On the Palindromic Reversal Process

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Abstract

When adding an integer and its reversal, one obtains another integer, which may also be added to its reversal, and so on. One iterates until it gives a palindromic result, that is to say a number reading the same backward as forward. 196 is known to be the first number that apparently never produces a palindrome. We present new properties which provide a more accurate interpretation of the palindromic reversal process and give a recursive algorithm whose derecursivation leads to the classical algorithm. We also exhibit a finer classification of numbers giving the same integers after several steps of the process.

1 Introduction

In previous studies of the palindromic reversal process, it was of great interest to consider pragmatic approaches of the algorithm: after having programmed the process, the first conjecture was that 196 never produced a palindrome. For instance, [Wal90] made a program that performed results over 196 on one million digits during three years. The process started on August 12, 1987 and stopped on May 24, 1990 by printing the message:

```
Stop point reached on pass 2415836.
Number contains 1000000 digits.
```

This means that after 2,415,836 iterations of the palindromic reversal process, 196 did not produce any palindromic word but grew up to a number of 1 million digits.

We also wrote different programs making a faster computation and we have attempted to classify numbers that apparently never produce a palindrome. This gives rise to the formalization we introduce below. Our work does not restrict to the decimal basis, but any.

2 Description of the Standard Algorithm

Given a basis $\flat$, say 10, we choose an integer $\mathcal{N}$ on $\flat$, whose leftmost digit is non null. When reading $\mathcal{N}$ from left to right, we can form a word w made of red digits. When reading it from right to left, we form another word $\overline{w}$, called the reversal of w. Both w and $\overline{w}$ may also be converted to the corresponding number in $\flat$ basis. Since $\mathcal{N}$ denotes the number associated to w, $\overline{\mathcal{N}}$ denotes the number associated to $\overline{w}$.

When $\mathcal{N}$ is added to $\overline{\mathcal{N}}$, we obtain another integer, say $\mathcal{M}$, which may also be added to $\overline{\mathcal{M}}$, and so on. This process is iterated while the word associated to $\mathcal{M}$ is not a palindrom, *i.e.* when it can be red the same backward as forward. When the former $\mathcal{N}$ corresponds to a palindrom, the process stops immediately. Notice that at each step the number of digits of $\mathcal{M}$ stays the same or increases by one.

It was first believed that all integers give rise to a palindrom through this process. Then, it was conjectured by computation that 196 never produces a palindrom. More precisely we checked that, between 1 and 9999, almost all numbers give a palindrom, iterating at most 24 times the above algorithm. But a few of them seem to make the algorithm loop (more than 4 millions iterations). The first of these is 196, the second 295, etc.

We present in the next section some properties showing that if "196 loops", it is also the case for 295, 394, 493, 592, 691, 790, 887, etc. We thus infer a classification of numbers (it was pragmatically made in [Yam97], but in a rougher manner), based on these properties.

3 Conservating Properties

Let us introduce some notations.

Let $\flat$ be a (decimal, binary, etc.) basis and $w = w_{n-1}..w_1w_0$ denote the word associated to the non-negative integer $\mathcal{N} = w_{n-1} \times \flat^{n-1} + \cdot\cdot + w_1 \times \flat + w_0$. $\mathcal{N}$ may also be written:

$$\mathcal{N} = \sum_{k=0}^{n-1} w_k \flat^k \quad ,$$

or, in a simpler way, $\mathcal{N} = \langle w \rangle_\flat$ (or $\langle w \rangle$ when there is no ambiguity).
This first arouses some remarks: $\sum_{k=0}^{n-1} w_k \flat^k = \sum_{k=0}^{p} w_k \flat^k$, where $p \geq n-1$. But $\underbrace{00...0}_{p-n+1} w \neq w$. This means that for a given integer $\mathcal{N}$, there is an infinity of w, such that $\mathcal{N} = \langle w \rangle$.

Definition 3.1 (Length) *Let $w = w_{n-1}..w_1 w_0$. Then the* length *of w is denoted by $|w|$ and is equal to n.*

Definition 3.2 (Integer representative) *Let $\mathcal{N}$ be a non null integer. Among all v such that $\mathcal{N} = \langle v \rangle$, there is only one w such that $w_{|w|-1} \neq 0$. $w = \downarrow\mathcal{N}\downarrow$ is called the* representative *of $\mathcal{N}$.*

Definition 3.3 (Reversal) *Let $w = w_{n-1}..w_1 w_0$. Then $\overline{w} = w_0 w_1..w_{n-1}$ is called the* reversal *of w.*

Definition 3.4 (p-reversal) *Let $w = w_{n-1}..w_1 w_0$, $w' = \overbrace{00..0w_{n-1}..w_1 w_0}^{p}$. Then $\overline{w}^p = \overline{w'}$ is called the p*-reversal *of w.*

We easily check that $\overline{w} = \overline{w}^{|w|}$.

In order to simplify notations, when $\mathcal{N} = \langle w \rangle$, $\overline{\mathcal{N}}$ will denote $\langle \overline{w} \rangle$.

3.1 Conservating properties in one step

In this section, we present some properties that indicate how to transform a word w into w', such that for $\mathcal{N} = \langle w \rangle$,$\mathcal{N}' = \langle w' \rangle$, $\mathcal{N}' + \overline{\mathcal{N}'} = \mathcal{N} + \overline{\mathcal{N}}$, that is to say how to produce a new word w' from w such that after one step of the palindromic reversal process, w and w' give the same word.

Let $w = w_{n-1}..w_1 w_0$. Then $\mathcal{N} = \sum_{k=0}^{n-1} w_k \flat^k$ and $\overline{\mathcal{N}} = \sum_{k=0}^{n-1} w_k \flat^{n-k-1}$. Hence:

$$\begin{aligned}
\mathcal{N} - \overline{\mathcal{N}} &= \sum_{k=0}^{n-1} w_k (\flat^k - \flat^{n-k-1}) \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor - 1} w_k (\flat^k - \flat^{n-k-1}) + \sum_{k=\lfloor n/2 \rfloor}^{n-1} w_k (\flat^k - \flat^{n-k-1}) \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor - 1} w_k (\flat^k - \flat^{n-k-1}) - \sum_{p=0}^{\lceil n/2 \rceil - 1} w_{n-p-1} (\flat^p - \flat^{n-p-1}) \quad ,
\end{aligned}$$

with $p = n - k - 1$. But if n is odd, then $\flat^{\lceil n/2 \rceil - 1} - \flat^{n - \lceil n/2 \rceil - 2} = 0$. Thus:

$$\mathcal{N} - \overline{\mathcal{N}} = \sum_{k=0}^{\lfloor n/2 \rfloor - 1} (w_k - w_{n-k-1})(\flat^k - \flat^{n-k-1}) \quad . \tag{1}$$

Equation (1) is interesting in the sense that each term $(w_k - w_{n-k-1})(\flat^k - \flat^{n-k-1})$ describes the swap between digits w_k and w_{n-k-1}.

Let $t_k = (w_{n-k-1} - w_k)(\flat^k - \flat^{n-k-1})$. More precisely, it is easy to check that $\downarrow\mathcal{N} + t_k\downarrow$ is equal to w in which w_k and w_{n-k-1} have been interchanged.

Proposition 3.5 *Let* $w = w_{n-1}..w_1w_0$ *and* $t_k = (w_{n-k-1} - w_k)(\flat^k - \flat^{n-k-1})$, $k \in \{0;..;\lfloor n/2 \rfloor - 1\}$. *Let* $\mathcal{N} = \langle w \rangle$, $\mathcal{M} = \mathcal{N} + t_k$. *We have:*

$$\mathcal{M} + \overline{\mathcal{M}} = \mathcal{N} + \overline{\mathcal{N}} \quad . \tag{2}$$

Proof. Componentwise, n being set. □

Considering that, equation (1) may be rewritten as:

$$\mathcal{N} - \overline{\mathcal{N}} = \sum_{k=0}^{\lfloor n/2 \rfloor - 1} \sum_{i=1}^{w_k - w_{n-k-1}} (\flat^k - \flat^{n-k-1}) \quad ,$$

where each $e_k = (\flat^{n-k-1} - \flat^k)$ may be seen as a single step towards the swap between w_k and w_{n-k-1}, that is to say a move of one unit from w_k to w_{n-k-1}.

For instance, when $\flat = 10$, $n = 4$, $k \in \{0; 1\}$, and for $w = 1495$, we have $e_0 = 1000 - 1 = 999$, $t_0 = (1 - 5)(-e_0) = 3996$. Thus:

$$\begin{aligned} 1495 + 999 &= 2494 \\ 2494 + 999 &= 3493 \\ 3493 + 999 &= 4492 \\ 4492 + 999 &= 5491 \quad . \end{aligned}$$

We deduce starting from $1495 + 3996 = 5491$, that $5491 + \overline{5491} = 1495 + \overline{1495}$ by proposition (3.5).

Proposition 3.6 *Let* $w = w_{n-1}..w_1w_0$, $e_k = \flat^k - \flat^{n-k-1}$, $k \in \{0;..;\lfloor n/2 \rfloor - 1\}$ *and* $w_k > 0, w_{n-k-1} < \flat - 1$. *Let* $\mathcal{N} = \langle w \rangle, \mathcal{M} = \mathcal{N} + e_k$. *We have:*

$$\mathcal{M} + \overline{\mathcal{M}} = \mathcal{N} + \overline{\mathcal{N}} \quad . \tag{3}$$

Proof. Componentwise, n being set. $\square$

Let us take an example.

Example 3.7 *Let* $\flat = 10$. *Then* $23456 + 2 \times 990 + 3 \times 9999 = 55433$, *i.e.* 2 *has been substracted from* 5 *and added to* 3 *in* w, *while* 3 *has been substracted from* 6 *and added to* 2. *It therefore follows that* $(5-2)+(3+2) = 5+3$ *and* $(6-3)+(2+3) = 6+2$, *and so,* $23456 + \overline{23456} = 55433 + \overline{55433}$.

3.2 Properties forecasting

We just saw how to deform a word w into a word w' such that for $\mathcal{N} = \langle w \rangle$ and $\mathcal{M} = \langle w' \rangle$ we have $\mathcal{N} + \overline{\mathcal{N}} = \mathcal{M} + \overline{\mathcal{M}}$. This concerns only a single step of the process. It is quite obvious to see that we also can deform a word w into w' such that $(\mathcal{N}+\overline{\mathcal{N}}) + \overline{\mathcal{N}+\overline{\mathcal{N}}} = (\mathcal{M}+\overline{\mathcal{M}}) + \overline{\mathcal{M}+\overline{\mathcal{M}}}$ without having $\mathcal{N} + \overline{\mathcal{N}} = \mathcal{M} + \overline{\mathcal{M}}$, and so on.

To obtain such a deformation of w, we shall determinate if, for a given $k \in \{0; .., |w|-1\}$, there exists a word v^k of length $n = |w|$ such that $\langle v^k \rangle + \overline{\langle v^k \rangle} = m.e_k$, where $m \leq \min(w_k, \flat - w_{n-k-1})$. Let then $\mathcal{N}^k = \mathcal{N} + \langle v^k \rangle$. We will have $\mathcal{N}^k + \overline{\mathcal{N}^k} = \mathcal{N} + \overline{\mathcal{N}} + m.e_k$, when $\overline{\mathcal{N} + \langle v^k \rangle} = \overline{\mathcal{N}} + \overline{\langle v^k \rangle}$ and thus $\mathcal{N}^k + \overline{\mathcal{N}^k} + \overline{\mathcal{N}^k + \overline{\mathcal{N}^k}} = \mathcal{N} + \overline{\mathcal{N}} + \overline{\mathcal{N} + \overline{\mathcal{N}}}$ by iterating m times proposition (3.6).

Example 3.8 *Let* $\flat = 10$, $w = 8089$, $k = 1$, $n = 5$, $v^1 = 0090$, *then* $\langle v^1 \rangle + \overline{\langle v^1 \rangle} = 1.e_1 = 990$, *where* $\downarrow e_1 \downarrow = 00990$. *Then* $\mathcal{N} + \overline{\mathcal{N}} = 17897$ *and* $17897 + \overline{17897} = 97768$. *On the other hand,* $\mathcal{N}' = \mathcal{N} + 90 = 8899$, *thus* $8899 + \overline{8899} = 18887$ *and* $18887 + \overline{18887} = 97768$.

We thus conclude that it is not necessary to launch the process on all numbers in order to obtain pragmatic results, since we may only consider classes (also called *kinsnumber* [Yam97], *i.e.* numbers giving the same result).

4 The Palindromic Reversal Process

4.1 Every Word is a Pseudo-palindrom...

We saw how to recognize words giving the same number by iterating the palindromic reversal process. This identification between words may be understood in terms of equivalence classes. In the iterating process, each word is potentially a palindrom. We thereafter present how to derive a *pseudo-palindrom* from each word, pointing out the fact that a palindrom is a particular case of pseudo-palindrom. We therefore introduce the notions of pseudo-digit, pseudo-word, and finally, pseudo-palindrom.

Definition 4.1 (Pseudo-digit) *A* pseudo-digit *is any (positive) rational number.*

Definition 4.2 (Pseudo-word) *A* pseudo-word w *is a word on pseudo-digits.*

Remark 4.3 *Note that $\langle w \rangle$ is always defined for any pseudo-word w.*

Definition 4.4 (Pseudo-palindrom) *A pseudo palindrom is a palindrom on pseudo-digits.*

In other words, a pseudo-palindrom is a pseudo-word that is a palindrom.

To each word $= w_{n-1}..w_1w_0$, we associate a pseudo-palindrom:

$$\widetilde{w} = \boxed{\tfrac{w_{n-1}+w_0}{2}}\boxed{\tfrac{w_{n-2}+w_1}{2}}\boxed{\dots}\boxed{\tfrac{w_{\lceil n-1/2\rceil}+w_{\lfloor n-1/2\rfloor}}{2}}\boxed{\dots}\boxed{\tfrac{w_1+w_{n-2}}{2}}\boxed{\tfrac{w_0+w_{n-1}}{2}}$$

Remark 4.5 *We frame pseudo-digits when necessary, since each may contain several classical digits.*

Since symmetric digits in $\widetilde{w}$ are the same, $\widetilde{w}$ is actually a pseudo-palindrom. $\widetilde{w}$ may be seen as the normal form of w.

Example 4.6 $\widetilde{196} = \widetilde{295} = .. = \widetilde{691} = \widetilde{790} = \boxed{\frac{7}{2}}\boxed{9}\boxed{\frac{7}{2}}$.
$\widetilde{1675} = \widetilde{1765} = .. = \widetilde{2674} = .. = \widetilde{5761} = \widetilde{5941} = \boxed{3}\boxed{\frac{13}{2}}\boxed{\frac{13}{2}}\boxed{3}$.

Proposition 4.7 *For any (pseudo-)word w, take $\mathcal{N} = \langle w \rangle$, $\overline{\mathcal{N}} = \langle \overline{w} \rangle$ and $\mathcal{M} = \langle \widetilde{w} \rangle$. Then:*

$$\mathcal{N} + \overline{\mathcal{N}} = 2\mathcal{M} \quad .$$

Proof. Immediate (componentwise, n being set). $\square$

4.2 Additionning a Word to its Reversal

While describing the palindromic reversal process is not so complex, it is quite difficult to find a good formulation of the expression $\mathcal{N} + \overline{\mathcal{N}}$.

Proposition (4.7) leads us to an explicit mathematical formula of $\mathcal{N} + \overline{\mathcal{N}}$. We describe in the sequel the different steps towards this formula.

Definition 4.8 (Fold and Mean) *Given a pseudo-word w of length n, we define the action of folding up w and divide componentwise by 2 ($\widehat{\widehat{w}}$) by:*

$$\begin{aligned}
\widehat{\widehat{w_i}} &= \frac{w_i}{2} + \frac{w_{n-i-1}}{2} \quad , \quad \forall i \in \{0, .., \lfloor n/2 \rfloor - 1\} \\
w_{(n/2-1)} &= \begin{cases} w_{\lceil n/2-1 \rceil}, & \text{if } n \text{ is odd,} \\ \varepsilon, & \text{otherwise.} \end{cases} \\
\widehat{\widehat{w}} &= (\widehat{\widehat{w_0}} .. w_{\widehat{\widehat{\lfloor n/2 \rfloor}}-1}; w_{(n/2-1)}) \quad .
\end{aligned}$$

That is to say $\widehat{\widehat{w}}$ is a couple of the form $(v; v')$ (v any pseudo-word, v' pseudo-word reduced to one or zero digit), such that $\widetilde{w} = vv'\overline{v}$. Here, $w_i + w_{n-i-1}$ is an abbreviation for $\downarrow \langle w_i \rangle + \langle w_{n-i-1} \rangle \downarrow$—we will widely use such notations, without loss of generality.

For example, $\widehat{\widehat{196}} = \left(\boxed{\tfrac{7}{2}}; 9\right)$, since $\frac{7}{2} = \frac{6+1}{2}$, and $\widehat{\widehat{1356}} = \left(\boxed{\tfrac{7}{2}}\boxed{4}; \varepsilon\right)$. Hence, the right projection of $\widehat{\widehat{w}}$ gives the medium pseudo-digit of w if any, ε otherwise (when $|w|$ is even).

Now, we are able to give a first mathematical formulation of the palindromic reversal process:

Definition 4.9 (ξ) *Let w be any (pseudo-)word in the $\flat$ basis, and $\mathcal{N} = \langle w \rangle$. Then:*

$$\xi(c_1; c_2) := 2 \times \begin{cases} c_2, & \textit{if } c_1 = \varepsilon, \\ (\flat^p - 1) \times c_1 + \flat^{p-1} \times c_2 + \xi(\widehat{\widehat{c_1}}), & \textit{otherwise} \end{cases} \quad ,$$

with $p = |c_1| + |c_2|$.

Proposition 4.10

$$\mathcal{N} + \overline{\mathcal{N}} = \xi(\widehat{\widehat{w}}) \quad . \tag{4}$$

Proof. The proof is postponed until proposition (4.14). $\square$

Let us take a short example. Remark that ξ is nœtherian and we actually have:

$$\begin{aligned}
\xi(\widehat{\widehat{196}}) &= 2 \times \left((10^2 - 1) \times \frac{7}{2} + 10 \times 9 + \xi\left(\widehat{\widehat{\frac{7}{2}}}\right) \right) \\
&= 2 \times \left(99 \times \frac{7}{2} + 90 + \xi\left(\varepsilon; \frac{7}{2}\right) \right) \\
&= 2 \times \frac{693 + 180 + 14}{2} \\
&= 887 \\
&= 196 + 691 \quad .
\end{aligned}$$

This new constructive definition of $\mathcal{N}+\overline{\mathcal{N}}$ is a formula that has the advantage to make the word reversal not explicitely intervene in the calculation.

As you may expect from the latter calculation, we can discard the powers of 2 in formula (4.9), both in factor and in numerators of pseudo-digits. We can also derecursivate ξ. This leads to the following definitions.

Definition 4.11 (Fold) *Let w be any pseudo-word. We define the action of folding up w by $\widehat{w} := 2 \times \widehat{\widehat{w}}$.*

That is to say, when $\widehat{\widehat{w}} = (c_1; c_2)$, $\widehat{w} := (2 \times c_1; 2 \times c_2)$. We leave as an easy proof to the reader that $\frac{1}{2}$ factorizes over the basis $\flat$ in c_1:

$$2 \times \left\langle \begin{array}{|c|c|c|c|}\hline \frac{w_{n-1}}{2} & \frac{w_{n-2}}{2} & .. & \frac{w_0}{2} \\ \hline \end{array} \right\rangle = \langle w_{n-1} w_{n-2} .. w_0 \rangle \quad ,$$

which we abbreviate, as previously done, by:

$$2 \times \begin{array}{|c|c|c|c|}\hline \frac{w_{n-1}}{2} & \frac{w_{n-2}}{2} & .. & \frac{w_0}{2} \\ \hline \end{array} = w_{n-1} w_{n-2} .. w_0 \quad .$$

Notation 4.12 *For any w, we have $\widehat{w} = (c_1; c_2)$. This time, we also can fold c_1 up, and so on. $\widehat{w}^2$ will denote $\widehat{c_1}$, etc. π_1 and π_2 will respectively denote the first and the second projection over couples.*

We thus have $\pi_1(\widehat{w}^1) = \pi_1(\widehat{w}) = c_1$, and $\pi_2(\widehat{w}) = c_2$.

Definition 4.13 (ζ) *Let w be any (pseudo-)word in the $\flat$ basis, and $\mathcal{N} = \langle w \rangle$. Then:*

$$\zeta(w) := \sum_{i=1}^{\lfloor \log_2(|w|) \rfloor + 1} (\flat^{p_i} - 1) \times \pi_1(\widehat{w}^i) + \flat^{p_i - 1} \times \pi_2(\widehat{w}^i) \quad ,$$

with $p_i = \left|\pi_1(\widehat{w}^i)\right| + \left|\pi_2(\widehat{w}^i)\right|$.

Proposition 4.14

$$\mathcal{N}+\overline{\mathcal{N}}=\zeta(w) \quad . \tag{5}$$

Proof . ζ is a possible derecursivation of the function ξ (set $\widehat{(c_1;c_2)}^0 = (c_1;c_2)$). Proving proposition (4.14) thus yields the demonstration of proposition (4.10). For sake of simplicity, words and numbers are identified.

By definition, $\widehat{w_i} = w_i + w_{n-i-1}, \forall i \in \{0,..,\lfloor n/2\rfloor - 1\}$, and thus $\widehat{w} = \left(\widehat{w_0}..\widehat{w_{\lfloor n/2\rfloor -1}}; 2\times w_{(n/2-1)}\right)$. Let $p = |\pi_1(\widehat{w})| + |\pi_2(\widehat{w})| = |c_1| + |c_2|$. By a quite labourous calculus, we get that $\zeta(w) = \flat^p \times \pi_1(\widehat{w}) + \flat^{p-1}\times c_2 + \overline{\pi_1(\widehat{w})}$. Hence, $\zeta(w) = \flat^p\left(\sum_{i=0}^{\lfloor n/2\rfloor-1}\flat^i(w_i+w_{n-i-1})\right) + \sum_{i=0}^{\lfloor n/2\rfloor -1}\flat^{\lfloor n/2\rfloor -1-i}(w_i + w_{n-i-1}) + 2\flat^{p-1}\times w_{(n/2-1)}$. $\zeta(w)$ may be rewritten as a single sum: $\zeta(w) = \sum_{i=0}^{n-1}\flat^i(w_i + w_{n-1-i})$, since $p = \lceil n/2\rceil$. $\square$

Example 4.15 *Let* $w = 1936425$, $c = \widehat{w} = \left(\boxed{6\,|\,11\,|\,7}; 12\right)$. *$i$ ranges over* $1..(\lfloor\log_2(7)\rfloor + 1 = 3)$. *Decomposing the sum* $\zeta(w)$, *we obtain:*

$$\begin{array}{llr}
i=1 & 9999\times\boxed{6\,|\,11\,|\,7} + 1000\times 12 = & 7181283\\
i=2 & 99\times\boxed{13} + 10\times 22 = & 1507\\
i=3 & 9\times\varepsilon + 1\times 26 = & 26\\
\hline
 & \sum = & 7182816
\end{array} \quad ,$$

where $\left\langle\boxed{6\,|\,11\,|\,7}\right\rangle = 6\times 10^2 + 11\times 10 + 7$.

And we thus verify that $\zeta(1936425) = 1936425 + \overline{1936425} = 7182816$.

In fact, $\zeta(w)$ amounts to process $\mathcal{N}+\overline{\mathcal{N}}$, for $\mathcal{N} = \langle w\rangle$, but it gives another view of the palindromic reversal process. The runtime complexity of ζ is also linear in $|w|$.

5 Conclusion

In this paper, we describe conservating properties of the palindromic reversal process. These properties essentially concern a single step of the process, but can be extended to several steps, thus enabling to describe classes of numbers producing the same result for a given number of digits. Understanding classes of such numbers amounts to describe $\mathcal{N} - \overline{\mathcal{N}}$ (where $\overline{\mathcal{N}}$ is the integer obtained by a backward reading of $\mathcal{N}$), for any integer $\mathcal{N}$, and also $\mathcal{N} + \overline{\mathcal{N}} - \overline{\mathcal{N} + \overline{\mathcal{N}}}$, and so on.

In fact, the $\mathcal{N} + \overline{\mathcal{N}}$ operation, *i.e.* the actual palindromic process, can be understood as a recursive mechanism on pseudo-words. By giving a recursive algorithm of the process, we hope that basic case and recursion steps for each processed number will allow to understand the general process in its whole complexity.

We are indebted to Pr. Giuseppe Pirillo (Univ. Firenze, Italy) who submitted us this problem.

Appendix A: Classifying Numbers

We give a finer classification of the one given in [Yam97] for the numbers from 1 to 10000 that apparently never produce a palindrom. By reading each subtable in the following table from left to right, we easily see that numbers only differ of one unit on symmetric digits, as stated in proposition (3.6). There are some exceptions in this classification, for instance in the class of 196, consider the subtable beginning with 4079: 4709 and 4799 are close to each other, although they do not differ of one unit on the second digit. This kind of exceptions reflect the property described in section "Properties forecasting" (3.2), *i.e.* $4709 + \overline{4709} + \overline{4709 + \overline{4709}} = 4799 + \overline{4799} + \overline{4799 + \overline{4799}}$.

In the same class, we jump from one subtable to the next one by applying one step of the palindromic reversal process to any number of the former subtable in order to obtain one uniq number in the latter one. For instance, for any $w \in \{196, 295, 394, 493, 592, 691, 790\}$, one step of the process gives 887, and we can thereafter derive the whole subclass of 887.

```
# Class of 196
196 295 394 493 592 691 790

689 788 887 986

1495 1585 1675 1765 1855 1945 2944 2854 2764 2674 2584 2494
3493 3583 3673 3763 3853 3943 4942 4852 4762 4672 4582 4492
5491 5581 5671 5761 5851 5941 6940 6850 6760 6670 6580 6490

4079 4169 4259 4349 4439 4529 4619 4709 4799 4889 4979
5078 5168 5258 5348 5438 5528 5618 5708 5798 5888 5978
6077 6167 6257 6347 6437 6527 6617 6707 6797 6887 6977
7076 7166 7256 7346 7436 7526 7616 7706 7796 7886 7976
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8075 8165 8255 8345 8435 8525 8615 8705 8795 8885 8975
9074 9164 9254 9344 9434 9524 9614 9704 9794 9884 9974

# Class of 879
879 978

1497 1587 1677 1767 1857 1947 2946 2856 2766 2676 2586 2496
3495 3585 3675 3765 3855 3945 4944 4854 4764 4674 4584 4494
5493 5583 5673 5763 5853 5943 6942 6852 6762 6672 6582 6492
7491 7581 7671 7761 7851 7941 8940 8850 8760 8670 8580 8490

8079 8169 8259 8349 8439 8529 8619 8709 8799 8889 8979
9078 9168 9258 9348 9438 9528 9618 9708 9798 9888 9978

# Class of 1997
1997 2996 3995 5993 6992 7991 8990

8089 8179 8269 8359 8449 8539 8629 8719 8809 8899 8989
9088 9178 9268 9358 9448 9538 9628 9718 9808 9898 9988

# Class of 7059
7059 7149 7239 7329 7419 7509 7599 7689 7779 7869 7959
8058 8148 8238 8328 8418 8508 8598 8688 8868 8958
9057 9147 9237 9327 9417 9507 9597 9687 9777 9867 9957
```

We also made this kind of classification for $\flat = 3$.

Appendix B: Lucky Numbers

In [Wal90], John Walkers assumes that "*in order for addition of a digits-reversed number to yield a palindrom, there must be no carries in the addition and hence each pair of digits must sum to 9 or less*".

This is not actually true, since some *lucky numbers* give a palindrom after having propagating one or more carries.

Definition 5.1 (Growing word) *A* growing word *w is a word such that if $\mathcal{N} = \langle w \rangle$, then $\left| \downarrow \mathcal{N} + \overline{\mathcal{N}} \downarrow \right| = |w| + 1$.*

Example 5.2 *Let $\flat = 10$, $w = 57$, then $\left|57 + \overline{57}\right| = 132 = |57| + 1$.*

Definition 5.3 (Lucky word) *Let $\flat > 2$ be a basis, $w = w_{n-1}..w_0$. w is said to be* lucky *if w is a growing word and for all $i \in \{0, .., \lfloor \frac{n-1}{2} \rfloor\}$, $w_{n-1-i} + w_i = \flat + 1$ or 0 if $i \neq 0$.*

Lemma 5.4 *The only words that give a palindrom after one step of the process, in which a carry has been produced are the lucky words.*

Proof. First of all, remark that given a word w and $\mathcal{N} = \langle w \rangle$, if there is at least one carry in $\mathcal{N} + \overline{\mathcal{N}}$ and $\mathcal{N} + \overline{\mathcal{N}}$ is a palindrom, then the number of digits of $\mathcal{N} + \overline{\mathcal{N}}$ increases by one with respect to $|w|$, *i.e.* w is a growing word. Thus, the leftmost digit of the result is 1, (and so is the first, since the result is a palindrom).

We first present the proof of the lemma for the two-digit case. Let $w = (x)(y)$ be such a word and $\mathcal{N} = \langle w \rangle$, then $\mathcal{N}+\overline{\mathcal{N}} = (1)(x+y+1-\flat)(x+y-\flat)$. But $(x_y - \flat) = (1)$, and so $x + y = \flat - 1$. Thus the only two-digit words that give a palindrom are $(i)(\flat + 1 - i)$, for $i = 2..\flat - 1$.

We leave the three-digit case to the reader and prove the lemma for the four-digit one. Let $w = (x)(y)(z)(t)$ and $\mathcal{N} = \langle w \rangle$. A carry may be produced either by $x + t$ or (inclusively) by $z + y$. But $x + t = \flat + 1$ as previously explained in the case of two digits. So, we must have either $y + z + 1 = x + t + 1$ or $y + z + 1 = x + t - \flat$, since $y + z + 1 = x + t$ and $y + z + 1 = x + t - \flat + 1$ are excluded (they do not lead to a palindrom).

In the generic case,

$$\begin{array}{cccccc} & x & \alpha & \dots & \beta & y \\ & y & \beta & \dots & \alpha & x \\ \hline 1 & ?_1 & ?_2 & \dots & ?_3 & 1 \end{array}$$

But now, $x + y = \flat + 1$, thus $?_1 = 1$ or 2. If $?_1 = 1$, then $?_3 = 1 + \alpha + \beta = 1$, since $\alpha + \beta < \flat$ $(?_1 = 1)$. So $?_1 = 1 \Rightarrow \alpha = \beta = 0$. If $?_1 = 2$, then $?_3 = \flat + 2$ and thus $\alpha + \beta = \flat + 1$.

Since the number of digits has increased, we can iterate this reasoning on the subword $\alpha..\beta$.

This proves that the only digits of w are $w_i = 0$, $i \neq 0$ or w_i such that $w_{n-1-i} + w_i = \flat + 1$, $i = 0..n - 1$. □

Remark 5.5 *In the case of* $\flat = 2$*, there are no lucky words* w *as described above, since* $w_i + w_{n-i-1}$ *will never reach* $\flat + 1$*. But a kind of lucky words may be found under the shape* 11...110 *(note that this kind of words can not appear for* $\flat > 2$*).*

Therefore, the quest of words not giving a palindrom for $\flat = 2$ *is reduced to the search of words such that the last step of the process will not give any carry, except those mentioned.*

References

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