Asymptotics of banded plane partitions: from $\exp(n^{1/2})$ to $\exp(n^{2/3})$

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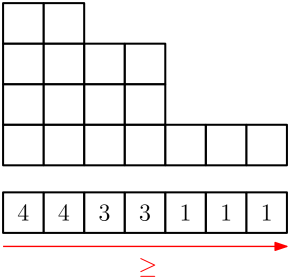
Partitions

Partition: squares tightly piled up on a corner,
Or: eventually zero decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$, Size $= \sum_i \lambda_i$.



Generating function (Euler):

$$P(z) = \sum_{p \text{ partition}} z^{|p|} = \prod_{k \geq 1} \frac{1}{1 - z^k}.$$

Asymptotics of partitions

$p(n)$ = #(partitions of size n).

Enumeration: Hardy-Ramanujan (1918):

$$p_n \sim \frac{1}{4 \cdot 3^{1/2} \cdot n} \exp\left(\frac{2^{1/2}\pi}{3^{1/2}} n^{1/2}\right).$$

Exact convergent series given by Rademacher (1937).

Explained in detail in *Analytic Combinatorics*.

Limit shape: Vershik (1996)

After a rescaling of $n^{1/2}$, the boundary becomes

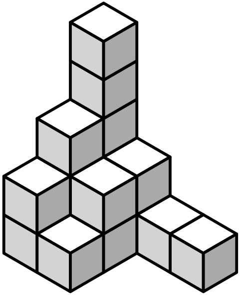
$$\exp\left(-\frac{x}{6^{1/2}\pi}\right) + \exp\left(-\frac{y}{6^{1/2}\pi}\right) = 1.$$

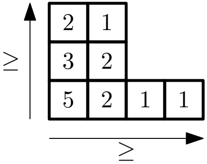
Typical length: $\Theta(n^{1/2} \log n)$.

Plane partitions

Plane partition: boxes tightly piled up on corner,
Or: filling of $\mathbb{N}^2$, decreasing upwards and rightwards, eventually zero.

Size = sum of fillings.





Generating function (MacMahon):

$$PP(z) = \sum_{p \text{ plane partition}} z^{|p|} = \prod_{k \geq 1} \left(\frac{1}{1 - z^k} \right)^k .$$

Asymptotics of unrestricted plane partitions

$pp(n)$ = #(plane partitions of size n).

Asymptotic enumeration: Wright (1931), Mutafchiev and Kamenov (2006)

$$pp_n \sim \frac{\zeta(3)^{7/36} e^{-\zeta'(-1)}}{2^{11/36}\sqrt{3\pi}} n^{-25/36} \exp\left(\frac{3\zeta(3)^{1/3}}{2^{2/3}} n^{2/3}\right).$$

Maximal: Pittel (2005)

Height, width and depth of a uniformly random plane partition of size n:

$$\frac{n^{1/3}}{2^{1/3}\zeta(3)^{1/3}}\left(\frac{2}{3}\log\frac{n}{2\zeta(3)} - d\right),$$

where d (iid for all three quantities) follows the Gumbel distribution $\mathbb{P}[d > x] = e^{-e^{-x}}$.

Typical length: $\Theta(n^{1/3}\log n)$, also from Mutafchiev (2018)

A phase transition?

Partitions = plane partitions of width ≤ 1, type $\exp(c \cdot n^{1/2})$

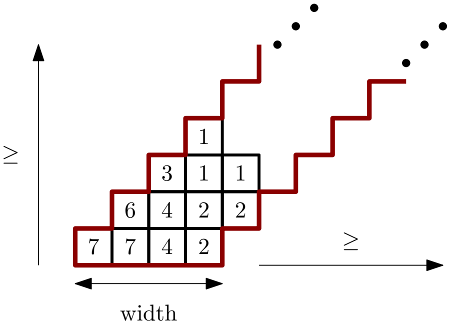
Plane partitions of width $\leq \infty$, type $\exp(c \cdot n^{3/2})$

Question: How the asymptotic changes if width varies with size?

Maybe on nice variants of plane partitions with a natural notion of width

Banded plane partitions

Banded plane partitions: a special case of *skew double shifted plane partition*, defined by Han and Xiong (2017).



Other than size n, it has width m.

- $m = 1 \Rightarrow$ partition, type $\exp(c \cdot n^{1/2})$
- $m \geq n \Rightarrow$ column-strict plane partition, type $\exp(c \cdot n^{2/3})$

What is known

$B_{n,m} = \#$ banded partitions of size n and width m.

Han and Xiong (2017):

Generating function for width m:

$$B_m(z) = \sum_{n\geq 0} B_{n,m} z^n = \prod_{k\geq 1} \frac{1}{1-z^k} \prod_{\substack{k\geq 0 \\ 1\leq h<j\leq m-1}} \frac{1}{1-z^{2mk+h+j}}.$$

Asymptotic: For fixed constant m,

$$B_{n,m} \sim D(m) n^{-1} \exp\left(\pi \left(\frac{m^2+m+2}{6m}\right)^{1/2} n^{1/2}\right),$$

where $D(m)$ is a constant depending on m:

$$D(m) = \left(\prod_{i=1}^{m-2} \prod_{j=i+1}^{m-i-1} \sin \frac{i+j}{2m}\pi\right)^{-1} \frac{(m^2+m+2)^{1/2}}{2^{(m^2-3m+14)/4} 3^{1/2} m^{1/2}}.$$

Our result

Theorem (F., Hwang, Kang (2019+))

Suppose that $m = m(n)$.

- (**Subcritical**) *If* $m = o(n^{1/3}(\log n)^{-2/3})$, *then*

$$\log B_{n,m} \sim c_1(nm)^{1/2} + (1+o(1))c_2 m^2.$$

- (**Critical**) *For* $m = xn^{1/3}$ *with* $x = \omega(n^{-d})$ *for any* $d > 0$,

$$\log B_{n,m} \sim c_3(x)n^{2/3} + (1+o(1))c_4(x)n^{1/3},$$

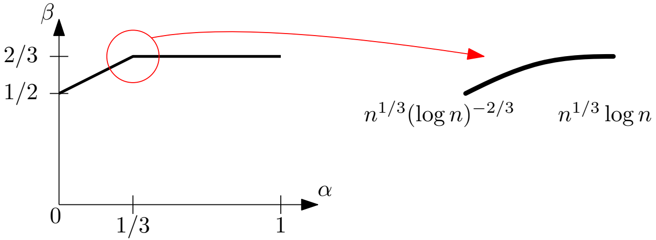
  with $c_3(x), c_4(x)$ *are continuous with the asymptotics:*
  - $x \to 0$: $c_3(x) = c_1 x^{1/2} + \Theta(x^2)$, $c_4(x) = \Theta(x^{-1/2})$.
  - $x \to \infty$: $c_3(x) \to c_5$, $c_4(x) \to c_6$.

- (**Supercritical**) *If* $m = \omega(n^{1/3}\log n)$, *then*

$$\log B_{n,m} \sim c_5 n^{2/3} + (1+o(1))c_6 n^{1/3}.$$

All constants are explicit.

In a graph



$$m = \Theta(n^{\alpha}) \Rightarrow \log B_{n,m} = \Theta(n^{\beta})$$

In the window:

- Subcritical end: subdominant term changes behavior
- Supercritical end: full saturation

Precise behavior is computed in the window.

Partition as a toy example

Generating function for partitions:

$$P(z) = \prod_{k \geq 1} \frac{1}{1 - z^k}.$$

Essential singularities dense on $|z| = 1$, no singularity analysis!

Saddle point method: Cauchy integral formula on the circle $|z| = e^{-r}$ with $r > 0$

Change of variable: $p(z) = \log P(z)$, $z = e^{-\tau}$

$$p_n = [z^n]P(z) = \frac{1}{2\pi i} \int_{r - i\pi}^{r + i\pi} \exp\left(n\tau + p(e^{-\tau})\right) d\tau.$$

Saddle point equation: $n + e^{-r} p'(e^{-r}) = 0 \Rightarrow r \to 0$

Aim: Behavior of $p(e^{-\tau})$ for $\tau = r + i\theta$ when $r \to 0$

A simple relation

When $r \to 0$, the function $p(e^{-\tau})$ gets close to essential singularities.

Tricky!

Miraculously we have

$$p(e^{-\tau}) = \frac{\pi^2}{6\tau} + \frac{1}{2}\log\frac{\tau}{2\pi} - \frac{\tau}{24} + p(e^{-4\pi^2\tau^{-1}}).$$

Related to the modularity of the Dedekind eta function.

But can be seen by Mellin transform.

Mellin transform

For analytic function h, its Mellin transform is given by

$$h^*(s) = \mathcal{M}[h](s) = \int_0^{+\infty} h(\tau)\tau^{s-1}d\tau.$$

If $h(\tau) = O(\tau^u)$ for $\tau \to 0$, and $h(\tau) = O(\tau^v)$ for $\tau \to \infty$, then $\mathcal{M}[h](s)$ is defined on the fundamental strip $-u < \mathrm{Re}(s) < -v$.

Transforming asymptotic behavior to singularities!

The inverse is given by

$$h(\tau) = \mathcal{M}^{-1}[h^*](\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^*(s)\tau^{-s}ds.$$

Here, $-u < c < -v$, that is, we integrate in the fundamental strip.

Why is Mellin transform nice?

$$h^*(s) = \mathcal{M}[h](s) = \int_0^{+\infty} h(\tau)\tau^{s-1}d\tau.$$

- Reading asymptotic behavior off singularities
- Linearity
- Rescaling rule: for $h_k(\tau) = h(k\tau)$, we have

$$\mathcal{M}[h_k](s) = k^{-s}\mathcal{M}[h](s).$$

Nice for so-called harmonic sums, *i.e.* sums of the form

$$g(\tau) = \sum_{k\geq 1} \alpha_k h(k\tau).$$

Its Mellin transform is simply

$$\mathcal{M}[g](s) = \sum_{k\geq 1} \alpha_k k^{-s}\mathcal{M}[h](s).$$

$\alpha_k = 1 \Rightarrow$ Riemann zeta function

Partitions as a harmonic sum

Let $h(\tau) = \log(1 - e^{-\tau})$, then

$$p(e^{-\tau}) = -\sum_{k \geq 1} \log\left(1 - e^{-k\tau}\right) = -\sum_{k \geq 1} h(k\tau).$$

A harmonic sum!

The Mellin transform of h is (Hint: expand by $e^{-\tau}$)

$$\mathcal{M}[h](s) = -\Gamma(s)\zeta(s+1).$$

The Mellin transform of $p(e^{-\tau})$ is thus

$$K(s) = -\sum_{k \geq 1} k^{-s} \mathcal{M}[h](s) = \zeta(s)\Gamma(s)\zeta(s+1),$$

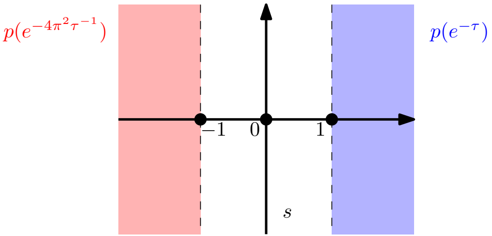
with the fundamental strip $\mathrm{Re}(s) > 1$. $\tau \to 0 \Rightarrow h(\tau) = \Theta(\tau^{-1})$

On the two sides

- Mellin transform of $p(e^{-\tau})$: $K(s) = \zeta(s)\Gamma(s)\zeta(s+1)$.
- Mellin transform of $p(e^{-4\pi^2\tau^{-1}})$: (reflection identities of $\zeta(s)$ and $\Gamma(s)$)

$$K_*(s) = (4\pi^2)^{-s}\zeta(-s)\Gamma(-s)\zeta(-s+1)$$
$$= \zeta(s)\Gamma(s)\zeta(s+1) = K(s),$$

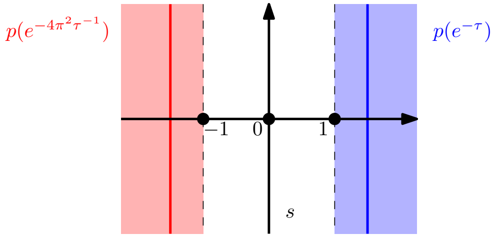
Same Mellin transform, different fundamental strip.



Mellin transform: $K(s) = \zeta(s)\Gamma(s)\zeta(s+1)$

And the big miracle

Inverse Mellin transform: integrate along a vertical line, with factor τ^{-s}



Mellin transform: $K(s) = \zeta(s)\Gamma(s)\zeta(s+1)$

From one to the other: passing through singularities $1, 0, -1$

$$p(e^{-\tau}) = \frac{\pi^2}{6\tau} + \frac{1}{2}\log\frac{\tau}{2\pi} - \frac{\tau}{24} + p(e^{-4\pi^2\tau^{-1}}).$$

Why is it nice?

$$p(e^{-\tau}) = \frac{\pi^2}{6\tau} + \frac{1}{2}\log\frac{\tau}{2\pi} - \frac{\tau}{24} + p(e^{-4\pi^2\tau^{-1}}).$$

For $\tau \to 0$, $p(e^{-4\pi^2\tau^{-1}}) \sim e^{-4\pi^2\tau^{-1}}$. So behavior is known!

$$p_n = [z^n]P(z) \approx \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \exp\left(n\tau + \frac{\pi^2}{6\tau} + \frac{1}{2}\log\frac{\tau}{2\pi} - \frac{\tau}{24}\right) d\tau.$$

Saddle point equation (approx): $n - (\pi^2/6)r^{-2} = 0 \Rightarrow r = 6^{-1/2}\pi n^{-1/2}$.

The rest is classical. Note that considering only τ near r suffices.

The case of banded plane partitions

Generating function of banded plane partitions of width m:

$$B_m(z) = \prod_{k\geq 1} \prod_{j=1}^{2m-1} \left(\frac{1}{1 - z^{2mk+j}} \right)^{w(j)},$$

with $w(j) = \lfloor \frac{m-1-|m-j|}{2} \rfloor$.

Let $b_m(z) = \log B_m(z)$. We have

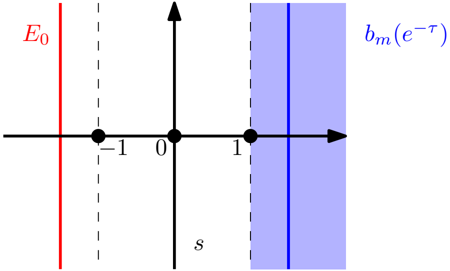
$$B_m(z) = \sum_{k\geq 1} \sum_{j=1}^{2m-1} w(j) \log \left(\frac{1}{1 - z^{2mk+j}} \right).$$

For Mellin transform, not Riemann zeta, but Hurwitz zeta:

$$\zeta(s,\beta) = \sum_{k\geq 0} (k+\beta)^{-s}.$$

A small miracle

$b_m(e^{-\tau}) =$ contributions from $\{1, 0, -1\} + E_0$



The integral E_0 involves Hurwitz zeta $\zeta(s, \beta)$, which still has a more complicated "reflection property".

With some computation, we can express E_0, thus also $b_m(e^{-\tau})$, in an exact form involving $p(e^{-\tau})$.

An equality for $b_m(e^{-\tau})$

$$
\begin{aligned}
&b_m(e^{-\tau})\\
&= -\frac{m^2-3m+4}{4}\log(2\pi) + \sum_{1\le j\le 2m-1}\left\lfloor \frac{m-1-|m-j|}{2}\right\rfloor \log\Gamma\left(\frac{j}{2m}\right)\\
&+ \frac{1}{2}p(e^{-\frac{4\pi^2}{m\tau}}) - \frac{1}{2}p(e^{-\frac{2\pi^2}{\tau}}) + \frac{m+2}{4}p(e^{-\frac{4\pi^2}{\tau}})\\
&+ \frac{1}{2}\log\tau + \left(\frac{m^3-7m^2+2}{96}\right)\tau + \frac{\pi^2(m^2+m+2)}{24m\tau}\\
&- \frac{1}{2m}\sum_{1\le \ell<m}\frac{\cos\frac{(2\ell-1)\pi}{m}}{1-\cos\frac{(2\ell-1)\pi}{m}}\sum_{k\ge 0}\frac{e^{-\left(k+\frac{2\ell-1}{2m}\right)\frac{4\pi^2}{\tau}}}{\left(k+\frac{2\ell-1}{2m}\right)\left(1-e^{-(k+\frac{2\ell-1}{2m})\frac{4\pi^2}{\tau}}\right)}.
\end{aligned}
$$

A small miracle to have an exact expression!

Analysis term by term

$$
\begin{aligned}
& b_m(e^{-\tau}) \\
&= -\frac{m^2-3m+4}{4}\log(2\pi) + \sum_{1\le j\le 2m-1}\left\lfloor \frac{m-1-|m-j|}{2}\right\rfloor \log\Gamma\left(\frac{j}{2m}\right) \\
&+ \frac{1}{2}p(e^{-\frac{4\pi^2}{m\tau}}) - \frac{1}{2}p(e^{-\frac{2\pi^2}{\tau}}) + \frac{m+2}{4}p(e^{-\frac{4\pi^2}{\tau}}) \\
&+ \frac{1}{2}\log\tau + \left(\frac{m^3-7m^2+2}{96}\right)\tau + \frac{\pi^2(m^2+m+2)}{24m\tau} \\
&- \frac{1}{2m}\sum_{1\le \ell<m}\frac{\cos\frac{(2\ell-1)\pi}{m}}{1-\cos\frac{(2\ell-1)\pi}{m}}\sum_{k\ge 0}\frac{e^{-\left(k+\frac{2\ell-1}{2m}\right)\frac{4\pi^2}{\tau}}}{\left(k+\frac{2\ell-1}{2m}\right)\left(1-e^{-(k+\frac{2\ell-1}{2m})\frac{4\pi^2}{\tau}}\right)}.
\end{aligned}
$$

Relatively easy to handle

Analysis term by term

$$
\begin{aligned}
&b_m(e^{-\tau})\\
&= -\frac{m^2-3m+4}{4}\log(2\pi) + \sum_{1\le j\le 2m-1}\left\lfloor\frac{m-1-|m-j|}{2}\right\rfloor \log\Gamma\left(\frac{j}{2m}\right)\\
&+\frac{1}{2}p(e^{-\frac{4\pi^2}{m\tau}}) - \frac{1}{2}p(e^{-\frac{2\pi^2}{\tau}}) + \frac{m+2}{4}p(e^{-\frac{4\pi^2}{\tau}})\\
&+\frac{1}{2}\log\tau + \left(\frac{m^3-7m^2+2}{96}\right)\tau + \frac{\pi^2(m^2+m+2)}{24m\tau}\\
&-\frac{1}{2m}\sum_{1\le \ell<m}\frac{\cos\frac{(2\ell-1)\pi}{m}}{1-\cos\frac{(2\ell-1)\pi}{m}}\sum_{k\ge 0}\frac{e^{-\left(k+\frac{2\ell-1}{2m}\right)\frac{4\pi^2}{\tau}}}{\left(k+\frac{2\ell-1}{2m}\right)\left(1-e^{-(k+\frac{2\ell-1}{2m})\frac{4\pi^2}{\tau}}\right)}.
\end{aligned}
$$

Negligible when $n\to\infty$, where $\tau\to 0$

Analysis term by term

$$
\begin{aligned}
&b_m(e^{-\tau}) \\
&= -\frac{m^2-3m+4}{4}\log(2\pi) + \sum_{1\le j\le 2m-1}\left\lfloor \frac{m-1-|m-j|}{2}\right\rfloor \log\Gamma\left(\frac{j}{2m}\right) \\
&+ \frac{1}{2}p(e^{-\frac{4\pi^2}{m\tau}}) - \frac{1}{2}p(e^{-\frac{2\pi^2}{\tau}}) + \frac{m+2}{4}p(e^{-\frac{4\pi^2}{\tau}}) \\
&+ \frac{1}{2}\log\tau + \left(\frac{m^3-7m^2+2}{96}\right)\tau + \frac{\pi^2(m^2+m+2)}{24m\tau} \\
&- \frac{1}{2m}\sum_{1\le\ell<m}\frac{\cos\frac{(2\ell-1)\pi}{m}}{1-\cos\frac{(2\ell-1)\pi}{m}}\sum_{k\ge 0}\frac{e^{-\left(k+\frac{2\ell-1}{2m}\right)\frac{4\pi^2}{\tau}}}{\left(k+\frac{2\ell-1}{2m}\right)\left(1-e^{-(k+\frac{2\ell-1}{2m})\frac{4\pi^2}{\tau}}\right)}.
\end{aligned}
$$

Depending on m, since it changes the saddle point r, thus behavior of $m\tau$

Subcritical phase

In this phase, $m = o(n^{1/3}(\log n)^{-2/3})$, making $mr \to 0$.

$$\begin{aligned}
&b_m(e^{-\tau}) \\
&= -\frac{m^2-3m+4}{4}\log(2\pi) + \sum_{1\le j\le 2m-1}\left\lfloor \frac{m-1-|m-j|}{2}\right\rfloor \log\Gamma\left(\frac{j}{2m}\right) \\
&+ \frac{1}{2}p(e^{-\frac{4\pi^2}{m\tau}}) - \frac{1}{2}p(e^{-\frac{2\pi^2}{\tau}}) + \frac{m+2}{4}p(e^{-\frac{4\pi^2}{\tau}}) \\
&+ \frac{1}{2}\log\tau + \left(\frac{m^3-7m^2+2}{96}\right)\tau + \frac{\pi^2(m^2+m+2)}{24m\tau} \\
&- \frac{1}{2m}\sum_{1\le \ell<m}\frac{\cos\frac{(2\ell-1)\pi}{m}}{1-\cos\frac{(2\ell-1)\pi}{m}}\sum_{k\ge 0}\frac{e^{-\left(k+\frac{2\ell-1}{2m}\right)\frac{4\pi^2}{\tau}}}{\left(k+\frac{2\ell-1}{2m}\right)\left(1-e^{-\left(k+\frac{2\ell-1}{2m}\right)\frac{4\pi^2}{\tau}}\right)}.
\end{aligned}$$

Saddle point $r \approx \sqrt{\frac{\pi^2(m^2+m+2)}{24mn}}$, with value $\approx \frac{\pi^2(m^2+m+2)n}{6m} \approx \frac{\pi^2}{6}mn$

We can also get lower order terms.

Supercritical phase

In this phase, $m = \omega(n^{1/3} \log n)$, making $mr \to \infty$.

$$\begin{aligned}
& b_m(e^{-\tau}) \\
&= -\frac{m^2 - 3m + 4}{4} \log(2\pi) + \sum_{1 \le j \le 2m-1} \left\lfloor \frac{m - 1 - |m - j|}{2} \right\rfloor \log \Gamma\left(\frac{j}{2m}\right) \\
&+ \frac{1}{2} p(e^{-\frac{4\pi^2}{m\tau}}) - \frac{1}{2} p(e^{-\frac{2\pi^2}{\tau}}) + \frac{m+2}{4} p(e^{-\frac{4\pi^2}{\tau}}) \\
&+ \frac{1}{2} \log \tau + \left(\frac{m^3 - 7m^2 + 2}{96}\right) \tau + \frac{\pi^2 (m^2 + m + 2)}{24 m \tau} \\
&- \frac{1}{2m} \sum_{1 \le \ell < m} \frac{\cos \frac{(2\ell-1)\pi}{m}}{1 - \cos \frac{(2\ell-1)\pi}{m}} \sum_{k \ge 0} \frac{e^{-\left(k + \frac{2\ell-1}{2m}\right) \frac{4\pi^2}{\tau}}}{\left(k + \frac{2\ell-1}{2m}\right) \left(1 - e^{-\left(k + \frac{2\ell-1}{2m}\right) \frac{4\pi^2}{\tau}}\right)}.
\end{aligned}$$

Problematic term: double sum for $mr \to \infty$

Dealing with the double sum

Idea: $\frac{\cos x}{1-\cos x} = 2x^{-2} + 5/6 + O(x^2)$

$$\frac{1}{2m} \sum_{1 \le \ell < m} \frac{\cos \frac{(2\ell-1)\pi}{m}}{1 - \cos \frac{(2\ell-1)\pi}{m}} \sum_{k \ge 0} \frac{e^{-\left(k+\frac{2\ell-1}{2m}\right)\frac{4\pi^2}{\tau}}}{\left(k + \frac{2\ell-1}{2m}\right)\left(1 - e^{-\left(k+\frac{2\ell-1}{2m}\right)\frac{4\pi^2}{\tau}}\right)}$$

$$\approx \frac{1}{2m} \sum_{1 \le \ell < m} \left(\frac{2m^2}{(2\ell-1)^2\pi^2} + \frac{5}{6} + O(\ell^2 m^{-2}) \right) \cdot \frac{e^{-\frac{2\pi^2(2\ell-1)}{m\tau}}}{\frac{2\ell-1}{2m}\left(1 - e^{-\frac{2\pi^2(2\ell-1)}{m\tau}}\right)}$$

$$= m^2\varphi_1(m\tau) + \varphi_2(m\tau) + O(m^{-2}\varphi_3(m\tau))$$

All φ_i can be expressed as an integral involving $\Gamma(s), \zeta(s)$, thus can be estimated at $mr \to \infty$

Supercritical phase (cont'd)

We plug in the estimates of red terms

$$
\begin{aligned}
& b_m(e^{-\tau}) \\
&= -\frac{m^2-3m+4}{4}\log(2\pi) + \sum_{1\le j\le 2m-1}\left\lfloor \frac{m-1-|m-j|}{2}\right\rfloor \log\Gamma\left(\frac{j}{2m}\right) \\
&+ \frac{1}{2}\log\tau + \left(\frac{m^3-7m^2+2}{96}\right)\tau + \frac{\pi^2(m^2+m+2)}{24m\tau} \\
&+ \frac{\zeta(3)}{2\tau^2} + \frac{7\zeta(3)m^2}{8\pi^2} - \left(\frac{m^3-7m}{96}\right)\tau - \frac{\pi^2(m^2+2)}{24m\tau} - \frac{11}{24}\log\frac{m\tau}{\pi} + \frac{1}{24}\log 2 \\
&+ o(1).
\end{aligned}
$$

A lot of terms cancels out!

Supercritical phase (cont'd)

After some computation...

$$b_m(e^{-\tau}) = \frac{\tau}{48} + \frac{\pi^2}{24\tau} + \frac{\zeta(3)}{2\tau^2} + \frac{1}{24}\log\tau + \frac{1}{2}\zeta'(-1) - \frac{1}{4}\log 2 + o(1).$$

A lot of terms cancels out, and no dependency on m!

This indicates a saturation.

Saddle point $r \approx \zeta(3)^{1/3} n^{-1/3}$, with value $\frac{3}{2}\zeta(3)^{1/3} n^{2/3}$

Final result agrees with that of column strict plane partitions, which are in bijection.

Critical phase

Key: the double sum, expressed in $\varphi_1, \varphi_2, \varphi_3$.

More complicated computations, but doable saddle point analysis

The transition is smooth.

Conclusion

- Unexpected nice(?) exact formula unrelated to modularity
- Detailed analysis of phase transition in plane partition variant
- Ongoing: other models

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Thank you!