Approximability of the Maximum Solution Problem for Certain Families of Algebras

Peter Jonsson⋆ and Johan Thapper

Department of Computer and Information Science
Linköpings universitet
SE-581 83 Linköping, Sweden
{petej, johth}@ida.liu.se

Abstract. We study the approximability of the *maximum solution problem*. This problem is an optimisation variant of the constraint satisfaction problem and it captures a wide range of interesting problems in, for example, integer programming, equation solving, and graph theory. The approximability of this problem has previously been studied in the two-element case [Khanna et al, 'The approximability of constraint satisfaction', SIAM Journal on Computing 23(6), 2000] and in some algebraically motivated cases [Jonsson et al, 'MAX ONES generalized to larger domains', SIAM Journal on Computing 38(1), 2008]. We continue this line of research by considering the approximability of MAX SOL for different types of constraints. Our investigation combined with the older results strengthens the hypothesis that MAX SOL exhibits a pentachotomy with respect to approximability.

Keywords: optimisation, approximability, constraint satisfaction, algebra, computational complexity

1 Introduction

We study the *maximum solution problem*, a problem perhaps most intuitively described as a generalisation of MAX ONES. The latter problem is that of finding an assignment from a set of variables to a domain $\{0, 1\}$ such that a given set of constraints are satisfied and the number of variables assigned to 1 are maximised. This type of constraint satisfaction problems are commonly parametrised by a *constraint language* Γ, i.e., a set of relations describing the structure of the constraints that are allowed to appear. Among the problems realisable by MAX ONES(Γ) one finds the MAX INDEPENDENT SET-problem for graphs and certain variants of MAX 0/1 PROGRAMMING. The maximum solution problem, or MAX SOL(Γ) for short, generalises the domain of the variable assignment from $\{0, 1\}$ to an arbitrary finite subset of the natural numbers. The measure of a solution is now the sum of a variable weight times its assigned value, taken over all variables. This allows us to capture a wider array of problems, including certain problems in integer linear programming, problems in multi-valued logic [8], and in equation solving over various algebraic structures [11]. The problem has also been studied (with respect to computational complexity) on undirected graphs [9], i.e., when Γ consists

⋆ Partially supported by the *Center for Industrial Information Technology* (CENIIT) under grant 04.01 and by the *Swedish Research Council* (VR) under grant 2006-4532.

of a single, symmetric binary relation. However, it is quite obvious that the systematic study of MAX SOL is still in its infancy; for example, given an arbitrary constraint language Γ, no plausible conjecture has been suggested for the approximability (or even the complexity) of MAX SOL(Γ).

The situation is quite different for MAX ONES: for any constraint language Γ, MAX ONES(Γ) is either polynomial-time solvable, **APX**-complete, **poly-APX**-complete, it is **NP**-hard to obtain a solution of nonzero measure, or it is **NP**-hard to obtain any solution. This classification and the borderlines between the different cases were presented by Khanna et al. in [10]. For MAX SOL(Γ), a similar classification of approximability for homogeneous constraint languages Γ was obtained in [7], together with a (conjectured complete) classification of the approximability for maximal constraint languages. Such classifications are obviously interesting from a theoretical point of view, but also from a practical point of view, where they can help identifying families of tractable constraints and algorithms for them.

A constraint language can be extended in a complexity-preserving way to a larger set of relations called a *relational clone*. Furthermore, the relational clones can be described by algebraic operations so it makes sense talking about MAX SOL$(\mathbf{A})$ where $\mathbf{A}$ is an algebra. In this way, we can consider complexity-theoretic problems from an algebraic angle and this algebraic approach [3, 6] has proved to be very fruitful when studying constraint problems. The study of MAX SOL using the algebraic approach was initiated in [7] and we continue the algebraic study of MAX SOL in this paper. We begin by providing approximability results for certain affine algebras and 2-element algebras; these results are used extensively in the 'main' classification results. These results appear to be useful for studying other algebras as well. For instance, a classification of para-primal algebras probably hinges on a classification of all affine algebras (via Corollary 4.12 in [14]). Our proof is based on combinatorial properties in the subspace lattices of finite vector spaces.

The first classification determines the approximability of MAX SOL$(\mathbf{A})$ when $\mathbf{A}$ is a strictly simple surjective algebra. Such an algebra has a very 'simple' structure: all its smaller homomorphic images and all its proper subalgebras are one-element. These algebras can be viewed as building blocks for more complex algebras and they are well-studied in the literature; an understanding of such algebras is probably needed in order to make further progress using the algebraic approach. We note, for example, that the proof of our second classification result is partly based on the results for strictly simple surjective algebras. Concrete examples include when $\mathbf{A}$ is a finite field of prime order or a Post algebra. Furthermore, these algebras generalise the two-element case nicely since every surjective two-element algebra is strictly simple. Our proof is based on Szendrei's characterisation of strictly simple surjective algebras. In each case of the characterisation, we can either use results from [7] or our new results for affine and 2-element algebras in order to determine the approximability. The corresponding classification of the CSP problem was carried out in [3].

The second classification considers algebras that are symmetric in the sense of [15]. Examples include algebras whose automorphism group contains the alternating group (i.e. the permutation group containing only even permutations) and certain three-element algebras with cyclic automorphism groups [15]. Well-known examples are the *homogeneous* algebras; an algebra $\mathbf{A}$ is homogeneous if its automorphism group $\mathrm{Aut}(\mathbf{A})$ is the full symmetric group. The approximability of MAX SOL($\mathbf{A}$) is known for all homogeneous algebras [7] and our result generalises this result. The proof is basically a mix of Szendrei's classification for symmetric algebras [15] and our approximability results for affine and strictly simple surjective algebras. It should be noted that our proof is considerably simpler than the original proof for homogeneous algebras (which is a fairly tedious case analysis). As a by-product of the proof, we also get a classification of CSP($\mathbf{A}$) for symmetric algebras (Theorem 12.)

In order to concretise, consider the equation $x = y + 1 \pmod 3$ over the domain $A = \{0, 1, 2\}$. For brevity, we define $R = \{(x, y) \mid x = y + 1 \pmod 3\}$ and note that $R = \{(x, \sigma(x)) \mid x \in A\}$ for the permutation $\sigma(0) = 2$, $\sigma(1) = 0$, and $\sigma(2) = 1$. Let Γ be *any* relational clone containing R. It is known that for every permutation $\pi : A \to A$, $\{(x, \pi(x)) \mid x \in A\} \in \Gamma$ if and only if $\pi(x) \in \mathrm{Aut}(\mathsf{Pol}(\Gamma))$ where $\mathsf{Pol}(\Gamma)$ denotes the algebra with universe A and the functions that preserve Γ. It is now easy to see that $\mathrm{Aut}(\mathsf{Pol}(\Gamma))$ contains every even permutation on A: the identity is always an automorphism and σ^{-1} is generated by σ. Thus, $\mathsf{Pol}(\Gamma)$ is symmetric and the approximability of MAX SOL(Γ) can be determined using Theorem 11.

2 Preliminaries

This section is divided into two parts: we begin by giving the formal definition of the constraint satisfaction and the maximum solution problems, and continue by reviewing algebraic techniques for analysing relations. We will assume basic familiarity with complexity and approximability classes (such as **PO**, **NPO**, **APX** and **poly-APX**), and reductions (such as AP-, S-, and L-reductions) [1, 10].

We formally define constraint satisfaction as follows: let A (*the domain*) be a finite set and let R_A denote the set of all finitary relations over A. A constraint language A is a subset $\Gamma \subseteq R_A$. The constraint satisfaction problem over the constraint language Γ, denoted CSP(Γ), is the decision problem with instance (V, A, C). Here, V is a set of variables, A is a finite set of values, and C is a set of constraints $\{C_1, \ldots, C_q\}$, in which each constraint C_i is a pair (s_i, ϱ_i) with s_i a list of variables of length m_i, called the constraint scope, and ϱ_i an m_i-ary relation over the set A, belonging to Γ, called the constraint relation. The question is whether or not there exists a function from V to A such that, for each constraint in C, the image of the constraint scope is a member of the constraint relation.

We define the maximum solution problem over a constraint language Γ (MAX SOL(Γ)) as the maximisation problem with

Instance A tuple (V, A, C, w), where A is a finite subset of $\mathbb{N}$, (V, A, C) is a CSP instance over Γ, and $w : V \to \mathbb{Q}^+$ is a weight function.
Solution An assignment $f : V \to A$ such that all constraints are satisfied.
Measure $\sum_{v \in V} w(v) \cdot f(v)$

Next, we consider clones and operations. As usual, let A be a domain. An operation on A is an arbitrary function $f : A^k \to A$ and the set of all finitary operations on A is denoted by O_A. A k-ary operation $f \in O_A$ can be extended to an operation on n-tuples $\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots, \boldsymbol{t_k}$ by $f(\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots, \boldsymbol{t_k}) =$

$$(f(\boldsymbol{t_1}[1], \boldsymbol{t_2}[1], \ldots, \boldsymbol{t_k}[1]), f(\boldsymbol{t_1}[2], \boldsymbol{t_2}[2], \ldots, \boldsymbol{t_k}[2]), \ldots f(\boldsymbol{t_1}[n], \boldsymbol{t_2}[n], \ldots, \boldsymbol{t_k}[n])),$$

where $\boldsymbol{t_j}[i]$ is the i-th component of $\boldsymbol{t_j}$. Let $\varrho \in R_A$. If f is an operation such that for all $\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots, \boldsymbol{t_k} \in \varrho_i$ $f(\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots, \boldsymbol{t_k}) \in \varrho_i$, then ϱ is *preserved* by f. If all constraint relations in Γ are preserved by f, then Γ is preserved by f. An operation f which preserves Γ is called a polymorphism of Γ and the set of polymorphisms is denoted $\mathsf{Pol}(\Gamma)$. Given a set of operations F, the set of all relations that are preserved by the operations in F is denoted $\mathsf{Inv}(F)$.

Sets of operations of the form $\mathsf{Pol}(\Gamma)$ are known as *clones*, and they are well-studied objects in algebra (cf. [12, 14]). We remark that the operators Inv and Pol form a Galois correspondence between the set of relations over A and the set of operations on A. A comprehensive study of this correspondence can be found in [12].

A first-order formula φ over a constraint language Γ is said to be *primitive positive* (we say φ is a pp-formula for short) if it is of the form $\exists \mathbf{x} : (\varrho_1(\mathbf{x}_1) \wedge \ldots \wedge \varrho_k(\mathbf{x}_k))$ where $\varrho_1, \ldots, \varrho_k \in \Gamma$ and $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are vectors of variables of size equal to the arity of the corresponding relation. Note that a pp-formula φ with m free variables defines an m-ary relation $\varrho \subseteq A^m$; ϱ is the set of all tuples satisfying the formula φ.

We define a closure operation $\langle \cdot \rangle$ such that $\varrho \in \langle \Gamma \rangle$ if and only if the relation ϱ can be obtained from Γ by pp-formulas. Sets of relations of the form $\langle \Gamma \rangle$ are called *relational clones*. The following theorem states that we have access to a handy S-reduction from Max Sol over finite subsets of $\langle \Gamma \rangle$ to Max Sol over Γ itself.

Theorem 1 ([7]). *Let Γ be a constraint language and $\Gamma' \subseteq \langle \Gamma \rangle$ a finite subset. Then,* Max Sol(Γ') *is S-reducible to* Max Sol(Γ).

The concept of a core of a constraint language Γ has previously shown its value when classifying the complexity of CSP(Γ). The analogous concept of a *max-core* for the optimization problem Max Sol(Γ) was defined in [8]: a constraint language Γ is a max-core if and only if there is no noninjective unary operation f in $\mathsf{Pol}(\Gamma)$ such that $f(a) \geq a$ for all $a \in A$. A constraint language Γ' is a max-core of Γ if and only if Γ' is a max-core and $\Gamma' = f(\Gamma)$ for some unary operation $f \in \mathsf{Pol}(\Gamma)$ such that $f(a) \geq a$ for all $a \in A$. We have the following result:

Lemma 2 ([8]). *If Γ' is a max-core of Γ, then* Max Sol(Γ) *and* Max Sol(Γ') *are equivalent under S-reductions.*

We will now introduce the concept of an algebra and some of the terminology related to it. For a coherent treatment of this subject, we refer to [14]. Let A be a domain. An *algebra* $\mathbf{A}$ over A is a tuple $(A; F)$, where $F \subseteq O_A$ is a family of operations on A. For the purposes of this paper, all algebras will be finite, i.e., the set A will be finite. An operation $f \in O_A$ is called a *term operation* of $\mathbf{A}$ if $f \in \mathsf{Pol}(\mathsf{Inv}(F))$. The set of all term operations of $\mathbf{A}$ will be denoted by $\mathsf{Term}(\mathbf{A})$. Two algebras over the same universe are called *term equivalent* if they have the same set of term operations. An operation f is called *idempotent* if $f(a, \ldots, a) = a$ for all $a \in A$. The set of all idempotent term operations of $\mathbf{A}$ will be denoted by $\mathsf{Term}_{id}(\mathbf{A})$. For a domain A, let $\mathcal{C}_A$ denote the constraint language consisting of all constant, unary constraints, i.e., $\mathcal{C}_A = \{\{(a)\} \mid a \in A\}$. We use algebras to specify constraint languages and we will often write MAX SOL$(\mathbf{A})$ for the problem MAX SOL$(\mathsf{Inv}(\mathsf{Term}(\mathbf{A})))$.

Next, we present some operations that will be important in the sequel. If $\overline{A}$ is an abelian group, then the *affine operation* $a_{\overline{A}}(a, b, c) : A^3 \to A$ satisfies $a_{\overline{A}}(a, b, c) = a - b + c$. The *discriminator operation* $t : A^3 \to A$ satisfies $t(a, b, c) = c$ if $a = b$ and $t(a, b, c) = a$ otherwise. The *dual discriminator operation* $d : A^3 \to A$ satisfies $d(a, b, c) = a$ if $a = b$ and $d(a, b, c) = c$ otherwise. Finally, the *switching operation* $s : A^3 \to A$ satisfies $s(a, b, c) = c$ if $a = b$, $s(a, b, c) = b$ if $a = c$, and $s(a, b, c) = a$ otherwise. We remind the reader that CSP$(\mathsf{Inv}(r)) \in \mathbf{P}$ when $r \in \{a_{\overline{A}}, t, d, s\}$, cf. [3, 6]. The following proposition is a summary of the results from [7] which we will need:

Proposition 3. *(1)* MAX SOL$(\mathsf{Inv}(t))$ *is in* **PO**. *(2) Let* $R = \{(a, a), (a, b), (b, a)\}$ *with* $a, b \in A$ *and* $0 < a < b$. *Then,* MAX SOL$(\{R\})$ *is* **APX**-*complete. (3) Let* $R = \{(0, 0), (0, b), (b, 0)\}$ *with* $b \in A$ *and* $0 < b$. *Then,* MAX SOL$(\{R\})$ *is* **poly-APX**-*complete.*

3 Affine algebras and two-element algebras

In this section, we look at certain affine algebras and constraint languages over 2-element domains. Let $\overline{A} = (A; +)$ be a finite abelian group. The finite-dimensional vector space on $\overline{A}$ over K will be denoted ${}_K\overline{A} = (A; +, K)$. An algebra $\mathbf{A}$ is said to be *affine with respect to an abelian group* $\overline{A}$ if (1) $\mathbf{A}$ and $\overline{A}$ have the same universe, (2) the 4-ary relation $Q_{\overline{A}} = \{(a, b, c, d) \in A^4 \mid a - b + c = d\}$ is in $\mathsf{Inv}(\mathbf{A})$, and (3) $a_{\overline{A}}$ is a term operation of $\mathbf{A}$. It is known that MAX SOL$(\mathbf{A})$ is in **APX** for all affine algebras $\mathbf{A}$ and that MAX SOL$(\mathbf{A})$ is **APX**-complete for the affine algebra $\mathbf{A} = (A; a_{\overline{A}})$ [7]. Here, we will extend the latter result to cover some affine algebras with a larger set of term operations, where the underlying group is a finite vector space.

Let ${}_K\overline{A}$ be an n-dimensional vector space over a finite field K of size q. Let $\Lambda_0({}_K\overline{A})$ be the constraint language consisting of all relations $\{(x_1, \ldots, x_n) \mid \sum_{i=1}^{n} c_i x_i = d\}$, for some $c_i \in K, d \in A$ and with $\sum_{i=1}^{n} c_i = 0$. The algebra over A with operations $\mathsf{Pol}(\Lambda_0({}_K\overline{A}))$ is affine and we have the following result:

Theorem 4. MAX SOL$(\Lambda_0({}_K\overline{A}))$ *is* **APX**-*hard for any finite dimensional vector space* ${}_K\overline{A}$ *over a finite field* K *of size* $q \geq 2$.

The proof of this theorem relies on Lemma 5 which will be presented below. The lemma uses the subspace structure in finite vector spaces and we will need some terminology and notation: an *affine hyperplane* in ${}_K\overline{A}$ is a coset $a + S$, where $a \in A$ and S is a codimension 1 subspace of ${}_K\overline{A}$. Let $\mathcal{H}$ be the set of affine hyperplanes in ${}_K\overline{A}$. Denote by V the q^n-dimensional vector space over $\mathbb{Q}$ with basis A. For any subset $B \subseteq A$, let $\chi(B)$ denote the characteristic vector of B, i.e., $\chi(B) = \sum_{a \in B} a$. Let $g : A \to \mathbb{Q}$ be any function from A to the rational numbers. We can then extend g to a linear transformation $g : V \to \mathbb{Q}$ by letting $g(v) = \sum_i v_i g(a_i)$, when $v = \sum_i v_i a_i$. In particular, $g(\chi(B)) = \sum_{a \in B} g(a)$.

Lemma 5. *If $g(\chi(H)) = C$ for all $H \in \mathcal{H}$ and some constant C, then $g(a) = C/q^{n-1}$ for all $a \in A$.*

Proof. We will show that the set $X = \{\chi(H) \mid H \in \mathcal{H}\}$ spans V. From this it follows that g is uniquely determined by its values on X. The *q-binomial coefficients* are defined by

$$b_q(n,k) = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

They count, among other things, the number of k-dimensional subspaces in an n-dimensional vector space over a finite field of size q. The number of codimension 1 subspaces (hyperplanes) containing a fixed 1-dimensional subspace (line) is counted by $b_q(n-1,1)$ (cf. [4].) We let $a \in A$ be fixed and for each $v \in A$, $v \neq a$, we count the number of (affine) hyperplanes through a that also contain v. They are exactly the hyperplanes containing the unique line through a and v and this number is $b_q(n-1,1) = \frac{q^{n-1}-1}{q-1}$. Thus, $\sum_{a \in H \in \mathcal{H}} \chi(H) = b_q(n, n-1) \cdot a + b_q(n-1,1) \cdot \chi(A - a) = q^{n-1} \cdot a + b_q(n-1,1) \cdot \chi(A)$. Now, choose an arbitrary affine hyperplane $H^1 \in \mathcal{H}$, let $H^2, \ldots, H^q$ denote its translations and note that $\sum_{i=1}^{q} \chi(H^i) = \chi(A)$. This implies that a can be written as the following linear combination of vectors in X: $a = q^{1-n} \left(\sum_{a \in H \in \mathcal{H}} \chi(H) - b_q(n-1,1) \cdot \sum_{i=1}^{q} \chi(H^i) \right)$. Apply g to both sides and use linearity to obtain $g(a) = q^{1-n} \left(b_q(n, n-1) \cdot C - b_q(n-1,1) \cdot q \cdot C \right) = q^{1-n} \left(\frac{q^n-1}{q-1} - \frac{q^n-q}{q-1} \right) \cdot C$, from which the lemma follows. □

The proof of Theorem 4 is a reduction from the problem MAX q-CUT. In this problem, one is given a graph $G = (V, E)$, and a solution σ to G is an assignment from V to some set K of size q. The objective is to maximise the number of edges $(u, v) \in E$ such that $\sigma(u) \neq \sigma(v)$. It is well known that MAX q-CUT is **APX**-complete for $q \geq 2$.

Proof (of Theorem 4). Let ${}_K\overline{A} = (A; +, K)$ be a vector space of size $|A| = q^n$. We present an L-reduction from MAX q-CUT to MAX SOL$(\Lambda_0({}_K\overline{A}))$, which proves that the latter is **APX**-hard.

Remember that we view the set of elements, A, as a subset of the natural numbers. In order to avoid ambiguity, we will use $a + b$ and $a - b$ to denote addition and subtraction in the group and $a \oplus b$ for the addition of the values of the group elements $a, b \in A$. Define $g : A \to \mathbb{N}$ as follows:

$$g(a) = \max_{x,y \in A} \{x \oplus y \mid x - y = a\} = \max_{x \in A} \{x \oplus (x - a)\}.$$

Extend g to arbitrary subsets $B \subseteq A$ through $g(B) = \sum_{a \in B} g(a)$. Let 0_A denote the zero vector and note that $g(0_A) = 2 \cdot \max A > g(a)$ for all $a \in A \setminus \{0_A\}$. Hence, g is nonconstant on A, and by Lemma 5, g must be nonconstant on the set of affine hyperplanes in ${}_K\overline{A}$. Therefore, we can let $H \subseteq A$ be a hyperplane (through the origin) in ${}_K\overline{A}$ and $e \in A$ an element such that $g(e + H) = \min\{g(a + H) \mid a \in A\} < g(A)/q$.

Let $I = (V, E)$ be an instance of MAX q-CUT. We create an instance $F(I)$ of MAX SOL$(\Lambda_0({}_K\overline{A}))$ as follows. For each $v_i \in V$, we create $|A|/q$ variables x_i^s, $s \in H$ and add the $(|A|/q) - 1$ equations $x_i^s - x_i^{0_A} = s$ to $F(I)$, for $s \in H \setminus \{0_A\}$. These equations ensure that in a solution σ' to $F(I)$ it must hold that $\{\sigma'(x_i^s) \mid s \in H\} = a + H$ for some $a \in A$. For each edge $(v_i, v_j) \in E$, we create $2|A|^2/q$ new variables $y_{ij}^{st,k}$ and $z_{ij}^{st,k}$ for $s, t \in H$, $k \in K$, and add the following $|A|^2/q$ equations:

$$e + k(x_i^s - x_j^t) = y_{ij}^{st,k} - z_{ij}^{st,k}. \tag{1}$$

Finally, we let $w(x_i^s) = 0$ and $w(y_{ij}^{(s,t)}) = w(z_{ij}^{(s,t)}) = 1$. From a solution σ' to $F(I)$, we derive a solution σ to I by fixing an element $h_\perp \in A \setminus H$ and letting $\sigma(v_i) = k \in K$ when $\{\sigma'(x_i^s) \mid s \in H\} = kh_\perp + H$. Note that the measure of σ is independent of the choice of $h_\perp$.

We will now determine the measure of σ'. Note that in any solution, due to (1) and the definition of g, we have

$$\sigma'(y_{ij}^{st,k}) \oplus \sigma'(z_{ij}^{st,k}) \leq g(e + k(\sigma'(x_i^s) - \sigma'(x_j^t))). \tag{2}$$

Assume that $\{\sigma'(x_i^s) \mid s \in H\} = a + H$ and $\{\sigma'(x_j^t) \mid t \in H\} = b + H$. Then,

$$\{e + k(\sigma'(x_i^s) - \sigma'(x_j^t)) \mid s \in H\} = e + k(a - b) + H,$$

for any fixed $t \in H$ and $k \in K$. Therefore,

$$\sum_{s,t \in H, k \in K} \sigma'(y_{ij}^{st,k}) \oplus \sigma'(z_{ij}^{st,k}) \leq \sum_{s,t \in H, k \in K} g(e + k(\sigma'(x_i^s) - \sigma'(x_j^t))) = q^{n-1} \sum_{k \in K} g(e + k(a - b) + H)$$

If $\sigma(v_i) = \sigma(v_j)$, then $a - b \in H$, so the right-hand side equals $C = q^n g(e + H)$. Otherwise, $a - b \notin H$ and the right-hand side equals $D = q^{n-1} g(A)$. Now, assume that the q-cut determined by σ contains $m(\sigma)$ edges. Then, the measure m' of the solution σ' to $F(I)$ is bounded by

$$m'(\sigma') \leq |E| \cdot C + m(\sigma) \cdot (D - C). \tag{3}$$

When σ' is an optimal solution, the inequality in (2) can be replaced by an equality and it follows that

$$\text{OPT}(F(I)) = |E| \cdot C + \text{OPT}(I) \cdot (D - C). \tag{4}$$

By a straightforward probabilistic argument, it follows that $\text{OPT}(I) \geq |E| \cdot (1 - 1/q)$, which in turn implies that

$$\text{OPT}(F(I)) = \text{OPT}(I) \left(\frac{|E| \cdot C}{\text{OPT}(I)} + (D - C) \right) \leq \text{OPT}(I) \cdot (C/(q-1) + D). \tag{5}$$

Note that both C and D are independent of the instance I. By subtracting (3) from (4) we get

$$\text{OPT}(F(I)) - m'(\sigma') \geq (\text{OPT}(I) - m(\sigma)) \cdot (D - C). \tag{6}$$

By the choice of H and e, we have $C = q^n g(e + H) < q^n g(A)/q = q^{n-1} g(A) = D$. Consequently, (5) and (6) shows that F is the desired L-reduction. □

The following consequence of Theorem 4 will be needed in the forthcoming proofs. The endomorphism ring of ${}_K\overline{A}$, i.e., the ring of linear transformations on ${}_K\overline{A}$, will be denoted $\mathsf{End}\ {}_K\overline{A}$. One can consider $\overline{A}$ as a module over $\mathsf{End}\ {}_K\overline{A}$ and this module will be denoted ${}_{(\mathsf{End}\ {}_K\overline{A})}\overline{A}$. The group of translations $\{x+a \mid a \in A\}$ will be denoted $T(\overline{A})$.

Corollary 6. *Let ${}_K\overline{A}$ be a finite dimensional vector space over a finite field K. Then,* MAX SOL$(\mathsf{Inv}(\mathsf{Term}_{id}({}_K\overline{A})))$ *and* MAX SOL$(\mathsf{Inv}(\mathsf{Term}_{id}({}_{(\mathsf{End}\ {}_K\overline{A})}\overline{A}) \cup T(\overline{A})))$ *are* **APX***-complete.*

We now use Theorem 4 to extend the classification of MAX ONES(Γ) by Khanna et al. [10]; this result will be needed several times in the sequel. Khanna et al. have proved a complete classification result for $D = \{0, 1\}$ and their proof is easy to generalise to the case when $D = \{0, a\}$, $a > 0$. If $D = \{a, b\}$, $0 < a < b$, then it is possible to exploit *Post's lattice* [13] for proving a similar result. The next lemma follows without difficulties by combining this lattice with results from Proposition 3 and Theorem 4.

Lemma 7. *Let Γ be a constraint language over a 2-element domain. Then,* MAX SOL(Γ) *is either in* **PO***,* **APX***-complete,* **poly-APX***-complete, it is* **NP***-hard to find a nonzero solution or it is* **NP***-hard to find a feasible solution.*

4 Strictly Simple Surjective Algebras

The strictly simple surjective algebras were classified by Szendrei in [16], and the complexity of constraint satisfaction over such algebras was studied in [3]. Here, we do the corresponding classification of the approximability of MAX SOL. First, we will need a few definitions to be able to state Szendrei's theorem.

Let $\mathbf{A} = (A; F)$ be an algebra and let $B \subseteq A$. Let $f|_B$ denote the restriction of f to B and let $F|_B = \{f|_B \mid f \in F\}$. If for every $f \in F$, it holds that $f|_B(B) \subseteq B$, then $\mathbf{B} = (B; F|_B)$ is called a *subalgebra* of $\mathbf{A}$ and B is said to *support* this subalgebra. If $|B| < |A|$, then $\mathbf{B}$ is called a *proper subalgebra* of $\mathbf{A}$.

Let I be an index set and let $\mathbf{A} = (A; F)$ and $\mathbf{B} = (B; F')$ be two algebras with $F = \{f_i : i \in I\}$ and $F' = \{f'_i : i \in I\}$ such that f_i and f'_i have the same arity k_i for all $i \in I$. Then, a map $h : A \to B$ is called a *homomorphism* if for all $i \in I$, $h(f_i(a_1, \ldots, a_{k_i})) = f'_i(h(a_1), \ldots, h(a_{k_i}))$. When h is surjective, $\mathbf{B}$ is called a *homomorphic image* of $\mathbf{A}$. An algebra is called *simple* if all its smaller homomorphic images are trivial (one-element) and *strictly simple* if, in addition, all its proper subalgebras are one-element. An algebra is called *surjective* if all of its term operations are surjective.

Let $\mathcal{I}$ be a family of bijections between subsets of a set A. By $\mathcal{R}(\mathcal{I})$ we denote the set of operations on A which preserve each relation of the form $\{(a, \pi(a)) \mid a \in A\}$ for $\pi \in \mathcal{I}$. By $\mathcal{R}_{id}(\mathcal{I})$ we denote the set of idempotent operations in $\mathcal{R}(\mathcal{I})$.

Let G be a permutation group on A. Then, G is called *transitive* if, for any $a, b \in A$, there exists $g \in G$ such that $g(a) = b$. G is called *regular* if it is transitive and each nonidentity member has no fixed point. G is called *primitive*, or is said to act primitively on A, if it is transitive and the algebra $(A; G)$ is simple.

Let a be some fixed element in A, and define the relation $X_k^a = \{(a_1, \ldots, a_k) \in A^k \mid a_i = a \text{ for at least one } i, 1 \leq i \leq k\}$. Let $\mathcal{F}_k^a$ denote the set of all operations preserving X_k^a, and let $\mathcal{F}_\omega^a = \bigcap_{k=2}^{\infty} \mathcal{F}_k^a$.

Theorem 8 **([16]).** *Let* $\mathbf{A}$ *be a finite strictly simple surjective algebra. If* $\mathbf{A}$ *has no one-element subalgebras, then* $\mathbf{A}$ *is term equivalent to one of the following: (a)* $(A; \mathcal{R}(G))$ *for a regular permutation group* G *acting on* A*; (b)* $(A; \mathsf{Term}_{id}(_{(\mathsf{End}\ _K\overline{A})}\overline{A}) \cup T(\overline{A}))$ *for some vector space* $_K\overline{A} = (A; +, K)$ *over a finite field* K*; or (c)* (A, G) *for a primitive permutation group* G *on* A*. If* $\mathbf{A}$ *has one-element subalgebras, then* $\mathbf{A}$ *is idempotent and term equivalent to one of the following algebras:*

- *(a°)* $(A; \mathcal{R}_{id}(G))$ *for a permutation group* G *on* A *such that every nonidentity member of* G *has at most one fixed point;*
- *(b°)* $(A; \mathsf{Term}_{id}(_{(\mathsf{End}\ _K\overline{A})}\overline{A}))$ *for some vector space* $_K\overline{A}$ *over a finite field* K*;*
- *(d)* $(A; \mathcal{R}_{id}(G) \cap \mathcal{F}_k^a)$ *for some* k $(2 \leq k \leq \omega)$*, some element* $a \in A$*, and some permutation group* G *acting on* A *such that* a *is the unique fixed point of every nonidentity member of* G*;*
- *(e)* $(A; F)$ *where* $|A| = 2$ *and* F *contains a semilattice operation; or*
- *(f) a two-element algebra with an empty set of basic operations.*

Using the results from Section 3, we can give the following classification of approximability of MAX SOL$(\mathbf{A})$ for finite strictly simple surjective algebras $\mathbf{A}$.

Theorem 9. *Let* $\mathbf{A}$ *be a finite strictly simple surjective algebra. Then,* MAX SOL$(\mathbf{A})$ *is either in* **PO***, it is* **APX***-complete, it is* **poly-APX***-complete, or it is* **NP***-hard to find a solution.*

Proof (sketch). If $\mathbf{A}$ is of type (c) or (f), then CSP($\mathbf{A}$) is **NP**-complete [3]. If $\mathbf{A}$ is of type (a) or (a°), then the discriminator operation $t(x, y, z)$ is a term operation of $\mathbf{A}$ and tractability follows from Proposition 3(1). If $\mathbf{A}$ is of type (b) or (b°), then **APX**-completeness follows from Corollary 6. If $\mathbf{A}$ is of type (d), then from [2, 3, 7], one can deduce tractability when $a = \max A$, membership in **poly-APX** when $a = 0 \in A$ and membership in **APX** otherwise. To prove **APX**- and **poly-APX**-hardness in the relevant cases, note that $X_k^a \in \mathsf{Inv}(\mathbf{A})$ for some $k \geq 2$ and that $\mathbf{A}$ is idempotent. The relation $r = (A \times \{a\}) \cup (\{a\} \times A)$ is pp-definable in $\{X_k^a\} \cup \mathcal{C}_A$ via $r(x, y) \equiv_{pp} \exists z : X_k^a(x, y, z, \ldots, z) \wedge \{\max A\}(z)$ and the max-core of r is the relation $\{(a, a), (a, \max A), (\max A, a)\}$. Thus, **APX**- and **poly-APX**-hardness follows from Lemma 2 combined with Proposition 3(2) and 3(3). Finally, if $\mathbf{A} = (A; F)$ is of type (e), then since $|A| = 2$, the result follows from Lemma 7. □

5 Symmetric Algebras

A bijective homomorphism from $\mathbf{A}$ to itself is called an automorphism. An algebra $\mathbf{A}$ is *symmetric* (in the sense of Szendrei [15]) if for every subalgebra $\mathbf{B} = (B; F)$ of $\mathbf{A}$, (1) the automorphism group of $\mathbf{B}$ acts primitively on B; and (2) for any set $C \subseteq A$ with $|C| = |B|$, C supports a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{B}$. Examples of symmetric algebras include homogeneous algebras and algebras whose automorphism group contains the alternating group. Condition (2) on symmetric algebras implies that if $\mathbf{B} = (B; F|_B)$ is a proper subalgebra of $\mathbf{A}$, then $(C; F|_C)$ is a subalgebra of $\mathbf{A}$ whenever C is a subset of A with $|C| \leq |B|$. Consequently, we can assign a number

$\nu(\mathbf{A})$, $0 \leq \nu(\mathbf{A}) \leq |A| - 1$, to every symmetric algebra such that a proper subset $B \subset A$ is the universe of a subalgebra of $\mathbf{A}$ if and only if $|B| \leq \nu(\mathbf{A})$. One may note that $\nu(\mathbf{A}) \geq 1$ if and only if $\mathbf{A}$ is idempotent.

We need some notation for describing symmetric algebras: a bijective homomorphism is called an *isomorphism* and an isomorphism between two subalgebras of an algebra $\mathbf{A}$ is called an *internal isomorphism* of $\mathbf{A}$. The set of all internal isomorphisms will be denoted $\mathsf{Iso}\,\mathbf{A}$. A $k \times l$ *cross* ($k, l \geq 2$) is a relation on A^2 of the form $X(B_1, B_2, b_1, b_2) = (B_1 \times \{b_2\}) \cup (\{b_1\} \times B_2)$, where $b_1 \in B_1$, $b_2 \in B_2$, $|B_1| = k$, and $|B_2| = l$. Let $\mathcal{D}_1$ denote the clone of all idempotent operations on A, and let $\mathcal{E}_1$ denote the subclone of $\mathcal{D}_1$ consisting of all operations which in addition preserve every relation $L_{a,b} = \{(a,a,a), (a,b,b), (b,a,b), (b,b,a)\}$ where $a, b \in A$ and $a \neq b$. For $2 \leq m \leq |A|$, let $\mathcal{D}_m$ be the clone of all operations in $\mathcal{D}_1$ preserving every $m \times 2$ cross. For $2 \leq m \leq |A|$, let $\mathcal{E}_m$ be the clone consisting of all operations $f \in \mathcal{D}_1$ for which there exists a projection p agreeing with f on every m-element subset B of A.

Theorem 10 ([15]). *Let $\mathbf{A}$ be a finite symmetric algebra. If $\mathbf{A}$ is not idempotent, then $|A|$ is prime and there is a cyclic group $\overline{A} = (A; +)$ such that $\mathbf{A}$ is term equivalent to $(A; \mathcal{R}(T(\overline{A})))$, $(A; \mathsf{Term}_{id}(\overline{A}) \cup T(\overline{A}))$, or $(A; T(\overline{A}))$. If $\mathbf{A}$ is idempotent, then $\mathbf{A}$ is term equivalent to one of the following algebras:*

1. *$(A; \mathcal{R}(\mathsf{Iso}\,\mathbf{A}) \cap \mathcal{D}_m)$ for some m with $1 \leq m \leq \nu(\mathbf{A})$ or $m = |A|$;*
2. *$(A; \mathcal{R}(\mathsf{Iso}\,\mathbf{A}) \cap \mathcal{E}_m)$ for some m with $1 \leq m \leq \nu(\mathbf{A})$ or $m = |A|$;*
3. *$(A; \mathsf{Term}_{id}({}_K\overline{A}))$ for a 1-dimensional vector space ${}_K\mathbf{A} = (A; +, K)$ over a finite field K; or*
4. *$(A; \mathsf{Term}_{id}(\overline{A}))$ for a 4-element abelian group $\overline{A} = (A; +)$ of exponent 2.*

Theorem 11. *Let $\mathbf{A}$ be a symmetric algebra. Then,* MAX SOL$(\mathbf{A})$ *is either in* **PO***, it is* **APX***-complete, it is* **poly-APX***-complete, or it is* **NP***-hard to find a solution.*

Proof (sketch). If $\mathbf{A}$ is not idempotent, then one can show that $\mathbf{A}$ is strictly simple and surjective. In this case, the the result follows from Theorem 9.

Assume instead that $\mathbf{A}$ is idempotent; cases 3 and 4 are now immediately covered by Corollary 6 so we suppose that $\mathbf{A} = (A; \mathcal{R}(\mathsf{Iso}\,\mathbf{A}) \cap \mathcal{D}_m)$. If $m = 1$, then t is a term operation and tractability follows from Proposition 3(1). When $m > 1$, then d is a term operation and the membership results in **APX** and **poly-APX** follow. Furthermore, all $m \times 2$ crosses are in $\mathsf{Inv}\mathbf{A}$ and hardness can be shown by utilising max-cores.

Finally assume that $\mathbf{A} = (A; \mathcal{R}(\mathsf{Iso}\,\mathbf{A}) \cap \mathcal{E}_m)$ for some $1 \leq m \leq \nu(\mathbf{A})$ or $m = |A|$. When $m = 1$, then membership in **APX** can be shown based on the fact that s is a term operation; hardness can be proved by using Lemma 7. If $m > 1$, then $\mathbf{A}$ contains 2-element subalgebras $\{a, b\}$ such that each operation in $\mathbf{A}$ restricted to $\{a, b\}$ is a projection. Consequently, MAX SOL$(\mathbf{A})$ is **NP**-hard for all $m \geq 2$. □

By following the proof of Theorem 11, the complexity of CSP$(\mathbf{A})$ can be determined, too. We note that this result agrees with Conjecture 7.5 in [3] on the source of intractability in finite, idempotent algebras.

Theorem 12. *Let $\mathbf{A}$ be an idempotent symmetric algebra. If there exists a nontrivial homomorphic image $\mathbf{B}$ of a subalgebra of $\mathbf{A}$ such that the operations of $\mathbf{B}$ are all projections, then* CSP$(\mathsf{Inv}(\mathbf{A}))$ *is* **NP***-complete. Otherwise,* CSP$(\mathsf{Inv}(\mathbf{A}))$ *is in* **P**.

6 Discussion

The results in this paper together with the approximability classifications in [7, 10] provide support for the following conjecture: for every constraint language Γ over a finite domain $D \subseteq \mathbb{N}$, MAX SOL(Γ) is either polynomial-time solvable, **APX**-complete, **poly-APX**-complete, it is **NP**-hard to obtain a solution of nonzero measure, or it is **NP**-hard to obtain any solution. Where the exact borderlines between the cases lie is largely unknown, though, and a plausible conjecture seems remote for the moment. Therefore, we present a selection of questions that may be considered before attacking the 'main' approximability classification for MAX SOL.

It has been observed that classifying the complexity of CSP for all strictly simple algebras could be seen as a possible *'base case for induction'* [3]. This is due to the necessary condition that for a tractable algebra, all of its subalgebras and homomorphic images must be tractable. Furthermore, it is sufficient to study surjective algebras with respect to CSP since the application of a unary polymorphism to a set of relations does not change the complexity of the set [5]. For MAX SOL, however, it is possible to turn an **APX**-hard problem into a problem in **PO** by applying a unary polymorphism f, unless it satisfies some additional condition, such as $f(a) \geq a$ for all $a \in A$. It therefore looks appealing to replace the property of the algebra being surjective by that of the constraint language being a max-core. It should be noted that being a max-core is not a purely algebraic property and that we do not know how, or if, it is possible to obtain a usable characterisation of such algebras.

Kuivinen [11] has given tight inapproximability bounds (provided that **P** $\neq$ **NP**) for the problem of solving systems of equations with integer coefficients over an arbitrary abelian group. In [7], this problem was shown to be **APX**-hard for cyclic groups of prime order. Theorem 4 extends this result to show **APX**-hardness when the underlying group is a finite vector space and the sum of the coefficients is 0. The next step would be to prove **APX**-hardness for arbitrary abelian groups. We note that the proof of Theorem 4 relies on a result which utilises the subspace structure of finite vector spaces. Informally, this is needed to be able to distinguish one affine hyperplane from the average of the others. Unfortunately, it is not hard to find an abelian group on $A = \{0, 1, 2, 4, 5, 6\}$ in which the sum of the elements in all cosets of the same nontrivial subgroup is the same: let $\overline{A}$ be the abelian group defined by the isomorphism $f : \mathbb{Z}_6 \to \overline{A}$, where $f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 6, f(4) = 5$ and $f(5) = 2$. It is easy to check that if $a + H$ is a coset with $|H| > 1$, then $\sum_{a' \in a+H} a' = \frac{|H|}{|A|} \sum_{a \in A} a = 3 \cdot |H|$. A deeper analysis of these abelian groups is thus needed to settle this case.

Algebras preserving relations obtained from unrestricted systems of equations over abelian groups are full idempotent reducts of affine algebras. Corollary 6 covers some nonidempotent cases as well, due to the restriction of the coefficients in the equations defining $\Lambda_0({}_K\overline{A})$. However, in the proof of Lemma 7 we find an example of a constraint language corresponding to an affine algebra which is $\max A$-valid. The question which arises is, whether or not MAX SOL($\mathbf{A}$) can be shown to be **APX**-hard for all affine algebras $\mathbf{A}$ such that $\mathsf{Inv}(\mathbf{A})$ is not $\max A$-valid. As observed in the introduction, this is probably a key question when it comes to deciding the approximability of MAX SOL($\mathbf{A}$) when $\mathbf{A}$ is, for example, para-primal.

References

1. G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, and M. Protasi. *Complexity and Approximation*. Springer, 1999.
2. A. Bulatov. Combinatorial problems raised from 2-semilattices. *Journal of Algebra*, 298:321–339, 2006.
3. A. Bulatov, P. Jeavons, and A. Krokhin. Classifying the computational complexity of constraints using finte algebras. *SIAM Journal on Computing*, 34(3):720–742, 2005.
4. J. Goldman and G.-C. Rota. On the foundations of combinatorial theory IV. Finite vector spaces and Eulerian generating functions. *Studies in Appl. Math*, 49:239–258, 1970.
5. P. Jeavons. On the algebraic structure of combinatorial problems. *Theoretical Computer Science*, 200(1–2):185–204, 1998.
6. P. Jeavons, D. Cohen, and M. Gyssens. Closure properties of constraints. *Journal of the ACM*, 44(4):527–548, 1997.
7. P. Jonsson, F. Kuivinen, and G. Nordh. MAX ONES generalized to larger domains. *SIAM Journal on Computing*, 38(1):329–365, 2008.
8. P. Jonsson and G. Nordh. Generalised integer programming based on logically defined relations. In *Proceedings of the 31st International Symposium on Mathematical Foundations of Computer Science (MFCS-2006)*, pages 549–560, 2006.
9. P. Jonsson, G. Nordh, and J. Thapper. The maximum solution problem on graphs. In *Proceedings of 32th International Symposium on Mathematical Foundations of Computer Science (MFCS-2007)*, pages 228–239, 2007.
10. S. Khanna, M. Sudan, L. Trevisan, and D. P. Williamson. The approximability of constraint satisfaction problems. *SIAM Journal on Computing*, 30(6):1863–1920, 2000.
11. F. Kuivinen. Tight approximability results for the maximum solution equation problem over Z_p. In *Proceedings of the 30th International Symposium on Mathematical Foundations of Computer Science (MFCS-2005)*, pages 628–639, 2005.
12. R. Pöschel and L. Kalužnin. *Funktionen- und Relationenalgebren*. DVW, Berlin, 1979.
13. E. Post. The two-valued iterative systems of mathematical logic. *Annals of Mathematical Studies*, 5:1–122, 1941.
14. Á. Szendrei. *Clones in Universal Algebra*, volume 99 of *Seminaires de Mathématiques Supérieures*. University of Montreal, 1986.
15. Á. Szendrei. Symmetric algebras. In *Contributions to General Algebra 6*, pages 259–280. Verlag Hölder-Pichler-Tempsky, Wien and Verlag Teubner, Stuttgart, 1989.
16. Á. Szendrei. Simple surjective algebras having no proper subalgebras. *J. Austral. Math. Soc. ser. A*, 48:434–454, 1990.