}_\delta$. Let also $v := 0^{\alpha_3-\alpha_2}$ so that $E_v \in \text{Cl}_{\bar{\delta}}$. Now, one has $E_u \circ_1 E_v = E_w$ with $w := 0^{\alpha_3-1}(\delta(\alpha_2) + 1)$. Since $\delta(\alpha_3) \geq \delta(\alpha_2) + 1$, we have $w \in \text{Cl}_\delta$, so that $E_w \notin \mathcal{V}_\delta$. This shows that $\mathcal{V}_\delta$ is not an operad ideal of $\mathbf{Cl}_{\bar{\delta}}$.

Conversely, assume that δ is unimodal. Let $u, v \in \text{Cl}_{\bar{\delta}}$ and $i \in [|u|]$ with $u \notin \text{Cl}_\delta$ or $v \notin \text{Cl}_\delta$. Let us set $E_w := E_u \circ_i E_v$. From these assumptions, we obtain that there is an index $j \geq 1$ and a letter a at position j of u or of v such that $a > \delta(j)$. Since $u \in \text{Cl}_{\bar{\delta}}$ and $v \in \text{Cl}_{\bar{\delta}}$, there is an index $j' < j$ such that $\delta(j') > \delta(j)$ and $a \leq \delta(j')$. Moreover, since δ is unimodal, for all $j'' \geq j$, $\delta(j) \geq \delta(j'')$. Now, due to the definition of the partial composition of $\text{Cl}_{\bar{\delta}}$, the letter a appears

in w at a certain position $j + k$ for a $k \geq 0$. Since $w(j + k) = a > \delta(j) \geq \delta(j + k)$, we have that $w \notin \mathbf{Cl}_\delta$ and thus $E_w \in \mathcal{V}_\delta$. Therefore, and since $E_e \notin \mathcal{V}_\delta$, $\mathcal{V}_\delta$ is an operad ideal of $\mathbf{Cl}_\delta$. $\square$

As a consequence of Proposition 2.1.1, one has the following result.

Theorem 2.1.2. *For any unimodal range map δ , the space $\mathbf{Cl}_\delta/\mathcal{V}_\delta$ is an operad.*

For any unimodal range map δ , we set $\mathbf{Cl}_\delta := \mathbf{Cl}_\delta/\mathcal{V}_\delta$. Since when δ is weakly increasing, $\mathcal{V}_\delta$ is the null space, this definition is consistent with the previous definition of $\mathbf{Cl}_\delta$ given in Section 2.1.1. By construction, the partial composition of $\mathbf{Cl}_\delta$ satisfies, for any $u, v \in \mathbf{Cl}_\delta$ and $i \in [|u|]$,

$$E_u \circ_i E_v = \chi_\delta(u \sqcup_i v) E_{u \sqcup_i v}, \quad (2.1.1)$$

where $\chi_\delta : \mathbb{N}^* \rightarrow \mathbb{K}$ is the map defined for any $w \in \mathbb{N}^*$ by $\chi_\delta(w) := 1$ if $w \in \mathbf{Cl}_\delta$ and by $\chi_\delta(w) := 0$ otherwise. For instance, in $\mathbf{Cl}_{1232^\omega}$ we have

$$E_{002} \circ_3 E_{10} = E_{00102}, \quad (2.1.2a)$$

$$E_{002} \circ_3 E_{1311} = 0. \quad (2.1.2b)$$

2.2. Fundamental and homogeneous bases. The aim of this section is to introduce two alternative bases of $\mathbf{Cl}_\delta$ in order to prove that this operad is a set-operad. The first of these is the fundamental basis, which is defined from the elementary basis by using the poset structure of δ -cliffs. The second is the homogeneous basis which is defined from the fundamental basis, again by using the same poset structure but in a different way. The partial composition of $\mathbf{Cl}_\delta$ is expressed here over these two alternative bases.

2.2.1. Fundamental basis. Let δ be a unimodal range map. For any $u \in \mathbf{Cl}_\delta$, let

$$F_w := \sum_{\substack{w' \in \mathbf{Cl}_\delta \\ w \preceq w'}} \mu_{\preceq}(w, w') E_{w'}, \quad (2.2.1)$$

where $\mu_{\preceq}$ is the Möbius function of the δ -cliff posets. Since this poset is a Cartesian product of total orders, for any $w, w' \in \mathbf{Cl}_\delta$ such that $w \preceq w'$, $\mu_{\preceq}(w, w') = \prod_{i \in [\ell(w)]} \mu(w(i), w'(i))$ where, for any $a \preceq a' \in \mathbb{N}$,

$$\mu(a, a') := \begin{cases} 1 & \text{if } a' = a, \\ -1 & \text{if } a' = a + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2.2)$$

For instance, in $\mathbf{Cl}_{224^\omega}$ we have

$$F_{1221} = E_{1221} - E_{1222} - E_{1231} - E_{2221} + E_{1232} + E_{2222} + E_{2231} - E_{2232}. \quad (2.2.3)$$

By Möbius inversion and triangularity, for any $u \in \mathbf{Cl}_\delta$,

$$E_w = \sum_{\substack{w' \in \mathbf{Cl}_\delta \\ w \preceq w'}} F_{w'}, \quad (2.2.4)$$

so that the set $\{F_w : w \in \mathbf{Cl}_\delta\}$ is a basis of $\mathbf{Cl}_\delta$, called *fundamental basis* (or *F-basis* for short).

Given $w \in \mathbf{Cl}_\delta$ and a map $f : [\ell(w)] \rightarrow \mathbb{N} \setminus \{0\}$, let $m_\delta^f(w)$ be the word on $\mathbb{N}$ satisfying, for any $j \in [\ell(w)]$,

$$(m_\delta^f(w))(j) = \begin{cases} \delta(f(j)) & \text{if } w(j) = \delta(j), \\ w(j) & \text{otherwise.} \end{cases} \quad (2.2.5)$$

For any $u, v \in \text{Cl}_\delta$ and $i \in [|u|]$, let

$$u \sqcup_i v := m_\delta^f(u) \sqcup_i m_\delta^g(v) \quad (2.2.6)$$

where $f(j) := j$ if $j \leq i - 1$ and $f(j) := j + \ell(v)$ otherwise, and $g(j) := j + i - 1$. For instance for $\delta := 11321^\omega$, one has

$$1022 \sqcup_3 101 = 10 \ 301 \ 21, \quad (2.2.7a)$$

$$1022 \sqcup_4 003 = 102 \ 001 \ 1. \quad (2.2.7b)$$

Observe from (2.2.7a) that even if 1022 and 101 are two δ -cliffs, 1030121 is not a δ -cliff.

Lemma 2.2.1. *Let δ be a range map, $u, v \in \text{Cl}_\delta$, and $i \in [|u|]$. If $u \sqcup_i v$ is a δ -cliff, then $u \sqcup_i v$ also is.*

Proof. Let $w := u \sqcup_i v$ and assume that $w \in \text{Cl}_\delta$. Thus, for all $j \in [\ell(w)]$, $w(j) \leq \delta(j)$. By setting $w' := u \sqcup_i v$, by definition of the operation $\sqcup_i$ and the previous hypothesis, $w'(j) \leq \max\{w(j), \delta(j)\}$. Therefore, $w' \in \text{Cl}_\delta$. $\square$

Lemma 2.2.2. *Let δ be a range map, $u, v \in \text{Cl}_\delta$, and $i \in [|u|]$ such that $u \sqcup_i v$ is a δ -cliff. We have $w \in [u \sqcup_i v, u \sqcup_i v]_{\preceq}$ if and only if the following three assertions hold.*

- (i) For any $j \in [i - 1]$, $w(j) = u(j)$.
- (ii) For any $j \in [i, \ell(v) + i - 1]$, if $v(j - i + 1) = \delta(j - i + 1)$ then $w(j) \in [v(j - i + 1), \delta(j)]$, and $w(j) = v(j - i + 1)$ otherwise.
- (iii) For any $j \in [\ell(v) + i, \ell(w)]$, if $u(j - \ell(v)) = \delta(j - \ell(v))$ then $w(j) \in [u(j - \ell(v)), \delta(j)]$, and $w(j) = u(j - \ell(v))$ otherwise.

Proof. This is a direct consequence of the definitions of the operations $\sqcup_i$ and $\sqcup_i$. $\square$

Proposition 2.2.3. *For any unimodal range map δ , the partial composition map $\circ_i$ on Cl_δ satisfies, for any $u, v \in \text{Cl}_\delta$ and $i \in [|u|]$,*

$$F_u \circ_i F_v = \chi_\delta(u \sqcup_i v) \sum_{w \in [u \sqcup_i v, u \sqcup_i v]_{\preceq}} F_w. \quad (2.2.8)$$

Proof. Let us denote by $\circ'_i$ the operation on Cl_δ defined in (2.2.8) and let us show that this operation is the same as $\circ_i$. First of all, by Lemma 2.2.1, for any $u, v \in \text{Cl}_\delta$, if $\chi_\delta(u \sqcup_i v) = 1$, then $[u \sqcup_i v, u \sqcup_i v]_{\preceq}$ is an interval of a δ -cliff poset. For this reason, (2.2.8) is well-defined. By Möbius inversion,

$$E_u \circ'_i E_v = \sum_{\substack{u', v' \in \text{Cl}_\delta \\ u \preceq u' \\ v \preceq v'}} F_{u'} \circ'_i F_{v'} = \sum_{\substack{u', v' \in \text{Cl}_\delta \\ u \preceq u' \\ v \preceq v'}} \chi_\delta(u' \sqcup_i v') \sum_{w \in [u' \sqcup_i v', u' \sqcup_i v']_{\preceq}} F_w. \quad (2.2.9)$$

Let $z := u \sqcup_i v$.

Assume first that $z \notin \text{Cl}_\delta$. Thus, there is $j \in [\ell(z)]$ such that $z(j) > \delta(j)$. For all $u', v' \in \text{Cl}_\delta$ such that $u \preceq u'$ and $v \preceq v'$, by setting $z' := u' \sqcup_i v'$, one has $z \preceq z'$. Therefore, we have $\delta(j) < z(j) \leq z'(j)$, showing that $z' \notin \text{Cl}_\delta$. By (2.2.9), $E_u \circ'_i E_v = 0$ in this case.

Assume now that $z \in \text{Cl}_\delta$. First of all, we immediately have that if a F_w appears in (2.2.9) with $w \in \text{Cl}_\delta$, then $u \sqcup_i v \preceq w$. Conversely, let $w \in \text{Cl}_\delta$ such that $u \sqcup_i v \preceq w$. By Lemma 2.2.2, there exists a unique pair $(u', v') \in \text{Cl}_\delta^2$ such that $u \preceq u'$, $v \preceq v'$, and $w \in [u' \sqcup_i v', u' \sqcup_i v']_{\preceq}$.

From all this, we deduce by (2.1.1) that

$$E_u \circ_i' E_v = \chi_\delta(u \sqcup_i v) \sum_{\substack{w \in \text{Cl}_\delta \\ u \sqcup_i v \preceq w}} F_w = \chi_\delta(u \sqcup_i v) E_{u \sqcup_i v} = E_u \circ_i E_v. \quad (2.2.10)$$

Therefore, the operations $\circ_i'$ and $\circ_i$ are the same, establishing (2.2.8). $\square$

For instance, in $\text{Cl}_{123454^\omega}$ we have

$$F_{10} \circ_2 F_{021} = F_{10210} + F_{10310}, \quad (2.2.11a)$$

$$F_{013} \circ_2 F_{103} = F_{010313} + F_{010314} + F_{010413} + F_{010414} + F_{020313} + F_{020314} + F_{020413} + F_{020414}. \quad (2.2.11b)$$

2.2.2. *Homogeneous basis.* For any $w \in \text{Cl}_\delta$, let

$$H_w := \sum_{\substack{w' \in \text{Cl}_\delta \\ w' \preceq w}} F_{w'}. \quad (2.2.12)$$

For instance, in Cl_{3221^ω} we have

$$H_{2101} = F_{0000} + F_{0001} + F_{0101} + F_{1001} + F_{1100} + F_{1101} + F_{2000} + F_{2001} + F_{2100} + F_{2101}. \quad (2.2.13)$$

By triangularity, the set $\{H_w : w \in \text{Cl}_\delta\}$ is a basis of Cl_δ , called *homogeneous basis* (or *H-basis* for short).

Proposition 2.2.4. *For any unimodal range map δ , the partial composition map $\circ_i$ on Cl_δ satisfies, for any $u, v \in \text{Cl}_\delta$ and $i \in [u]$,*

$$H_u \circ_i H_v = H_{r_\delta(u \sqcup_i v)}. \quad (2.2.14)$$

Proof. By Proposition 2.2.3,

$$H_u \circ_i H_v = \sum_{\substack{u', v' \in \text{Cl}_\delta \\ u' \preceq u \\ v' \preceq v}} F_{u'} \circ_i F_{v'} = \sum_{\substack{u', v' \in \text{Cl}_\delta \\ u' \preceq u \\ v' \preceq v}} \chi_\delta(u' \sqcup_i v') \sum_{w \in [u' \sqcup_i v', u' \sqcup_i v']_{\preceq}} F_w. \quad (2.2.15)$$

First of all, we immediately have that if a F_w appears in (2.2.15) with $w \in \text{Cl}_\delta$, then $w \preceq r_\delta(u \sqcup_i v)$. Conversely, let $w \in \text{Cl}_\delta$ such that $w \preceq r_\delta(u \sqcup_i v)$. By Lemma 2.2.2, there exists a unique pair $(u', v') \in \text{Cl}_\delta$ such that $u' \preceq u$, $v' \preceq v$, and $w \in [u' \sqcup_i v', u' \sqcup_i v']_{\preceq}$. For this reason, from (2.2.15), we obtain

$$H_u \circ_i H_v = \sum_{\substack{w \in \text{Cl}_\delta \\ w \preceq r_\delta(u \sqcup_i v)}} F_w = H_{r_\delta(u \sqcup_i v)}, \quad (2.2.16)$$

showing the statement of the proposition. $\square$

For instance, in Cl_{22342^ω} we have

$$H_{01} \circ_3 H_{221} = H_{01341}, \quad (2.2.17a)$$

$$H_{2033} \circ_3 H_{12} = H_{201422}. \quad (2.2.17b)$$

Recall that a basis of an operad is a *set-operad basis* if any partial composition of two basis elements is a basis element. An operad having a set-operad basis is a *set-operad*. As a consequence of Proposition 2.2.4, one has the following result.

Theorem 2.2.5. *For any unimodal range map δ , the operad Cl_δ is a set-operad and its H-basis is a set-operad basis.*

2.3. Generators and relations. We provide here some results about minimal generating families of $\mathbf{Cl}_\delta$, the existence of finite such families for these operads, and the nonexistence of finite families of nontrivial relations for these operads. To explore the structure of the operads $\mathbf{Cl}_\delta$, we work through the E-basis since the partial composition expressed over this basis (see Equation (2.1.1)) is very elementary.

2.3.1. Prime cliffs. A nonempty δ -cliff w is δ -prime if the relation $w = u \square_i v$ with $u, v \in \mathbf{Cl}_\delta$ and $i \in [|u|]$ implies $(u, v) \in \{(w, \epsilon), (\epsilon, w)\}$. For instance, for $\delta := 1223321^\omega$, the δ -cliffs 10033 and 121332 are δ -prime, while $11222 = 122 \square_2 12$ is not. We denote by $\mathcal{P}_\delta$ the graded subset of $\mathbf{Cl}_\delta$ of all δ -prime δ -cliffs.

If δ is a nonconstant range map, let us denote by $c(\delta)$ the smallest index $k \geq 1$ such that $\delta(k) \neq \delta(k+1)$. For instance, $c(2) = 1$ and $c(2224445^\omega) = 3$. If δ is 1-dominated, we denote by $d(\delta)$ the smallest index $k \geq 1$ such that for all $k' \geq k$, $\delta(1) \geq \delta(k')$. For instance, $d(3) = 1$ and $d(22334110^\omega) = 6$.

Proposition 2.3.1. *Let δ be a unimodal range map.*

- (i) *If δ is weakly decreasing, then $\mathcal{P}_\delta = \llbracket \delta(1) \rrbracket$.*
- (ii) *Otherwise, if δ is 1-dominated, then for any $w \in \mathcal{P}_\delta$, $\ell(w) \leq d(\delta) - 1$.*
- (iii) *Otherwise, δ is not 1-dominated and for any $k \geq c(\delta) + 1$, $\delta(1, k) \in \mathcal{P}_\delta$.*

Proof. Assume that δ is weakly decreasing. It is immediate that any δ -cliff having 1 as length is δ -prime. Moreover, let $w \in \mathbf{Cl}_\delta$ with $\ell(w) \geq 2$. We have $w = aw'$ with $a \in \llbracket \delta(1) \rrbracket$ and w' is a word of length $\ell(w) - 1$. Therefore, we have $w = a \square_2 w'$, and since δ is weakly decreasing, $w' \in \mathbf{Cl}_\delta$. This shows that $w \notin \mathcal{P}_\delta$ and implies (i).

Assume that δ is not weakly decreasing and is 1-dominated. Therefore $d(\delta) \geq 2$. Let $w \in \mathbf{Cl}_\delta$ such that $\ell(w) \geq d(\delta)$. One has

$$w = w(1, d(\delta) - 1) \square_{d(\delta)} w(d(\delta)). \quad (2.3.1)$$

By construction, $w(d(\delta)) \leq \delta(d(\delta)) \leq \delta(1)$. For this reason, $w(d(\delta))$ is a δ -cliff. Moreover, since $w(1, d(\delta) - 1)$ is a prefix of a δ -cliff, this last word is also a δ -cliff. This shows that w is not δ -prime and implies (ii).

Finally, assume that δ is not 1-dominated and let $k \geq c(\delta) + 1$. Assume that there are $u, v \in \mathbf{Cl}_\delta$ and $i \in [|u|]$ such that $\delta(1, k) = u \square_i v$. The fact that there is an index $j \geq c(\delta) + 1$ such that $\delta(j) > \delta(1)$ implies that $u = \epsilon$ or $v = \epsilon$. Indeed, if $u \neq \epsilon$ and $v \neq \epsilon$, there would be an index $j' < j$ such that $u(j') = \delta(j)$ or $v(j') = \delta(j)$, contradicting the fact that u and v are δ -cliffs. This implies (iii). $\square$

2.3.2. Minimal generating sets. Let us denote by $\mathfrak{G}_\delta$ the family $\{E_w : w \in \mathcal{P}_\delta\}$ of elements of $\mathbf{Cl}_\delta$. For instance,

$$\mathfrak{G}_{21^\omega} = \{E_0, E_1, E_2\}, \quad (2.3.2a)$$

$$\mathfrak{G}_{1221^\omega} = \{E_0, E_1, E_{02}, E_{12}, E_{022}, E_{122}\}, \quad (2.3.2b)$$

$$\begin{aligned} \mathfrak{G}_{11320^\omega} = \{E_0, E_1, E_{002}, E_{003}, E_{012}, E_{013}, E_{102}, E_{103}, E_{112}, E_{113}, E_{0022}, E_{0032}, \\ E_{0122}, E_{0132}, E_{1022}, E_{1032}, E_{1122}, E_{1132}\}, \end{aligned} \quad (2.3.2c)$$

$$\mathfrak{G}_{112^w} = \{E_0, E_1, E_{002}, E_{012}, E_{102}, E_{112}, E_{0022}, E_{0122}, E_{1022}, E_{1122}, E_{00222}, E_{01222}, E_{10222}, E_{11222}, \dots\}, \quad (2.3.2d)$$

By Proposition 2.3.1, the families (2.3.2a), (2.3.2b), and (2.3.2c) are finite, and the family (2.3.2d) is infinite.

Proposition 2.3.2. *For any unimodal range map δ , $\mathfrak{G}_\delta$ is a minimal generating set of the operad $\mathbf{Cl}_\delta$.*

Proof. Let us first prove that $\mathfrak{G}_\delta$ is a generating set of $\mathbf{Cl}_\delta$. By induction of the arity, it appears that any $w \in \mathbf{Cl}_\delta$ writes as an expression involving only δ -prime cliffs and operations $\square_i$. Therefore, due to the partial composition of $\mathbf{Cl}_\delta$ over the E-basis (see (2.1.1)), any E_w writes as an expression involving only elements of $\mathfrak{G}_\delta$ and operations $\circ_i$. This shows the first assertion.

Finally, the minimality of $\mathfrak{G}_\delta$ follows from the fact each $E_w \in \mathfrak{G}_\delta$ cannot be obtained as partial compositions of elements of $\mathfrak{G}_\delta \setminus \{E_w\}$. $\square$

Theorem 2.3.3. *For any unimodal range map δ , the generating set $\mathfrak{G}_\delta$ of the operad $\mathbf{Cl}_\delta$ is finite if and only if δ is 1-dominated.*

Proof. This is a consequence of Proposition 2.3.2 and Points (ii) and (iii) of Proposition 2.3.1. $\square$

2.3.3. Nontrivial relations. Let us denote by $\mathcal{R}_\delta$ the space of the nontrivial relations of $\mathbf{Cl}_\delta$. We have here only a sufficient condition for the fact that $\mathcal{R}_\delta$ is not finitely generated.

Proposition 2.3.4. *For any unimodal range map δ , if δ is not 1-dominated, then the space $\mathcal{R}_\delta$ is not finitely generated.*

Proof. When δ is not 1-dominated, by Point (iii) of Proposition 2.3.1, for any $k \geq c(\delta) + 1$, $\delta(1, k) \in \mathcal{P}_\delta$ so that $E_{\delta(1, k)} \in \mathfrak{G}_\delta$. Moreover, one has $\delta(1, k) \square_1 0 = 0\delta(1, k) = 0 \square_2 \delta(1, k)$ so that

$$E_{\delta(1, k)} \circ_1 E_0 = \chi_\delta(0\delta(1, k))E_{0\delta(1, k)} = E_0 \circ_2 E_{\delta(1, k)}. \quad (2.3.3)$$

Since both $E_{\delta(1, k)}$ and E_0 belong to $\mathfrak{G}_\delta$, the element $E_{\delta(1, k)} \circ_1 E_0 - E_0 \circ_2 E_{\delta(1, k)}$ is an element of $\mathcal{R}_\delta$. Since one has such a generating relation for each $k \geq c(\delta) + 1$, this implies the statement of the proposition. $\square$

3. QUOTIENT OPERADS

As exposed in Section 1.1, some subsets $\mathcal{S}$ of δ -cliffs satisfying some conditions are in one-to-one correspondence with other graded sets (in particular, c -rectangular paths and m -Dyck paths). We describe here a generic way to construct quotient operads of $\mathbf{Cl}_\delta$ whose bases are indexed by $\mathcal{S}$.

3.1. General construction. We construct now a quotient of $\mathbf{Cl}_\delta$ by identifying some of its elements over the F-basis with zero. In order to obtain a quotient operad, $\mathcal{S}$ needs to satisfy a condition which is stated now.

3.1.1. Closure by subword reduction. Let δ be a unimodal range map and $\mathcal{S}$ be a nonempty graded subset of $\mathbf{Cl}_\delta$. The graded set $\mathcal{S}$ is *closed by subword reduction* if for any $w \in \mathcal{S}$, for all subwords w' of w (that are sequences of not necessarily contiguous letters of w), $r_\delta(w') \in \mathcal{S}$. Remark that the fact that $\mathcal{S}$ is nonempty implies in this case that $\epsilon \in \mathcal{S}$.

3.1.2. Quotient operad. Let the quotient space $\mathbf{Cl}_S := \mathbf{Cl}_\delta / \mathcal{V}_S$ where $\mathcal{V}_S$ is the linear span of the set $\{F_w : w \in \mathbf{Cl}_\delta \setminus S\}$. This set is the *fundamental basis* (or *F-basis* for short) of $\mathbf{Cl}_S$.

Proposition 3.1.1. *Let δ be a unimodal range map δ and S be a nonempty graded subset of $\mathbf{Cl}_\delta$. If S is closed by subword reduction, then $\mathcal{V}_S$ is an operad ideal of $\mathbf{Cl}_\delta$. Therefore, in this case, $\mathbf{Cl}_S$ is a quotient operad of $\mathbf{Cl}_\delta$.*

Proof. Let us prove that $\mathcal{V}_S$ is an operad ideal of $\mathbf{Cl}_\delta$. For this, let $F_u, F_v \in \mathbf{Cl}_\delta$ and $i \in [|u|]$, and set $f := F_u \circ_i F_v$. We rely on the expression provided by Proposition 2.2.3 to compute the partial composition of two elements of the F-basis in $\mathbf{Cl}_\delta$.

- (1) Assume by contradiction that $u \notin S$ and that there is a $w \in S$ such that F_w appears in f . By Lemma 2.2.2, $r_\delta(w(1, i-1) w(i + \ell(v), \ell(w))) = u$. Since S is closed by subword reduction, this would imply that $u \in S$, which contradicts our hypothesis.
- (2) Similarly, assume now by contradiction that $v \notin S$ and that there is a $w \in S$ such that F_w appears in f . By Lemma 2.2.2, $r_\delta(w(i, \ell(v) - 1)) = v$. Since S is closed by subword reduction, this would imply that $v \in S$, which contradicts our hypothesis.

Therefore, f belongs in both cases to $\mathcal{V}_S$. Since moreover $e \in S$, $F_e \notin \mathcal{V}_S$, implying that $\mathcal{V}_S$ is an operad ideal of $\mathbf{Cl}_\delta$. $\square$

3.2. Partial composition maps. We provide now expressions to compute the partial composition maps on different bases for the quotient $\mathbf{Cl}_S$ of $\mathbf{Cl}_\delta$.

We shall use in the sequel the canonical projection map $\theta_S : \mathbf{Cl}_\delta \rightarrow \mathbf{Cl}_S$ satisfying, for any $w \in \mathbf{Cl}_\delta$,

$$\theta_S(F_w) = \begin{cases} F_w & \text{if } w \in S, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2.1)$$

3.2.1. Over the fundamental basis. We first need a property of S depending upon the fact that each $S(n)$, $n \geq 1$, forms a sublattice of the δ -cliff poset.

Lemma 3.2.1. *Let δ be a range map and S be a graded subset of $\mathbf{Cl}_\delta$ such that for any $n \geq 1$, $S(n)$ is a sublattice of $\mathbf{Cl}_\delta(n)$. For any $w \in \mathbf{Cl}_\delta(n)$,*

- (i) *the set $\{w' \in S : w \preceq w'\}$ admits at most one minimal element;*
- (ii) *the set $\{w' \in S : w' \preceq w\}$ admits at most one maximal element.*

Proof. Let $u, v \in X$, where X is the set considered in (i). Since $S(n)$ is a sublattice of $\mathbf{Cl}_\delta(n)$, the meet w' of u and v is an element of S . Moreover, by definition of X , w is a lower bound of $\{u, v\}$. Therefore, since w' is the greatest lower bound of $\{u, v\}$, we have $w \preceq w'$. For this reason, $w' \in X$. This implies (i). Similar arguments show (ii). $\square$

As a consequence of Lemma 3.2.1, when S satisfies the given prerequisites, for any $u, v \in \mathbf{Cl}_\delta(n)$ such that $u \preceq v$, $[u, v]_\preceq \cap S$ is empty or is an interval of $S(n)$. Moreover, let us denote by $\wedge_S(w)$ (resp. $\vee_S(w)$) the unique minimal (resp. maximal) element of the set described in (i) (resp. (ii)) when it is nonempty.

Theorem 3.2.2. *Let δ be a unimodal range map and S be a nonempty graded subset of Cl_δ such that S is closed by subword reduction. For any $u, v \in S$ and $i \in [|u|]$,*

$$F_u \circ_i F_v = \chi_\delta(u \square_i v) \sum_{w \in [u \square_i v, u \square_i v]_{\preceq} \cap S} F_w. \quad (3.2.2)$$

Moreover, when for any $n \geq 1$, $S(n)$ is a sublattice of $\text{Cl}_\delta(n)$, if (3.2.2) is different from 0, the support of this element is the interval $[\wedge_S(u \square_i v), \vee_S(u \square_i v)]_{\preceq}$ of S .

Proof. Since by Proposition 3.1.1, Cl_S is a quotient operad of Cl_δ , $F_u \circ_i F_v = \theta_S(F_u \circ_i F_v)$, where the partial composition in the left-hand side (resp. right-hand side) is the one of Cl_S (resp. Cl_δ). Expression (3.2.2) follows now from Proposition 2.2.3 and the linearity of the canonical projection map θ_S .

The second part of the statement of theorem is implied by the first part and by Lemma 3.2.1. $\square$

3.2.2. *Over the elementary basis.* For any $w \in S$, let

$$E_w := \theta_S(E_w) = \sum_{\substack{w' \in S \\ w \preceq w'}} F_{w'}, \quad (3.2.3)$$

where the second occurrence of E_w is an element of Cl_δ . By triangularity, the set $\{E_w : w \in S\}$ is a basis of Cl_S , called *elementary basis* (or *E-basis* for short).

Proposition 3.2.3. *Let δ be a unimodal range map and S be a nonempty graded subset of Cl_δ such that S is closed by subword reduction, and for any $n \geq 1$, $S(n)$ is a sublattice of $\text{Cl}_\delta(n)$. For any $u, v \in S$ and $i \in [|u|]$,*

$$E_u \circ_i E_v = \begin{cases} \chi_\delta(u \square_i v) E_{\wedge_S(u \square_i v)} & \text{if } \{w \in S : u \square_i v \preceq w\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2.4)$$

Proof. Since θ_S is an operad morphism, by (3.2.3) and (2.1.1),

$$\begin{aligned} E_u \circ_i E_v &= \theta_S(E_u) \circ_i \theta_S(E_v) = \theta_S(E_u \circ_i E_v) \\ &= \theta_S(\chi_\delta(u \square_i v) E_{u \square_i v}) \\ &= \chi_\delta(u \square_i v) \sum_{\substack{w \in S \\ u \square_i v \preceq w}} F_w. \end{aligned} \quad (3.2.5)$$

By Lemma 3.2.1, when $u \square_i v$ is a δ -cliff and $\{w \in S : u \square_i v \preceq w\} \neq \emptyset$, $\wedge_S(u \square_i v)$ is a well-defined element of S . In this case, the last term of (3.2.5) is equal to $E_{\wedge_S(u \square_i v)}$. Otherwise, when $\{w \in S : u \square_i v \preceq w\} = \emptyset$, the last term of (3.2.5) is zero. This establishes (3.2.4). $\square$

In the forthcoming Section 4, we shall consider minimal generating sets and nontrivial relations of Cl_S expressed on the E-basis. For this reason, we denote by $\mathfrak{G}_S$ the unique minimal generating set of Cl_S which is a subset of the E-basis, and by $\mathfrak{R}_S$ the space of the nontrivial relations of Cl_S .

3.2.3. *Over the homogeneous basis.* For any $w \in S$, let

$$H_w := \theta_S(H_w) = \sum_{\substack{w' \in S \\ w' \preceq w}} F_{w'}, \quad (3.2.6)$$

where the second occurrence of H_w is an element of $\mathbf{Cl}_\delta$. By triangularity, the set $\{H_w : w \in S\}$ is a basis of $\mathbf{Cl}_S$, called *homogeneous basis* (or *H-basis* for short).

Proposition 3.2.4. *Let δ be a unimodal range map and S be a nonempty graded subset of $\mathbf{Cl}_\delta$ such that S is closed by subword reduction, and for any $n \geq 1$, $S(n)$ is a sublattice of $\mathbf{Cl}_\delta(n)$. For any $u, v \in S$ and $i \in [|u|]$,*

$$H_u \circ_i H_v = \begin{cases} H_{\vee_S(r_\delta(u \square_i v))} & \text{if } \{w \in S : w \preceq u \square_i v\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2.7)$$

Proof. Since θ_S is an operad morphism, by (3.2.6) and Proposition 2.2.4,

$$\begin{aligned} H_u \circ_i H_v &= \theta_S(H_u) \circ_i \theta_S(H_v) = \theta_S(H_u \circ_i H_v) \\ &= \theta_S(H_{r_\delta(u \square_i v)}) \\ &= \sum_{\substack{w \in S \\ w \preceq r_\delta(u \square_i v)}} F_w. \end{aligned} \quad (3.2.8)$$

By Lemma 3.2.1, when $\{w \in S : w \preceq u \square_i v\} \neq \emptyset$, $\vee_S(u \square_i v)$ is a well-defined element of S . In this case, the last term of (3.2.8) is equal to $H_{\vee_S(r_\delta(u \square_i v))}$. Otherwise, when $\{w \in S : w \preceq u \square_i v\} = \emptyset$, the last term of (3.2.8) is zero. This establishes (3.2.7). $\square$

4. SOME PARTICULAR CONSTRUCTIONS

We study in this last part some operads $\mathbf{Cl}_\delta$ and quotients $\mathbf{Cl}_S$ for some concrete range maps and graded sets S of δ -cliffs.

4.1. **Operads on cliffs.** Let us begin by providing some properties about the operads $\mathbf{Cl}_{\underline{c}}$, $c \geq 0$, and $\mathbf{Cl}_{\underline{m}}$, $m \geq 0$.

4.1.1. *On constant range maps.* For any $c \geq 0$, the operad $\mathbf{Cl}_{\underline{c}}$ is by construction the interstice operad $\mathbf{I}(\llbracket c \rrbracket)$. Therefore, $\mathbf{Cl}_{\underline{c}}$ has the properties presented in Section 1.2.3. As a particular case, the map comp (see Section 1.1.3) allows us to interpret any $\underline{1}$ -cliff as an integer composition. Therefore, $\mathbf{Cl}_{\underline{1}}$ can be seen as an operad on integers compositions. For instance,

$$E_{(1,2,1,2,2)} \circ_5 E_{(2,3,1,1)} = E_{(1,2,1,2,3,1,2,2)}, \quad (4.1.1a)$$

$$F_{(1,2,1,2,2)} \circ_5 F_{(2,3,1,1)} = F_{(1,2,1,2,3,1,2,2)}, \quad (4.1.1b)$$

$$H_{(1,2,1,2,2)} \circ_5 H_{(2,3,1,1)} = H_{(1,2,1,2,3,1,2,2)}. \quad (4.1.1c)$$

It is possible to prove, by using Propositions 2.2.3 and 2.2.4, that the constant structures of $\mathbf{Cl}_{\underline{1}}$ are the same in the E, F, and H-bases.

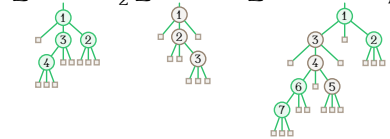
4.1.2. *On arithmetic range maps.* Let us study the operads $\mathbf{Cl}_m$, $m \geq 0$. The map perm (see Section 1.1.4) allows us to interpret any $\mathbf{1}$ -cliff as a permutation. Therefore, $\mathbf{Cl}_1$ can be seen as an operad on permutations. For instance,

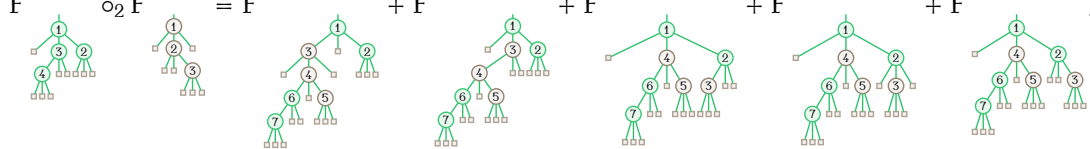
$$E_{25143} \circ_3 E_{3142} = E_{215369487}, \quad (4.1.2a)$$

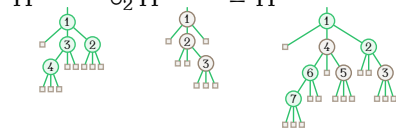
$$\begin{aligned} F_{25143} \circ_3 F_{3142} = & F_{215369487} + F_{251369487} + F_{521369487} + F_{235169487} + F_{253169487} + F_{523169487} \\ & + F_{325196487} + F_{352169487} + F_{532169487}, \end{aligned} \quad (4.1.2b)$$

$$H_{25143} \circ_3 H_{3142} = H_{532169487}. \quad (4.1.2c)$$

In the same way, the map tree (see Section 1.1.5) allows us to interpret any $\mathbf{m}$ -cliff as an m -increasing tree. Therefore, $\mathbf{Cl}_m$ can be seen as an operad on m -increasing trees. For instance, in $\mathbf{Cl}_2$,

$$E \circ_2 E = E, \quad (4.1.3a)$$


$$F \circ_2 F = F + F + F + F + F + F, \quad (4.1.3b)$$


$$H \circ_2 H = H. \quad (4.1.3c)$$


Proposition 2.3.2 allows us to list the first elements of the minimal generating sets $\mathfrak{G}_m$, $m \geq 0$, of $\mathbf{Cl}_m$. Here are the lists of these generators for $m \in \llbracket 2 \rrbracket$, up to arity 4:

$$E_0, \quad m = 0, \quad (4.1.4a)$$

$$E_0, \quad E_{01}, \quad E_{002}, E_{011}, E_{012}, \quad m = 1, \quad (4.1.4b)$$

$$E_0, \quad E_{01}, E_{02}, \quad E_{003}, E_{004}, E_{011}, E_{012}, E_{013}, E_{014}, E_{021}, E_{022}, E_{023}, E_{024}, \quad m = 2. \quad (4.1.4c)$$

By Proposition 2.3.1, $\mathfrak{G}_0$ is finite while $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are infinite.

In order to enumerate $\mathfrak{G}_m$, we need the following small result leading to a recursive description of $\mathbf{m}$ -prime $\mathbf{m}$ -cliffs.

Lemma 4.1.1. *For any $m \geq 0$, if w is a nonempty $\mathbf{m}$ -cliff which is not $\mathbf{m}$ -prime, then there exists an $\mathbf{m}$ -cliff w' and $i \in \llbracket |w'| \rrbracket$ such that $w = w' \square_i 0$.*

Proof. Since w is not $\mathbf{m}$ -prime, there exist two nonempty $\mathbf{m}$ -cliffs u and v , and $i \in \llbracket |u| \rrbracket$ such that $w = u \square_i v$. If $i = 1$, then one has the decomposition $w = v \square_{|v|} u$ so that, since $|v| \geq 2$, we

can assume that $i \geq 2$. Now, since v is an $\mathbf{m}$ -cliff, we have $v(1) = 0$ and

$$\begin{aligned} w &= u \square_i v \\ &= u(1, i-1) \ v \ u(i, \ell(u)) \\ &= u(1, i-1) \ 0 \ v(2, \ell(v)) \ u(i, \ell(u)) \\ &= u(1, i-1) \ v(2, \ell(v)) \ u(i, \ell(u)) \square_i 0. \end{aligned} \tag{4.1.5}$$

The fact that u and v are $\mathbf{m}$ -cliffs and $i \geq 2$ implies that $u(1, i-1) \ v(2, \ell(v)) \ u(i, \ell(u))$ also is. This establishes the stated property. $\square$

Lemma 4.1.2. *Let, for an $m \geq 0$, w be an $\mathbf{m}$ -cliff decomposing as $w = w'a$ with $w' \in \text{Cl}_{\mathbf{m}}$ and $a \in \mathbb{N}$. Then, w is $\mathbf{m}$ -prime if and only if one of the following two assertions is satisfied:*

- (i) $w' \notin \mathcal{P}_{\mathbf{m}}$ and $a \geq (\ell(w) - 2)m + 1$;
- (ii) $w' \in \mathcal{P}_{\mathbf{m}}$ and $a \neq 0$.

Proof. Assume first that (i) holds and assume that there are $u, v \in \text{Cl}_{\mathbf{m}}$ and $i \in [|u|]$ such that $w = u \square_i v$. If u and v are both nonempty, the letter a of w appears either in u or in v but at a position smaller than the one it has in w . Since $a \geq (\ell(w) - 2)m + 1$, either u or v would not be an $\mathbf{m}$ -cliff. Therefore, $u = v = \epsilon$ and w is prime. Assume now that (ii) holds and, again, assume that there are $u, v \in \text{Cl}_{\mathbf{m}}$ and $i \in [|u|]$ such that $w = u \square_i v$. If the letter a of w is in u , we have $u = u'a$ where $u' \in \text{Cl}_{\mathbf{m}}$ and $w'a = u'a \square_i v$. Otherwise, the letter a of w is in v and we have $v = v'a$ where $v' \in \text{Cl}_{\mathbf{m}}$ and $w'a = u \square_i v'a$. Therefore, w' decomposes respectively as $w' = u' \square_i v$ and $w' = u \square_i v'$. Since w' is $\mathbf{m}$ -prime, these decompositions are trivial. This implies that $u' = \epsilon$ so that $u = a$, or that $v = \epsilon$, or that $u = \epsilon$, or that $v' = \epsilon$ so that $v = a$. Since $a \neq 0$, a is not a $\mathbf{m}$ -cliff. Therefore, $v = \epsilon$ or $u = \epsilon$, implying that w is $\mathbf{m}$ -prime.

Conversely, assume that the negations of (i) and (ii) hold at the same time. Therefore, at least one of the following assertions holds.

- (A1) $w' \in \mathcal{P}_{\mathbf{m}}$ and $w' \notin \mathcal{P}_{\mathbf{m}}$;
- (A2) $w' \in \mathcal{P}_{\mathbf{m}}$ and $a = 0$;
- (A3) $a < (\ell(w) - 2)m + 1$ and $w' \notin \mathcal{P}_{\mathbf{m}}$;
- (A4) $a < (\ell(w) - 2)m + 1$ and $a = 0$.

Assertion (A1) is absurd so that this situation cannot occurs. If (A2) or (A4) holds, then $a = 0$ and w decomposes as $w = w' \square_{|w'|} 0$, showing that w is not $\mathbf{m}$ -prime. If (A3) holds, by Lemma 4.1.1, there exists $w'' \in \text{Cl}_{\mathbf{m}}$ and $i \in [|w''|]$ such that $w' = w'' \square_i 0$. Therefore, $w = w'a = w''a \square_i 0$. Since $a < (\ell(w) - 2)m + 1$, $w''a$ is an $\mathbf{m}$ -cliff, showing that w is not $\mathbf{m}$ -prime. We have shown that in all possible situations, w is not $\mathbf{m}$ -prime, establishing the equivalence of the statement of the lemma. $\square$

Theorem 4.1.3. *For any $m \geq 0$, $\#\mathfrak{G}_{\mathbf{m}}(1) = 0$, $\#\mathfrak{G}_{\mathbf{m}}(2) = 1$, and, for any $n \geq 3$,*

$$\#\mathfrak{G}_{\mathbf{m}}(n) = \frac{m}{m+1} (\#\text{Cl}_{\mathbf{m}}(n)). \tag{4.1.6}$$

Proof. First, since $\mathfrak{G}_{\mathbf{m}}(1) = \emptyset$ and $\mathfrak{G}_{\mathbf{m}}(2) = \{E_0\}$, the first part of the statement holds. By using the recursive description of $\mathbf{m}$ -prime $\mathbf{m}$ -cliffs provided by Lemma 4.1.2, for any $n \geq 3$,

$$\#\mathcal{P}_{\mathbf{m}}(n) = m(\#\text{Cl}_{\mathbf{m}}(n-1) - \#\mathcal{P}_{\mathbf{m}}(n-1)) + m(n-2)\#\mathcal{P}_{\mathbf{m}}(n-1). \tag{4.1.7}$$

Therefore,

$$\#\mathcal{P}_m(n) = m(\#\mathbf{Cl}_m(n-1) + (n-3)\#\mathcal{P}_m(n-1)). \quad (4.1.8)$$

By induction on $n \geq 3$, we obtain from (4.1.8) that $\#\mathcal{P}_m(n) = \frac{m}{m+1}(\#\mathbf{Cl}_m(n))$. Finally, since by Proposition 2.3.2, for any $n \geq 1$, $\#\mathcal{P}_m(n) = \#\mathfrak{G}_m(n)$, (4.1.6) follows. $\square$

By Theorem 4.1.3, the sequences of the numbers of elements of $\mathfrak{G}_m$, $m \in \llbracket 2 \rrbracket$, counted w.r.t. their arities start with

$$0, 1, 0, 0, 0, 0, 0, 0, \quad m = 0, \quad (4.1.9a)$$

$$0, 1, 1, 3, 12, 60, 360, 2520, 20160, \quad m = 1, \quad (4.1.9b)$$

$$0, 1, 2, 10, 70, 630, 6930, 90090, 1351350, \quad m = 2. \quad (4.1.9c)$$

Observe that the minimal generating set $\mathfrak{G}_1$ of $\mathbf{Cl}_1$ is in one-to-one correspondence with the set of even permutations. The second and third sequences are, respectively, Sequences **A001710** and **A293962** of [Slo].

Here are the list of the first elements of generating families of the relation spaces $\mathcal{R}_m$, $m \in \llbracket 2 \rrbracket$, up to arity 4:

$$E_0 \circ_1 E_0 - E_0 \circ_2 E_0, \quad m = 0, \quad (4.1.10a)$$

$$E_0 \circ_1 E_0 - E_0 \circ_2 E_0, \quad E_{01} \circ_1 E_0 - E_0 \circ_2 E_{01}, \quad E_0 \circ_2 E_{01} - E_{01} \circ_2 E_0, \quad E_0 \circ_1 E_{01} - E_{01} \circ_3 E_0, \quad m = 1, \quad (4.1.10b)$$

$$\begin{aligned} &E_0 \circ_1 E_0 - E_0 \circ_2 E_0, \\ &E_{02} \circ_1 E_0 - E_0 \circ_2 E_{02}, \quad E_{02} \circ_2 E_0 - E_0 \circ_2 E_{02}, \quad E_0 \circ_1 E_{02} - E_{02} \circ_3 E_0, \\ &E_{01} \circ_1 E_0 - E_0 \circ_2 E_{01}, \quad E_{01} \circ_2 E_0 - E_0 \circ_2 E_{01}, \quad E_0 \circ_1 E_{01} - E_{01} \circ_3 E_0, \quad m = 2. \end{aligned} \quad (4.1.10c)$$

The space $\mathcal{R}_0$ is finitely generated while, by Proposition 2.3.4, $\mathcal{R}_1$ and $\mathcal{R}_2$ are not. Despite what these lists of nontrivial relations suggest, for any $m \geq 1$, $\mathbf{Cl}_m$ is not a quadratic operad. Indeed, for any $m \geq 1$, $\mathcal{R}_m$ contains the nontrivial relation

$$E_{002} \circ_3 E_{01} - (E_0 \circ_2 E_0) \circ_3 E_{012} \quad (4.1.11)$$

of arity 6 which is nonhomogeneous in terms of degrees and nonquadratic. These spaces $\mathcal{R}_m$, $m \geq 1$, seem hard to describe. With the help of the computer, we obtain that the sequences of the dimensions of $\mathcal{R}_m$, $m \in \llbracket 2 \rrbracket$, begin by

$$0, 0, 1, 0, 0, 0, 0, \quad m = 0, \quad (4.1.12a)$$

$$0, 0, 1, 3, 13, 65, 372, 2424, \quad m = 1, \quad (4.1.12b)$$

$$0, 0, 1, 6, 44, 378, 3788, \quad m = 2. \quad (4.1.12c)$$

For the time being, the last two sequences do not appear in [Slo].

4.2. Operads on hills. We provide here some properties about quotient operads of $\mathbf{Cl}_\delta$ whose bases are indexed by δ -hills.

4.2.1. General properties. Let us begin by presenting some general properties of δ -hills and of the quotient operads $\mathbf{Cl}_\delta$ where δ is a set of δ -hills and δ is a weakly increasing range map.

Lemma 4.2.1. *For any weakly increasing range map δ , $\mathbf{Hi}_\delta$ is closed by subword reduction.*

Proof. This is a straightforward consequence of the fact that any subword w' of a δ -hill w is weakly increasing and of the fact that since δ is weakly increasing, $r_\delta(w')$ remains weakly increasing. $\square$

By Proposition 3.1.1 and Lemma 4.2.1, $\mathbf{Hi}_\delta := \mathbf{Cl}_{\mathbf{Hi}_\delta}$ is an operad. For instance, in $\mathbf{Hi}_{1334^\omega}$,

$$E_{0234} \circ_2 E_{112} = E_{0112234}, \quad (4.2.1a)$$

$$F_{0234} \circ_2 F_{112} = F_{0112234} + F_{0112244}, \quad (4.2.1b)$$

$$H_{0234} \circ_2 H_{112} = H_{0112244}, \quad (4.2.1c)$$

$$E_{0234} \circ_3 E_{112} = E_{0222234}, \quad (4.2.2a)$$

$$F_{0234} \circ_3 F_{112} = 0, \quad (4.2.2b)$$

$$H_{0234} \circ_3 H_{112} = H_{0111244}. \quad (4.2.2c)$$

By Theorem 3.2.2, the support of any partial composition over the F-basis in this operad is an interval of the δ -hill poset $(\mathbf{Hi}_\delta(n), \preceq)$, $n \geq 1$, introduced in [CG20, CG22]. As shown here, this poset is also a sublattice of $(\mathbf{Cl}_\delta(n), \preceq)$. In order to express the partial composition of $\mathbf{Hi}_\delta$ over the E-basis and the H-basis, let us introduce the following notations. For any $w \in \mathbf{Cl}_\delta$, let $\bar{w}$ (resp. $\underline{w}$) be the δ -hill defined for any $j \in [\ell(w)]$ by $\bar{w}(j) := \max\{w(1), \dots, w(j)\}$ (resp. $\underline{w}(j) := \min\{w(j), \dots, w(\ell(w))\}$).

Proposition 4.2.2. *Let δ be a weakly increasing range map. For any $u, v \in \mathbf{Hi}_\delta$ and $i \in [|u|]$,*

$$E_u \circ_i E_v = E_{\overline{u \sqcup_i v}} \quad \text{and} \quad H_u \circ_i H_v = H_{\underline{u \sqcup_i v}}. \quad (4.2.3)$$

Proof. For any $w \in \mathbf{Cl}_\delta$, let $X_w := \{w' \in \mathbf{Hi}_\delta : w \preceq w'\}$. Observe that since X_w contains $\delta(1, \ell(w))$, $X_w \neq \emptyset$. Moreover, since δ is weakly increasing, $u \sqcup_i v$ is a δ -cliff. For these reasons, and due to the fact that $\mathbf{Cl}_\delta(n)$, $n \geq 1$, is a sublattice of $\mathbf{Hi}_\delta(n)$, by Proposition 3.2.3,

$$E_u \circ_i E_v = E_{\wedge_{\mathbf{Hi}_\delta}(X_{\overline{u \sqcup_i v}})}. \quad (4.2.4)$$

The stated formula for the partial composition of two elements expressed over the E-basis of $\mathbf{Hi}_\delta$ is now the consequence of the fact that for any $w \in \mathbf{Cl}_\delta$, $\wedge_{\mathbf{Hi}_\delta}(X_w) = \bar{w}$, which follows by induction on the length of w .

The formula for the partial composition of two elements expressed over the H-basis follows from similar arguments. $\square$

Proposition 4.2.2 implies that $\mathbf{Hi}_\delta$ is a set-operad.

4.2.2. *On constant range maps.* Let us study the operads $\mathbf{Hi}_c$, $c \geq 0$. The map path (see Section 1.1.6) allows us to interpret any c -hill as a c -rectangular path. Therefore, $\mathbf{Hi}_c$ can be seen as an operad on c -rectangular paths. For instance, in $\mathbf{Hi}_2$,

$$E \circ_2 E = E, \quad (4.2.5a)$$

$$F \circ_2 F = F, \quad (4.2.5b)$$

$$H \circ_2 H = H, \quad (4.2.5c)$$

$$E \circ_3 E = E, \quad (4.2.6a)$$

$$F \circ_3 F = 0, \quad (4.2.6b)$$

$$H \circ_3 H = H. \quad (4.2.6c)$$

The next result provides a finite presentation by generators and relations of $\mathbf{Hi}_c$.

Theorem 4.2.3. *For any $c \geq 0$, the set $\{E_0, \dots, E_c\}$ is a minimal generating set of the operad $\mathbf{Hi}_c$. The space of the nontrivial relations of $\mathbf{Hi}_c$ is generated by*

$$E_a \circ_1 E_b - E_b \circ_2 E_{a'}, \quad b \in \llbracket c \rrbracket, \quad a, a' \in \llbracket b \rrbracket, \quad (4.2.7a)$$

$$E_b \circ_1 E_a - E_a \circ_2 E_b, \quad b \in \llbracket c \rrbracket, \quad a \in \llbracket b-1 \rrbracket. \quad (4.2.7b)$$

Moreover, $\mathbf{Hi}_c$ is a Koszul operad.

Proof. By Proposition 4.2.2, for any $wa \in \mathbf{Hi}_c$ such that $w \in \mathbf{Hi}_c$ and $a \in \llbracket c \rrbracket$, one has $E_w \circ_{|w|} E_a = E_{wa}$. Therefore, it follows by induction on the arity that the stated set is a generating family of $\mathbf{Hi}_c$. Its minimality follows from the fact that no E_a , $a \in \llbracket c \rrbracket$, can be written as a partial composition of other elements of this family.

Let us now show that the space of the nontrivial relations of $\mathbf{Hi}_c$ is generated by (4.2.7a) and (4.2.7b). Since the evaluations in $\mathbf{Hi}_c$ of these two families of expressions is 0, they belong to the space of the nontrivial relations of $\mathbf{Hi}_c$. Let $\mathcal{O}$ be the quotient of the free operad generated by the stated minimal generating set by the operad ideal generated by (4.2.7a) and (4.2.7b). From these relations, it appears that a basis of $\mathcal{O}$ is formed by the set $T := \{t_{(\alpha_0, \dots, \alpha_c)} : \alpha_i \in \mathbb{N}, i \in \llbracket c \rrbracket\}$ of expressions defined by

$$t_{(\alpha_0, \dots, \alpha_c)} := \underbrace{E_c \circ_1 \dots \circ_1 E_c}_{\alpha_c \text{ terms}} \circ_1 \underbrace{E_{c-1} \circ_1 \dots \circ_1 E_{c-1}}_{\alpha_{c-1} \text{ terms}} \circ_1 \dots \circ_1 \underbrace{E_0 \circ_1 \dots \circ_1 E_0}_{\alpha_0 \text{ terms}}. \quad (4.2.8)$$

For any $n \geq 1$, the bases of $\mathbf{Hi}_c$ and of $\mathcal{O}$ restrained to arity n are in one-to-one correspondence: each c -hill w is in correspondence with the sequence $(\alpha_0, \dots, \alpha_c)$ such that $w = 0^{\alpha_0} \dots c^{\alpha_c}$. Therefore, the operads $\mathcal{O}$ and $\mathbf{Hi}_c$ are isomorphic and the stated property holds.

Finally, the set T forms a Poincaré-Birkhoff-Witt basis of $\mathbf{Hi}_c$, which implies that this operad is Koszul [Hof10]. $\square$

By Theorem 4.2.3, for any $c \geq 0$ and any $b \in \llbracket c \rrbracket$, $E_b \circ_1 E_b = E_b \circ_2 E_b$, so that any algebra on the operad $\mathbf{Hi}_c$ has $c + 1$ associative binary products. This property is shared with some algebras appearing in [Gir16a, Gir16b] (multiassociative algebras), and in [ZGG20] (matching associative algebras).

Besides, Theorem 4.2.3 shows that $\mathbf{Hi}_c$ is a binary and quadratic operad. Therefore, $\mathbf{Hi}_c$ admits a Koszul dual operad $\mathbf{Hi}_c^!$. Let us study this operad now.

Proposition 4.2.4. *For any $c \geq 0$, the operad $\mathbf{Hi}_c^!$ is isomorphic to the quotient of the free operad generated by the set $\{E_0^*, \dots, E_c^*\}$ of symbols of arity 2 by the operad ideal generated by*

$$\sum_{a \in \llbracket b \rrbracket} E_a^* \circ_1 E_b^* - E_b^* \circ_2 E_a^*, \quad b \in \llbracket c \rrbracket, \quad (4.2.9a)$$

$$E_b^* \circ_1 E_a^* - E_a^* \circ_2 E_b^*, \quad b \in \llbracket c \rrbracket, \quad a \in \llbracket b - 1 \rrbracket. \quad (4.2.9b)$$

Proof. This is a straightforward computation based upon the presentation by generators and relations of $\mathbf{Hi}_c$ provided by Theorem 4.2.3. The generating family formed by (4.2.9a) and (4.2.9b) for the space of the nontrivial relations of $\mathbf{Hi}_c^!$ is obtained as the annihilator of the linear span of (4.2.7a) and (4.2.7b) w.r.t. to an appropriate linear map (see [Gir18] for instance). $\square$

Proposition 4.2.5. *For any $c \geq 0$ and any $n \geq 1$, $\dim \mathbf{Hi}_c^!(n) = \text{cat}_c(n)$.*

Proof. By Theorem 4.2.3, $\mathbf{Hi}_c$ is a Koszul operad. Hence, by [GK94], its Hilbert series $G(t)$ and the Hilbert series $F(t)$ of $\mathbf{Hi}_c^!$ satisfy $F(-G(-t)) = t = G(-F(-t))$. By (1.1.7), one has

$$G(t) = \frac{t}{(1-t)^{c+1}} \quad (4.2.10)$$

so that

$$G(-F(-t)) = \frac{-F(-t)}{(1 - (-F(-t)))^{c+1}} = t. \quad (4.2.11)$$

Therefore, $F(t)$ satisfies

$$F(t) = t(1 + F(t))^{c+1}. \quad (4.2.12)$$

This expression for $F(t)$ shows that $F(t)$ is the generating series of the graded set of all planar rooted trees such that each internal node has $c + 1$ children, where the size is given by the number of internal nodes. Hence, $F(t) = \sum_{n \geq 1} \text{cat}_c(n) t^n$, establishing the statement of the proposition. $\square$

Let us consider for all $b \in \llbracket c \rrbracket$ the elements

$$K_b^* := \sum_{a \in \llbracket b \rrbracket} E_a^* \quad (4.2.13)$$

of the operad $\mathbf{Hi}_c^!$. By triangularity, $\{K_0^*, \dots, K_c^*\}$ is a minimal generating set of $\mathbf{Hi}_c^!$.

Proposition 4.2.6. *For any $c \geq 0$, the operad $\mathbf{Hi}_c^!$ is isomorphic to the quotient of the free operad generated by the set $\{K_0^*, \dots, K_c^*\}$ of symbols of arity 2 by the operad ideal generated by*

$$K_b^* \circ_1 K_a^* - K_a^* \circ_2 K_b^*, \quad b \in \llbracket c \rrbracket, \quad a \in \llbracket b \rrbracket. \quad (4.2.14)$$

Proof. For any $b \in \llbracket c \rrbracket$ and $a \in \llbracket b \rrbracket$, we have

$$K_b^* \circ_1 K_a^* - K_a^* \circ_2 K_b^* = \sum_{\substack{b' \in \llbracket b \rrbracket \\ a' \in \llbracket a \rrbracket}} E_{b'}^* \circ_1 E_{a'}^* - E_{a'}^* \circ_2 E_{b'}^*, \quad (4.2.15)$$

implying that (4.2.14) expresses as a linear combination of (4.2.9a) and (4.2.9b). Moreover, the space generated by (4.2.9a) and (4.2.9b) has dimension $\binom{c+2}{2}$. We can observe that the space generated by (4.2.14) has the same dimension. Therefore, these two spaces are equal. Finally, since by Proposition 4.2.4, (4.2.9a) and (4.2.9b) form a generating family of the nontrivial relations of $\mathbf{Hi}_c^!$, this is also the case for (4.2.14). This leads to the stated presentation by generators and relations of $\mathbf{Hi}_c^!$. $\square$

By Proposition 4.2.6, $\mathbf{Hi}_1^!$ is the duplicial operad [BF03] and $\mathbf{Hi}_2^!$ is the triplicial operad [Ler11]. For any $c \geq 0$, we call *c-supplcial operad* each operad $\mathbf{Hi}_c^!$. These operads are therefore natural generalizations of the duplicial and triplicial operads.

4.2.3. On arithmetic range maps. Let us study the operads $\mathbf{Hi}_m$, $m \geq 0$. The map dyck (see Section 1.1.7) allows us to interpret any $\mathbf{m}$ -hill as an m -Dyck path. Therefore, $\mathbf{Hi}_m$ can be seen as an operad on m -Dyck paths. For instance, in $\mathbf{Hi}_1$ we have

$$E \circ_3 E = E, \quad (4.2.16a)$$

$$F \circ_3 F = F + F + F, \quad (4.2.16b)$$

$$H \circ_3 H = H, \quad (4.2.16c)$$

$$E \circ_3 E = E, \quad (4.2.17a)$$

$$F \circ_3 F = 0, \quad (4.2.17b)$$

$$H \circ_3 H = H. \quad (4.2.17c)$$

Here are the lists of the elements of the minimal generating sets $\mathfrak{G}_{\mathbf{Hi}_m}$ of $\mathbf{Hi}_m$ for $m \in \llbracket 2 \rrbracket$, up to arity 5:

$$E_0, \quad m = 0, \quad (4.2.18a)$$

$$E_0, \quad E_{01}, \quad E_{002}, E_{012}, \quad E_{0003}, E_{0013}, E_{0023}, E_{0113}, E_{0123}, \quad m = 1, \quad (4.2.18b)$$

$$E_0, \quad E_{01}, E_{02}, \quad E_{003}, E_{004}, E_{012}, E_{013}, E_{014}, E_{023}, E_{024},$$

$$\begin{aligned} & E_{0005}, E_{0006}, E_{0015}, E_{0016}, E_{0025}, E_{0026}, E_{0034}, E_{0035}, E_{0036}, E_{0045}, E_{0046}, E_{0115}, E_{0116}, E_{0123}, E_{0124}, \\ & E_{0125}, E_{0126}, E_{0134}, E_{0135}, E_{0136}, E_{0145}, E_{0146}, E_{0225}, E_{0226}, E_{0234}, E_{0235}, E_{0236}, E_{0245}, E_{0246}, \quad m = 2. \end{aligned} \quad (4.2.18c)$$

Proposition 4.2.7. *One has $\mathfrak{G}_{\mathbf{Hi}_1} = \{E_w : w \in \mathbf{Hi}_1 \text{ and } w(\ell(w)) = \ell(w) - 1\}$.*

Proof. Let us denote by G the set described in the statement of the proposition. Let $E_w \in G$ and assume that there exist $u, v \in \text{Hi}_1$ and $i \in [|u|]$ such that $E_w = E_u \circ_i E_v$. By Proposition 4.2.2, $w = \overline{u \square_i v}$. Therefore, there is in u or in v a letter of value $\ell(w) - 1$, implying that if it occurs in u , then $v = \epsilon$, or if it occurs in v that $u = \epsilon$. For this reason, E_w admits no nontrivial decompositions, so that $E_w \in \mathfrak{G}_{\text{Hi}_1}$.

Let us now prove that for any $w \in \text{Hi}_1$, E_w belongs to the suboperad of Hi_1 generated by G . We proceed by recurrence on $|w|$. If $|w| = 1$, then $w = 0$, and since $E_0 \in G$, the property holds. Otherwise, we have $w = w'a$ where $w' \in \text{Hi}_1$ and $a \in \mathbb{N}$. If $a = \ell(w) - 1$, $E_w \in G$ so that the property holds. Otherwise, let v be the subword of w' of length a obtained by scanning w' from right to left and by keeping a letter only if it small enough so that v is a **1**-hill. For instance, for $w := 001222356$, we have $w' = 00122235$, $a = 6$, and $v = 012235$. By construction, one has $E_{va} \in G$. Moreover, since w is a **1**-hill, the letters that are in w which have not been selected to form va have values that are necessarily in va . For this reason, and due to the Proposition 4.2.2, E_w can be expressed by partial compositions involving E_{va} and multiple occurrences of E_0 . Since $E_0 \in G$, this shows that E_w belongs to the suboperad of Hi_1 generated by G . $\square$

The sequences of the numbers of elements of $\mathfrak{G}_{\text{Hi}_m}$, $m \in [2]$, counted w.r.t. their arities start with

$$0, 1, 0, 0, 0, 0, 0, 0, 0, \quad m = 0, \quad (4.2.19a)$$

$$0, 1, 1, 2, 5, 14, 42, 132, 429, \quad m = 1 \quad (4.2.19b)$$

$$0, 1, 2, 7, 29, 133, 654, 3383, 18179, \quad m = 2. \quad (4.2.19c)$$

We deduce from Proposition 4.2.7 that $\#\mathfrak{G}_{\text{Hi}_1}(1) = 0$ and for any $n \geq 2$, $\#\mathfrak{G}_{\text{Hi}_1}(n) = \text{cat}_1(n - 2)$. The sequence of the cardinalities of $\mathfrak{G}_{\text{Hi}_2}$ does not appear for the time being in [Slo].

Here are the list of the first elements of the generating families of the relation spaces $\mathcal{R}_{\text{Hi}_m}$, $m \in [2]$, up to arity 4:

$$E_0 \circ_1 E_0 - E_0 \circ_2 E_0, \quad m = 0, \quad (4.2.20a)$$

$$E_0 \circ_1 E_0 - E_0 \circ_2 E_0, \quad E_{01} \circ_1 E_0 - E_{01} \circ_2 E_0, \quad E_0 \circ_2 E_{01} - E_{01} \circ_2 E_0, \quad E_0 \circ_1 E_{01} - E_{01} \circ_3 E_0, \quad m = 1, \quad (4.2.20b)$$

$$E_0 \circ_1 E_0 - E_0 \circ_2 E_0, \quad E_{01} \circ_1 E_0 - E_{01} \circ_2 E_0, \quad E_0 \circ_2 E_{01} - E_{01} \circ_2 E_0, \quad E_0 \circ_1 E_{01} - E_{01} \circ_3 E_0, \\ E_{02} \circ_1 E_0 - E_{02} \circ_2 E_0, \quad E_0 \circ_2 E_{02} - E_{02} \circ_2 E_0, \quad E_0 \circ_1 E_{02} - E_{02} \circ_3 E_0, \quad m = 2. \quad (4.2.20c)$$

The space $\mathcal{R}_{\text{Hi}_0}$ is finitely generated. When $m \geq 1$, due to the description of the partial composition map of Hi_m provided by Proposition 4.2.2, it appears that by setting $w := 0 \dots 0 \mathbf{m}(\ell(w))$, $E_w \in \mathfrak{G}_{\text{Hi}_m}$. Moreover, since $E_0 \circ_1 E_w = E_w \circ_{|w|} E_0$,

$$E_0 \circ_1 E_w - E_w \circ_{|w|} E_0 \quad (4.2.21)$$

is an element of a minimal generating family of $\mathcal{R}_{\text{Hi}_m}$. This shows that $\mathcal{R}_{\text{Hi}_m}$ is not finitely generated. Besides, despite what these lists of nontrivial relations suggest, for any $m \geq 1$, $\mathcal{R}_{\text{Hi}_m}$ is not a quadratic operad. Indeed, for any $m \geq 1$, $\mathcal{R}_{\text{Hi}_m}$ contains the nontrivial relation

$$(E_0 \circ_1 E_0) \circ_1 E_{01} - E_{01} \circ_3 E_{01} \quad (4.2.22)$$

of arity 5 which is nonhomogeneous in terms of degrees and nonquadratic. These spaces $\mathcal{R}_{\text{Hi}_m}$, $m \geq 1$, seem hard to describe. With the help of the computer, we obtain that the sequences of the dimensions of $\mathcal{R}_{\text{Hi}_m}$, $m \in \llbracket 2 \rrbracket$, begin by

$$0, 0, 1, 0, 0, 0, \quad m = 0, \quad (4.2.23a)$$

$$0, 0, 1, 3, 10, 35, 126, 462, \quad m = 1, \quad (4.2.23b)$$

$$0, 0, 1, 6, 35, 206, 1231, \quad m = 2. \quad (4.2.23c)$$

The second seems Sequence **A001700** of [Slo] and for the time being, the third sequence does not appear in [Slo].

CONCLUSION AND OPEN QUESTIONS

This work endows various set of combinatorial families (integer compositions, permutations, m -increasing trees, c -rectangular paths, m -Dyck paths, and more generally δ -cliffs and δ -hills) with operad structures, all being suboperads or quotients of the interstice operad on nonnegative integers. The main considered operads of this work fit into the diagram

$$\begin{array}{ccccc} & & \text{Cl}_\delta & \longrightarrow & \text{Cl}_\delta & \longrightarrow & \text{Hi}_\delta \\ & \swarrow & & & & & \\ \text{I}(\mathbb{N}) & \longleftarrow & & & & & \\ & \nwarrow & & & & & \\ & & \text{Cl}_m & \longrightarrow & & \longrightarrow & \text{Hi}_m \\ & \swarrow & & & & & \\ & & \text{Cl}_c & \longrightarrow & & \longrightarrow & \text{Hi}_c \end{array} \quad (4.2.24)$$

of injective ($\rightarrow$) and surjective ($\twoheadrightarrow$) operad morphisms, where δ is any unimodal range map. As shown, even if interstice operads have a very simple algebraic structure, the operads constructed here have more intricate ones since some of them do not admit finite minimal generating sets, have an infinite family of nontrivial relations, or have some nontrivial relations which are nonhomogeneous and nonquadratic.

Here is a list of open questions raised by this research:

- (1) **(Place of Cl_δ in the world of combinatorial operads)** — This consists in exploring other suboperads and quotients of Cl_δ and see if Cl_δ contains as substructures some already known operads. Moreover, this axis also consists in searching operad morphisms between Cl_δ and other operads. More specifically, one can ask for instance about such relations between Cl_m and the operad **As**.
- (2) **(Nontrivial relations of Cl_δ)** — Proposition 2.3.4 provides a sufficient condition for the fact that $\mathcal{R}_\delta$ is not finitely generated. We can ask about a necessary condition for this property. Moreover, the description and the enumeration arity by arity of a minimal generating set for $\mathcal{R}_m$, $m \geq 1$, is open.
- (3) **(Presentation of Hi_m)** — We know that the minimal generating set $\mathfrak{G}_{\text{Hi}_1}$ of Hi_1 is enumerated by a shifted version of Catalan numbers (see Section 4.2.3). The question of the description and the enumeration of the minimal generating set $\mathfrak{G}_{\text{Hi}_m}$ of Hi_m for $m \geq 2$ is open. The analogous question for a minimal generating set $\mathcal{R}_{\text{Hi}_m}$ is also open for $m \geq 1$. We conjecture in particular that $\mathcal{R}_{\text{Hi}_1}$ is enumerated by Sequence **A001700** of [Slo].

REFERENCES

- [AL07] M. Aguiar and M. Livernet. The associative operad and the weak order on the symmetric groups. *J. Homotopy Relat. Str.*, 2(1):57–84, 2007. 2, 7
- [BF03] C. Brouder and A. Frabetti. QED Hopf algebras on planar binary trees. *J. Algebra*, no. 1:298–322, 2003. 2, 3, 22
- [CG20] C. Combe and S. Giraudo. Three interacting families of Fuss-Catalan posets. *Formal Power Series and Algebraic Combinatorics, Séminaire Lotharingien de Combinatoire*, 84B(22), 2020. 2, 3, 5, 19
- [CG22] C. Combe and S. Giraudo. Three Fuss-Catalan posets in interaction and their associative algebras. *Combinatorial Theory*, 2, 2022. 2, 3, 4, 5, 19
- [Den13] H. Denoncourt. A refinement of weak order intervals into distributive lattices. *Ann. Comb.*, 17(4):655–670, 2013. 4
- [DM47] A. Dvoretzky and Th. Motzkin. A problem of arrangements. *Duke Math. J.*, 14(2):305–313, 1947. 5
- [Gir12] S. Giraudo. Algebraic and combinatorial structures on pairs of twin binary trees. *J. Algebra*, 360:115–157, 2012. 2
- [Gir15] S. Giraudo. Combinatorial operads from monoids. *J. Algebr. Comb.*, 41(2):493–538, 2015. 2
- [Gir16a] S. Giraudo. Operads from posets and Koszul duality. *Eur. J. Combin.*, 56C:1–32, 2016. 21
- [Gir16b] S. Giraudo. Pluriassociative algebras II: The polydendriform operad and related operads. *Adv. Appli. Math.*, 77:43–85, 2016. 21
- [Gir18] S. Giraudo. *Nonsymmetric Operads in Combinatorics*. Springer Nature Switzerland AG, 2018. ix+172. 6, 21
- [GK94] V. Ginzburg and M. M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994. 21
- [Hof10] E. Hoffbeck. A Poincaré-Birkhoff-Witt criterion for Koszul operads. *Manuscripta Math.*, 131(1-2):87–110, 2010. 20
- [Knu06] D. Knuth. *The Art of Computer Programming. Volume 4, Fascicle 4. Generating all trees — History of combinatorial generation*. Addison Wesley Longman, 2006. 5
- [Leh60] D. H. Lehmer. Teaching combinatorial tricks to a computer. In *Proc. Sympos. Appl. Math.*, Vol. 10, pages 179–193. American Mathematical Society, Providence, R.I., 1960. 4
- [Ler11] P. Leroux. L-algebras, triplicial-algebras, within an equivalence of categories motivated by graphs. *Comm. Algebra*, 39(8):2661–2689, 2011. 3, 22
- [LNPRR20] D. López N., L.-F. Préville-Ratelle, and M. Ronco. A simplicial complex splitting associativity. *J. Pure Appl. Algebra*, 224(5):106222, 2020. 2, 3
- [LR02] J.-L. Loday and M. O. Ronco. Order structure on the algebra of permutations and of planar binary trees. *J. Algebr. Comb.*, 15(3):253–270, 2002. 2
- [LV12] J.-L. Loday and B. Vallette. *Algebraic Operads*, volume 346 of *Grundlehren der mathematischen Wissenschaften*. Springer, Heidelberg, 2012. xxiv+634. 6
- [MR95] C. Malvenuto and C. Reutenauer. Duality between quasi-symmetric functions and the Solomon descent algebra. *J. Algebra*, 177(3):967–982, 1995. 2
- [Mé15] M. Méndez. *Set Operads in Combinatorics and Computer Science*. Springer International Publishing, 2015. xv+129. 6
- [Pir03] T. Pirashvili. Sets with two associative operations. *Cent. Eur. J. Math.*, 1(2):169–183, 2003. 7
- [Slo] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. <https://oeis.org/>. 18, 23, 24
- [Sta75] R. P. Stanley. The Fibonacci lattice. *Fibonacci Quart.*, 13(3):215–232, 1975. 3, 5
- [ZGG20] Y. Zhang, X. Gao, and L. Guo. Matching Rota-Baxter algebras, matching dendriform algebras and matching pre-Lie algebras. *J. Algebra*, 552:134–170, 2020. 7, 21