A generalization of the quadrangulation relation to constellations and hypermaps

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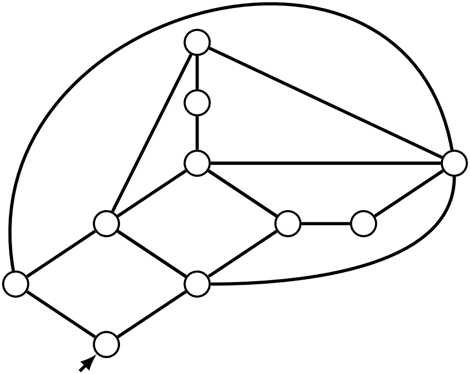
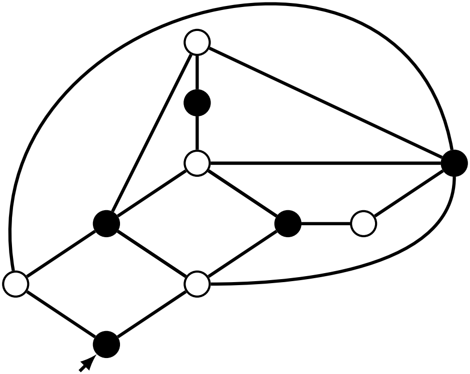
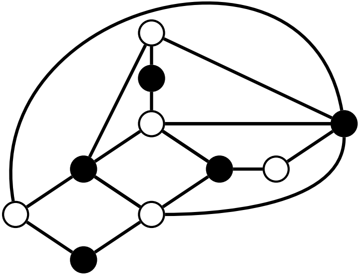
A planar quadrangulation ...



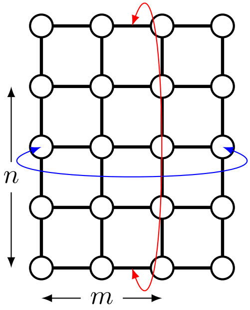
… is always bipartite, …



... which is not true in higher genus.



Planar case



Case $g = 1$ (on a torus)
(bipartite iff m, n are even)

Quadrangulation relation

Let $Q_n^{(g)}$ and $B_n^{(g,k)}$ be the number of quadrangulations (resp. bipartite quadrangulations with marked vertices) with:

- n edges,
- g as genus,
- k marked black vertices.

Theorem (The quadrangulation relation (Jackson and Visentin, 1990))

We have the following relation.

$$Q_n^{(g)} = 2^{2g} B_n^{(g,0)} + 2^{2g-2} B_n^{(g-1,2)} + 2^{2g-4} B_n^{(g-2,4)} \ldots$$

For the planar case, we have $Q_n^{(0)} = B_n^{(0,0)}$.

Obtained using algebraic method, can be generalized to general bipartite maps (Jackson and Visentin (1999)).

Asymptotic behavior of quadrangulations

We admit that the number of bipartite quadragulation of fixed genus g grows as $\Theta(n^{\frac{5}{2}(g-1)}12^n)$. (*c.f.* Bender and Canfield (1986))

Theorem (The quadrangulation relation (Jackson and Visentin, 1990))

We have the following relation.

$$Q_n^{(g)} = 2^{2g} B_n^{(g,0)} + 2^{2g-2} B_n^{(g-1,2)} + 2^{2g-4} B_n^{(g-2,4)} \ldots$$

The first term dominates, and we have $Q_n^{(g)} \sim 2^{2g} B_n^{(g,0)}$.

Corollary

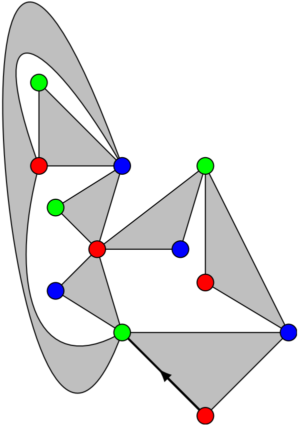
For any fixed g, the probability for a quadrangulation of genus g with n edges to be bipartite converges to 2^{-2g} when $n \to \infty$.

Constellations and hypermaps

The m-**constellations** can be seen as a generalization of bipartite maps.

bipartite map	m-**constellation**
2 colors	m colors
edges	hyperedges (black)
faces	hyperfaces (white)
even degree	degree divisible by m

C.f. Lando and Zvonkin (2004), also Bousquet-Mélou and Schaeffer (2000), Bouttier, Di Francesco and Guitter (2004).
We define m-**hypermaps** as the counterpart of ordinary maps for m-constellations, *i.e.* without coloring.



Our generalization

Let $H_{n,m}^{(g)}$ and $C_{n,m}^{(g,l_1,\ldots,l_{m-1})}$ be the number of m-hypermaps (resp. m-constellations with marked vertices) with:

- n hyperedges,
- g as genus,
- l_i marked vertices with color i.

Theorem (Our generalized relation)

We have the following relation:

$$H_{n,m}^{(g)} = \sum_{i=0}^{g} m^{2g-2i} \sum_{l_1+\ldots+l_{m-1}=2i} c_{l_1,\ldots,l_{m-1}}^{(m)} C_{n,m}^{(g-i,l_1,\ldots,l_{m-1})}.$$

Here, the coefficients $c_{l_1,\ldots,l_{m-1}}^{(m)}$ are all positive integers with explicit expression.

Some examples

Corollary (Our generalized relation, case $m = 2, 3, 4$)

$$H_{n,2}^{(g)} = \sum_{i=0}^{g} 2^{2g-2i} C_{n,2}^{(g-i,l,2i-l)},$$

$$H_{n,3}^{(g)} = \sum_{i=0}^{g} 3^{2g-2i} \sum_{l=0}^{2i} \frac{2 \cdot 2^l + (-1)^l}{3} C_{n,3}^{(g-i,l,2i-l)},$$

$$H_{n,4}^{(g)} = \sum_{i=0}^{g} 4^{2g-2i} \sum_{\substack{l_1,l_2 \geq 0 \\ l_1+l_2 \leq 2i}} \frac{2(3^{l_1}2^{l_2} + 2^{l_2}(-1)^{l_1})}{4} C_{n,4}^{(g-i,l_1,l_2,2i-l_1-l_2)}.$$

Application: asymptotic counting

In Chapuy (2009), the number $C_{n,m}^{(g)} = C_{n,m}^{(g,0,\ldots,0)}$ of m-constellations behaves as $\Theta(n^{\frac{5}{2}(g-1)}\rho_m^n)$ when n tends to infinity.
We recover the following result given by Chapuy (2009).

Corollary (Asymptotique behavior of m-hypermaps)

When n tends to infinity,

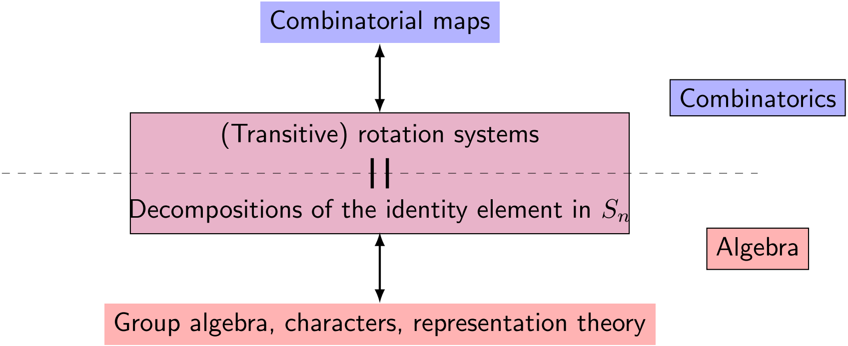
$$H_{n,m}^{(g)} \sim m^{2g} C_{n,m}^{(g)}.$$

Our relation can be viewed as a "higher order development" of this corollary.

Corollary

For any fixed g, the probability for an m-hypermap of genus g with n hyperedges to be an m-constellation converges to m^{-2g} when $n \to \infty$.

Algebraic approach of maps

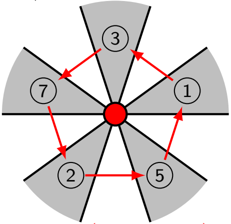


Example : Goupil and Schaeffer (1998), Goulden and Jackson (2008), Poulalhon and Schaeffer (2002), Goulden, Guay-Paquet and Novak (2012), etc...

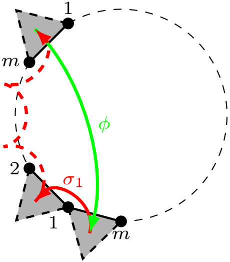
Maps as decompositions

Map model	Decomposition form	Group
m-constellation with n hyperedges	$\sigma_1\sigma_2\cdots\sigma_m\phi = id$	S_n (hyperedges)
m-hypermap with n hyperedges	$\sigma_\bullet\sigma_\circ\phi = id$ with $\sigma_\bullet$ of cycle type $[m^n]$ and $\sigma_\circ$ of cycle type $m\mu$	S_{mn} (edges)

For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, we note $m\mu$ the scaled partition $(m\mu_1, m\mu_2, \ldots, m\mu_k)$.



cycle: $(1, 3, 7, 2, 5)$



Counting decompositions

By the general representation theory, the number of decompositions of the form $\sigma_1\sigma_2\cdots\sigma_m = id$, with σ_i of cycle type $\lambda^{(i)}$, can be expressed with characters evaluated at each $\lambda^{(i)}$.

Frobenius formula

The number of such decompositions is

$$\sum_{\theta\vdash n} \frac{1}{dim(V_\theta)^m \# S_n} \left(\prod_{i=1}^{m} \# C_{\lambda^{(i)}}\right) \prod_{i=1}^{m} \chi^{\theta}_{\lambda^{(i)}}.$$

Here C_λ is the set of permutations with cycle type λ.
Then, for m-hypermaps, we need to evaluate characters in S_{mn} at $m\mu$.
How to exploit?

Key algebraic result - factorization of $\chi^{\theta}_{m\mu}$

In fact, we can express a character of the form $\chi^{\theta}_{m\mu}$ of S_{mn} with characters in smaller groups. The following theorem generalizes results in Jackson and Visentin (1990) and in the book of James and Kerber (1981).

Theorem (Factorization of certain characters (W.F.))

Let m, n be positive integers, and $\mu \vdash n$, $\theta \vdash mn$ two partitions. We have

$$\chi^{\theta}_{m\mu} = z_{\mu} \mathrm{sgn}(\pi_{\theta}\pi'_{\theta}) \sum_{\mu^{(1)} \uplus \cdots \uplus \mu^{(m)} = \mu} \prod_{i=1}^{m} \chi^{\theta^{(i)}}_{\mu^{(i)}} z^{-1}_{\mu^{(i)}}.$$

Here $z_{\mu} = \#S_n / \#C_{\mu}$.

Two possible approaches

There are two different approaches to obtain this result.

- **Algebraic approach**
  Using the Jacobi-Trudi identity, we can express $\chi^{\theta}_{m\mu}$ with a determinant, which has a block structure, resulting in the wanted factorization.
- **Combinatorial approach**
  There is a combinatorial interpretation of $\chi^{\theta}_{m\mu}$ using ribbon tableaux. In the framework of the boson-fermion correspondence, it gives the wanted character factorization.

Algebraic approach via an example : $m = 3$

We try to evaluate $\chi^{\theta}_{3\mu}$, with $\theta = (6, 6, 4, 4, 4, 3, 3)$.

$$\chi^{\theta}_{3\mu} = z_{3\mu}[p_{3\mu}]s_{\theta}$$

Here, $p_{3\mu}$ is the powersum symmetric funtion indexed by 3μ, s_θ the Schur function indexed by the partition θ, and $z_\lambda = \#S_n / \#C_\lambda$ for $\lambda \vdash n$. This is a consequence of the change of basis from Schur functions to powersum functions in the symmetric function ring.

Algebraic approach via an example : $m = 3$

We try to evaluate $\chi^{\theta}_{3\mu}$, with $\theta = (6, 6, 4, 4, 4, 3, 3)$.

$$\chi^{\theta}_{3\mu} = z_{3\mu}[p_{3\mu}] \det \begin{bmatrix} h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} \\ h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} \\ h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 & h_4 \\ 0 & 0 & 0 & h_0 & h_1 & h_2 & h_3 \end{bmatrix}$$

This is an application of the Jacobi-Trudi formula. Here h_k is the homogeneous symmetric function of degree k.

Algebraic approach via an example : $m = 3$

We try to evaluate $\chi^{\theta}_{3\mu}$, with $\theta = (6, 6, 4, 4, 4, 3, 3)$.

$$\chi^{\theta}_{3\mu} = z_{3\mu}[p_{3\mu}] \det \begin{bmatrix} h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} \\ h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} \\ h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 & h_4 \\ 0 & 0 & 0 & h_0 & h_1 & h_2 & h_3 \end{bmatrix}$$

Gray terms don't contribute, because $[p_{3\mu'}]h_m = 0$ for all μ' if $3 \nmid m$.

Algebraic approach via an example : $m = 3$

We try to evaluate $\chi^{\theta}_{3\mu}$, with $\theta = (6,6,4,4,4,3,3)$.

$$\chi^{\theta}_{3\mu} = z_{3\mu}[p_{3\mu}] \det \begin{bmatrix} h_6 & 0 & 0 & h_9 & 0 & 0 & h_{12} \\ 0 & h_6 & 0 & 0 & h_9 & 0 & 0 \\ 0 & h_3 & 0 & 0 & h_6 & 0 & 0 \\ 0 & 0 & h_3 & 0 & 0 & h_6 & 0 \\ h_0 & 0 & 0 & h_3 & 0 & 0 & h_6 \\ 0 & 0 & h_0 & 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_0 & 0 & 0 & h_3 \end{bmatrix}$$

Since gray terms don't contribute, we can replace them by 0.

Algebraic approach via an example : $m = 3$

We try to evaluate $\chi^{\theta}_{3\mu}$, with $\theta = (6, 6, 4, 4, 4, 3, 3)$.

$$\chi^{\theta}_{3\mu} = z_{3\mu}[p_{3\mu}] \det \begin{bmatrix} h_6 & 0 & 0 & h_9 & 0 & 0 & h_{12} \\ 0 & h_6 & 0 & 0 & h_9 & 0 & 0 \\ 0 & h_3 & 0 & 0 & h_6 & 0 & 0 \\ 0 & 0 & h_3 & 0 & 0 & h_6 & 0 \\ h_0 & 0 & 0 & h_3 & 0 & 0 & h_6 \\ 0 & 0 & h_0 & 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_0 & 0 & 0 & h_3 \end{bmatrix}$$

The remaining terms can be divided into three groups.

Algebraic approach via an example : $m = 3$

We try to evaluate $\chi^{\theta}_{3\mu}$, with $\theta = (6, 6, 4, 4, 4, 3, 3)$.

$$\chi^{\theta}_{3\mu} = z_{3\mu}[p_{3\mu}] \det \begin{bmatrix} h_6 & h_9 & h_{12} & 0 & 0 & 0 & 0 \\ h_0 & h_3 & h_6 & 0 & 0 & 0 & 0 \\ 0 & h_0 & h_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_6 & h_9 & 0 & 0 \\ 0 & 0 & 0 & h_3 & h_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_3 & h_6 \\ 0 & 0 & 0 & 0 & 0 & h_0 & h_3 \end{bmatrix}$$

We can rearrange them into blocks.

Algebraic approach via an example : $m = 3$

We try to evaluate $\chi^{\theta}_{3\mu}$, with $\theta = (6, 6, 4, 4, 4, 3, 3)$.

$$\chi^{\theta}_{3\mu} = z_{3\mu}[p_{3\mu}] \left(\det \begin{bmatrix} h_6 & h_9 & h_{12} \\ h_0 & h_3 & h_6 \\ 0 & h_0 & h_3 \end{bmatrix} \det \begin{bmatrix} h_6 & h_9 \\ h_3 & h_6 \end{bmatrix} \det \begin{bmatrix} h_3 & h_6 \\ h_0 & h_3 \end{bmatrix} \right)$$

We thus obtain a factorization where factors are similar to the determinant in the Jacobi-Trudi formula. These factors can also be expressed with characters.

Proof ideas of main result

- Character factorization $\Rightarrow$ the series of hypermaps as product of copies of the series of constellations
- Imposing connectedness by taking $\log \Rightarrow$ the product transforming into a sum
- Direct extraction of coefficient while controlling the genus
- Positivity of coefficients require some more work.

Combinatorial proof?

Corollary (Our generalized relation, case $m = 3, 4$)

$$H_{n,3,D}^{(g)} = \sum_{i=0}^{g} 3^{2g-2i} \sum_{l=0}^{2i} \frac{2 \cdot 2^l + (-1)^l}{3} C_{n,3,D}^{(g-i,l,2i-l)},$$

$$H_{n,4,D}^{(g)} = \sum_{i=0}^{g} 4^{2g-2i} \sum_{\substack{l_1,l_2 \geq 0 \\ l_1+l_2 \leq 2i}} \frac{2(3^{l_1}2^{l_2} + 2^{l_2}(-1)^{l_1})}{4} C_{n,4,D}^{(g-i,l_1,l_2,2i-l_1-l_2)}.$$

Is there a combinatorial proof?
What is the combinatorial meaning of the coefficients?

Thank you for your attention.