Algorithms for the Maximum Hamming Distance Problem

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Abstract. We study the problem of finding two solutions to a constraint satisfaction problem which differ on the assignment of as many variables as possible—the Max Hamming Distance problem for CSPs—a problem which can, among other things, be seen as a domain independent way of quantifying "ignorance." The first algorithm we present is an $\mathcal{O}(1.7338^n)$ microstructure based algorithm for Max Hamming Distance 2-SAT, improving the previously best known algorithm for this problem, which has a running time of $\mathcal{O}(1.8409^n)$. We also give algorithms based on enumeration techniques for solving both Max Hamming Distance l-SAT, and the general Max Hamming Distance (d, l)-CSP, the first non-trivial algorithms for these problems. The main results here are that if we can solve l-SAT in $\mathcal{O}(a^n)$ and (d, l)-CSP in $\mathcal{O}(b^n)$, then the corresponding Max Hamming problems can be solved in $\mathcal{O}((2a)^n)$ and $\mathcal{O}(b^n(1+b)^n)$, respectively.

1 Introduction

In its most basic form, a constraint satisfaction problem (CSP) consists of a collection of variables taking values from some domain, and a collection of constraints restricting the values different variables can simultaneously assume. The question here is: Can we find an assignment of values to the variables which does not violate any of the constraints? While this is certainly the most thoroughly studied problem for CSPs, there are a number of alternative, equally interesting, questions one can ask about a CSP. The question we will study in this paper asks us to find two solutions that are as far away from each other as possible; i.e. we want to find two satisfying assignments that disagree on the values for

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as many variables as possible. This is known as the MAX HAMMING DISTANCE problem, and was first introduced in Crescenzi & Rossi [3], where it was suggested as a domain independent measure of ignorance, quantifying how much we do not know of the world.

We present three different algorithms.The first one is a microstructure based algorithm for the special case when the domains have size two and we have binary constraints, denoted MAX HAMMING DISTANCE $(2,2)$-CSP. (We will exclusively consider CSPs over finite domains, denoted (d,l)-CSP, where d is the domain size and l the arity of the constraints.) Even in this restricted form, the problem is NP-complete. In the microstructure graph of a CSP [9], a vertex corresponds to an assignment of a value to a variable in the original problem (see Section 2 for definitions.) The algorithm exploits this by searching for a set of vertices where each vertex either does not have an edge to any other vertex—and thus can be interpreted as an assignment—or is part of a connected component with 2 or 4 vertices. Each vertex (i.e. assignment) in this set is then given a weight, and the original instance together with these weights is given to a weighted 2-SAT solver. This algorithm returns a solution with maximum weight W, and we can then construct a solution which differs on W variables.

By using the weighted 2-SAT algorithm from [4], we arrive at a running time of $\mathcal{O}(1.7338^n)$, where n is the number of variables in the problem. This is an improvement over the MAX HAMMING DISTANCE $(2,2)$-CSP algorithm presented in [1], which runs in $\mathcal{O}(1.8409^n)$.

When we allow domains with more than 2 elements, or constraints with arity higher than 2, it turns out that microstructures are not as successful, and, consequently, the algorithms we present for these cases are quite different. Intuitively, the algorithm for MAX HAMMING DISTANCE (d,l)-CSP works as follows:

1. Pick a subset of the variables which should assume different values in the two solutions, duplicate and rename them and the constraints they are involved in.
2. Add a constraint for each of these new variables, preventing it from assuming the same value as the one it is a copy of.
3. Solve this new, larger, instance.

Starting with assuming all variables are different in the two solutions, and then working downwards, the algorithm will, by trying out the different possible subsets of variables, arrive at a pair of solutions with maximum hamming distance. Given that we can solve the (d,l)-CSP in the last step in time $\mathcal{O}(a^n)$, the entire algorithm will have a running time of $\mathcal{O}(a^n(1+a)^n)$.

The final algorithm is for the case when the domain has two elements and the constraints have arity l, MAX HAMMING DISTANCE $(2,l)$-CSP. Here, we note that since there are only two possible choices of values for a variable, it is unnecessary to duplicate the variables that should take different values—instead, only the constraints they are involved in are duplicated, and then any occurrence of a variable which should assume different values in the two solutions is replaced by its negation in these constraints. The resulting algorithm will have

a running time of $\mathcal{O}((2a)^n)$, where $\mathcal{O}(a^n)$ is the time needed to solve the $(2,l)$-CSP problem in each step.

Overview of the paper: Section 2 contains most of the definitions we will need in the discussion. For convenience, it has been split into three parts, where Section 2.1 contains the definitions related to CSPs, Section 2.2 the graph and microstructure definitions, while Section 2.3 formally defines the problem we will be studying, MAX HAMMING DISTANCE. The algorithm for MAX HAMMING DISTANCE $(2,2)$-CSP, together with its analysis, is presented in Section 3, while Section 4 contains the algorithms for MAX HAMMING DISTANCE (d,l)-CSP and MAX HAMMING DISTANCE $(2,l)$-CSP.

2 Preliminaries

This section is divided into three parts in order simplify the search for a particular definition. In Section 2.1 we have the definitions related to constraint satisfaction problems, while Section 2.2 is devoted to graphs and the microstructure of CSPs. Finally, in Section 2.3, we define the problem we will be discussing in this paper; MAX HAMMING DISTANCE (d,l)-CSP. Note that Section 3 contains additional definitions specific to that part of the paper.

2.1 Constraint satisfaction problems

A *(d,l)-constraint satisfaction problem ((d,l)-CSP)* is a triple (X,D,C) where

- X is a finite set of variables,
- D a finite set of domain values, with $|D| = d$, and
- C is a set of constraints $\{c_1, c_2, \ldots, c_k\}$.

Each constraint $c_i \in C$ is a structure $R(x_{i_1}, \ldots, x_{i_j})$ where $j \leq l, x_{i_1}, \ldots, x_{i_j} \in X$ and $R \subseteq D^j$. A *solution* to a CSP instances is a function $f : X \to D$ s.t. for each constraint $R(x_{i_1}, \ldots, x_{i_j}) \in C$, $(f(x_{i_1}, \ldots, f(x_{i_j}))) \in R$. Given a (d,l)-CSP, the basic computational problem is to decide whether it has a solution or not—to determine if it is *satisfiable*.

The special case when $d = 2$ and we have binary constraints, i.e. $(2,2)$-CSP, will often be viewed as 2-SAT formulae. A 2-SAT formula is a conjunction of a number of clauses, where each clause is on one of the forms $(p \vee q)$, $(\neg p \vee q)$, $(\neg p \vee \neg q)$, (p), $(\neg p)$. The set of variables of a formula F is denoted $\mathrm{Var}(F)$, and an occurrence of a variable or its complement in a formula is termed a *literal*. Determining whether a 2-SAT formula is satisfiable can be done in polynomial time [2], while, in contrast, the more general l-SAT (i.e, the clauses consist of at most l literals) is known to be NP-complete for $l \geq 3$ [7].

Definition 1 ([4]). *Let F be a 2-SAT formula, and let L be the set of all literals for all variables occurring in F. Given a vector $\boldsymbol{w}$ of weights and a model M for F, we define the weight $W(M)$ of M as*

$$W(M) = \sum_{\{l \in L \mid l \text{ is true in } M\}} \boldsymbol{w}(l)$$

The problem of finding a maximum weighted model for F is denoted 2-SAT$_w$.

In [4], an algorithm for counting the number of maximum weighted solutions to 2-SAT instances is presented which has a running time of $\mathcal{O}(1.2561^n)$, and it can easily be modified to return one of the solutions.

2.2 Graphs and microstructures

A *graph* G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, where each element of $E(G)$ is an unordered pair of vertices. The *neighbourhood* of a vertex $v \in V(G)$ is the set of all vertices adjacent to v, excluding v itself, and is denoted $N_G(v)$, $N_G(v) := \{u \in V(G) \mid (v, w) \in E(G)\}$. If, by following the edges of the graph, we can get from a vertex v to v', then v' is *reachable* from v. The *connected components* of a graph are the equivalence classes of vertices under the "is reachable from" relation.

Definition 2 ([9]). *Given a binary CSP $\Theta = (X, D, C)$, the microstructure of Θ is an undirected graph G, defined as follows:*

1. *For each variable $x \in X$, and domain value $d \in D$, there is a vertex $x[d]$ in G.*
2. *There is an edge $(x[d], y[e]) \in E(G)$ iff (d, e) satisfies the constraint between x and y.*

We assume that there is exactly one constraint between any pair of variables, and variables with no explicit constraint between them is assumed to be constrained by the universal constraint which allows all values.

For convenience, we will work exclusively with the complement of the graph in Definition 2. The complement of a (microstructure) graph G is a graph containing exactly those edges which are not present in G (excluding loops), i.e. a graph with edge set $\{(v, u) \mid v \neq u \wedge (v, u) \notin E(G)\}$.

A variable with domain size d will in the microstructure graph be a clique of size d. When the domain has two elements and we have a clique of size 2, we let $x[i]$ denote an arbitrary value for x, and use $x[1-i]$ to denote the other possible value. For example, if we look at the 2-SAT formula $(x \vee y) \wedge (\neg x \vee z)$, with domain values 0 and 1, it has the microstructure graph shown in Fig. 1. One independent set the graph is $\{x[0], y[1], z[0]\}$ which corresponds to the satisfying assignment $\{x \mapsto 0, y \mapsto 1, z \mapsto 0\}$.

2.3 Hamming distance of CSPs

The algorithms we present in this paper are all designed to solve different variants of the Max Hamming Distance problem for constraint satisfaction problems [3]. Since our algorithms are not limited to problems with two valued domains, the following definition differs somewhat from the one given in [3]:

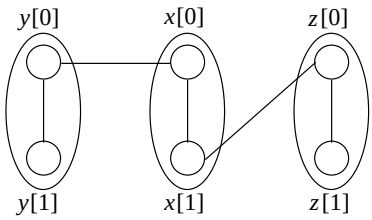

Fig. 1. The microstructure graph of $(x \vee y) \wedge (\neg x \vee z)$.

algorithm *MH1*(α, G, Θ)

1. **if** $\delta(x) \in \{(3,1),(2,2),(2,1),(1,1)\}$ for all variables x in G **then**
2. **return** *MH2*(α, G, Θ)
3. **end if**
4. Choose a variable x in G with $\delta(x) \in \{(\geq 3, \geq 2),(\geq 4,1)\}$
5. $(\alpha_0, \beta_0) =$ *MH1*$(\alpha \cup \{x[0]\}, G - N_G(x[0]) - \{x[0]\}, \Theta)$
6. $(\alpha_1, \beta_1) =$ *MH1*$(\alpha \cup \{x[1]\}, G - N_G(x[1]) - \{x[1]\}, \Theta)$
7. **return** $(\alpha_i, \beta_i), i \in \{0,1\}$ maximising $d_H(\alpha_i, \beta_i)$

Fig. 2. The main algorithm for Max Hamming Distance $(2,2)$-CSP.

Definition 3. *Given a set of variables X over finite domains, the* Hamming distance *between a pair f_1 and f_2 of assignments of values to the variables in X, denoted $d_H(f_1, f_2)$, is the number of variables on which f_1 and f_2 disagree.*

For example, consider the 2-SAT formula $(x \vee y) \wedge (\neg x \vee z)$, and the two assignments $f_1 = \{x \mapsto 0, y \mapsto 1, z \mapsto 0\}$, $f_2 = \{x \mapsto 1, y \mapsto 1, z \mapsto 1\}$. Clearly, both f_1 and f_2 satisfy the formula, and their Hamming distance is 2, since they disagree on the values for x and z.

The following formalises the problem:

Definition 4 (Maximum Hamming Distance of (d,l)-CSPs). *Let $\Theta = (X, D, C)$ be an instance of (d,l)-CSP. The* Max Hamming Distance *(d,l)-CSP problem is to find two satisfying assignments f and g to Θ which maximises $d_H(f,g)$.*

A naïve enumeration algorithm for this problem would have a time complexity of $\mathcal{O}\left(d^{2n}\right)$. In the following sections we will present ways to significantly improve this running time.

3 Algorithm for Max Hamming Distance $(2,2)$-CSP

algorithm *MH2*(α, G, Θ)

1. **if** $\delta(x) \in \{(2,1),(1,1)\}$ for all variables x in G **then**
2. **return** *MH3*(α, G, Θ)
3. **end if**
4. **if** G contains a cycle **then**
5. **if** all variables x has $\delta(x) = (2,2)$ in a cycle **then**
6. Choose x in this cycle
7. **else if** there is a variable z with $\delta(z) = (2,2)$ in a cycle **then**
8. Choose x in a cycle s.t $\delta(x) = (3,1)$ and $x[i]$ has a neighbour y with $\delta(y) = (2,2)$
9. **else**
10. Choose x with $\delta(x) = (3,1)$ in a cycle
11. **end if**
12. **else** *% G is cycle-free*
13. Choose x which is two variables from the end of a chain, if possible, otherwise, choose x one variable from the end of a chain
14. **end if**
15. $(\alpha_0, \beta_0) =$ *MH2*$(\alpha \cup \{x[0]\}, G - N_G(x[0]) - \{x[0]\}, \Theta)$
16. $(\alpha_1, \beta_1) =$ *MH2*$(\alpha \cup \{x[1]\}, G - N_G(x[1]) - \{x[1]\}, \Theta)$
17. **return** $(\alpha_i, \beta_i), i \in \{0,1\}$ maximising $d_H(\alpha_i, \beta_i)$

Fig. 3. The helper function *MH2*.

In this section we will discuss and analyse our algorithm for MAX HAMMING DISTANCE $(2,2)$-CSP. Since the formulae for the time complexity of the algorithm can be rather lengthy, the final step, that of calling a weighted 2-SAT solver for every leaf in the search tree, has been left out (unless otherwise noted.) Furthermore, we will omit polynomial factors in the time complexities.

Before we start the discussion of the algorithms, we will need some additional definitions: The *degree* of a vertex v in a graph, usually denoted $\deg(v)$, is the size of its neighbourhood, i.e. $|N_G(v)|$. However, we are not really interested in the degree of a single vertex, but rather in the degrees of the two vertices that make up a variable. Thus let $\Theta = (X, D, C)$ be a $(2,2)$-CSP and, for $x \in X$, define the *variable degree* $\delta(x)$ as a tuple $(\deg(x[i]), \deg(x[1-i]))$, where $x[i]$ is the vertex with highest degree. If we are interested in variables with degrees higher than a certain number, we write $\delta(x) = (\geq i, \geq j)$.

In the analysis of the algorithm in this section, we will often encounter recursions on the form $T(n) = \sum_{i=0}^{k} T(n - r_i) + p(n)$, where $p(n)$ is a polynomial in n and $r_i \in \mathbb{Z}^+$. These equations satisfy $T(n) \in \mathcal{O}(\tau(r_1, r_2, \ldots, r_k)^n)$, where $\tau(r_1, r_2, \ldots, r_k)$ is the largest real-valued solution to the equation $1 - \sum_{i=1}^{k} x^{-r_i} = 0$ (see Kullman [10].) Note that this bound depends on neither $p(n)$ nor the boundary conditions $T(1) = b_1, \ldots, T(k) = b_k$. We will sometimes refer to τ as the *work factor* (in the sense of [5].)

With that in mind, we are now ready to discuss the algorithm, which consists of three functions: *MH1*, the main algorithm, which calls *MH2* once no variable

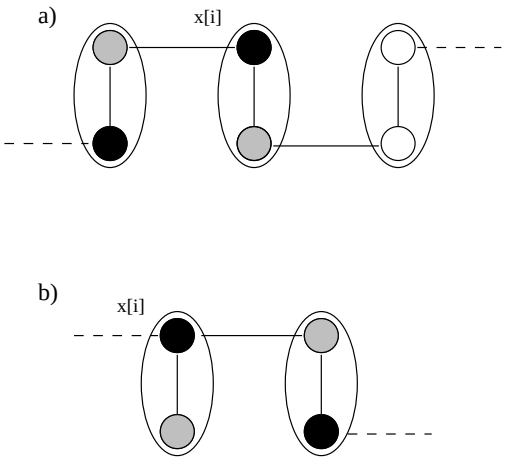

Fig. 4. Branching on $x[i]$ will remove the shaded values and force the black values.

is involved in more than three constraints, which, in turn, calls *MH3* when every variable is involved in at most one constraint.

MH1: The main algorithm, *MH1*, given in Fig. 2, takes as input a partial assignment α, a microstructure graph G, and the original problem instance Θ. If every variable in the microstructure is involved in less than 3 constraints, the helper function $MH2$ is called. In the graph, this translates to every variable x having $\delta(x)$ in the set $\{(3,1),(2,2),(2,1),(1,1)\}$. Otherwise, a variable involved in more than 3 constraints is chosen, and the algorithm branches on the two possible values. We note that for $\delta(x) = (3,2)$, there will be at least 3 variables less in one branch and 2 variables less in the second branch, and for $\delta(x) = (4,1)$, there are at least 4 and 1 variables less, respectively. Consequently, we get work factors of $\tau(3,2)$ and $\tau(4,1)$ for these cases, where $\tau(4,1)$ clearly dominates.

Example 1. Consider the 2-SAT formula $\Theta = (x \vee y) \wedge (\neg x \vee y)$, which has the microstructure shown in Fig. 1. The variables in Θ have degrees $\delta(x) = (2,2), \delta(y) = (2,1)$ and $\delta(z) = (2,1)$, thus we immediately jump to algorithm *MH2*. Had this not been the case, a variable with a degree of $(\geq 3, \geq 2)$ or $(\geq 4, 1)$ would have been chosen, and the algorithm would have branched on either of the possible values for it.

MH2: The first helper function, *MH2*, shown in Fig. 3, takes over when every variable is involved in zero, one or two constraints. Unless there are no variables involved in two constraints (in which case we jump straight to algorithm *MH3*), we start with checking for cycles. Any cycles we encounter need to be broken, which is done on lines 4 to 11.

First of all, if there is a cycle where every variable has a degree of $(2,2)$, then selecting one value for a variable in this cycle will propagate through the entire

cycle, as is shown in the top part of Fig. 4. On line 8, by choosing a variable x with $\delta(x) = (3,1)$ with a neighbour y with $\delta(y) = (2,2)$, one of the values for x will propagate through y. (See Fig. 4b.) Consequently, 4 variables are removed in one branch, and one in the other, giving a work factor of $\tau(4,1)$ for this case. The obvious exception is when the cycle contains only 3 variables, as is shown in Fig. 7a. Note that the coloured vertex $x[i]$ is the only possible choice—the other assignment would lead to an inconsistency.

Now if every variable x in the cycle has $\delta(x) = (3,1)$, we get a number of different possibilities, but before we discuss them, we need to make some observations. Once a variable has no neighbours (Fig. 6a), or is part of a component consisting of only two variables and one edge between them, a 'hurdle,' (Fig. 6b), we need no longer consider it. The first case is obvious, since if a variable has no neighbours, it is not involved in any constraints, and we can choose its value freely, while the second case is somewhat harder; We will get back to it when we discuss algorithm *MH3*. Consequently, when a component of $(3,1)$ variables has at most 3 variables, as in Fig. 7b, when choosing such a variable, in effect, the entire component is removed from the problem and need no longer be considered—in one case we get a unique assignment for the remaining (black) vertices, and in the other case we get a hurdle. If there are no cycles in the component, e.g we have a 'comb-like' structure, as in Fig. 8, then choosing any of the three variables to branch on will, again, remove the entire component, giving a work factor of $\tau(3,3)$. This also holds for cycle-free components of size 4 and 5. When there are more than 5 variables in the component, by choosing a variable which is two variables removed from the end of the comb (the marked variable in Fig. 8), the chain is broken and we remove 3 variables in one branch and 4 in the other. As was seen in the case for cycles where all variables have degree $(2,2)$, the number of removed variables increases if a neighbour of the branching variable has this property. Consequently, we will focus on the combs and merely note that the time complexity will not be worse if we have more variables with degree $(2,2)$.

Getting back to discussing cycles; When we reach line 10 of algorithm *MH2*, every cycle consists exclusively of variables with degree $(3,1)$, and since no vertex in the graph has degree higher than 3, there can be at most one cycle in a component. The case with cycles containing 3 variables was discussed earlier, and for the case with 4 variables we get one branch where the entire component is removed, and one where we get a comb with 3 variables, which can be removed in its entirety when we branch. There can be no more than $n/4$ cycles with 4 variables in the graph at this point. For each of these cycles, we choose one variable to branch on, and in one branch the entire component is removed, while in the other, we get a component with 3 variables. Since we want to look at all of these cycles, and both branches, this is equivalent to selecting k cycles where we remove the entire component, and then examine the remaining $n/4-k$ components. In other words, it will require

$$\sum_{k=0}^{n/4} \binom{n/4}{k} \left(1^k \cdot \tau(3,3)^{3(n/4-k)}\right)$$

algorithm *MH3*(α, G, Θ)

1. Let $\boldsymbol{w}$ be a vector of weights, initially all set to 0
2. **for each** $x[i] \in \alpha$ **do**
3. add weight $\boldsymbol{w}(x[1-i]) := 1$
4. **for each** connected component of G **do**
5. Add weights to $\boldsymbol{w}$, as shown in Fig. 6.
6. $(\beta, W) :=$ *2-SAT*$_w(\Theta, \boldsymbol{w})$
7. **for each** variable x in G **do**
8. **if** $x[i]$ in β **then**
9. If possible, add $x[1-i]$ to α, otherwise, add $x[i]$.
10. **end if**
11. **end for**
12. **return** (α, β)

Fig. 5. The helper function *MH3*.

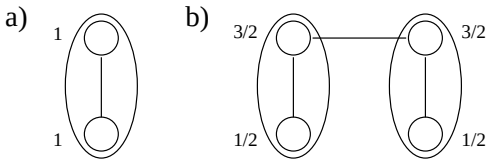

Fig. 6. Variables with no or exactly one neighbour, and the weights they are given by algorithm *MH3*.

steps to examine all the cycles. Using the binomial theorem, this can be simplified to $(1 + \tau(3,3)^3)^{n/4}$.

For cycles with 5 variables, the situation is similar, but for 6 we no longer remove the entire component in one of the branches. Instead, we get one branch with 5 variables, and one with 3, which, using the same reasoning as above, gives

$$\sum_{k=0}^{n/6} \binom{n/4}{k} \left(\tau(3,3)^{3k} \cdot \tau(5,5)^{5(n/6-k)} \right) = (\tau(3,3)^3 + \tau(5,5)^5)^{n/6}.$$

Similarly, for cycles of length 7, we get $(\tau(4,4)^4 + \tau(3,4)^6)^{n/7}$. In general, if we have cycles of length c, one branch will have one variable less, and the other three variables less, giving the following general running time:

$$\sum_{k=0}^{n/c} \binom{n/c}{k} \left(\tau(3,4)^{(c-3)k} \cdot \tau(3,4)^{(c-1)(n/c-k)} \right) =$$

$$= \left(\tau(3,4)^{c-1} + \tau(3,4)^{c-3} \right)^{n/c} < (2\tau(3,4)^c)^{n/c} = (2^{1/c}\tau(3,4))^n$$

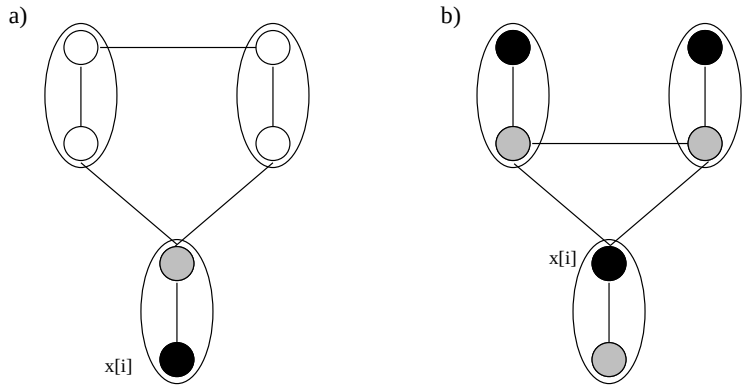

Fig. 7. The case when a component is a cycle with 3 variables.

Example 2 (cont'd). When we reach algorithm *MH2* with our (still unchanged) microstructure graph, we note that for x, $\delta(x) = (2, 2)$, thus we will continue past the test on line 1. Since the graph is cycle-free, we will choose a variable which is not on the end of a chain—in our case the only choice is x, and branch on the two possible values for x.

1. The branch with $x[0]$ removes $y[0]$ and $x[1]$ from the graph, forcing $y[1]$ and leaving z unconstrained.
2. For $x[1]$, $x[0]$ and $z[0]$ are removed, forcing $z[1]$ and leaving y unconstrained.

Consequently, both of these branches will result in *MH3* being called in the next recursive call (since we have a variable degree of $(1, 1)$ for the unconstrained variables left in the graph), with $\alpha = \{x[0], y[1]\}$ in the first case, and $\{x[1], z[1]\}$ in the second.

MH3: Finally, when algorithm *MH3* (see Fig. 5) is called, the graph G only contains variables involved in zero or one constraint, i.e. every variable will be of one of the forms found in Fig. 6. The weights shown in the figure is now added to the corresponding assignments in Θ, the original problem, and the resulting weighted 2-SAT problem is given to a *2-SAT*$_w$ solver. If the solution β returned by the solver has weight W, this means that we can add assignments (i.e. vertices) to α and create a solution which differs from β on W assignments in the following way: First of all, since all assignments in α are given weight 0, if any of these are chosen, they will not add anything to the distance, while the other possible value for all these variables will add one to the distance (and are consequently given a weight of 1 on line 3.) For the unconstrained variables in G, i.e. all vertices x with $\delta(x) = (1, 1)$, we can choose freely which value they should assume, and thus we can always find an assignment which adds one to the distance from β by choosing the other value for α. The remaining components then consist of pairs of variables with one edge between them, i.e. hurdles. If β contains both

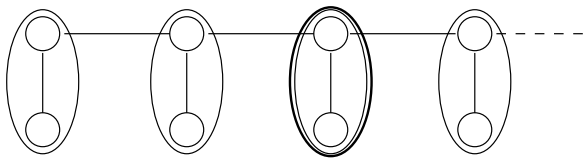

Fig. 8. Choosing a variable in a comb with more than 4 variables.

assignments with weight 1/2, then obviously, we have to add one of them to α, since not both assignments with weight 3/2 are allowed simultaneously—and thus we get a distance of 1, which is the sum of the weights in β. On the other hand, if β contains one 3/2 and one 1/2 assignments, then we can choose the opposing value for both of these and get a distance of 2. Consequently, the pair returned on line 12 will have a Hamming distance equal to the weight of β, and with α and G given, no pair with greater Hamming distance can exist.

Except for the call to *2-SAT*$_w$ on line 6, every step of algorithm *MH3* can be carried out in polynomial time, thus the time complexity is fully determined by that of the *2-SAT*$_w$ algorithm.

Example 3 (cont'd). Assuming we reach algorithm *MH3* with $\alpha = \{x[0], y[1]\}$, i.e. from branch 1 in Example 2 earlier, the algorithm will now assign weights to the assignments in the original problem. Each assignment in α is given weight 0, while its negation is given weight 1. In our case, we get $\boldsymbol{w}(x[0]) = \boldsymbol{w}(y[1]) = 0$, and $\boldsymbol{w}(x[1]) = \boldsymbol{w}(y[0]) = 1$. We have no "hurdles" in our graph (see Fig. 6b), but we have one free variable, z, which we assign weights $\boldsymbol{w}(z[0]) = \boldsymbol{w}(z[1]) = 1$.

Next, we call the weighted 2-SAT solver with the original instance together with the weight vector. It is easy to see that in our case, the maximum weight of a model will be 3; $\beta = \{x[1], y[0], z[1]\}$ has this weight, for instance. Consequently, since z is unconstrained in our graph, we can simply add $z[0]$ to α and we have two satisfying assignment α, β with a hamming distance of 3.

Theorem 1. *Algorithm MH1 correctly solves* Max Hamming Distance $(2,2)$*-CSP and has a running time of* $\mathcal{O}\left((a \cdot 1.3803)^n\right)$*, where* n *is the number of variables in the problem, and* $\mathcal{O}\left(a^n\right)$ *is the time complexity of solving a weighted 2-SAT problem.*

Proof. The highest work factor in algorithms *MH1, MH2, MH3* is $\tau(4,1)$, giving a running time of $\mathcal{O}\left(1.3803^n\right)$. The call to the weighted 2-SAT algorithm is done for every leaf in the search tree, and thus we get a total time complexity of $\mathcal{O}\left((a \cdot 1.3803)\right)$ if we assume we can solve weighted 2-SAT in $\mathcal{O}\left(a^n\right)$.

In every case, the algorithm branches on both values for a variable, thus from the correctness of the *2-SAT*$_w$ algorithm we know that the two solutions returned will be at a maximum hamming distance from each other. □

Corollary 1. Max Hamming Distance $(2,2)$*-CSP can be solved in* $\mathcal{O}\left(1.7338^n\right)$.

Proof. Dahllöf et al. [4] presents an algorithm for solving weighted 2-SAT in $\mathcal{O}(1.2561^n)$, and this together with Theorem 1 gives the result. □

4 Algorithm for Max Hamming Distance (d, l)-CSP

For problems where the arity of the relations is greater than 2, microstructures are not as convenient and we have to find a different approach.

Let us first consider the following problem: Given a CSP instance $\Theta = (X, D, C)$, can we find a pair of solutions with Hamming distance equal to k? One obvious way of doing this is the following:

1. Pick a subset Y of X with $|Y| = k$
2. Create a copy $\Theta' = (X', D, C')$ of Θ over variables X'
3. For each $x \in Y$, add the constraint $x \neq x'$ to C', and
4. for each $x \notin Y$, add the constraint $x = x'$ to C'.
5. If Θ' is satisfiable with solution s
   - Solve the instance $(X \cup X', D, C')$, giving a satisfying assignment s.
   - For each $x \in X$, add $s(x)$ to α
   - For each $x' \in X'$, add $s(x')$ to β
   - Return (α, β)

There are 2^n ways to choose Y on the first line, so if we can solve the satisfiability problem for Θ in time $\mathcal{O}(h(n))$, then, since the number of variables in Θ' is twice that of Θ, we can find a pair of solutions with maximum Hamming distance in $\mathcal{O}(2^n h(2n))$. For example, since 2-SAT can be solved in linear time, we would, using this approach, get a running time of $\mathcal{O}(2^n)$ for the Max Hamming Distance $(2, 2)$-CSP. This does give a slower running time than the algorithm we presented in the previous section, but it can be applied to CSP instances with domain size and constraint arity greater than 2.

Example 4. Again, consider the instance $(x \vee y) \wedge (\neg x \vee z)$. Following the algorithm in Fig. 9, we begin with trying to determine if there are two solutions with a distance of 3. We get a new, larger instance, which looks as follows:

$$(x \vee y) \wedge (\neg x \vee z) \wedge (x' \vee y') \wedge (\neg x' \vee z') \wedge$$
$$(x \neq x') \wedge (y \neq y') \wedge (z \neq z')$$

This instance has solution $\{x \mapsto 1, y \mapsto 0, z \mapsto 1, x' \mapsto 0, y' \mapsto 1, z' \mapsto 0\}$ and consequently, there are two solutions with a hamming distance of 3—we get one from reading the values of x, y, z and the other from the values of x', y', z'. Had this instance been unsatisfiable, we would have had to move on to try hamming distance 2, etc.

Actually, it is unnecessary to make a copy of *all* the variables. Having selected k variables that should be different in the two solutions, we only need to make copies of those, leaving the remaining $n - k$ variables unchanged. Thus, we get the algorithm for Max Hamming Distance (d, l)-CSP given in Fig. 9.

algorithm MAX HAMMING DISTANCE (d, l)-CSP $(\Theta = (X, D, C))$

```
1.  for k := |X| down to 0 do
2.      for each χ ⊆ X, |χ| = k do
3.          Let Θ' = (X', D, C') be a copy of Θ
4.          Let γ ⊆ C be all constraints involving variables from χ
5.          Create γ' by exchanging all variables not in χ with
            their counterparts from X'
6.          C' := C' ∪ γ'
7.          for each x ∈ χ do
8.              C' := C' ∪ {x ≠ x'}
9.          if (X ∪ X', D, C') is satisfiable then
10.             Let α, β be the two assignments found in a solution to Θ'
11.             return (α, β)
12.         end if
13.     end for
14. end for
```

Fig. 9. Algorithm for MAX HAMMING DISTANCE (d, l)-CSP.

Theorem 2. *If we can solve (d, l)-CSP in $\mathcal{O}(a^n)$, then there exists an algorithm for* MAX HAMMING DISTANCE *(d, l)-CSP which runs in $\mathcal{O}((a(1+a))^n)$.*

Proof. In the algorithm presented in Fig. 9, the instance Θ' will contain $2n - k$ variables, and there are $\binom{n}{k}$ ways of choosing χ. Consequently, given that we can solve (d, l)-CSP in $\mathcal{O}(a^n)$, the algorithm has a total running time of

$$\mathcal{O}\left(\sum_{k=0}^{n}\binom{n}{k}a^{2n-k}\right) = \mathcal{O}\left(a^n\sum_{k=0}^{n}\binom{n}{k}a^{n-k}\right) = \mathcal{O}\left(a^n(1+a)^n\right)$$

and the result follows. □

In Example 4 we saw how the algorithm for MAX HAMMING DISTANCE (d, l)-CSP worked. On instances of l-SAT we can actually do better than this. Since there are only two possible domain values, and we force x' to always assume the opposite of x, there is no reason to create new variables. Instead, we duplicate the clauses containing variables on which the two solutions should differ, and among these clauses, we replace every literal containing one of these variables with its negation. In the example, we would get:

$$(x \vee y) \wedge (\neg x \vee z) \wedge (\neg x \vee \neg y) \wedge (x \vee \neg z)$$

This formula has a solution $\{x \mapsto 0, y \mapsto 1, z \mapsto 0\}$, and we can easily derive a solution to the original formula which differ on the assignment of all three variables.

As can be seen in Fig. 10, the algorithm for MAX HAMMING DISTANCE $(2, l)$-CSP is similar to the one for the general case, but it does not add any variables to the problem.

algorithm MAX HAMMING DISTANCE $(2,l)$-CSP (F)

```
1.  for k := |Var(F)| down to 0 do
2.      for each χ ⊆ Var(F) with |χ| = k do
3.          Let γ be all the clauses of F containing variables from χ
4.          Create γ by negating all occurrences of a variable x ∈ χ in γ
5.          Let F+ := F ∪ {γ}
6.          if F+ is satisfiable then
7.              Let α, β be the two found in a solution to F+.
8.              return (α, β)
9.          end if
10.     end for
11. end for
```

Fig. 10. The MAX HAMMING DISTANCE $(2,l)$-CSP algorithm.

Theorem 3. *If we can solve $(2,l)$-CSP in $\mathcal{O}(a^n)$, then there exists an algorithm for solving* MAX HAMMING DISTANCE *$(2,l)$-CSP which runs in $\mathcal{O}((2a)^n)$.*

Proof. The algorithm in Fig. 10 considers all subsets of variables of the problem, as discussed in this section. Consequently, it will deliver a solution in $\mathcal{O}((2a)^n)$ time. □

There exist a number of algorithms for special cases of (d,l)-CSPs, and we can use them in conjunction with Theorems 2 and 3 to get the following corollary:

Corollary 2. *There exist algorithms for solving* MAX HAMMING DISTANCE *(d,l)-CSP with running times*

1. $\mathcal{O}(3.2264^n)$ *for* $d = 3, l = 2$,
2. $\mathcal{O}((d(0.4518 + 0.2042d))^n)$ *for* $4 \leq d \leq 10, l = 2$,
3. $\mathcal{O}\left((d!^{1/d}(1 + d!^{1/d}))^n\right)$ *for* $d \geq 11, l = 2$,
4. $\mathcal{O}(2.6604^n)$ *for* $d = 2, l = 3$,
5. $\mathcal{O}((4 - 4/l + \epsilon)^n)$, *for* $d = 2, l \geq 4$,
6. $\mathcal{O}\left(((d - d/l)^2 + d - d/l + \epsilon)^n\right)$ *for* $d \geq 3, l \geq 5$.

where $\epsilon > 0$ is an arbitrarily small constant.

Proof. Combine either of Theorems 2 and 3 with

1. the $(3,2)$-CSP algorithm by Eppstein [5],
2. the $(d,2)$-CSP algorithm by Eppstein [5],
3. the $(d,2)$-CSP algorithm by Feder & Motwani [6],
4. the 3-SAT algorithm by Hofmeister *et al.* [8],
5. the l-SAT algorithm by Schöning [11],
6. the (d,l)-CSP algorithm by Schöning [11].

and the result follows.

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