FWF

Der Wissenschaftsfonds.

Phase transitions of block-weighted planar maps

Séminaire CALIN
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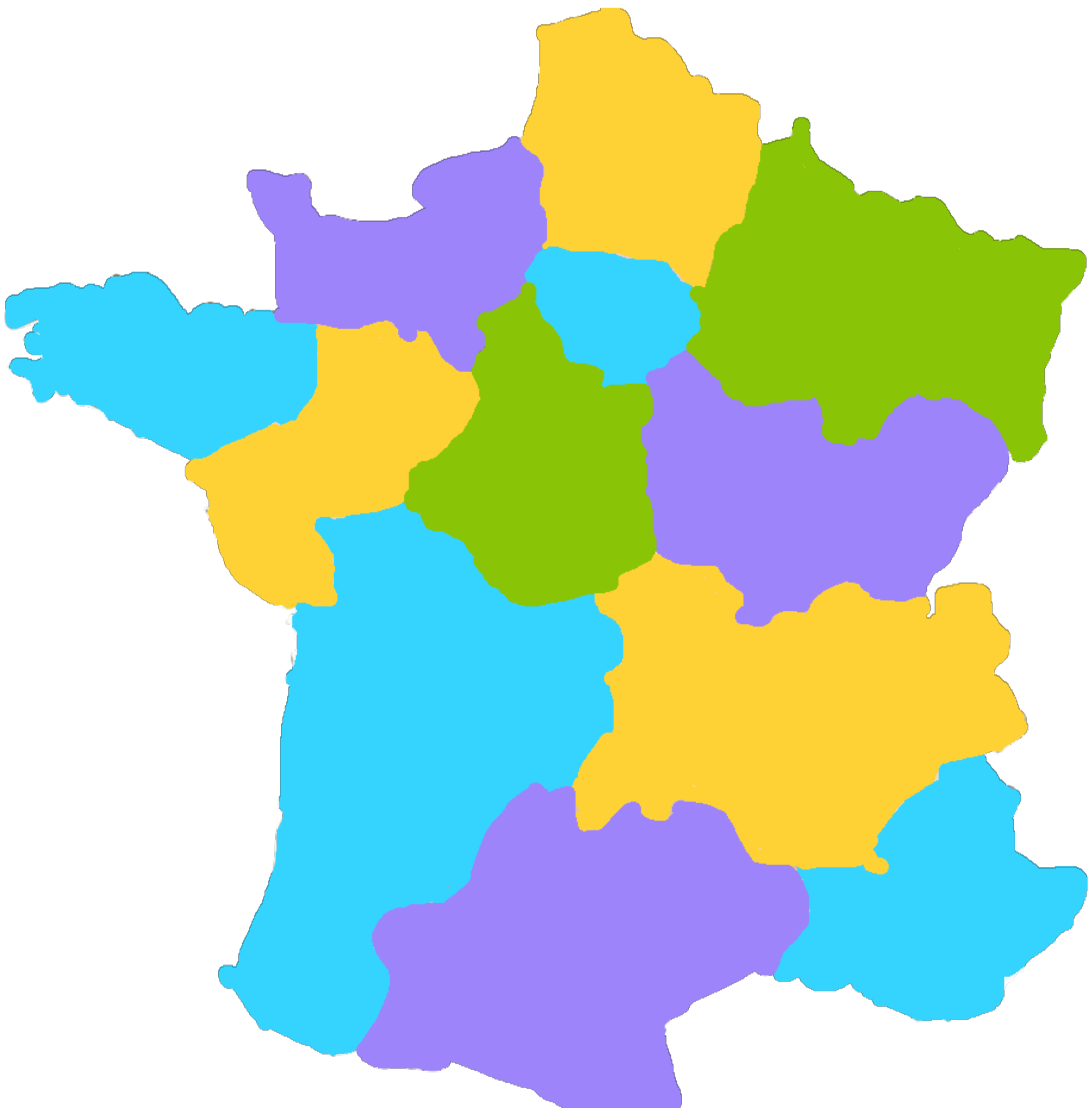
TU Wien, Austria
PhD in Université Gustave Eiffel under the supervision of Marie Albenque and Éric Fusy

Planar maps



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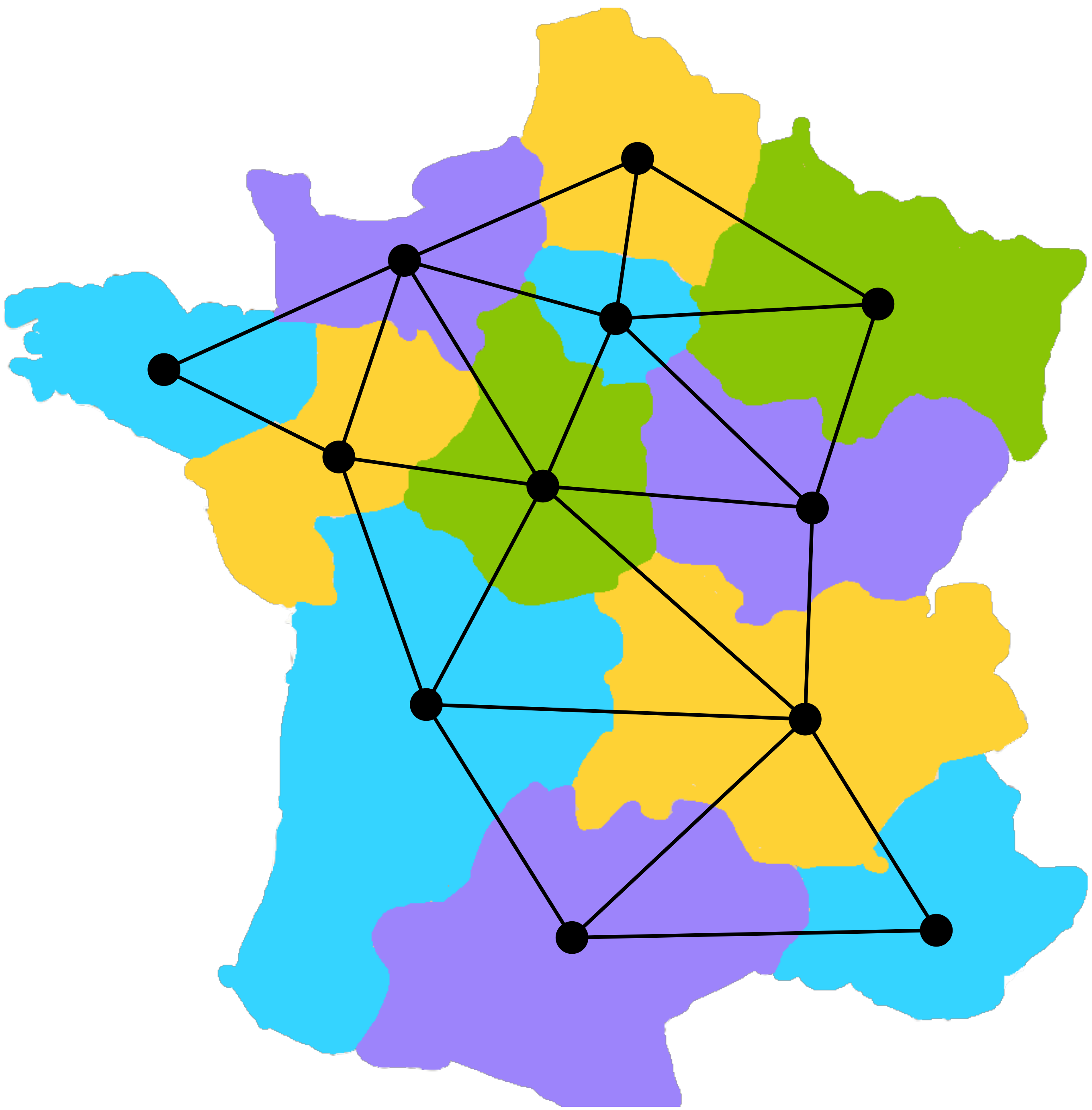
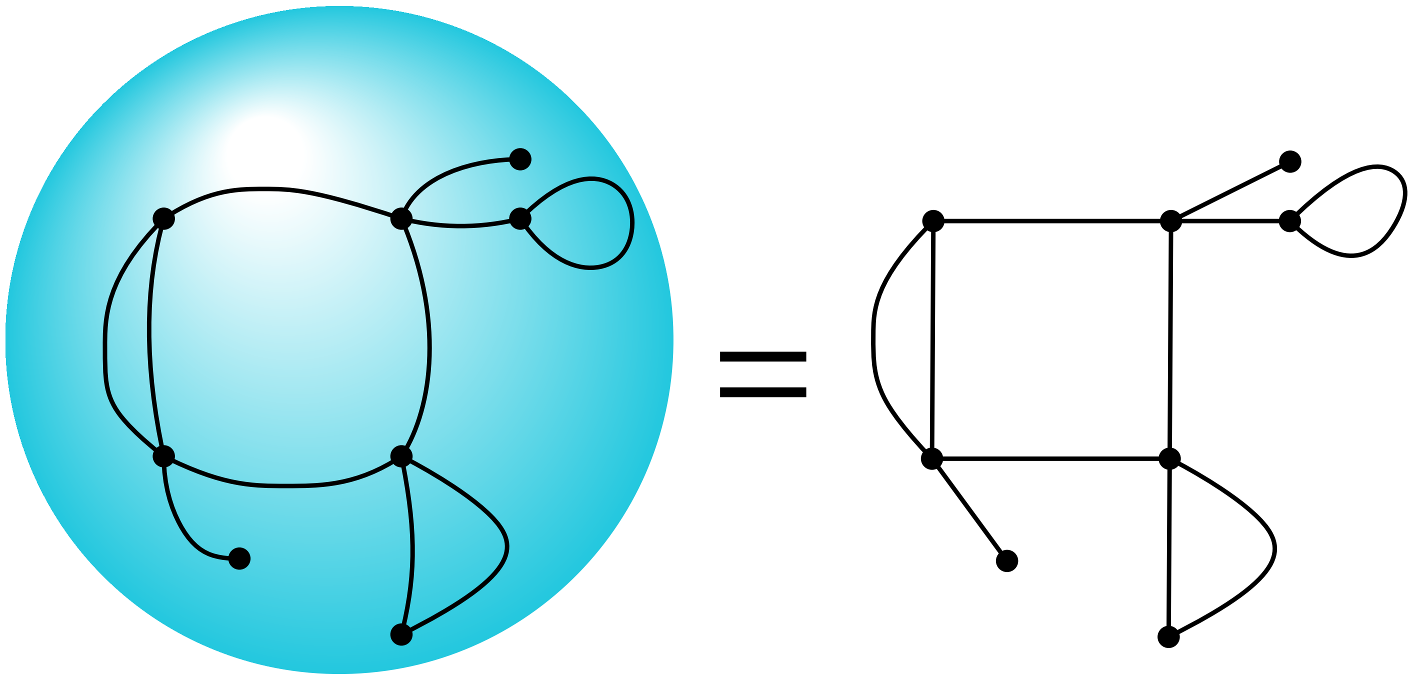


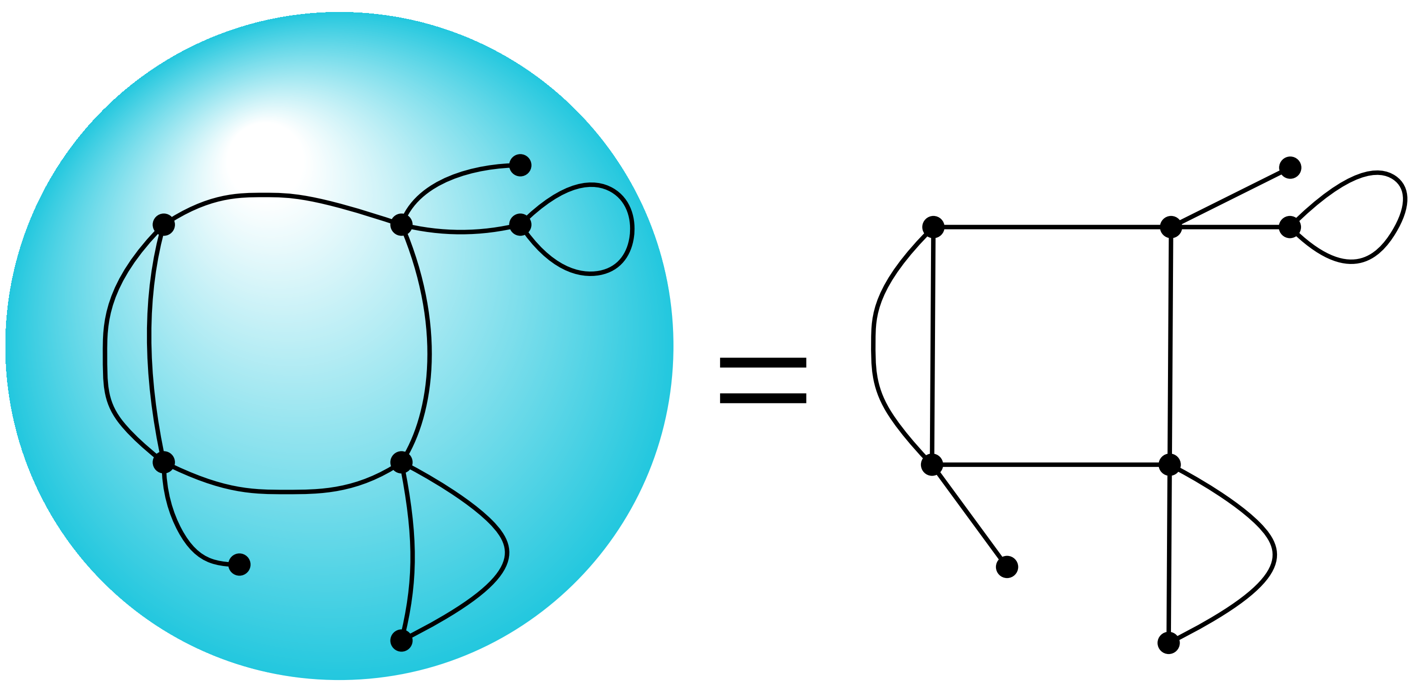
Planar maps

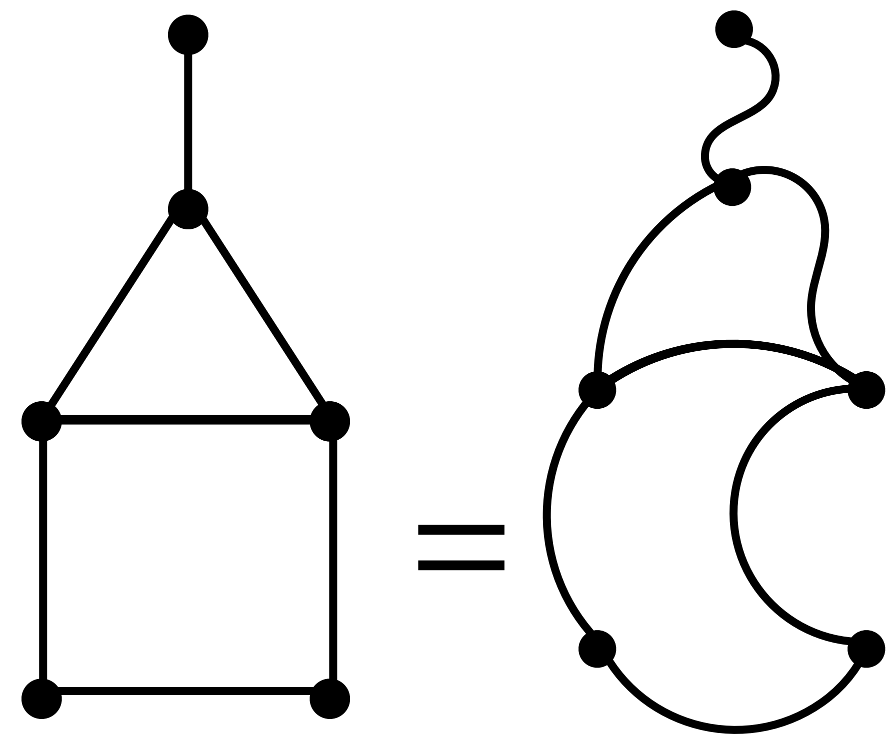
Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph,



Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms





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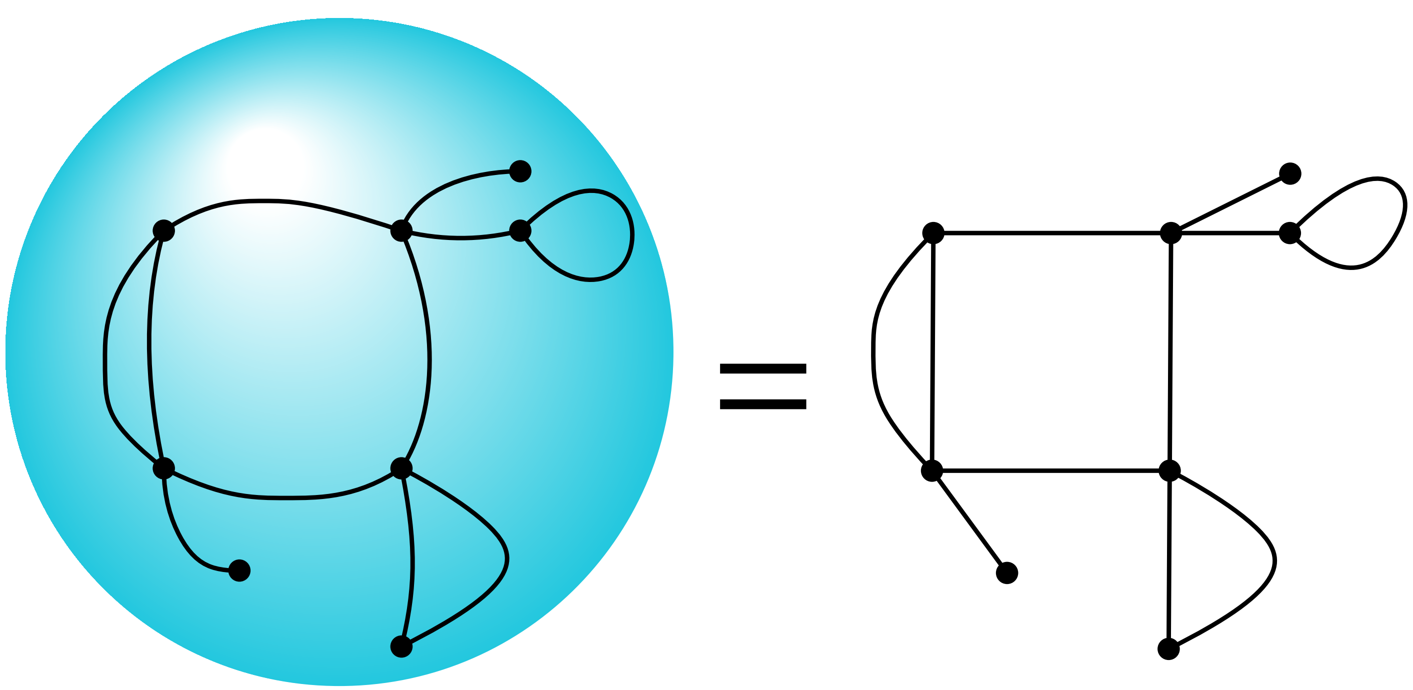
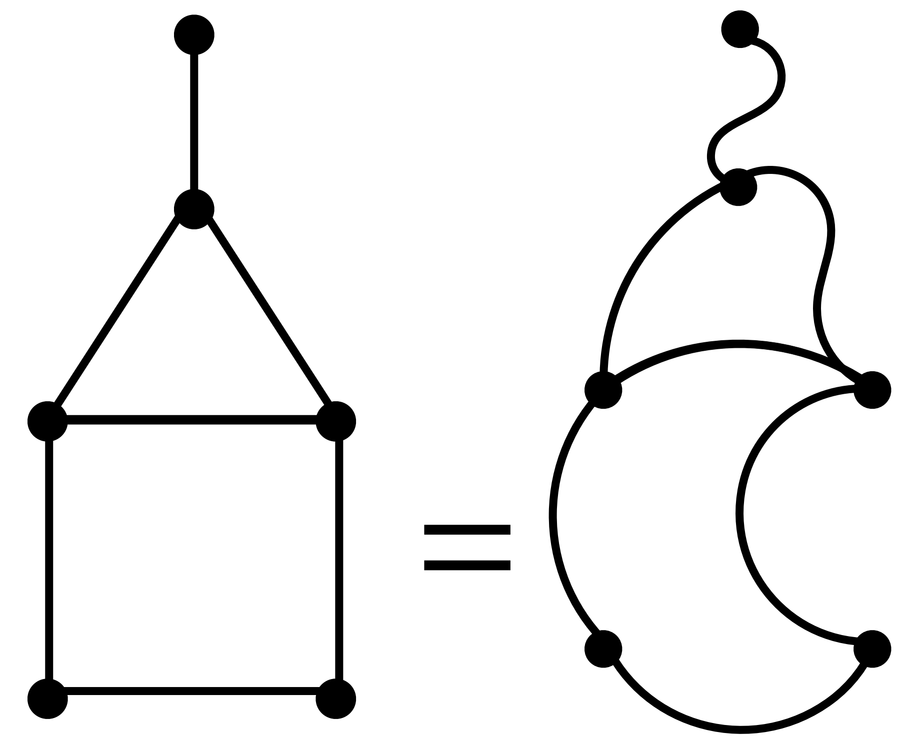
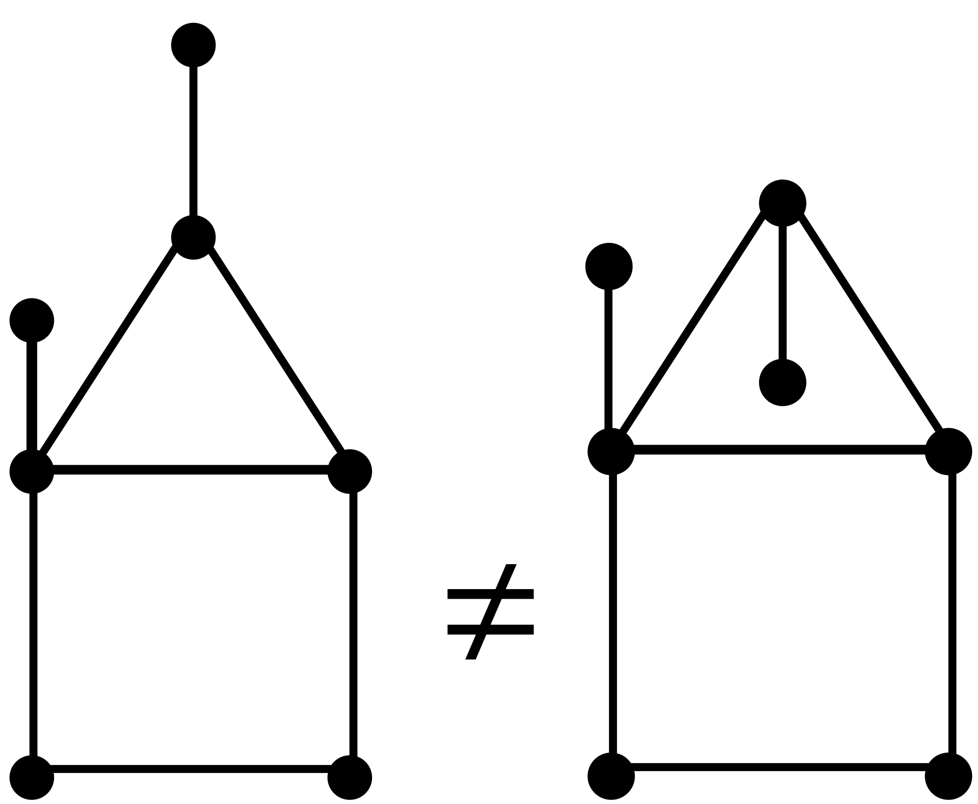
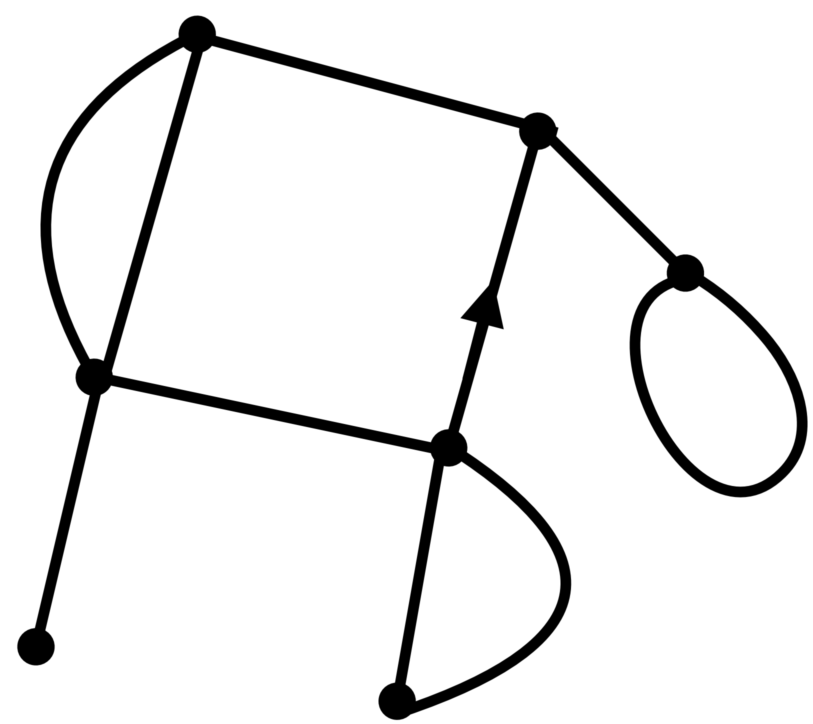






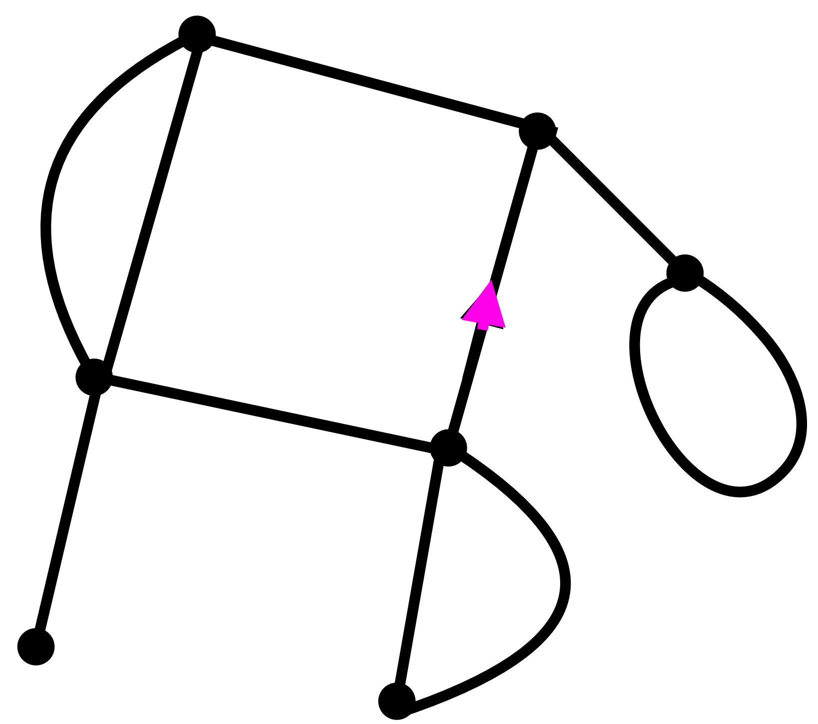
Planar map = planar graph + cyclic order on neighbours

Terminology on planar maps



- Rooted planar map = map endowed with a marked oriented edge (represented by an arrow)
- Size $|\mathfrak{m}|$ = number of edges
- Corner (does not exist for graphs!) = space between two consecutive edges around a vertex (trigonometric order)
- Distance = minimum number of edges in-between

Terminology on planar maps



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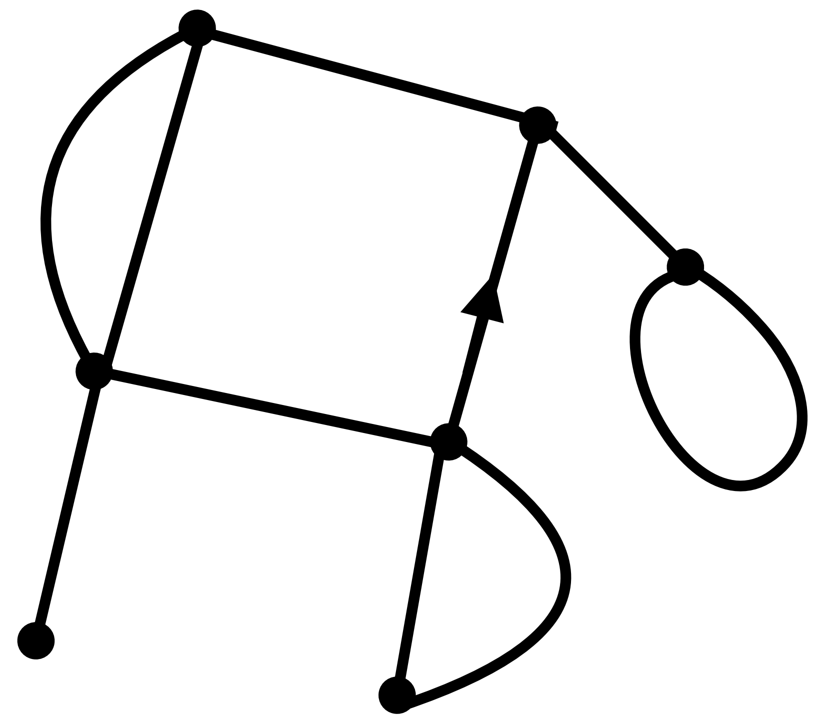
Terminology on planar maps

$|\mathfrak{m}| = 10$



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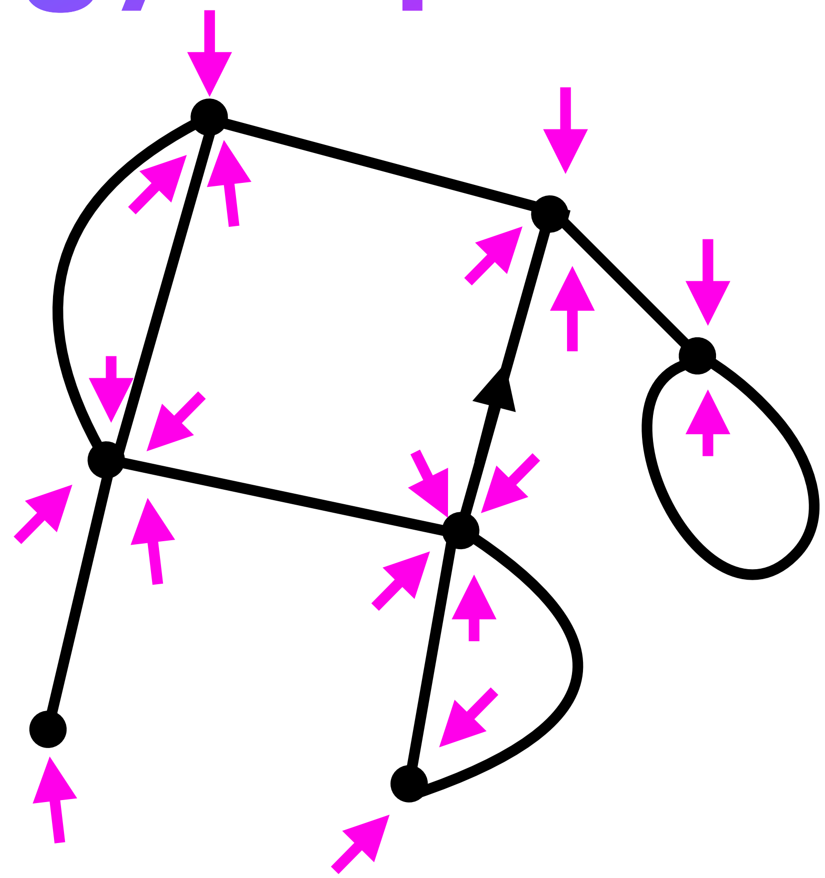

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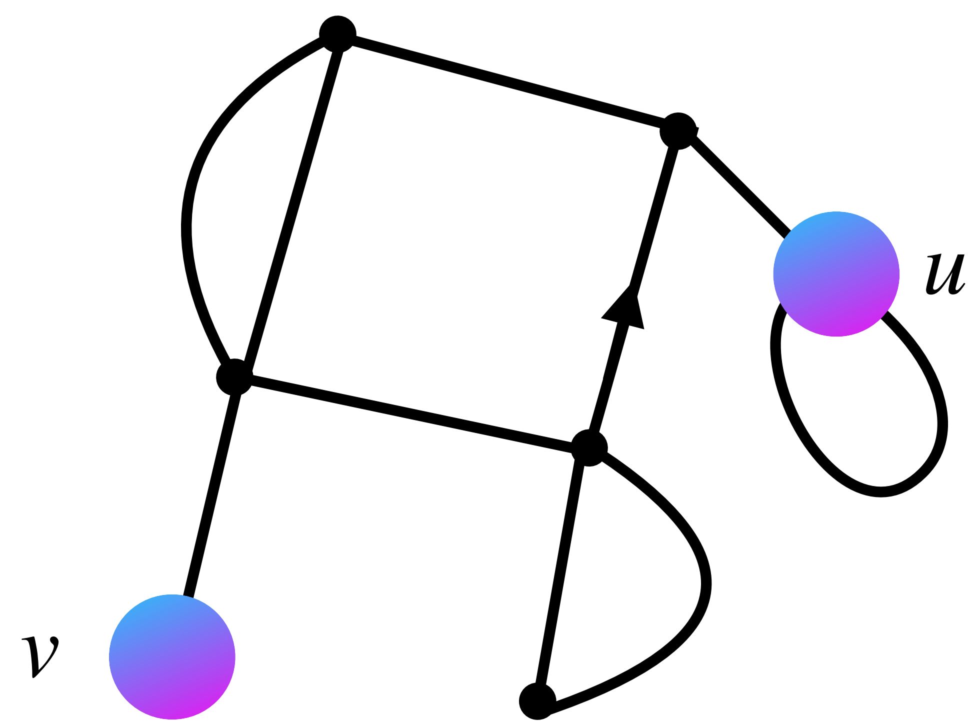

$d(u, v) = 4$

- Rooted planar map = map endowed with a marked oriented edge (represented by an arrow)
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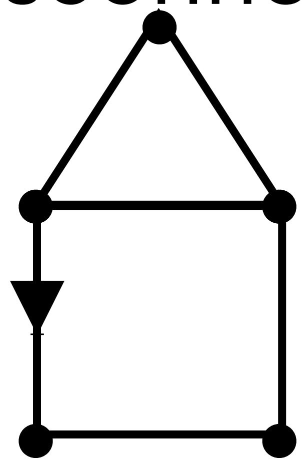
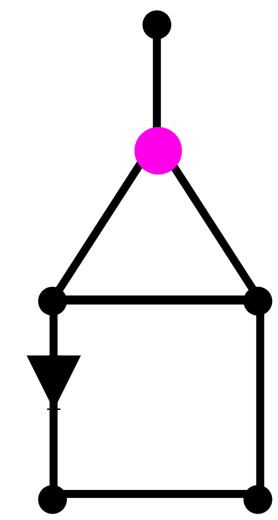
Families of maps

2-connected map = at least two vertices must be removed to disconnect





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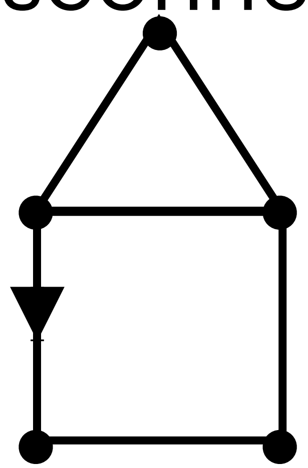
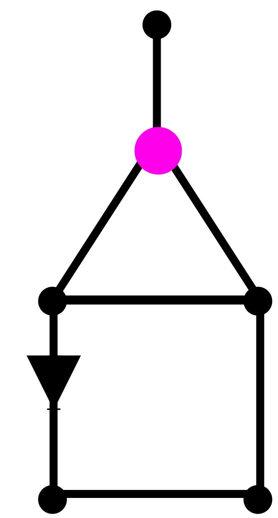
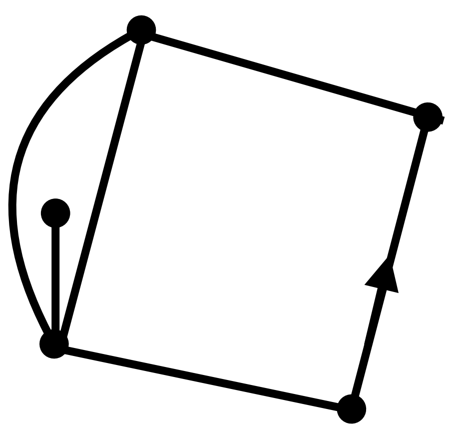
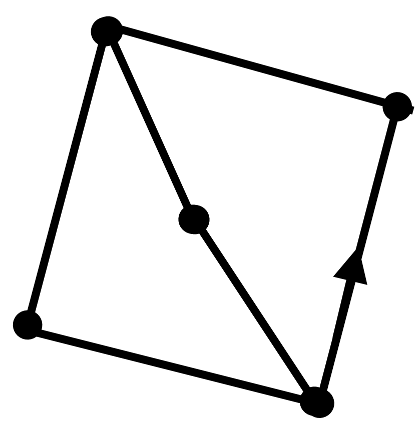


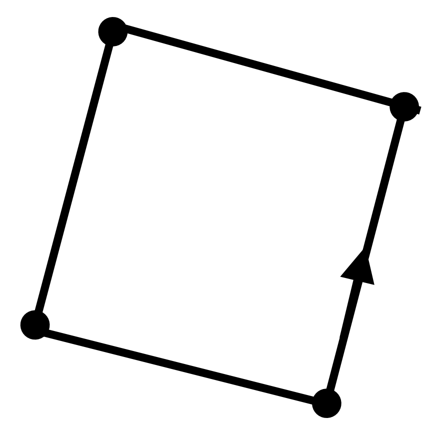


→ *cut vertex* => not 2-connected

Quadrangulation = map with all faces of degree 4 (= 4 sides of edges border the face)







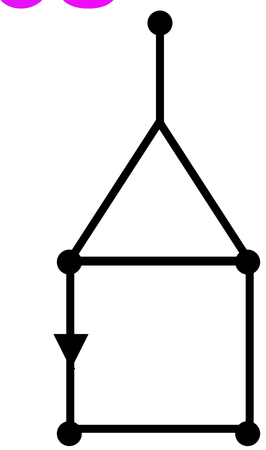
Simple quadrangulation = no multiple edges

Triangulation = map with all faces of degree 3

Enumeration of maps

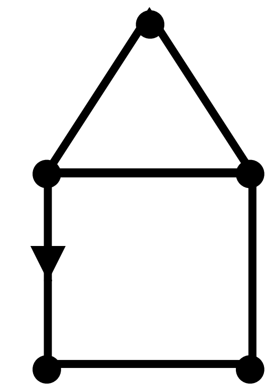
Maps with n edges

$$m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$$



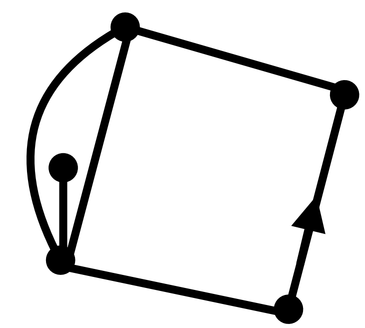
2-connected maps with n edges

$$b_n = \frac{2}{(n+1)(2n+1)} \binom{3n}{n}$$



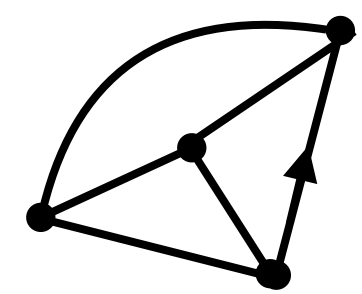
Quadrangulations with n faces

$$q_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$$



2-connected triangulations with $2n$ faces

$$t_n = \frac{2^n}{(n+1)(2n+1)} \binom{3n}{n}$$



[Tutte 1963; Mullin 1965]

Bijective study of maps

- Tutte's bijection between maps and quadrangulations [Tutte 1963];
- Cori–Vauquelin–Schaeffer bijection between quadrangulations and labelled trees [Cori, Vauquelin 1981; Schaeffer 1998];
- Bijection between 2-connected maps and skew ternary trees [Jacquard, Schaeffer 1998];
- Bouttier–Di Francesco–Guitter bijection between bipartite maps and mobiles [Bouttier, Di Francesco, Guitter 2004];
- Unified bijective scheme between maps and decorated trees [Bernardi, Fusy 2012; Albenque, Poulalhon 2015]...

=> Very active domain Bonichon, Bousquet-Mélou, Chapuy, Fang, Miermont...

Behaviour of a random quadrangulation

- Typical distance between two random vertices in a quadrangulation of size n: $\Theta\left(n^{1/4}\right)$

& distance profile converges [Chassaing, Schaeffer 2004];

- Definition of the Brownian sphere [Marckert, Mokkadem 2006];
- Scaling limit: Brownian sphere [Le Gall 2013; Miermont 2013].

Brownian Sphere $\mathcal{S}_e$

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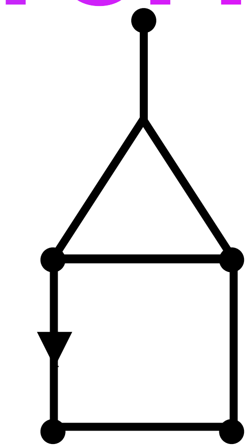
Universality of the Brownian sphere as scaling limit: random triangulations & $2q$-angulations [Le Gall 2013], general maps [Bettinelli, Jacob, Miermont 2014], simple quadrangulations and simple triangulations [Addario-Berry, Albenque 2017]

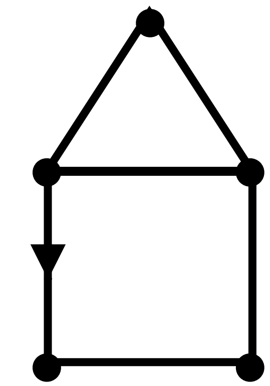
& results by Carrance, Curien, Fusy, Kortchemski, Lehéricy, Marzouk, Stufler

Universality in enumeration of maps

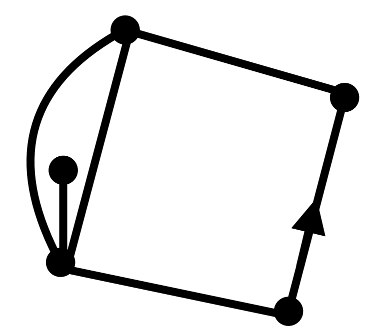
Maps with n edges $m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$



2-connected maps with n edges $b_n = \frac{2}{(n+1)(2n+1)} \binom{3n}{n}$



Quadrangulations with n faces $q_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$



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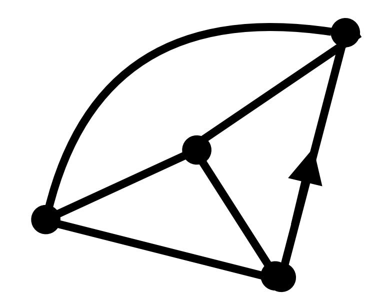

[Tutte 1963; Mullin 1965]

Universality in enumeration of maps

Maps with n edges	$m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$	$\sim \frac{2}{\sqrt{\pi}} 12^n n^{-5/2}$
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[Tutte 1963; Mullin 1965]

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[Tutte 1963; Mullin 1965]

Universality in enumeration of maps

Maps with n edges $$m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} \sim \kappa 12^n n^{-5/2}$$

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Universality in enumeration of maps

Maps with n edges $m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} \sim \kappa \rho^{-n} n^{-5/2}$

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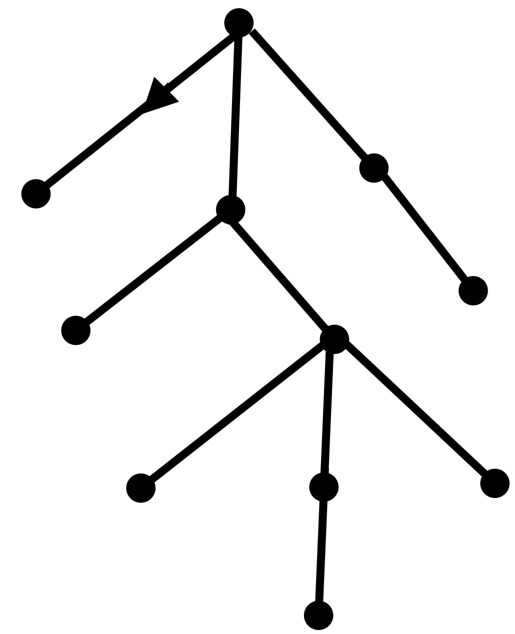
=> Universality phenomenon

[Tutte 1963; Mullin 1965] [Drmota, Noy, Yu 2022; Schaeffer 2023; Duchi Schaeffer 2024+]

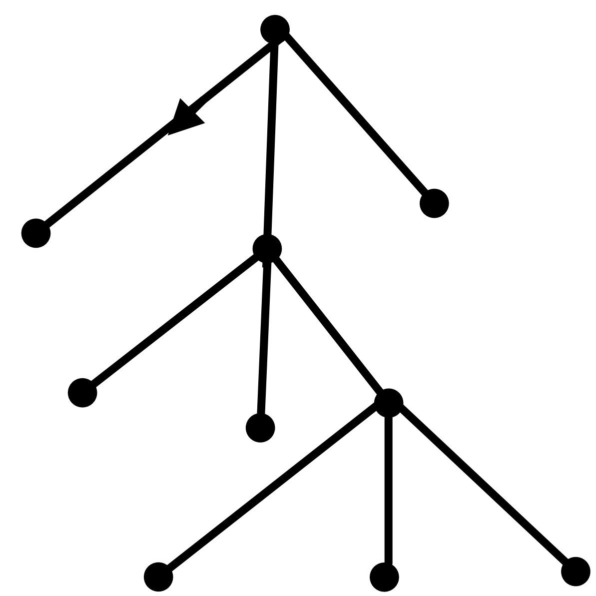
Background. Trees

Universality for tree families

(Rooted plane) tree = tree where children are ordered



Trees of size n $$\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$$



Ternary trees of size $3n$ $$\frac{1}{2n+1}\binom{3n}{n}$$

Universality for tree families

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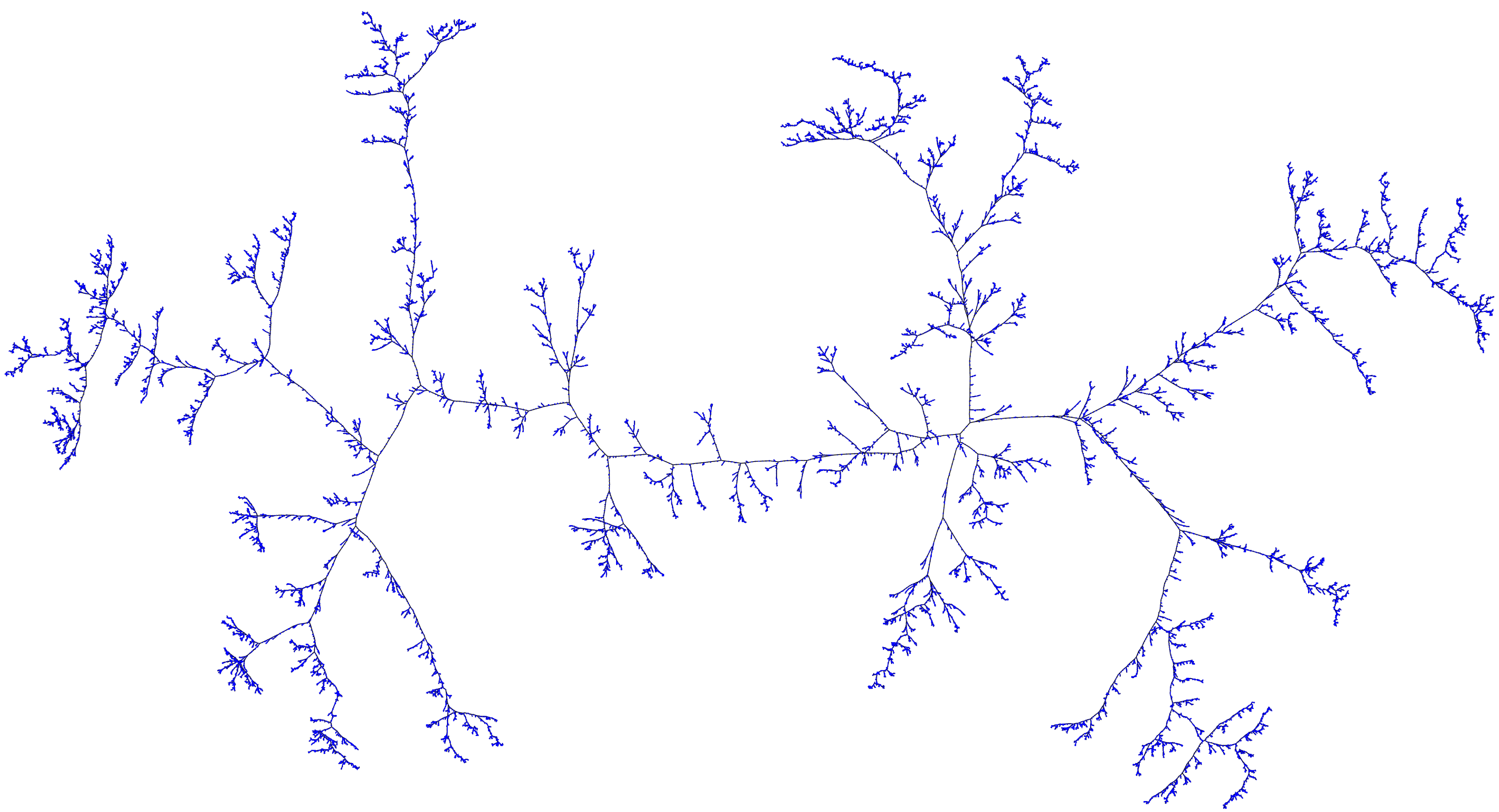
Ternary trees of size $3n$ $$\frac{1}{2n+1}\binom{3n}{n} \sim \kappa\rho^{-n}n^{-3/2}$$

Behaviour of a random tree

- Diameter & height of a uniform binary tree of size n: $\Theta\left(n^{1/2}\right)$ [Flajolet, Odlyzko 1982]
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006]

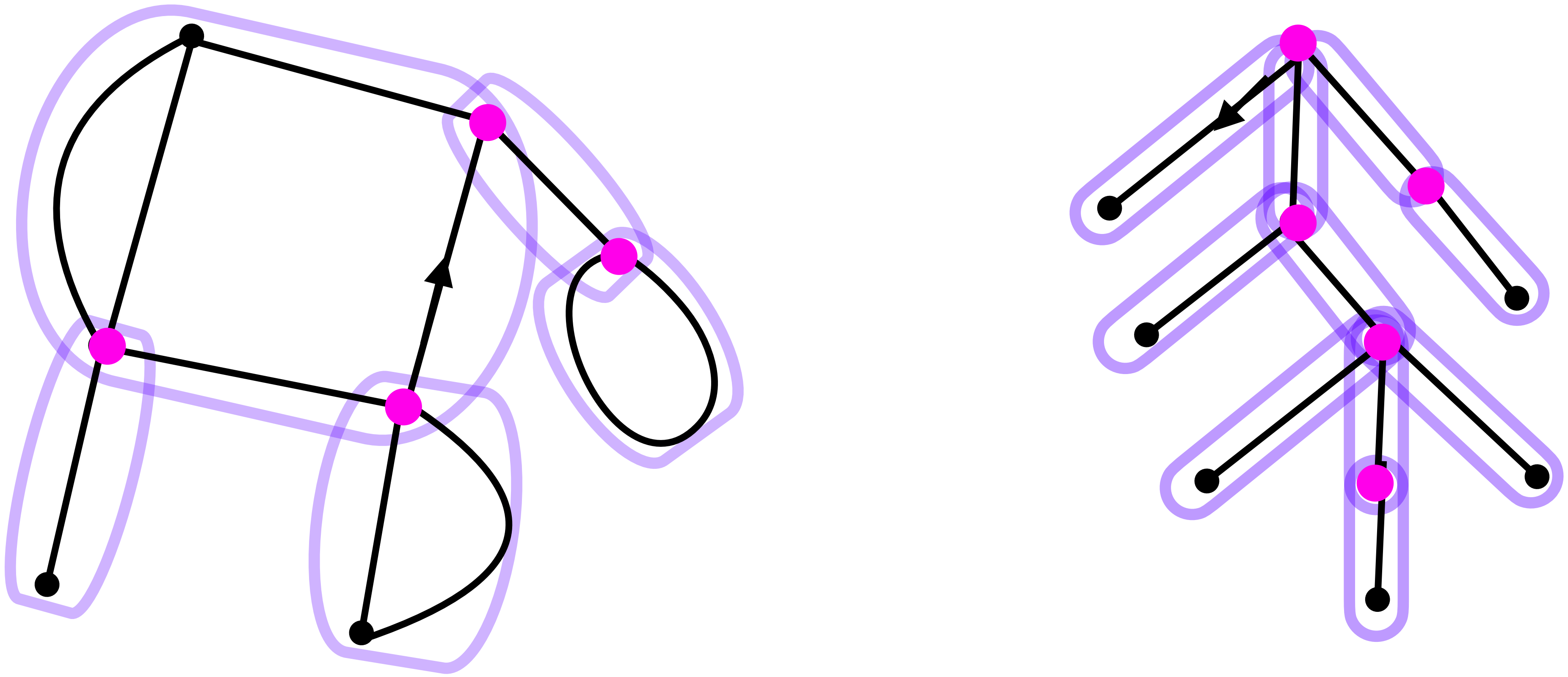


Brownian Tree $\mathscr{T}_2$

Escaping universality *via* blocks

2-c map = at least two vertices must be removed to disconnect

Block = maximal (for inclusion) 2-connected submap



Outline

Phase transitions of block-weighted planar maps

I. Model

II. Block tree of a map and its applications

III. Scaling limits

→ *W. Fleurat & Z. S. (Electronic Journal of Probability, 2024)*

IV. Extension to other families of maps

→ *Z. S. (Eurocomb'23)*

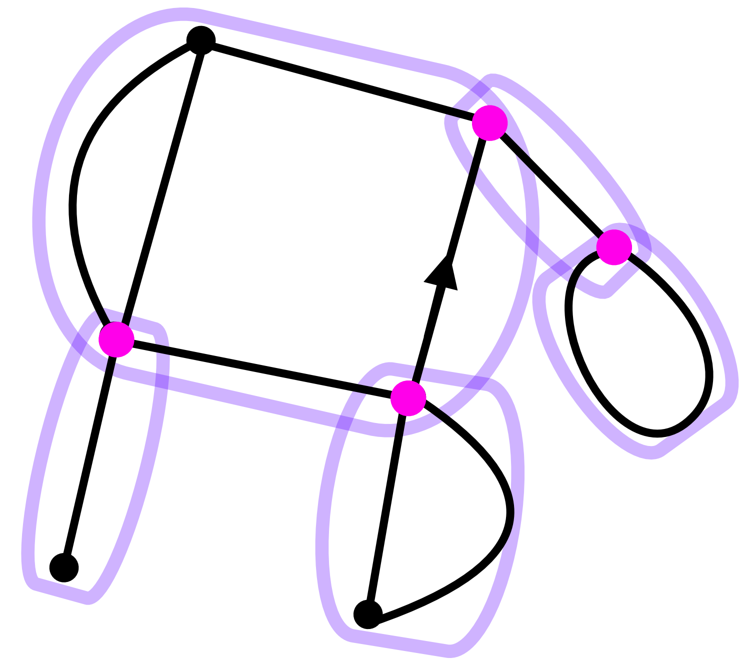
V. Extension to tree-rooted maps

→ *M. Albenque, É. Fusy & Z. S. (AofA'24)*

VI. Perspectives

I. Model

Model



Fix $u > 0$, define

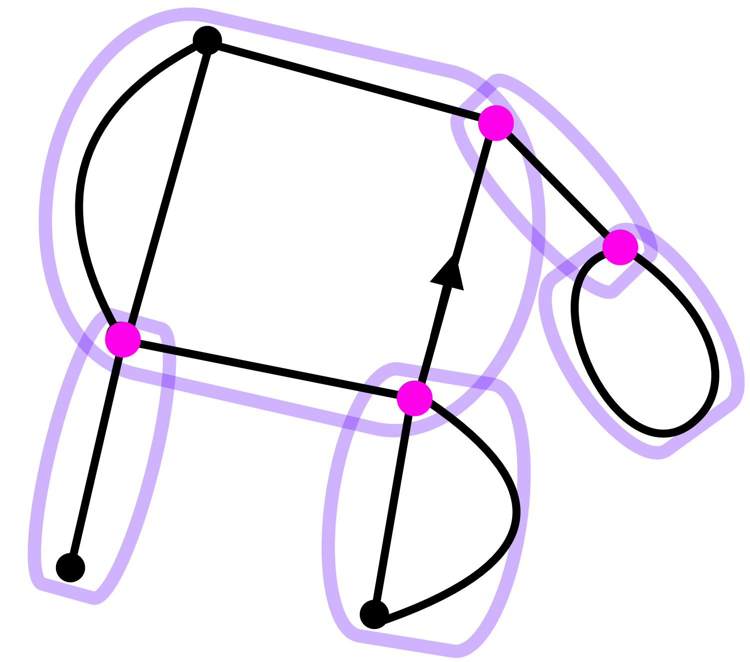
$$\mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$$

$\mathfrak{m} \in \{\text{maps of size } n\}$,

$Z_{n,u} =$ normalisation.

[Bonzom, Delepouve, Rivasseau 2015; Stufler 2020]

Model



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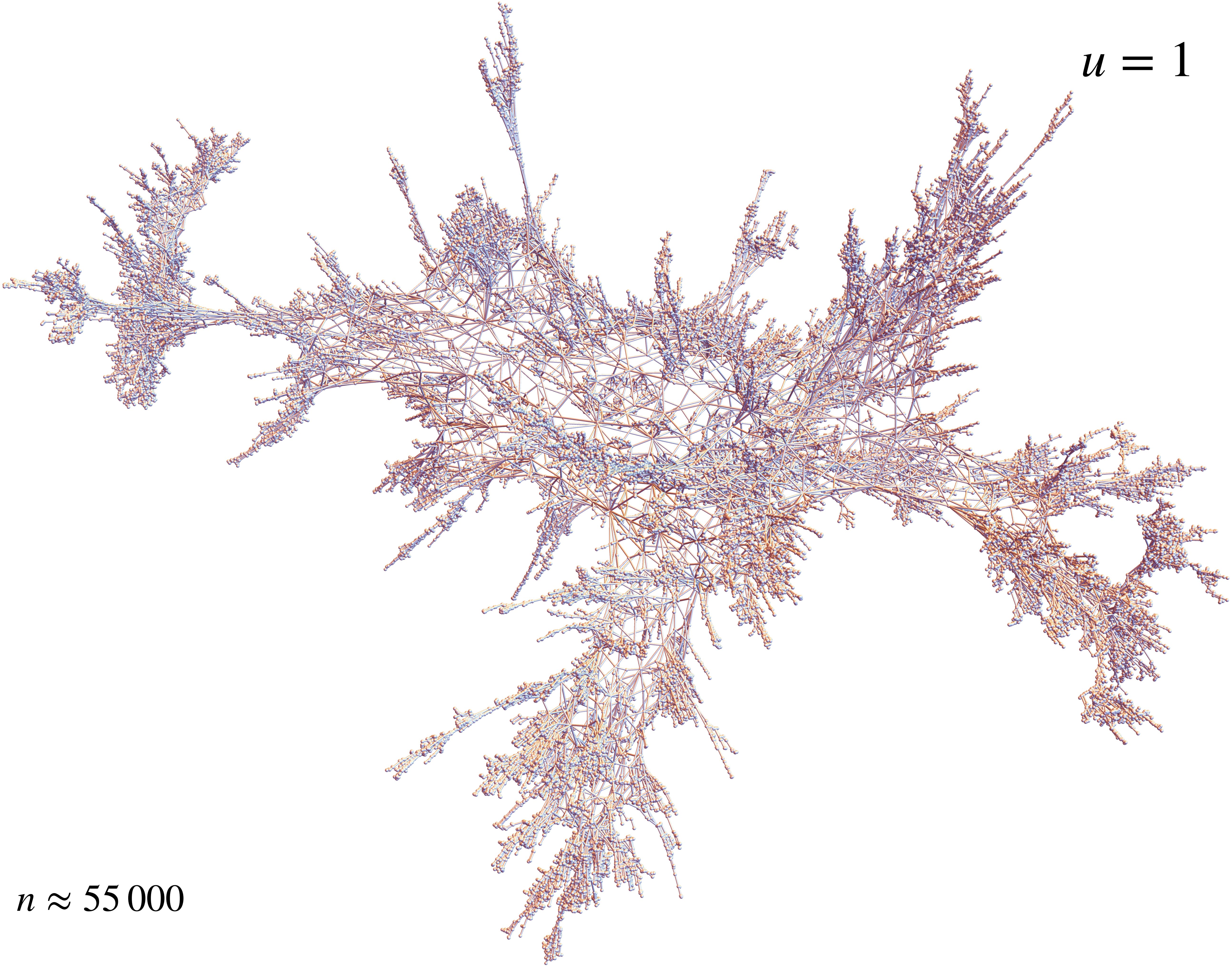
$\mathfrak{m} \in$ {maps of size n},

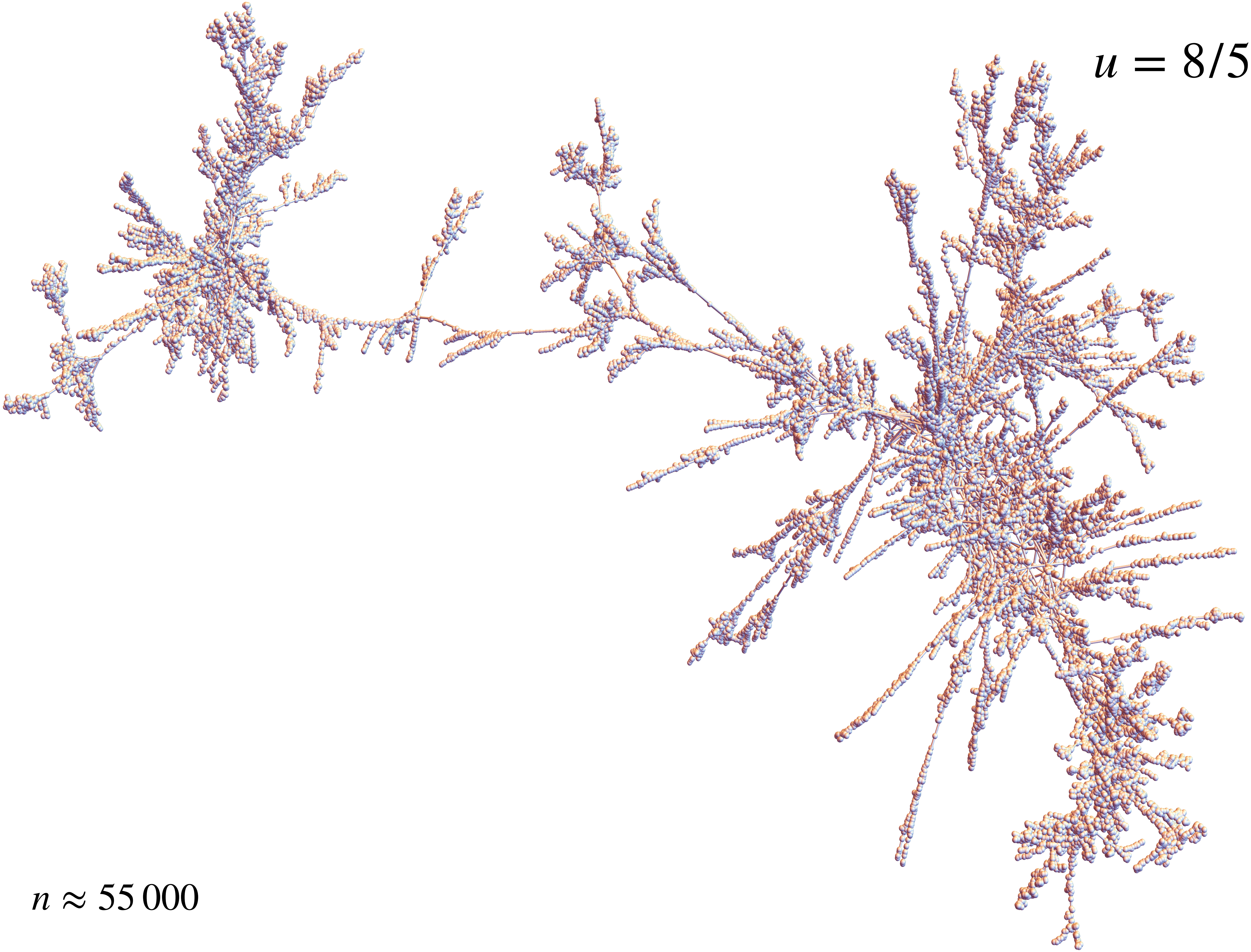
$Z_{n,u}$ = normalisation.

[Bonzom, Delepouve, Rivasseau 2015; Stufler 2020]

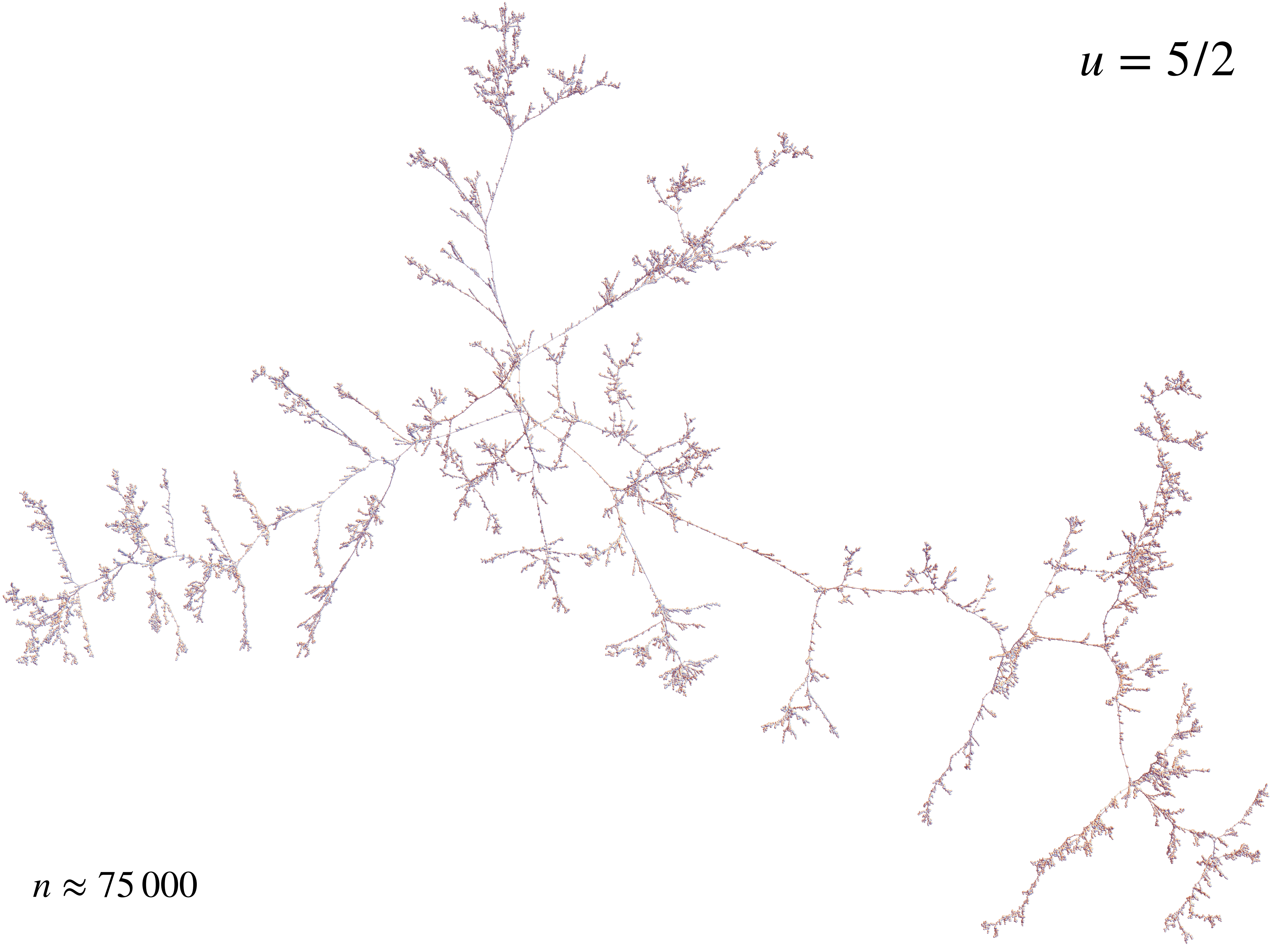
- $u = 1$: uniform distribution on maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected maps);
- $u \to \infty$: maximising the number of blocks (= trees!).

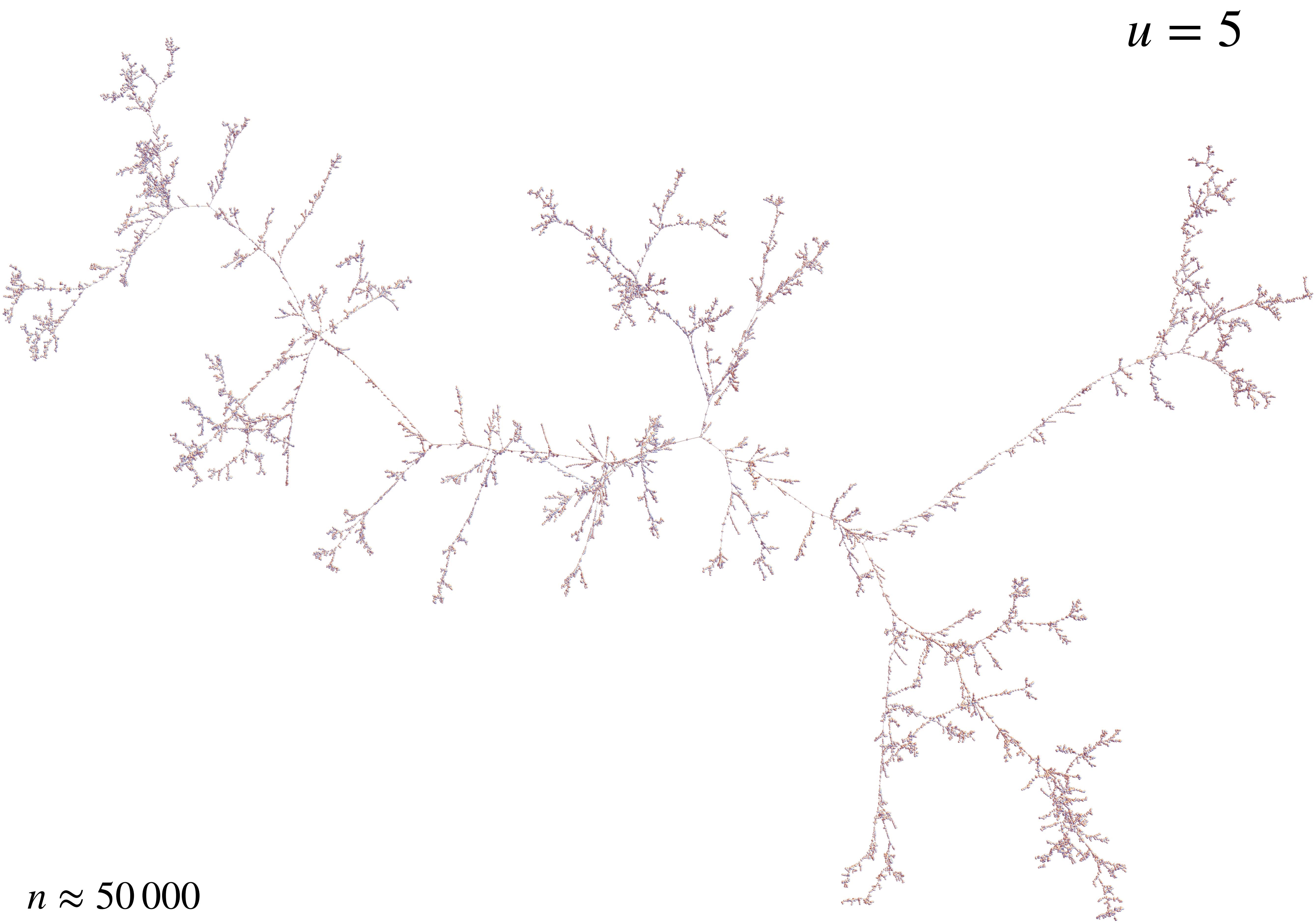
Given u, asymptotic behaviour when $n \to \infty$?

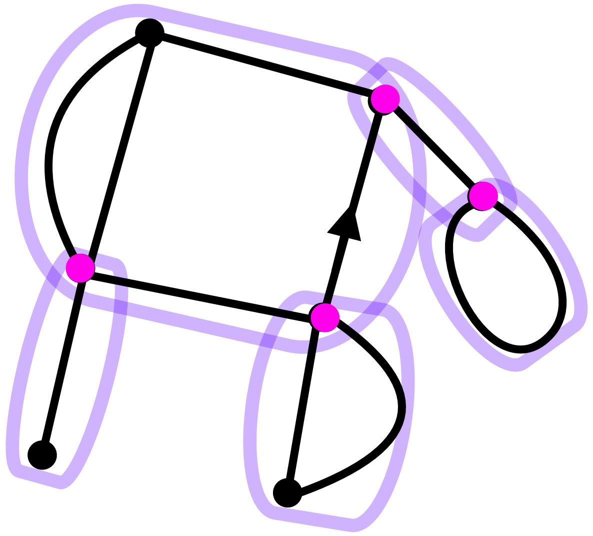

$u = 1$
$n \approx 55\,000$


$u = 8/5$
$n \approx 55\,000$


$u = 9/5$
$n \approx 80\,000$


$u = 5/2$
$n \approx 75\,000$


$u = 5$
$n \approx 50\,000$



Phase transition

Theorem [Fleurat, S. 24] Model exhibits a phase transition at $u = 9/5$. When $n \to \infty$:

- Subcritical phase $u < 9/5$: "general map phase" one macroscopic block;
- Critical phase $u = 9/5$: a few large blocks;
- Supercritical phase $u > 9/5$: "tree phase" only small blocks.

We obtain explicit results on enumeration, limit laws for the size of the largest blocks and scaling limits in each case.

Results

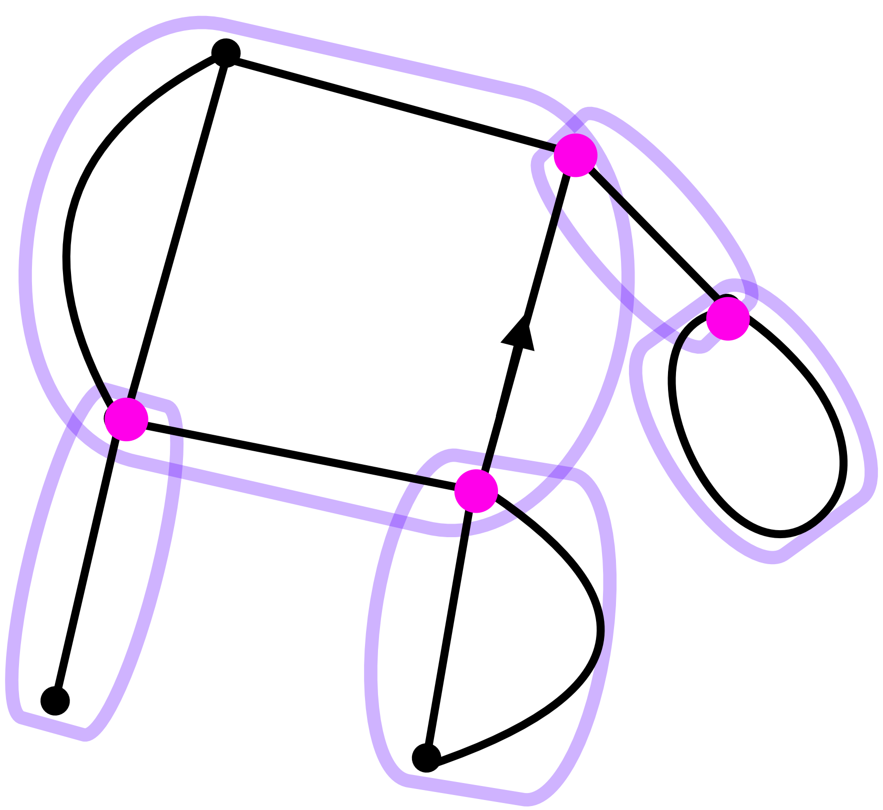
For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration			
Size of - the largest block - the second one			
Scaling limit of M_n			

A focus on the uniform case, *i.e.* $u = 1$

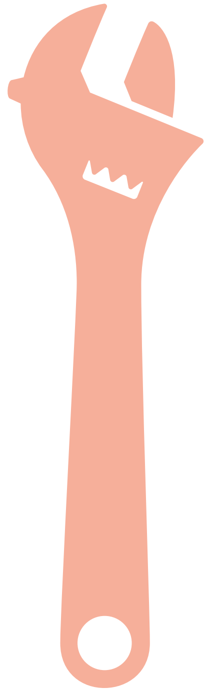
Studied by

- Analytic combinatorics methods [Banderier, Flajolet, Schaeffer, Soria 2001];
- Probability methods [Addario-Berry 2019];

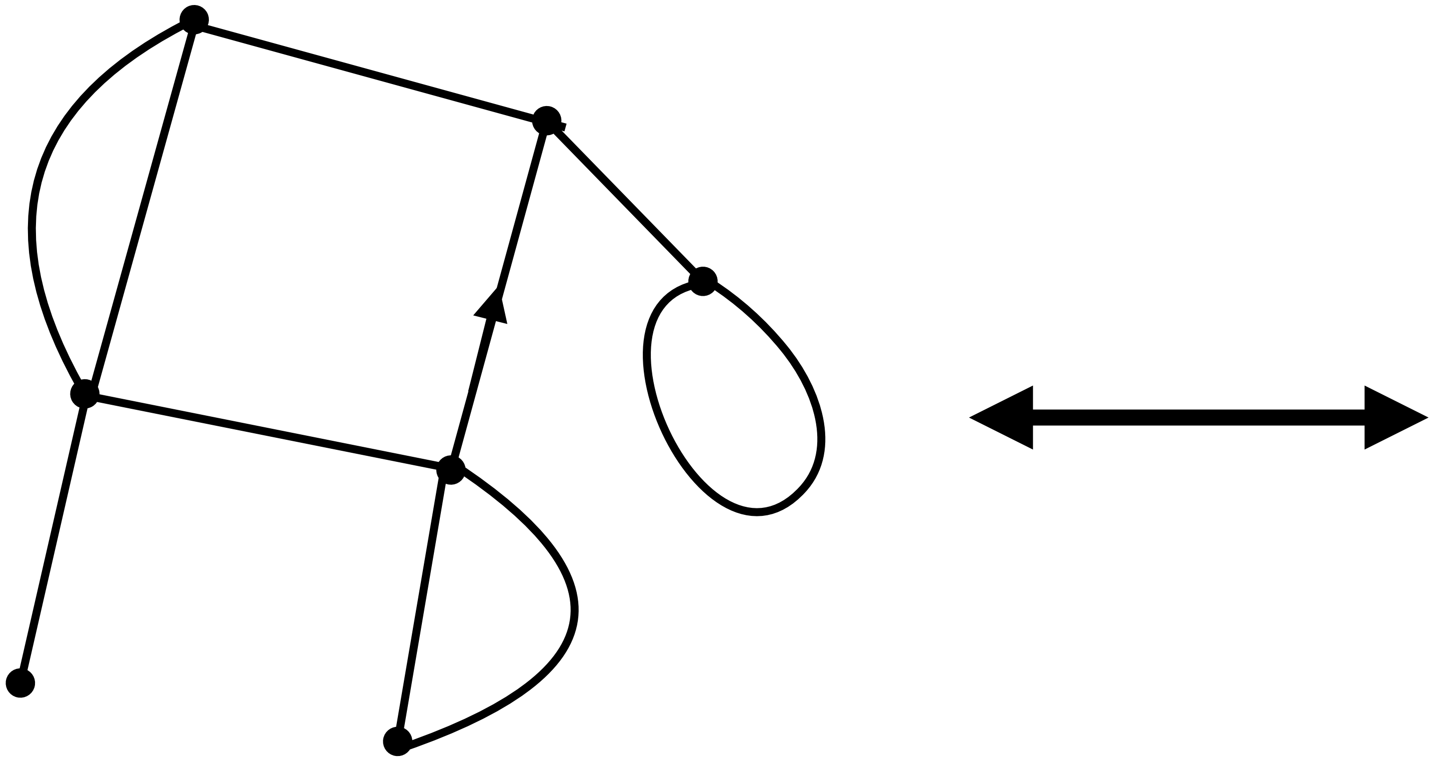
=> Show condensation phenomenon: a large block concentrates a macroscopic part of the mass.



II. Block tree of a map and its applications



Decomposition of a map into blocks (1/2)



[Tutte 1963]

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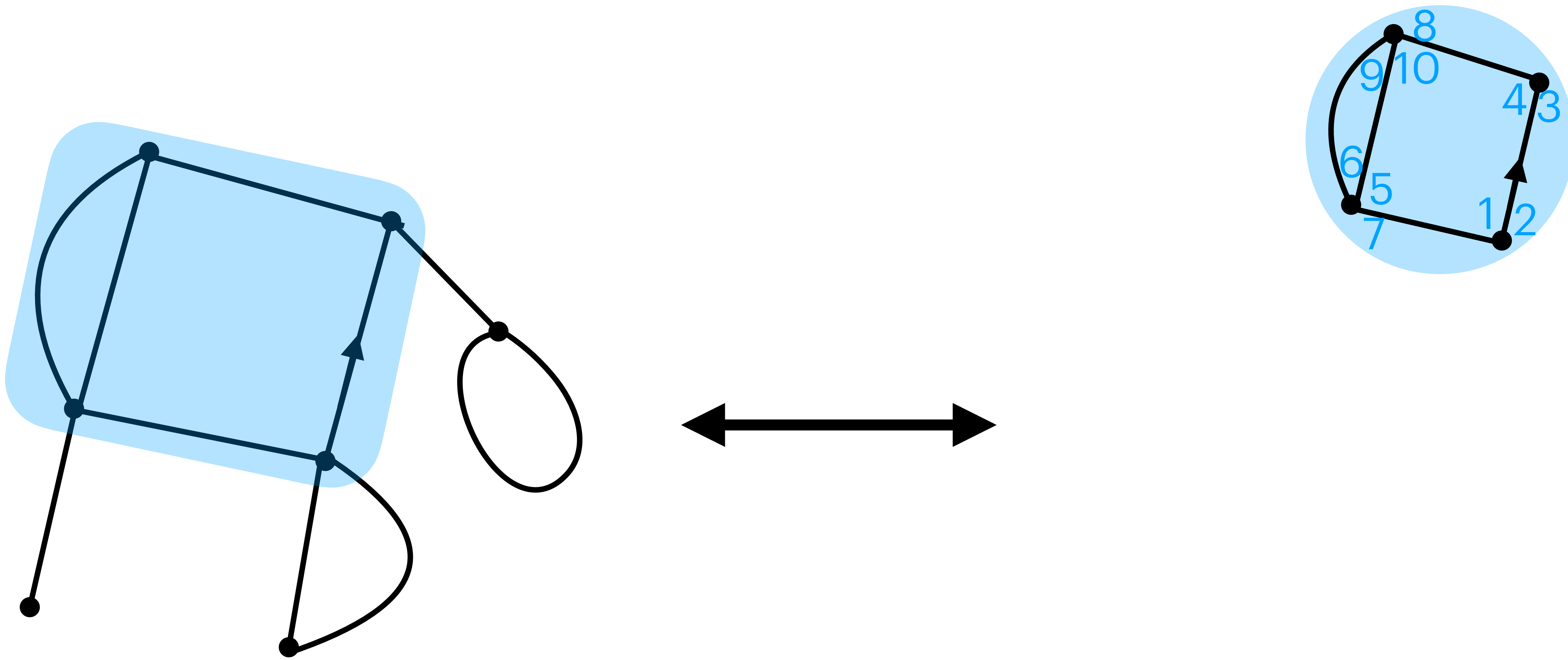

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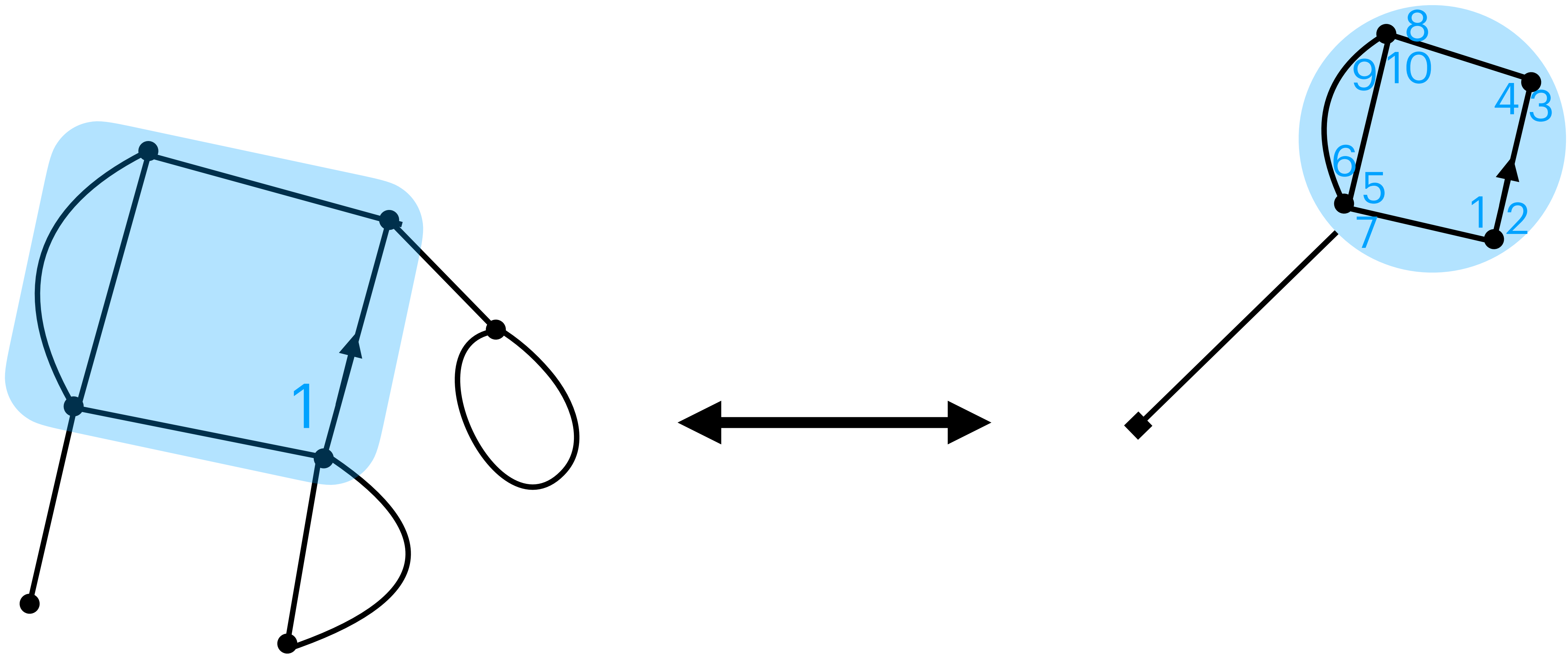

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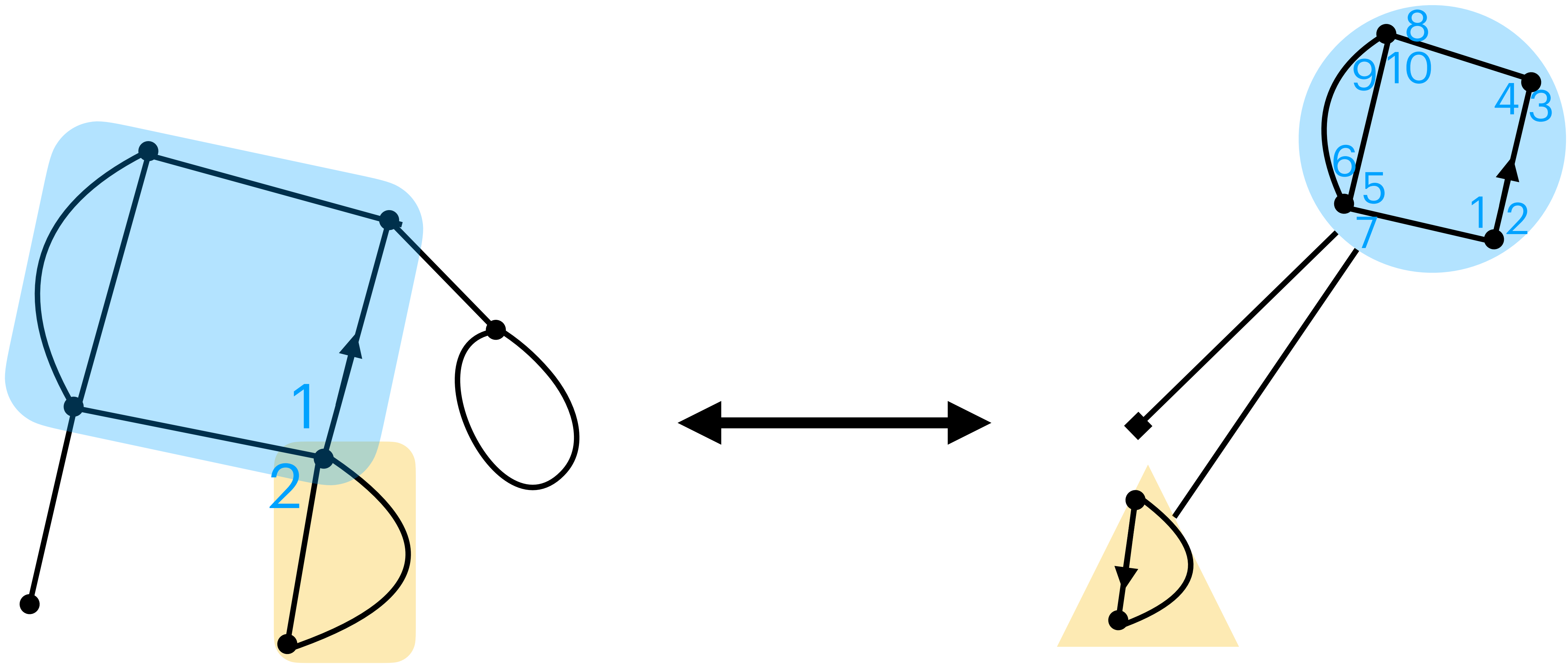

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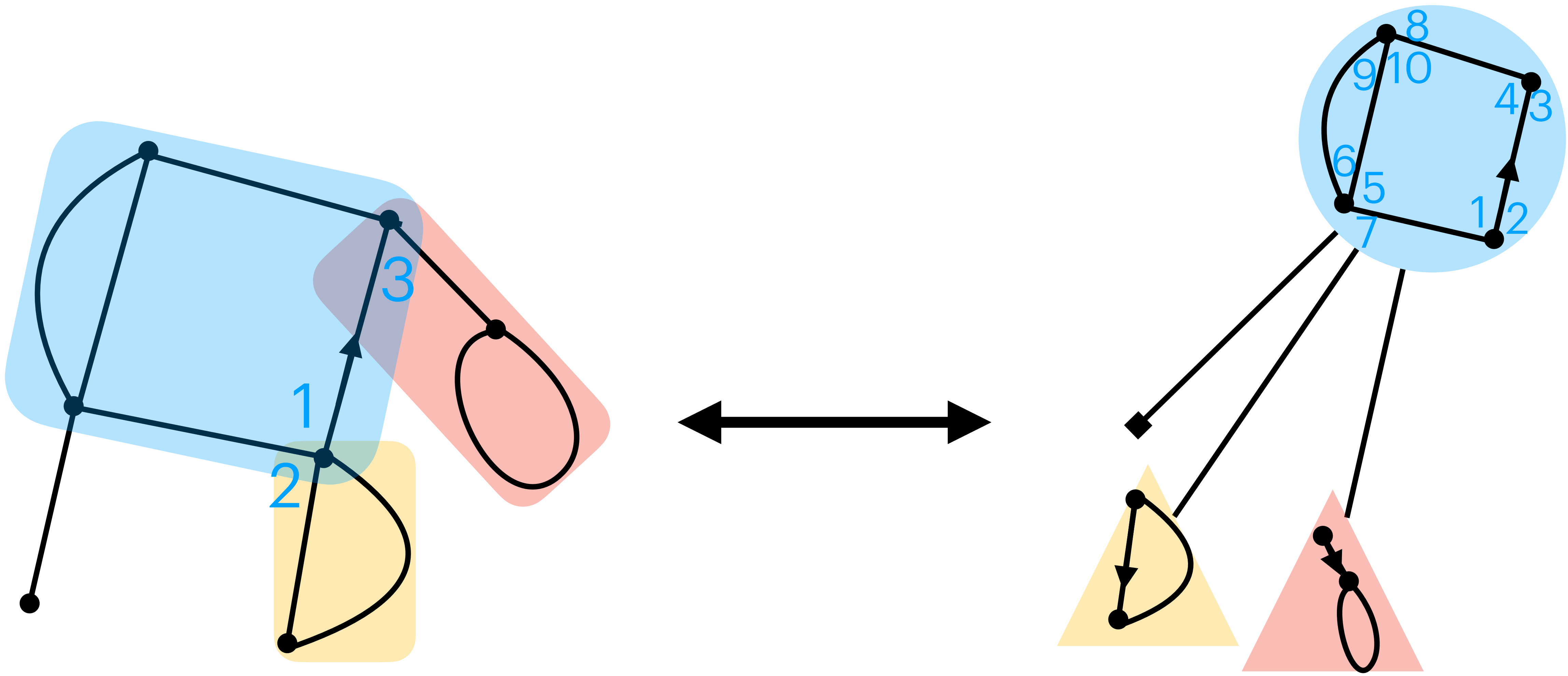

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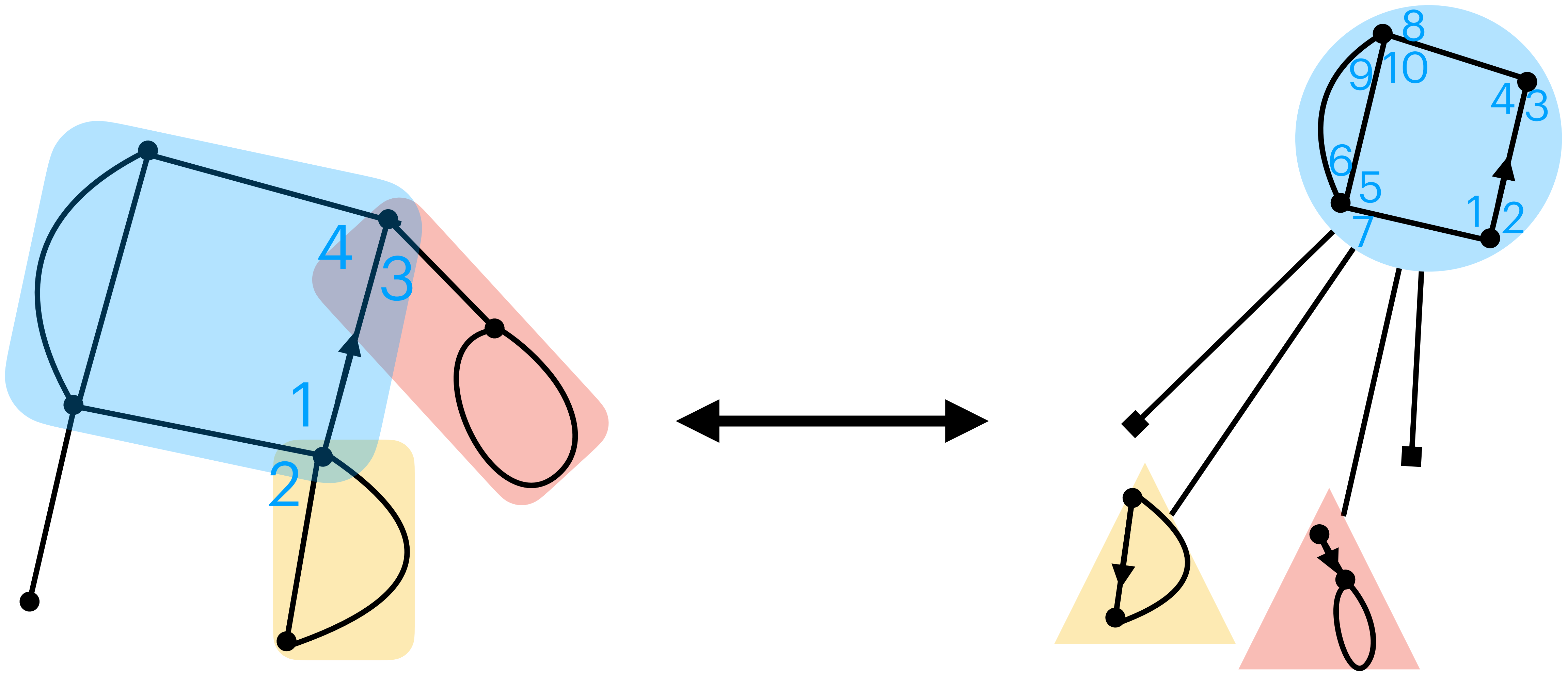

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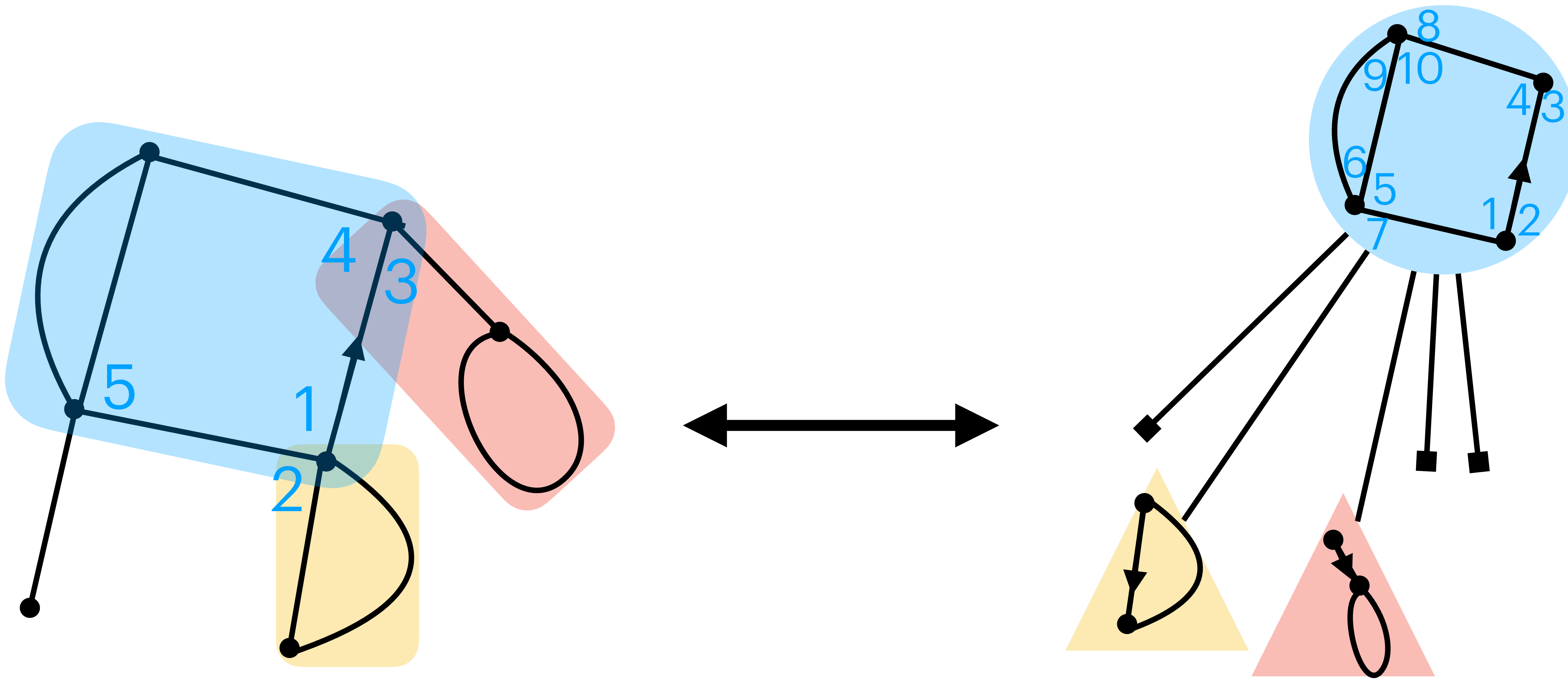

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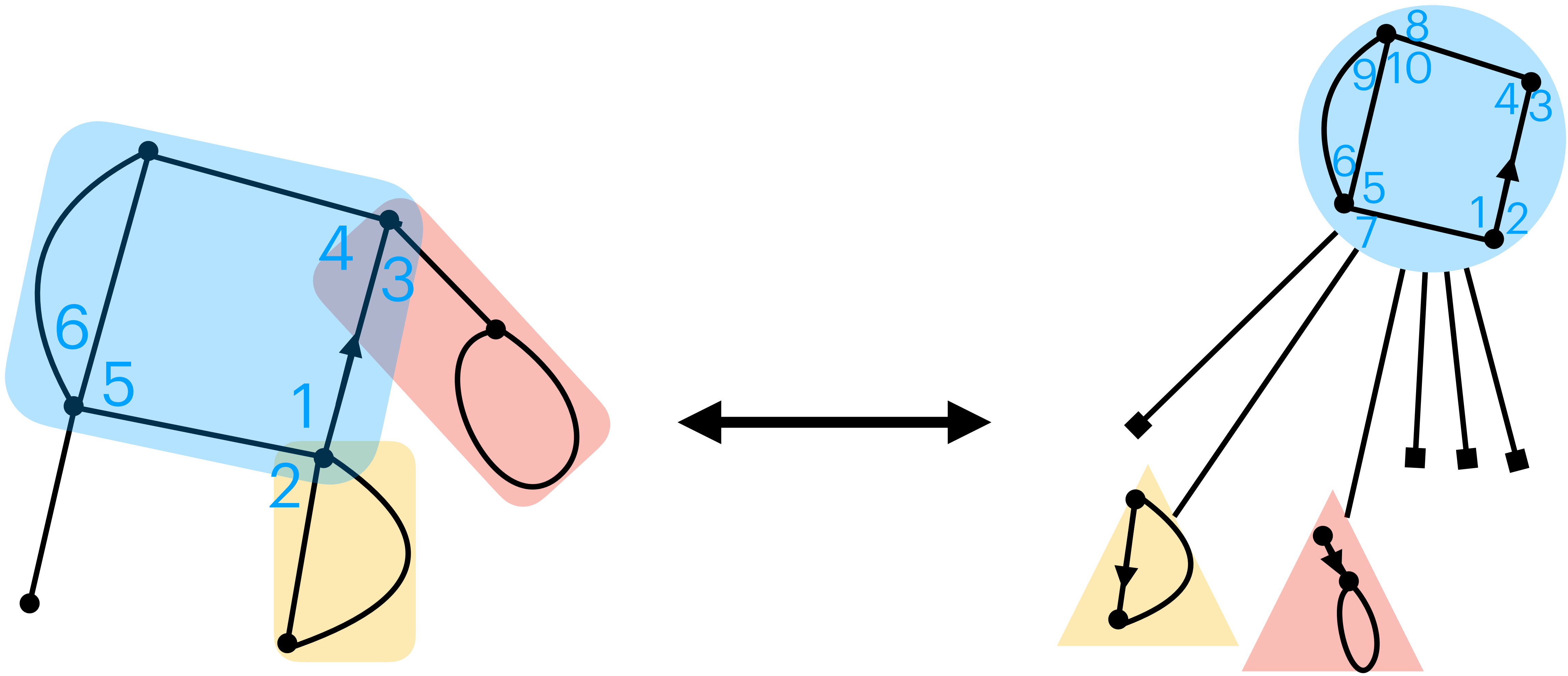

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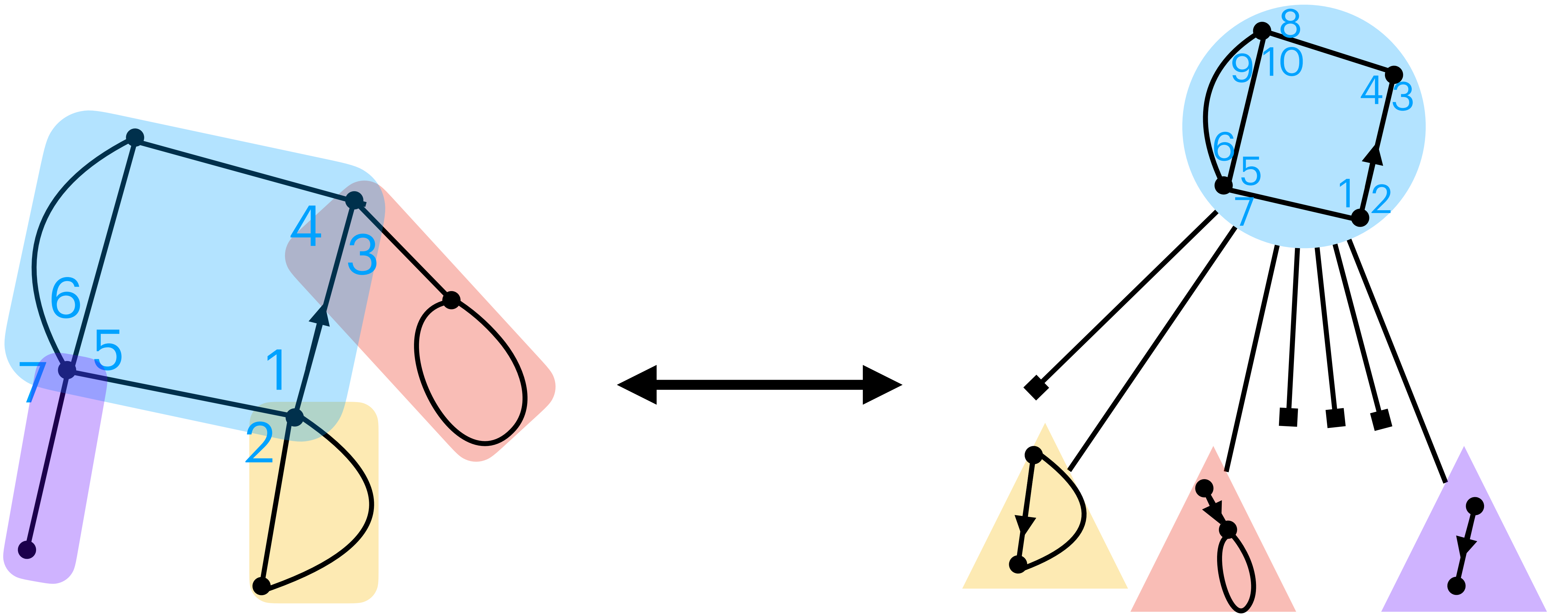

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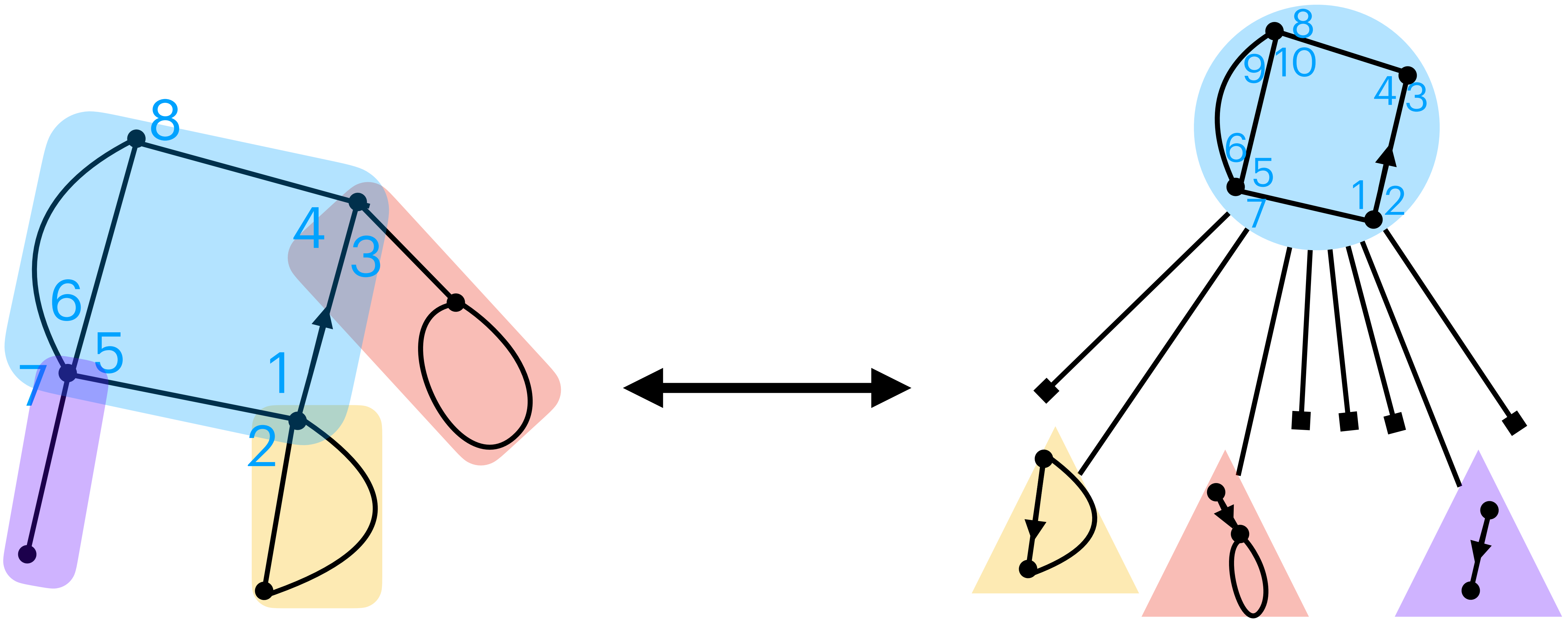

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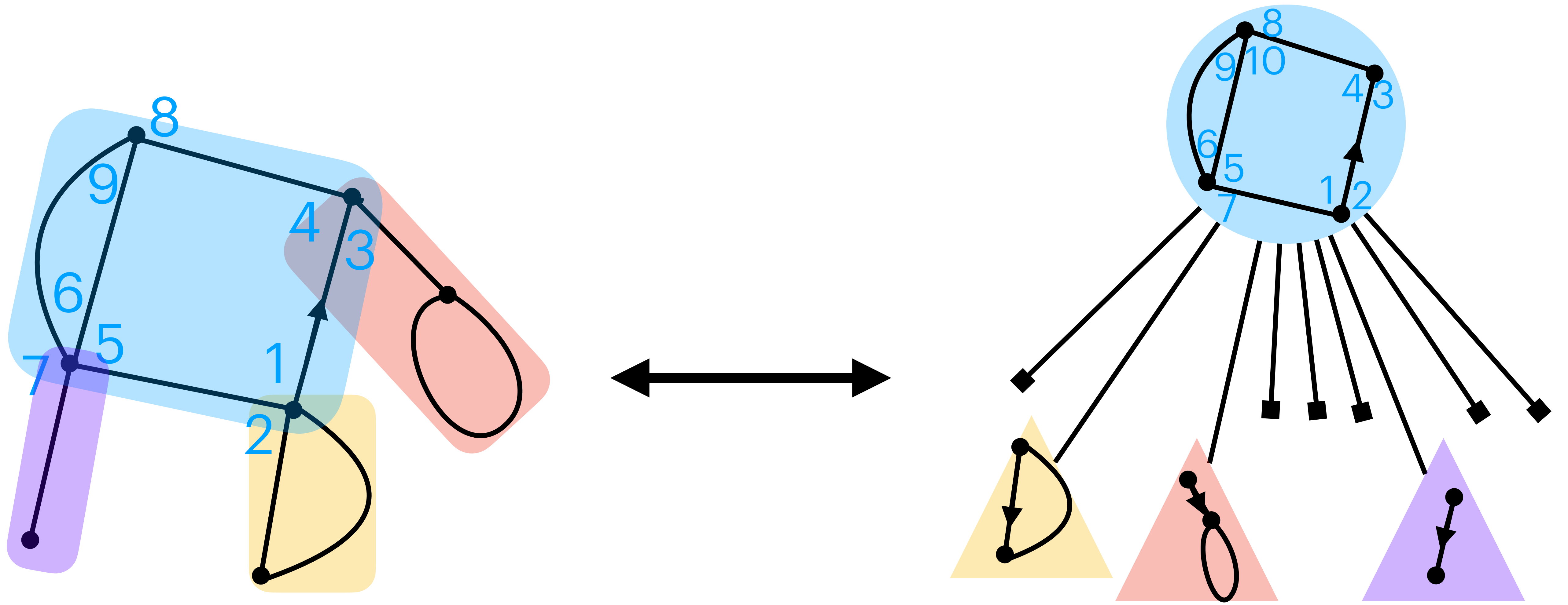

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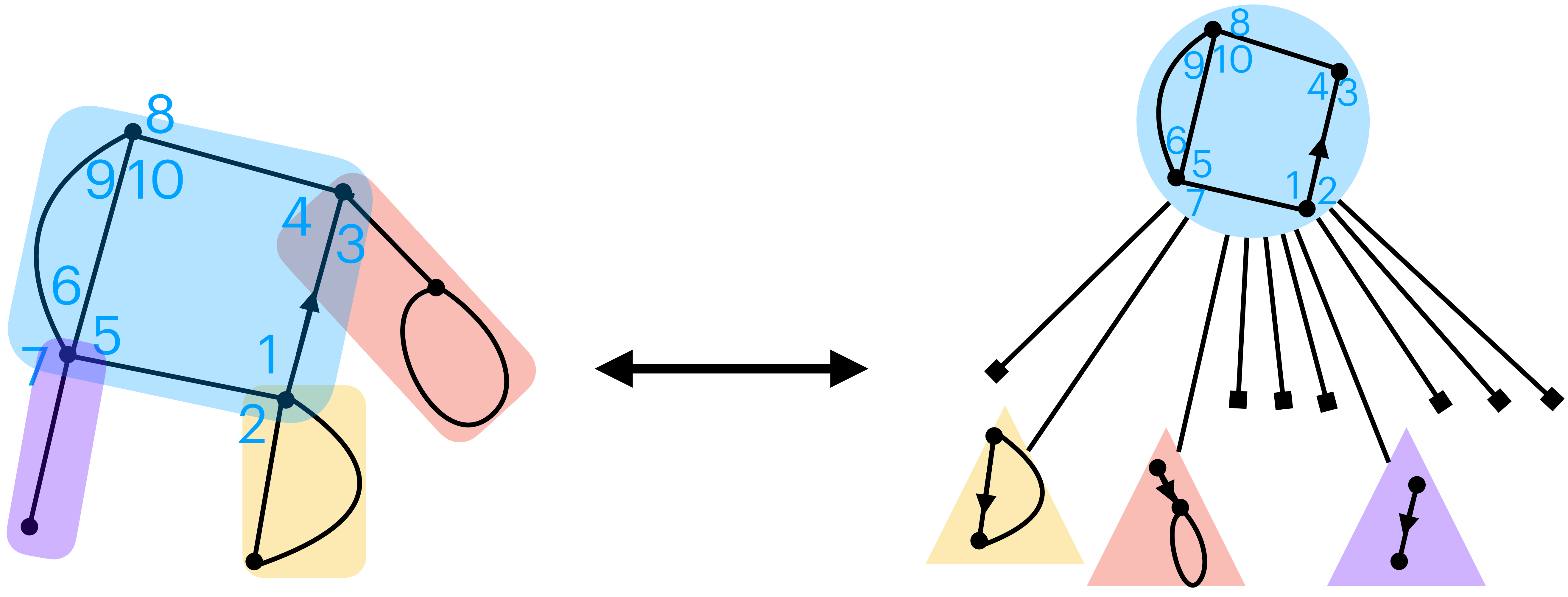

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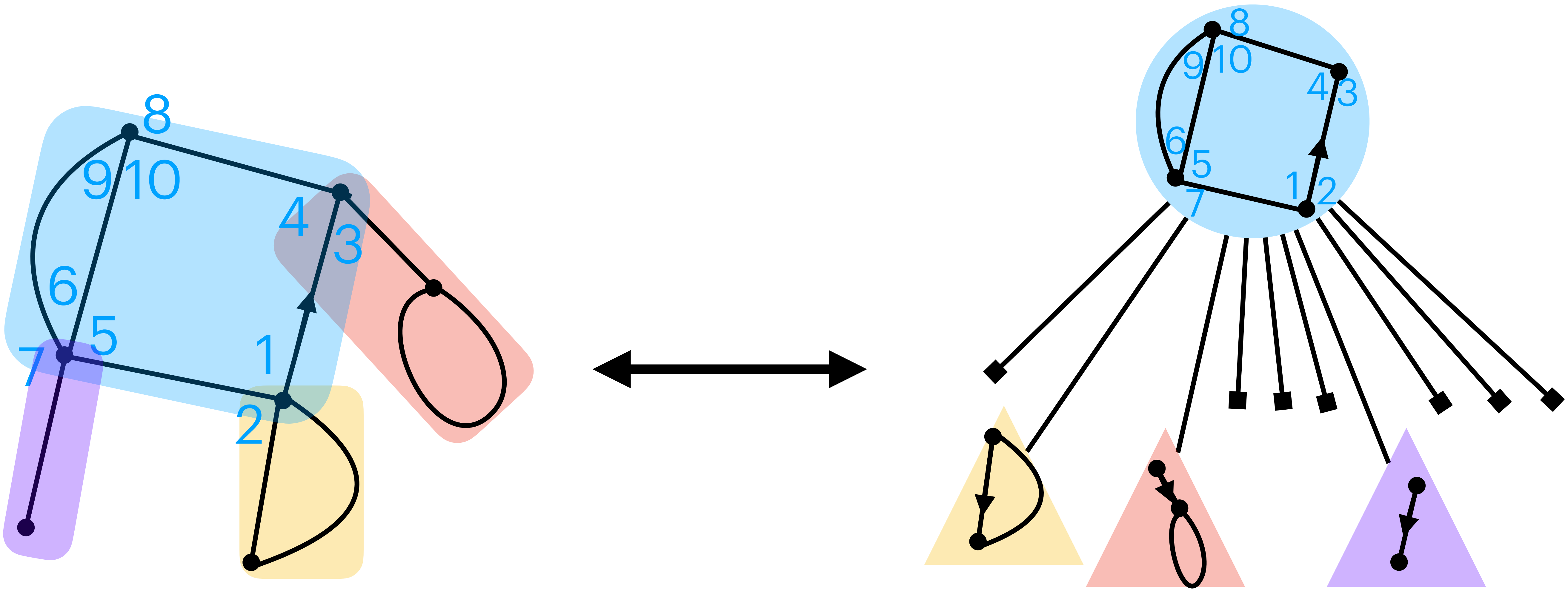

$$M(z) = B(zM^2(z))$$

GS of maps

GS of 2-connected maps

[Tutte 1963]

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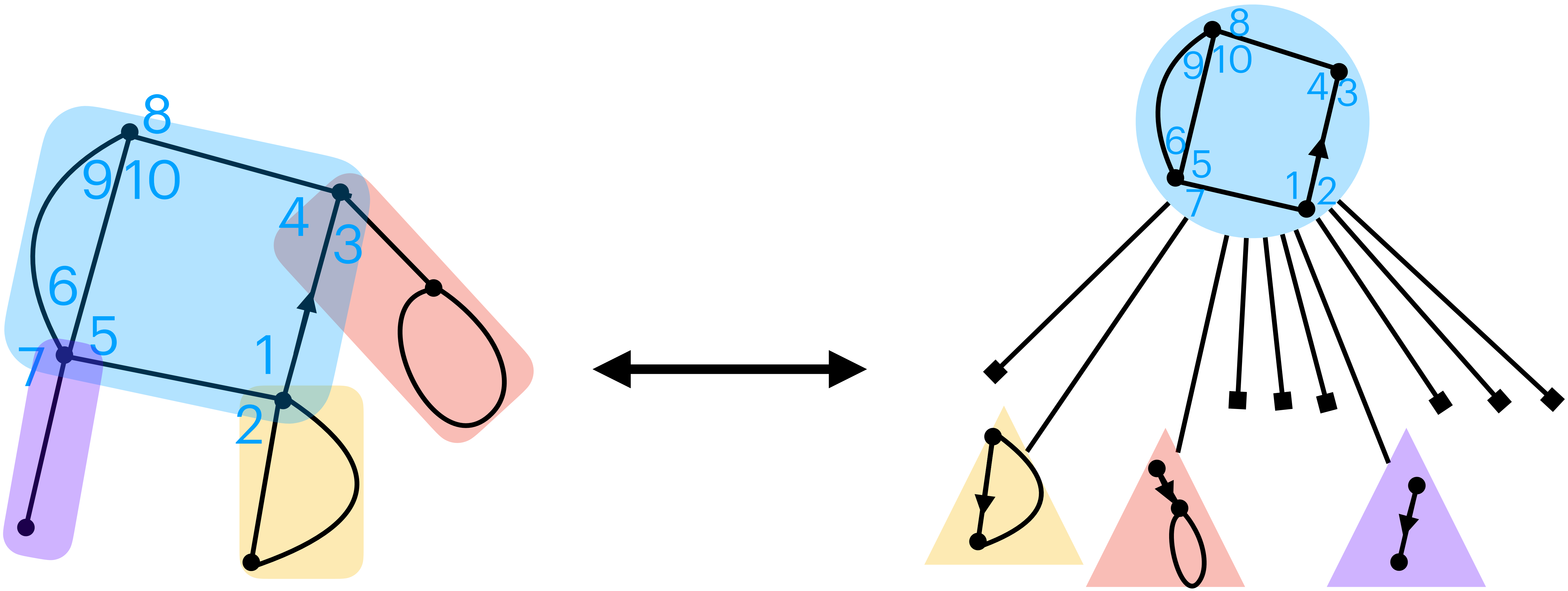


$$M(z) = B(zM^2(z))$$

GS of maps

GS of 2-connected maps

For $M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$

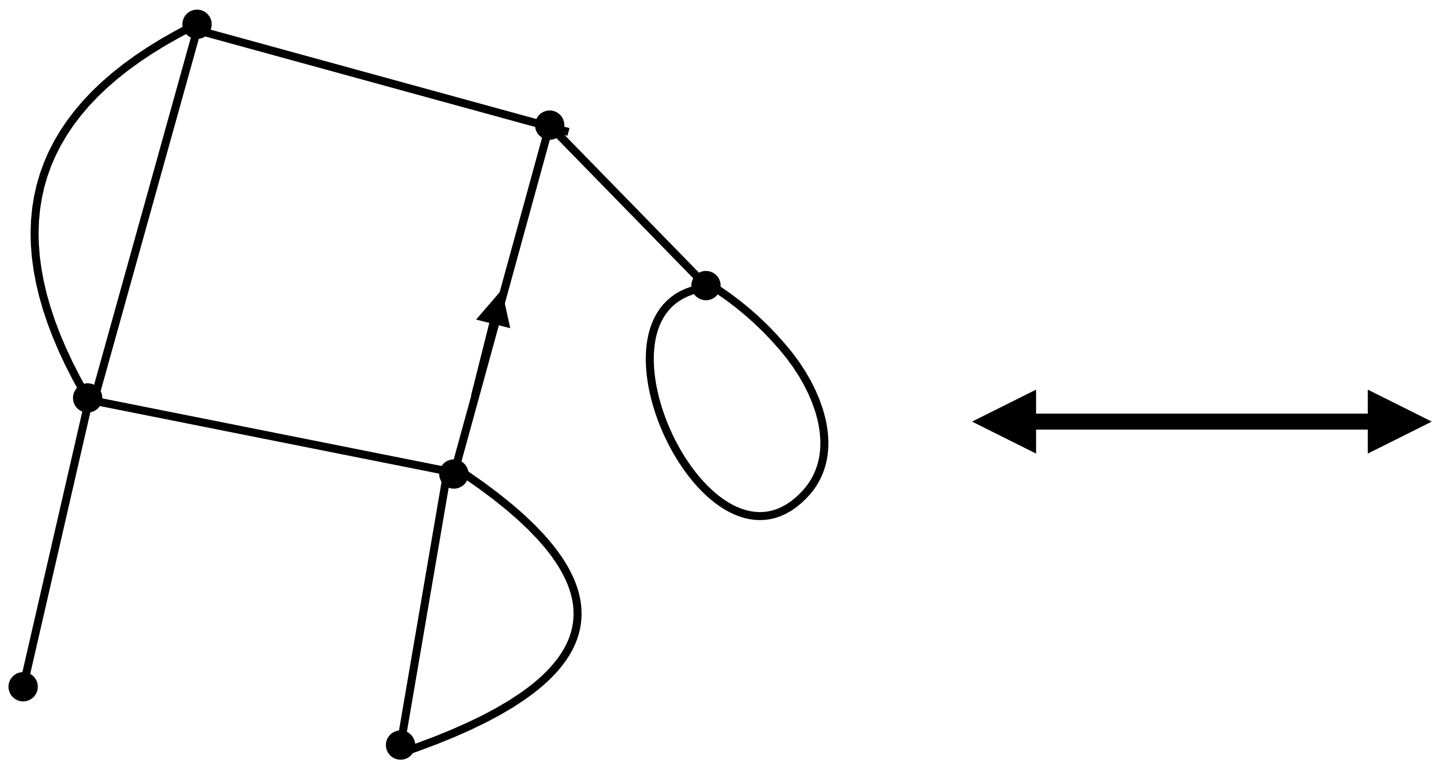
$$M(z,u) = uB(zM^2(z,u)) + 1 - u$$

[Tutte 1963]

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one			
Scaling limit of M_n			

Decomposition of a map into blocks (2/2)



⇒ Underlying block tree structure.

[Tutte 1963]

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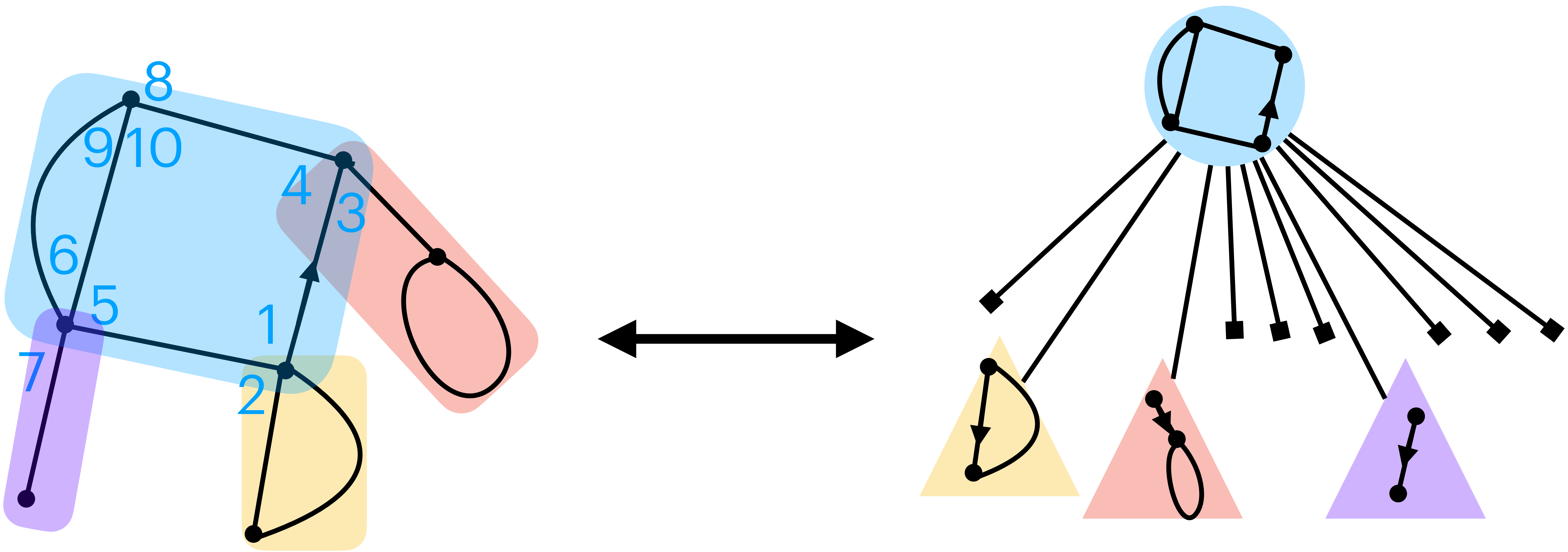

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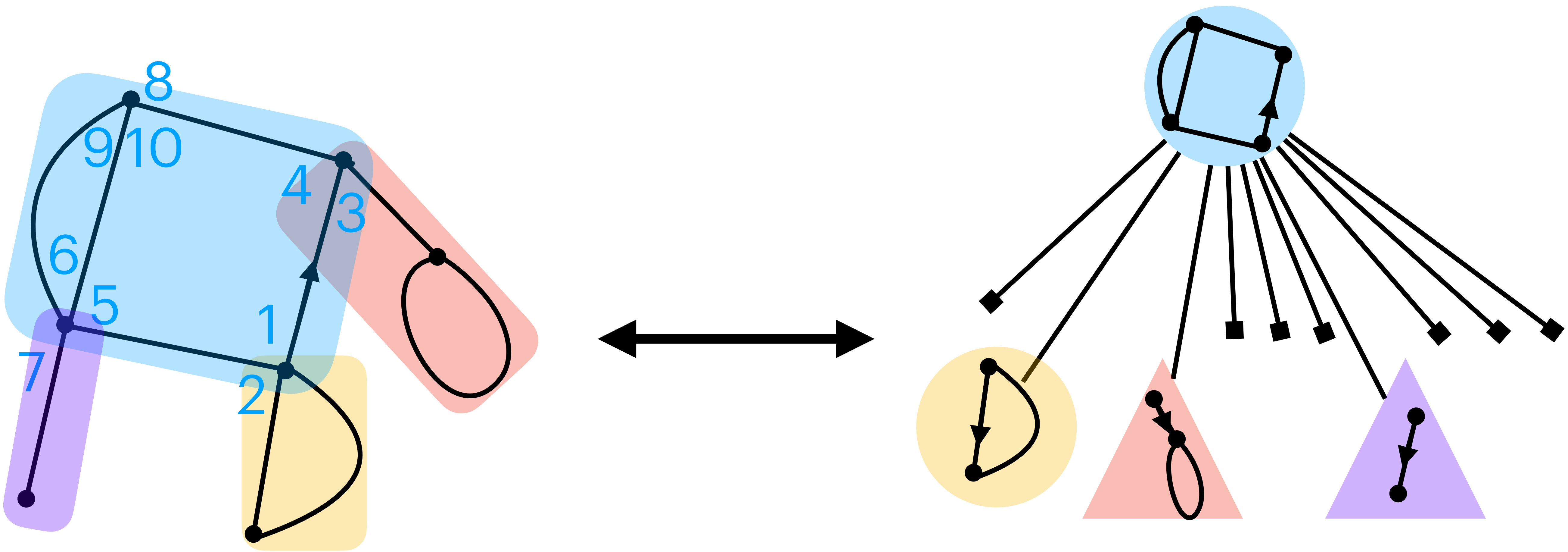

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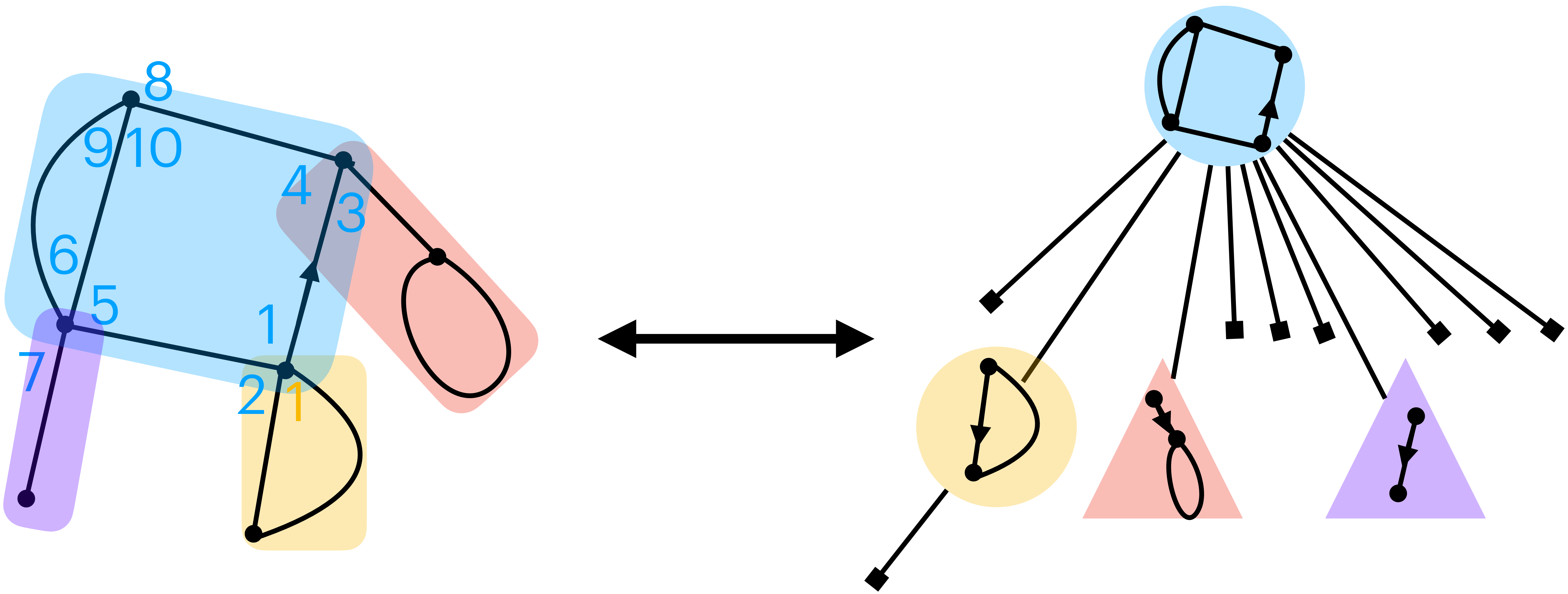

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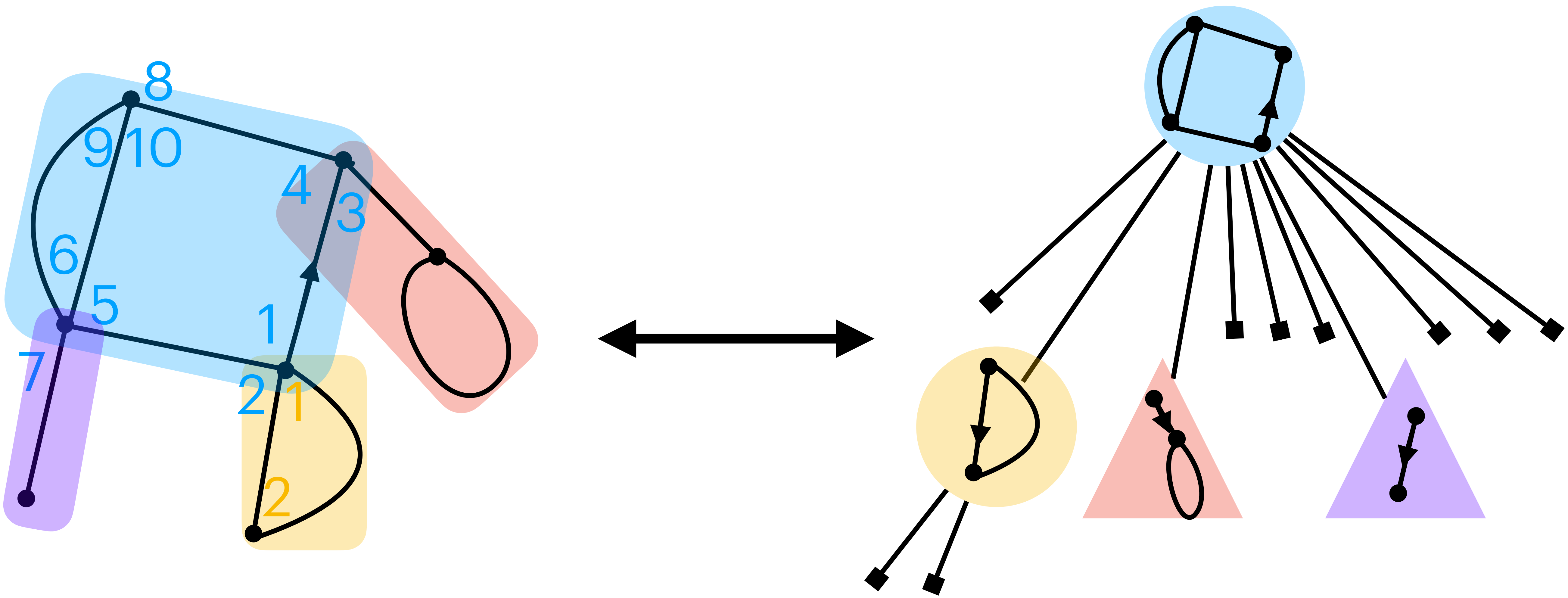

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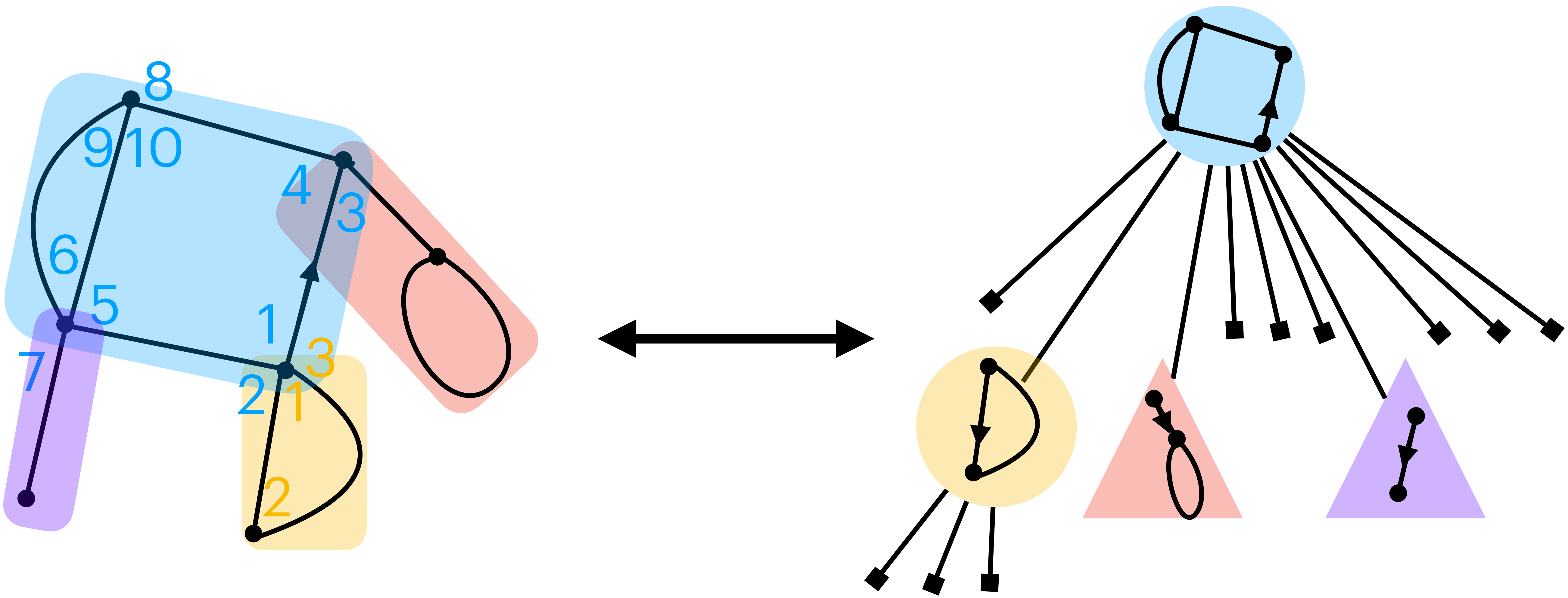

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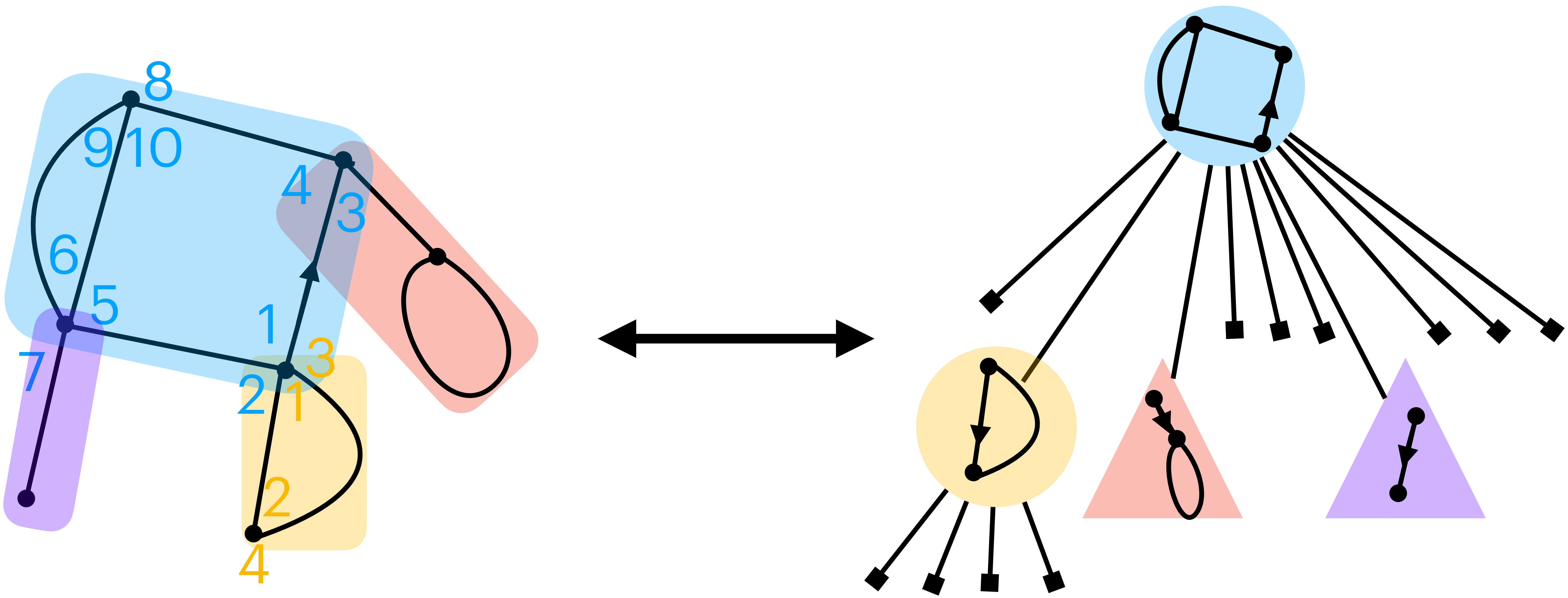

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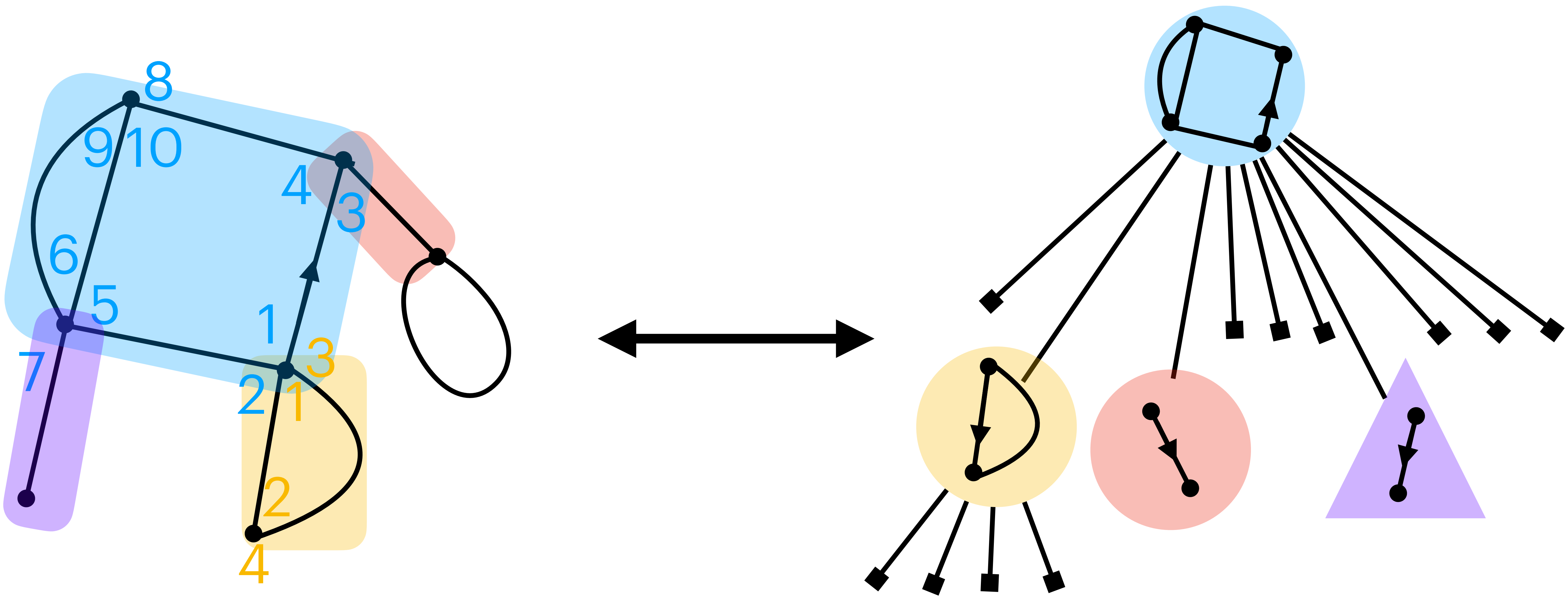

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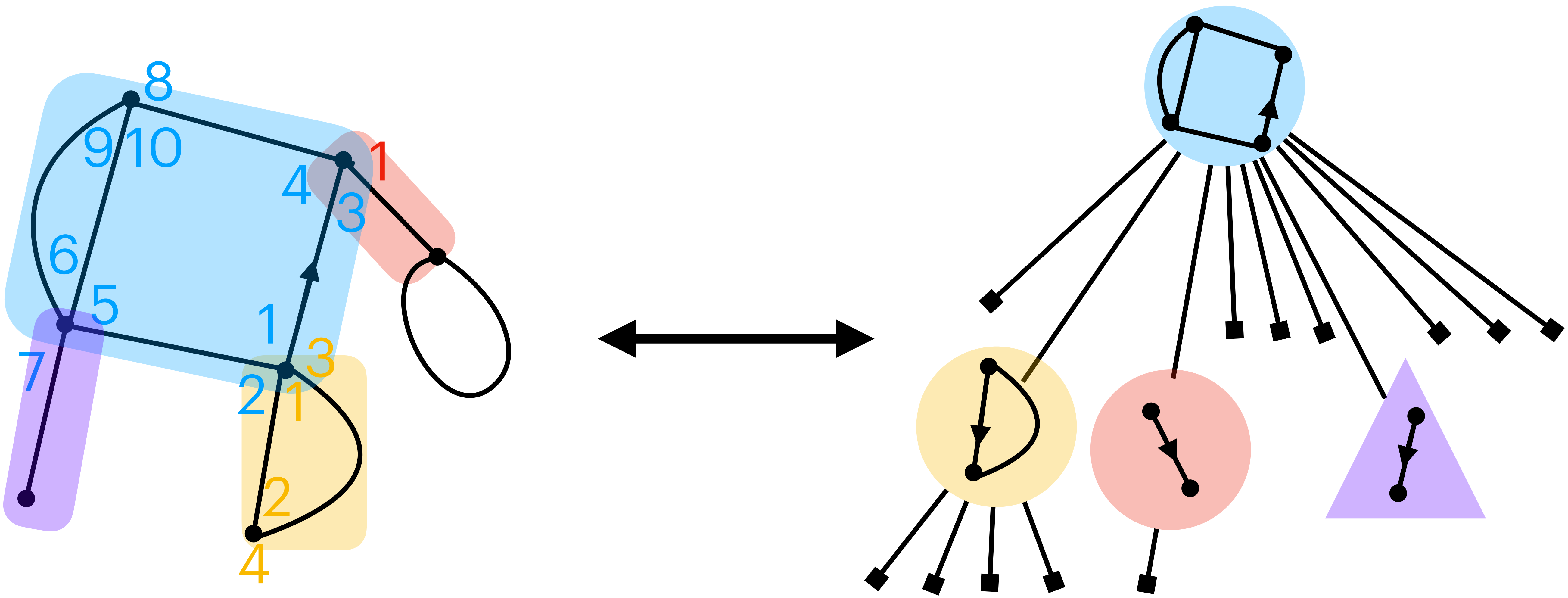

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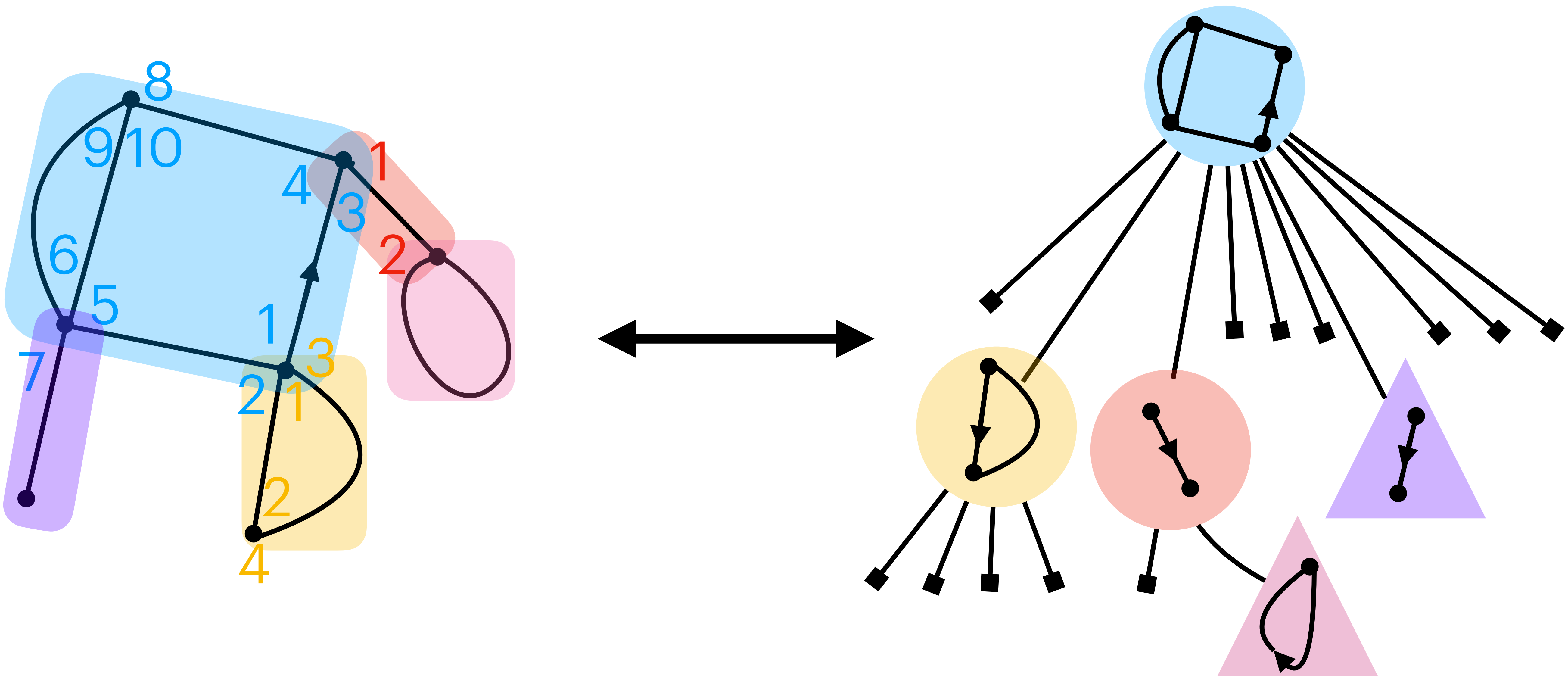

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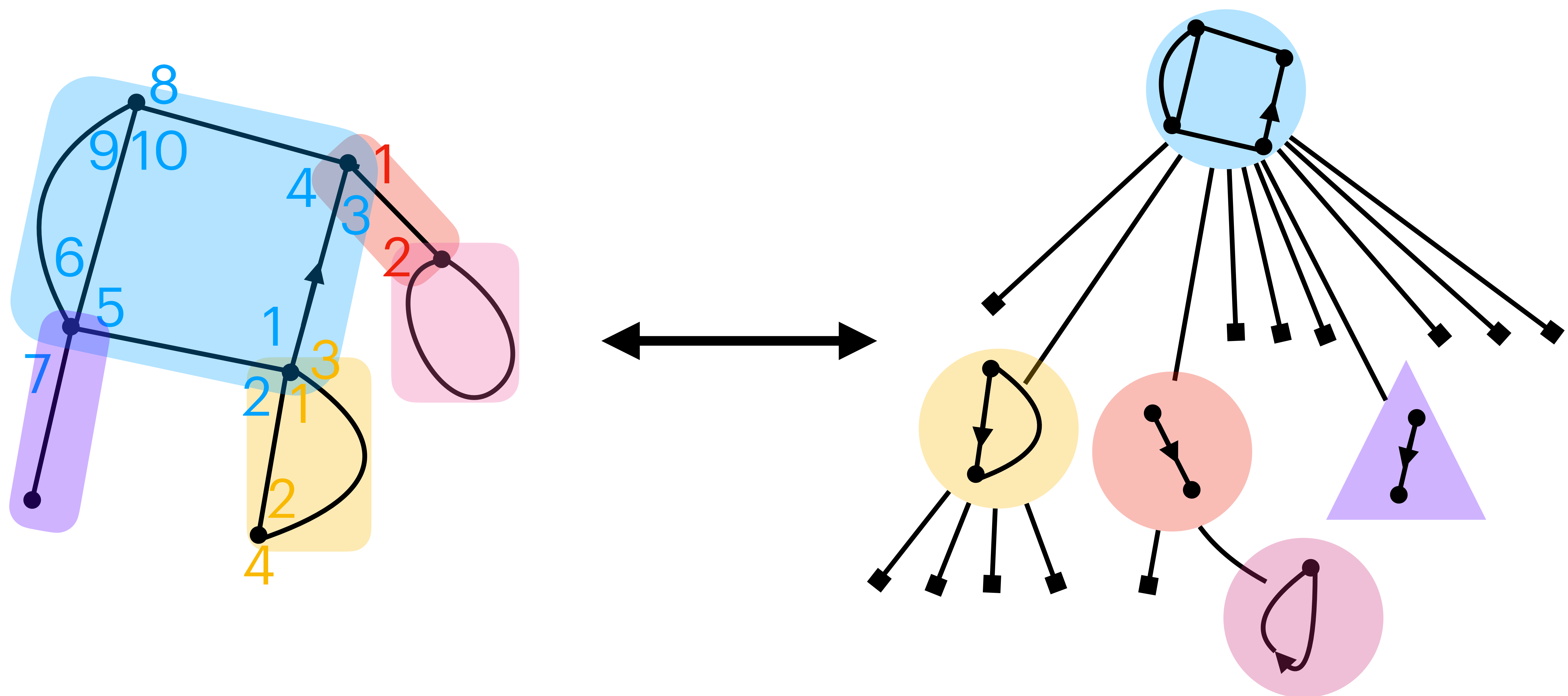

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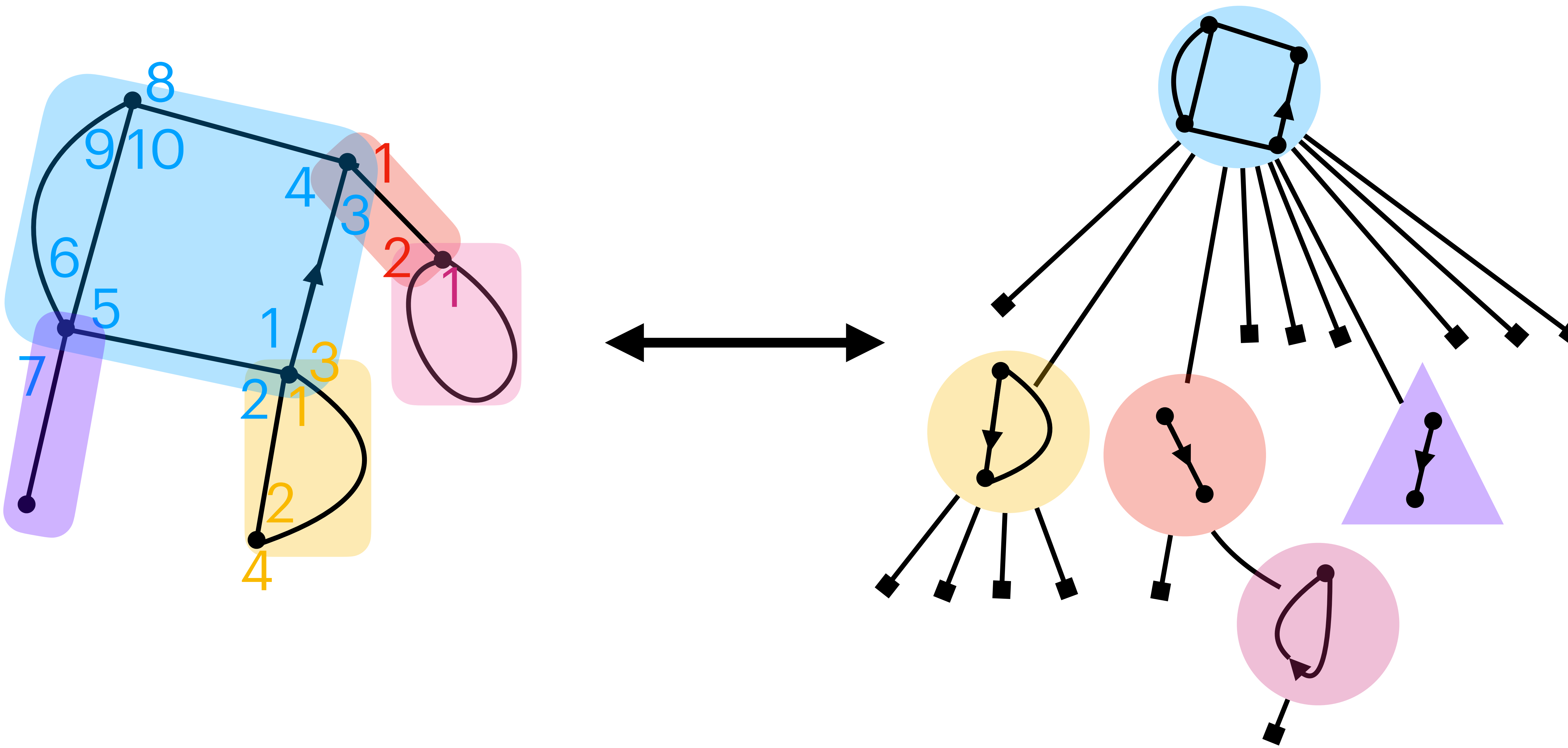

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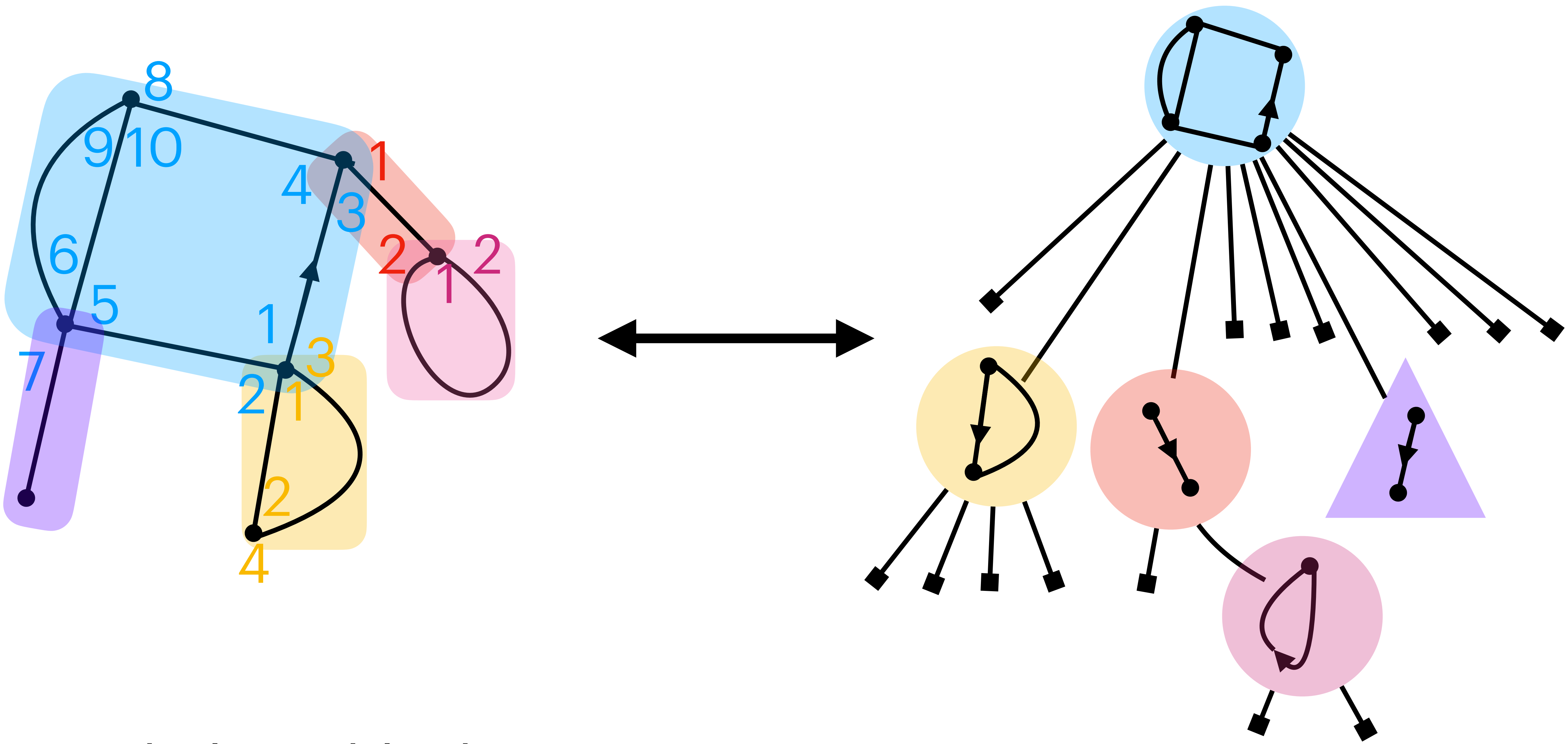

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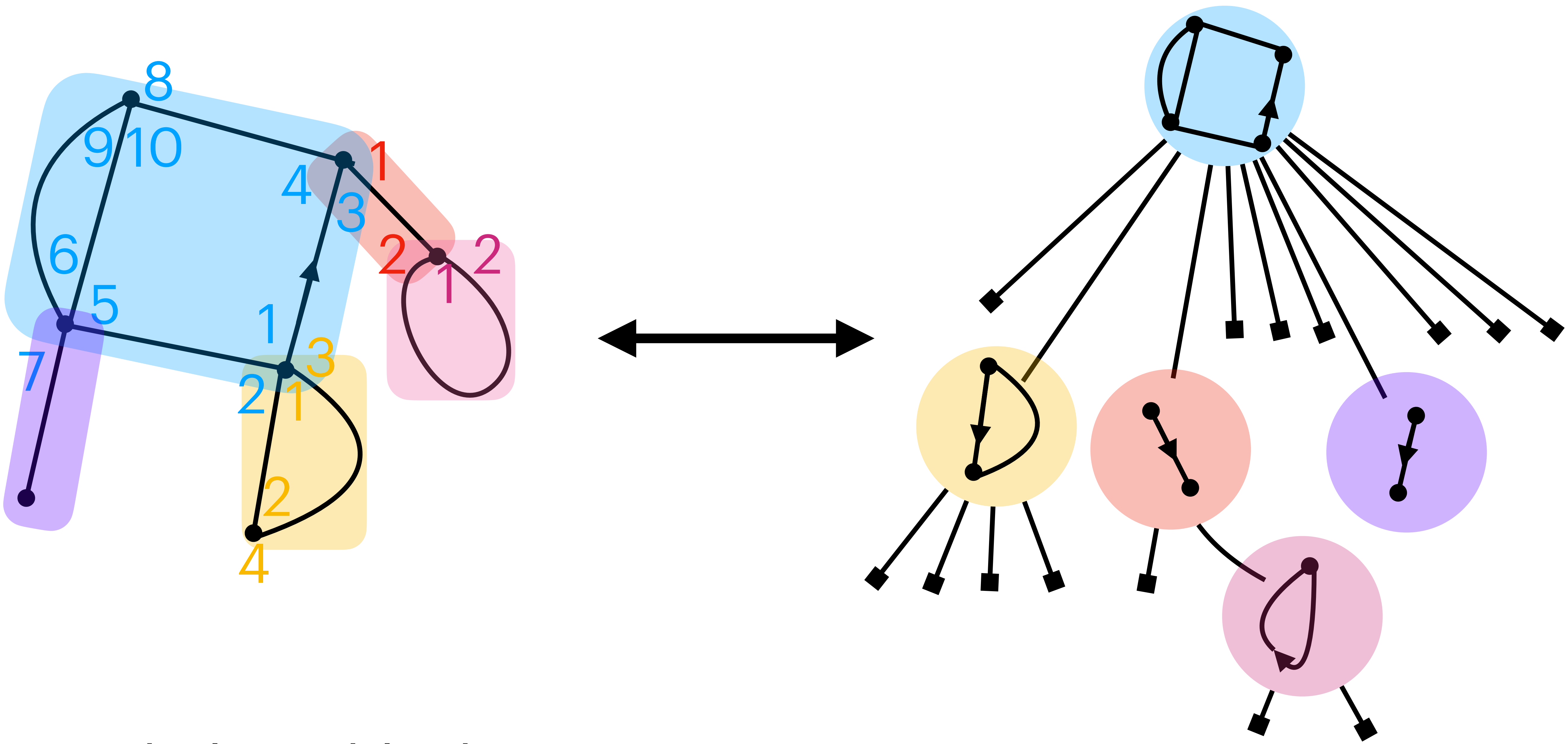

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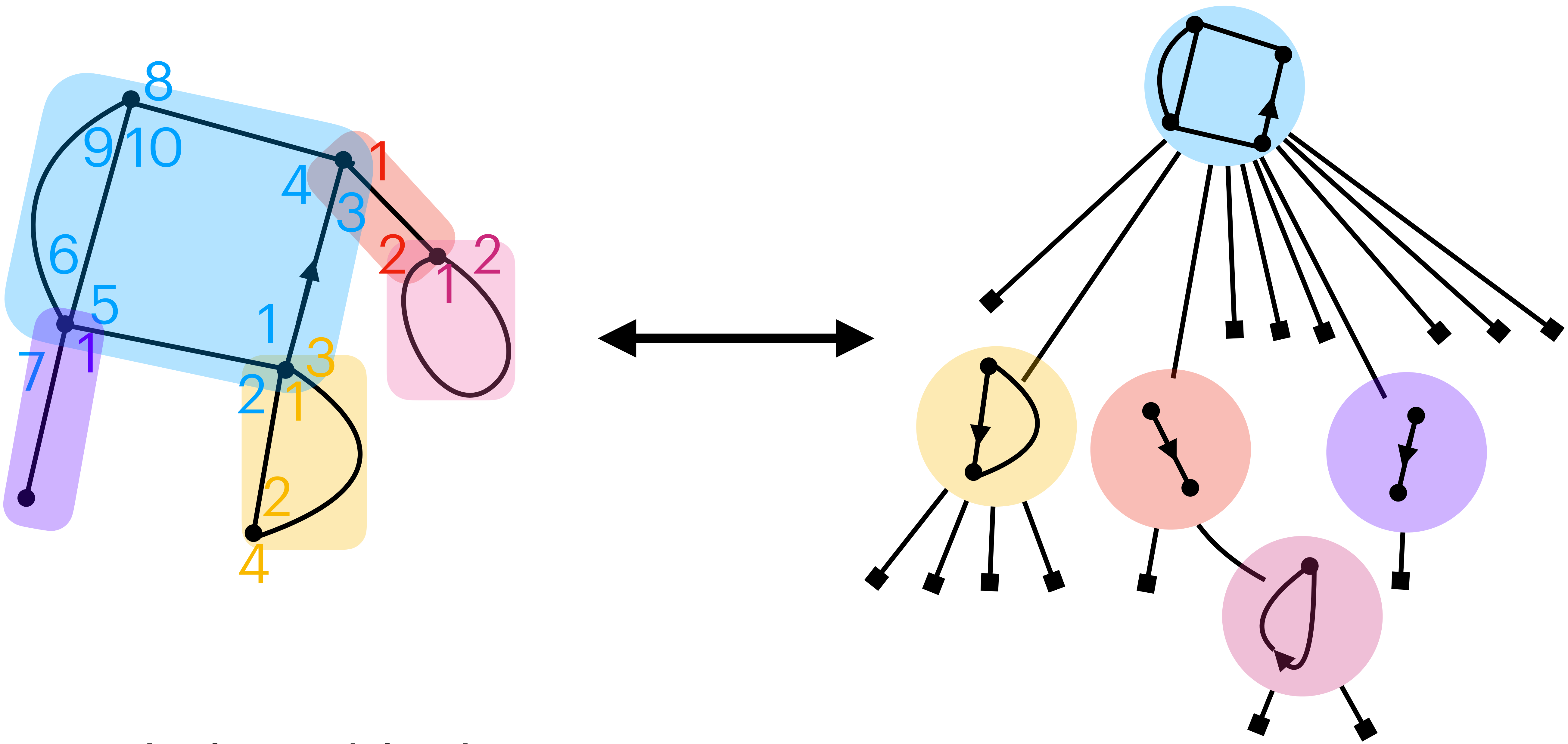

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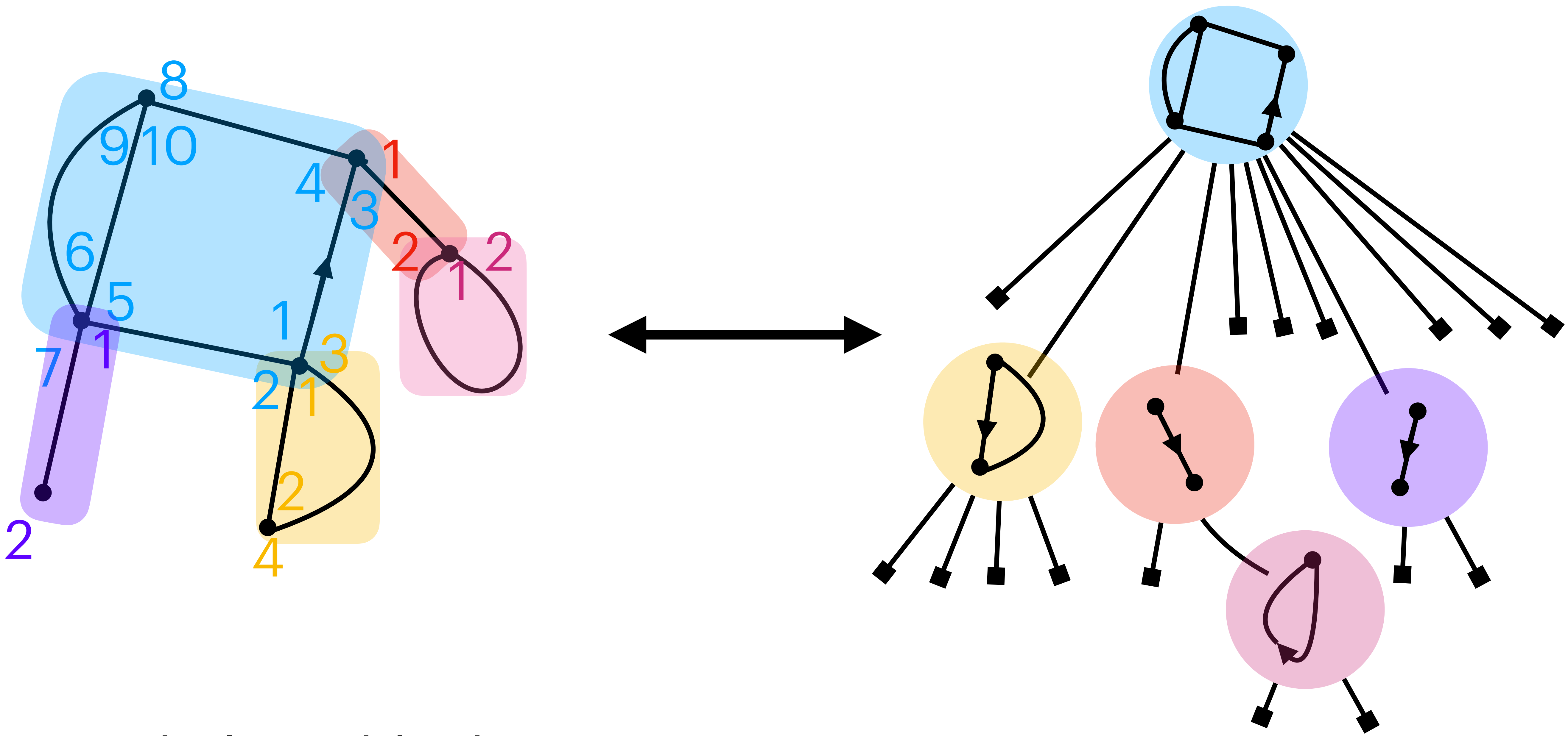

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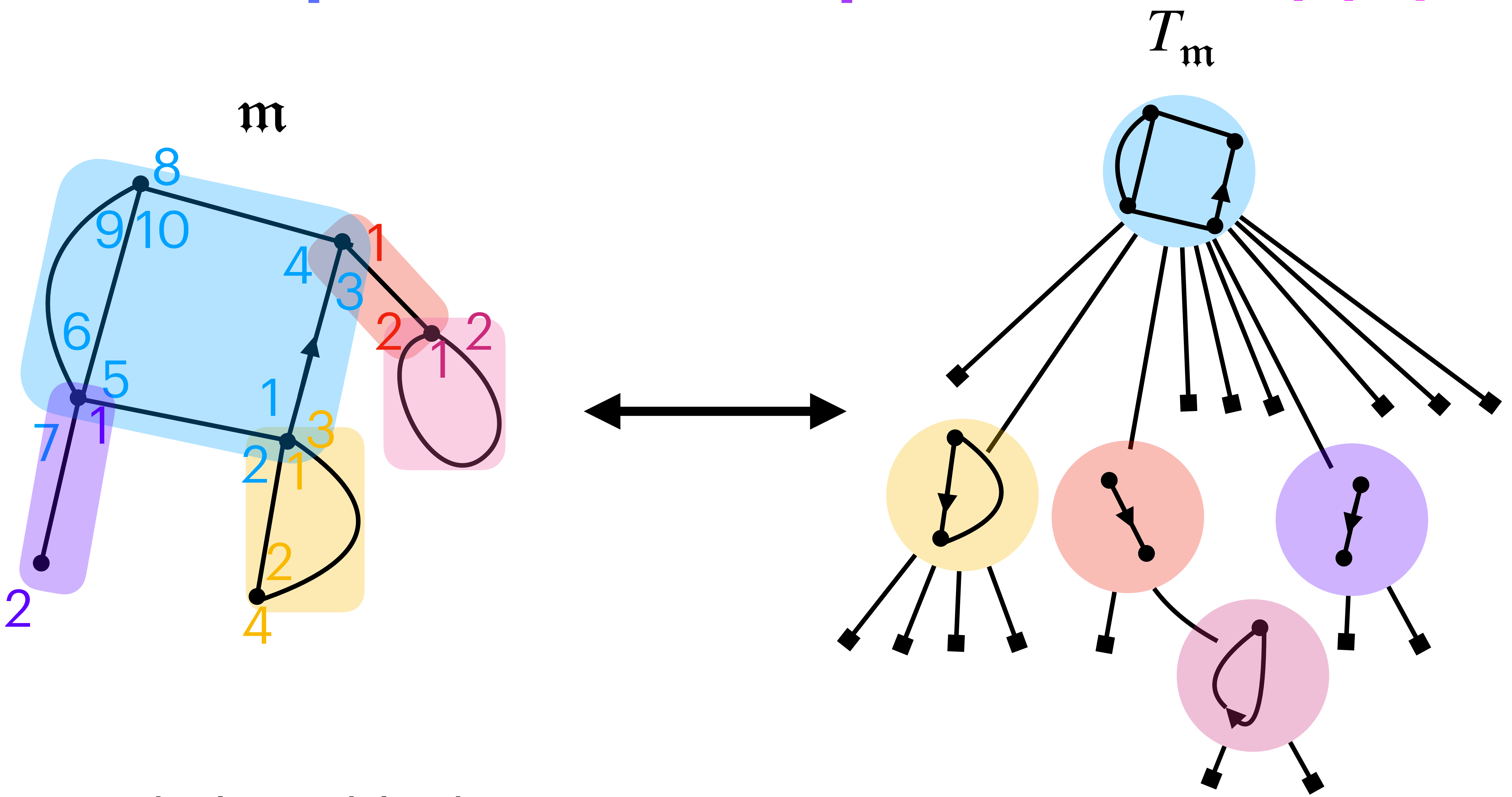

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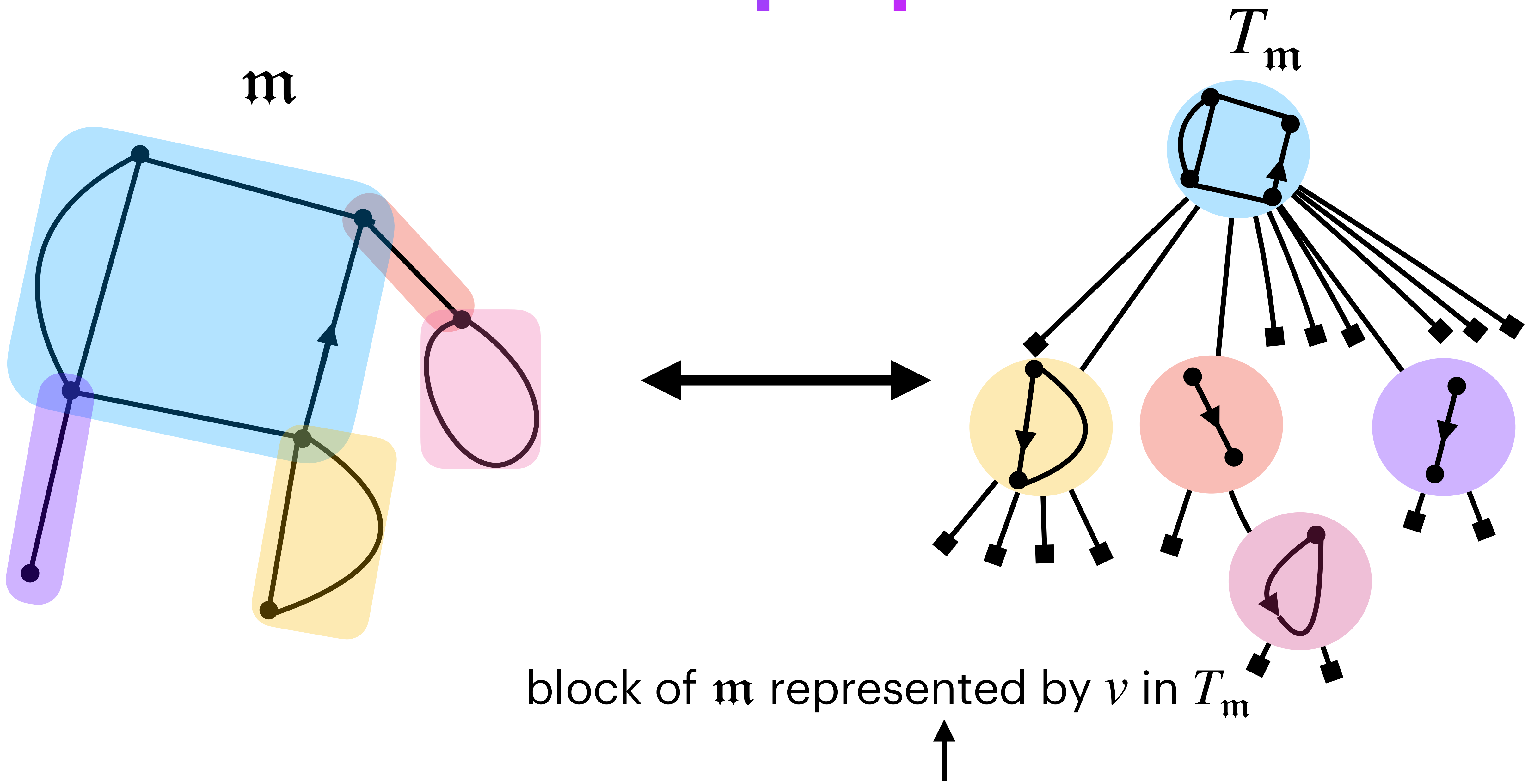
Block tree: properties



- $\mathfrak{m}$ is entirely determined by $T_{\mathfrak{m}}$ and $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$
- Internal node (with k children) of $T_{\mathfrak{m}} \leftrightarrow$ block of $\mathfrak{m}$ of size $k/2$

T_{M_n} gives the block sizes of a random map M_n

Block trees are BGW-trees

For μ probability law on $\mathbb{N}$, μ-Bienaymé-Galton-Watson (BGW) tree: random tree where the number of children of each node is given by μ independently.

Theorem [Fleurat, S. 24] $u > 0$

If $M_n \hookrightarrow \mathbb{P}_{n,u}$ and $y \in (0, \rho_B]$, then T_{M_n} has the law of a BGW tree of reproduction law $\mu^{y,u}$ conditioned to be of size $2n$, with

$$\mu^{y,u}(\{2k\}) = \frac{b_k y^k u^{\mathbf{1}_{k \neq 0}}}{uB(y) + 1 - u}$$

Generalisation of [Addario-Berry 2019]

Conditioning the BGW trees

When is the probability of the tree having size $2n$ not exponentially small, *cf* can one have y s.t. $\mathbb{E}(\mu^{y,u}) = 1$? (critical BGW)

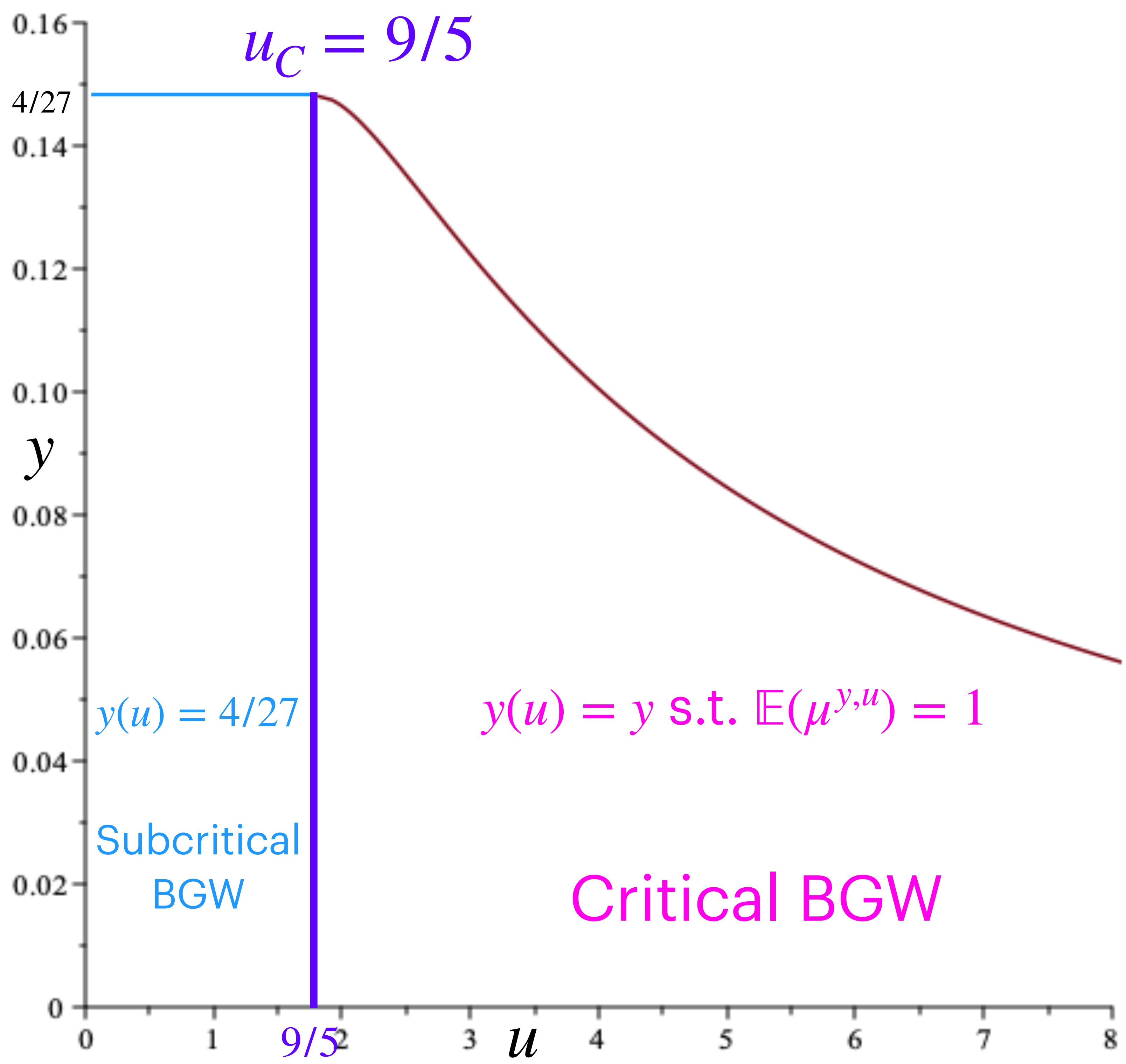
$$\mathbb{E}(\mu^{y,u}) = 1 \Leftrightarrow u = \frac{1}{2yB'(y) - B(y) + 1}$$

covers $[9/5, +\infty)$ when y covers $(0, \rho_B = 4/27]$

=> $\mathbb{E}(\mu^{y,u}) = 1$ is possible iff $u \geq 9/5$

Phase transition for $y(u)$

Set $y = y(u)$ in the following way:



Largest degrees of a BGW tree

	$u < 9/5$	$u = 9/5$	$u > 9/5$
BGW tree	subcritical	critical	
$\mu^{y(u),u}(\{2k\})$	$\sim c_u k^{-5/2}$	$\sim c_u k^{-5/2}$	$\sim c_u \pi_u^k k^{-5/2}$ $\pi_u < 1$
Variance	∞	∞	$< \infty$

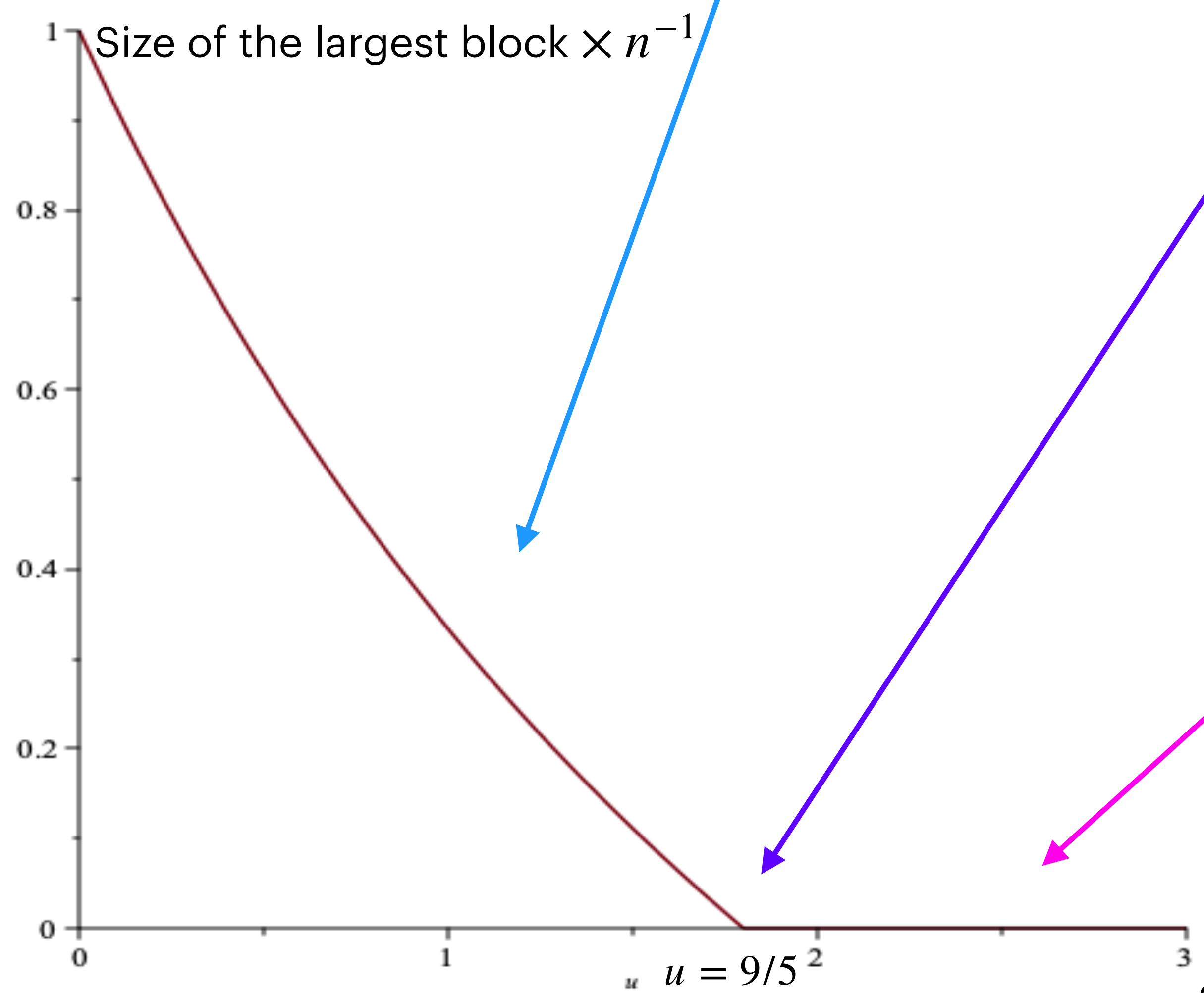
Condensation phenomenon

Largest degree behaves as maximum of independent variables with geometric tail

[Jonsson, Stefánsson 2011; Janson 2012]

Size $L_{n,k}$ of the k-th largest block

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
$L_{n,1}$	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ [Stufler 2020]	$\Theta(n^{2/3})$	$\dfrac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \dfrac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
$L_{n,2}$	$\Theta(n^{2/3})$ [Stufler 2020]		



Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

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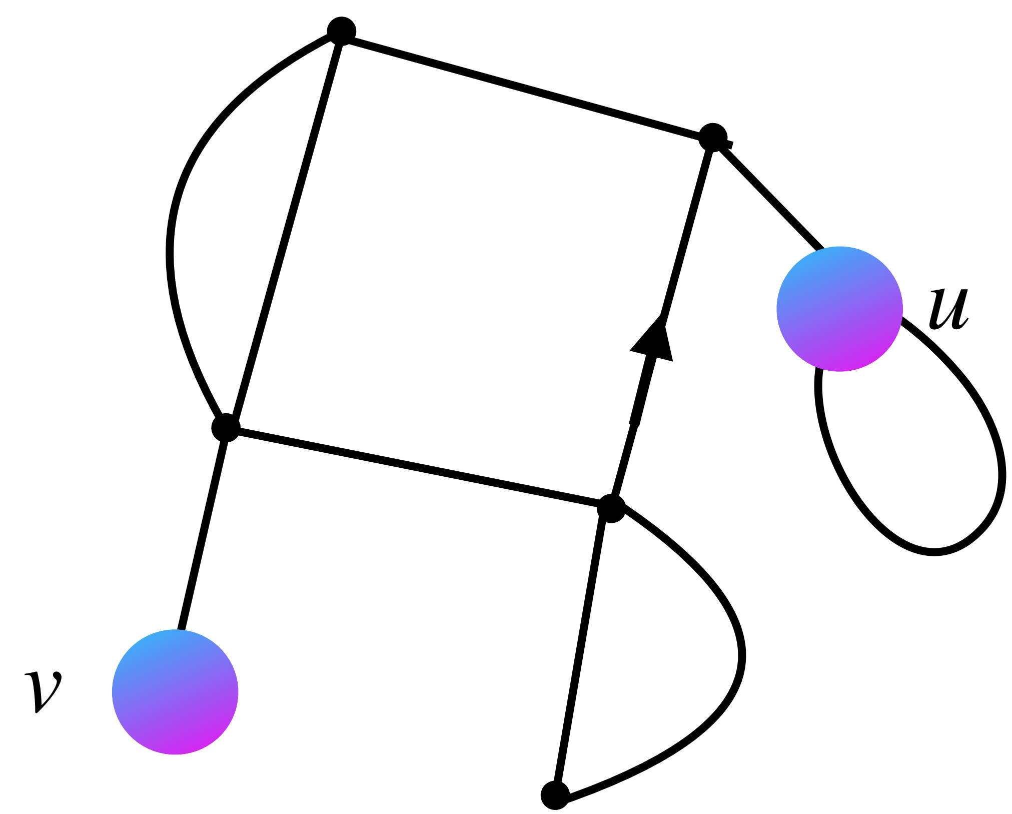
Scaling limit of M_n			

Ordered atoms of a Poisson Point Process

III. Scaling limits

Scaling limits

Convergence of the whole object considered as a (compact) metric space (with the graph distance), after renormalisation.



$d(u, v) = 4$

$M_n \hookrightarrow \mathbb{P}_{n,u}$

What is the limit of the sequence of metric spaces $((M_n,\ d/n^\gamma))_{n\in\mathbb{N}}$?

(Convergence for Gromov-Hausdorff-Prokhorov topology)

Scaling limit of supercritical and critical maps

Lemma For $M_n \hookrightarrow \mathbb{P}_{n,u}$

- If $u > 9/5$,

$$\frac{c(u)}{n^{1/2}} T_{M_n} \xrightarrow{(d)} \mathcal{T}_2$$

Brownian tree

- If $u = 9/5$,

$$\frac{c}{n^{1/3}} T_{M_n} \xrightarrow{(d)} \mathcal{T}_{3/2}$$

Stable tree

Proof Known scaling limits of critical BGW trees

- with finite variance [Aldous 1993, Le Gall 2006];
- infinite variance and polynomial tails [Duquesne 2003].

Scaling limit of supercritical and critical maps

Lemma For $M_n \hookrightarrow \mathbb{P}_{n,u}$

- If $u > 9/5$,

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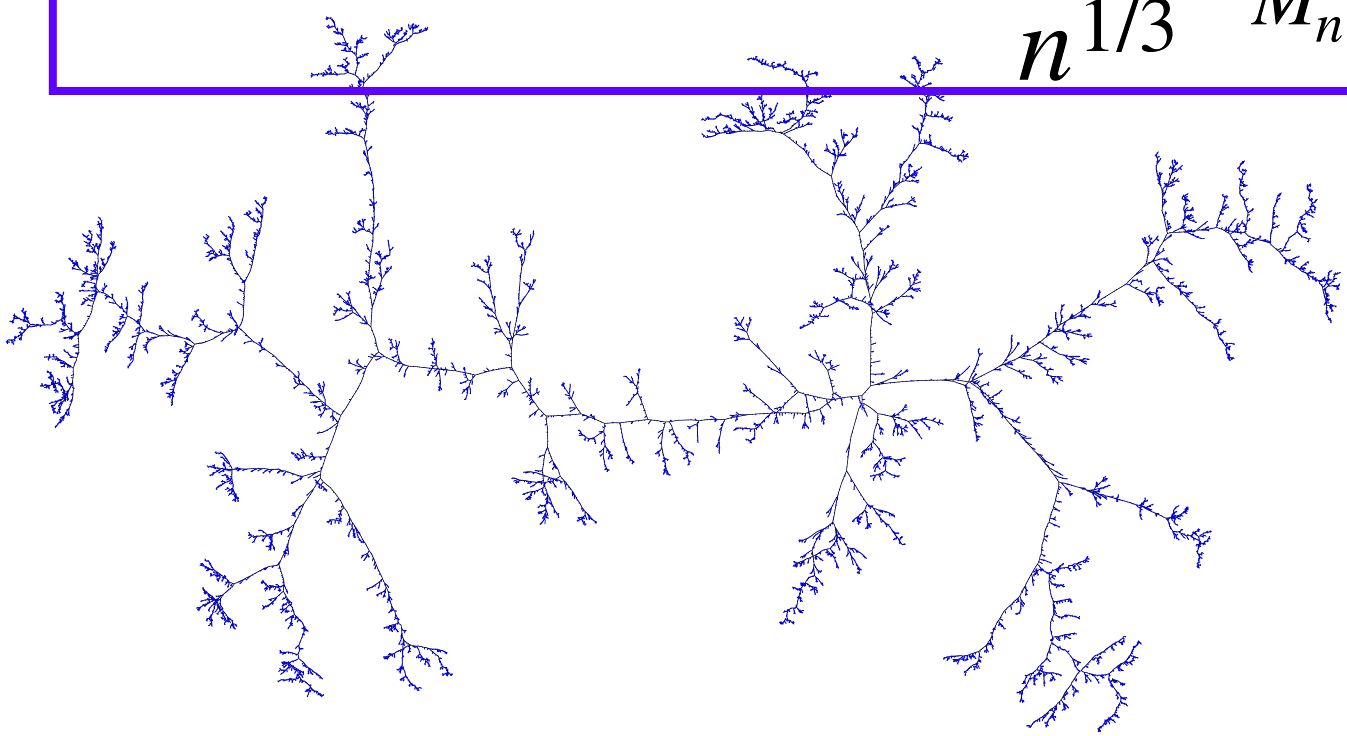
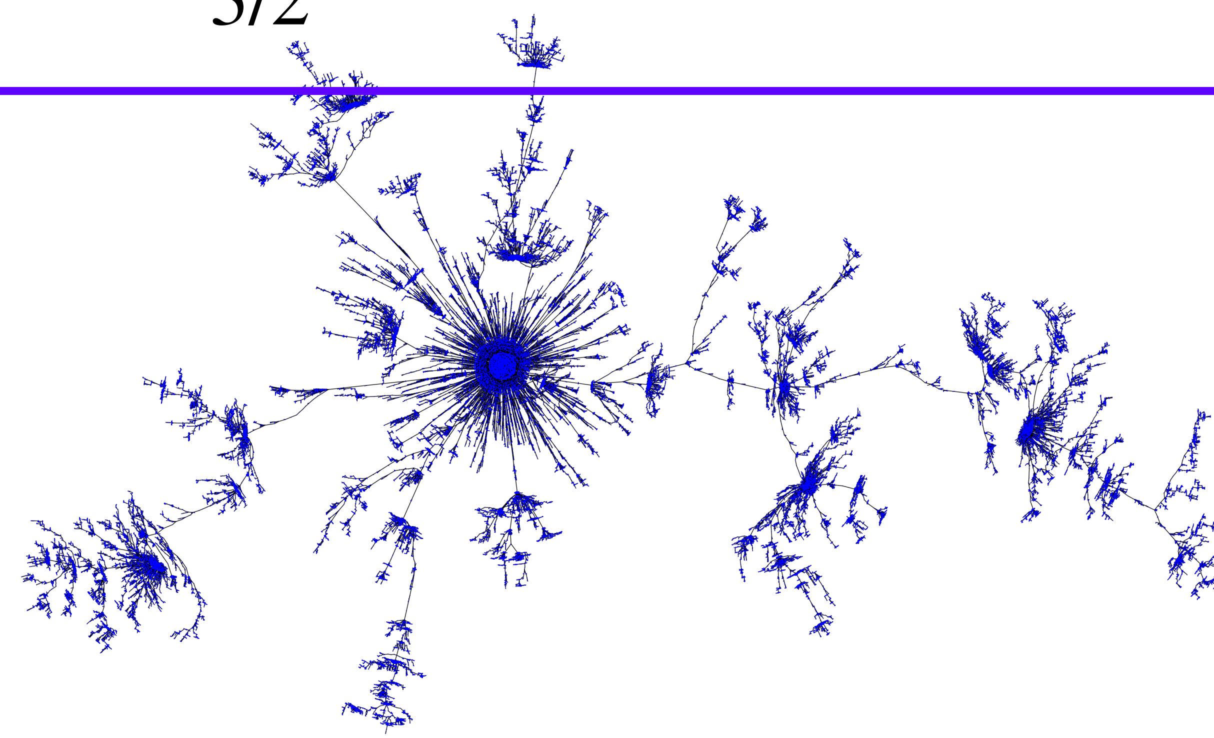


Brownian Tree $\mathcal{T}_2$



Stable Tree $\mathcal{T}_{3/2}$

Scaling limit of supercritical and critical maps

Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u}$

- [Stufler 2020] If $u > 9/5$,

$$\frac{c(u)}{n^{1/2}} T_{M_n} \xrightarrow{(d)} \mathcal{T}_2 \qquad \frac{C(u)}{n^{1/2}} M_n \xrightarrow{(d)} \mathcal{T}_2$$

- [Fleurat, S. 24] If $u = 9/5$,

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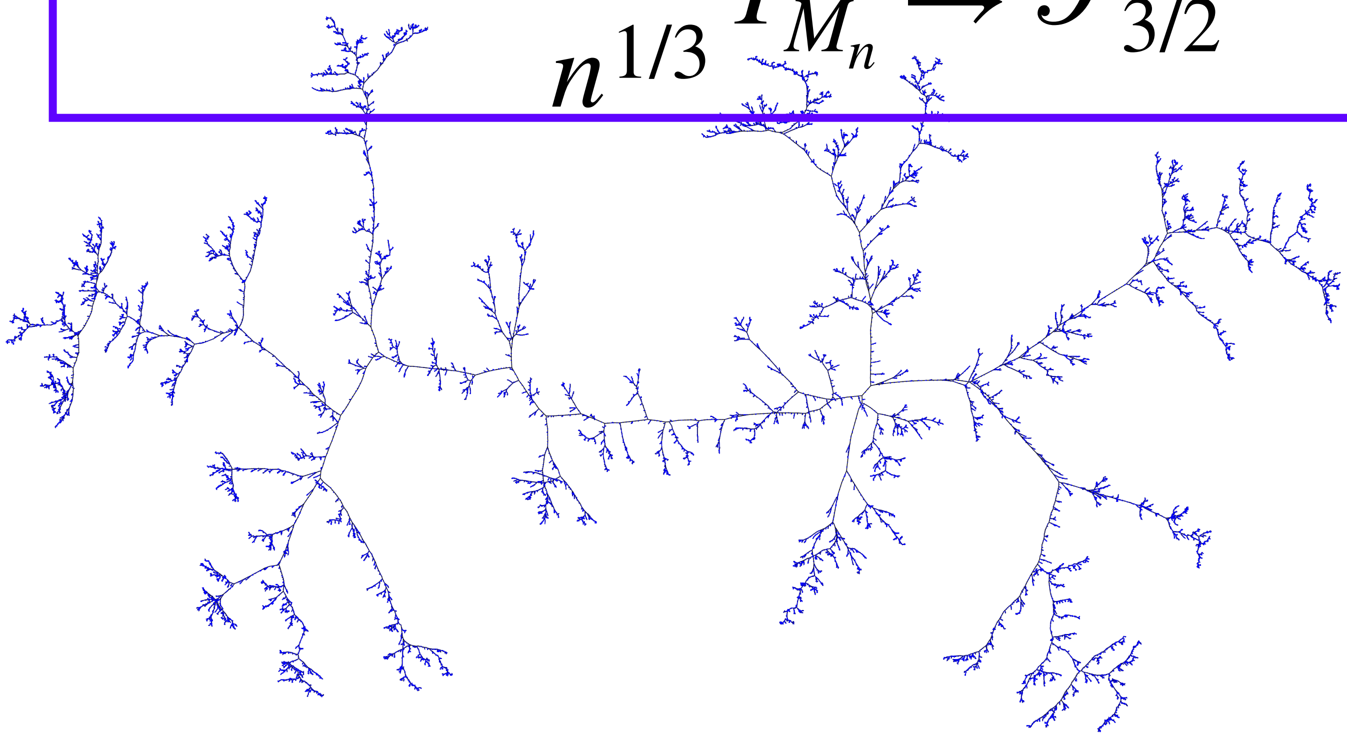

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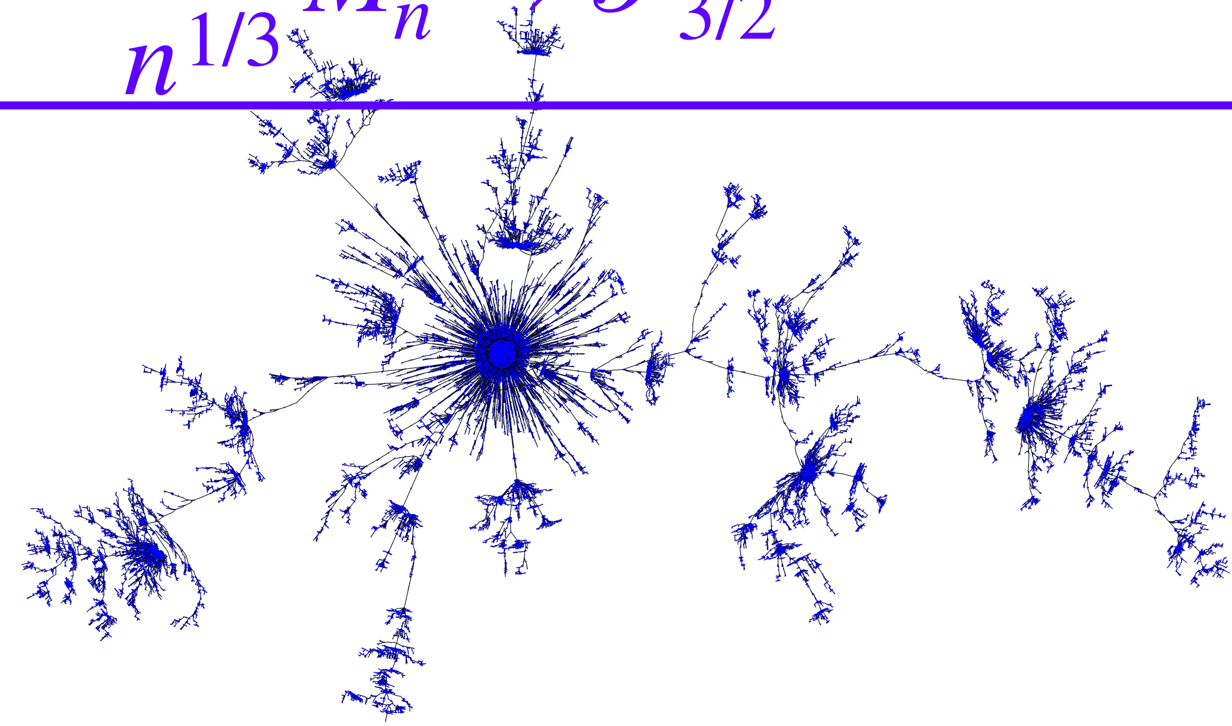

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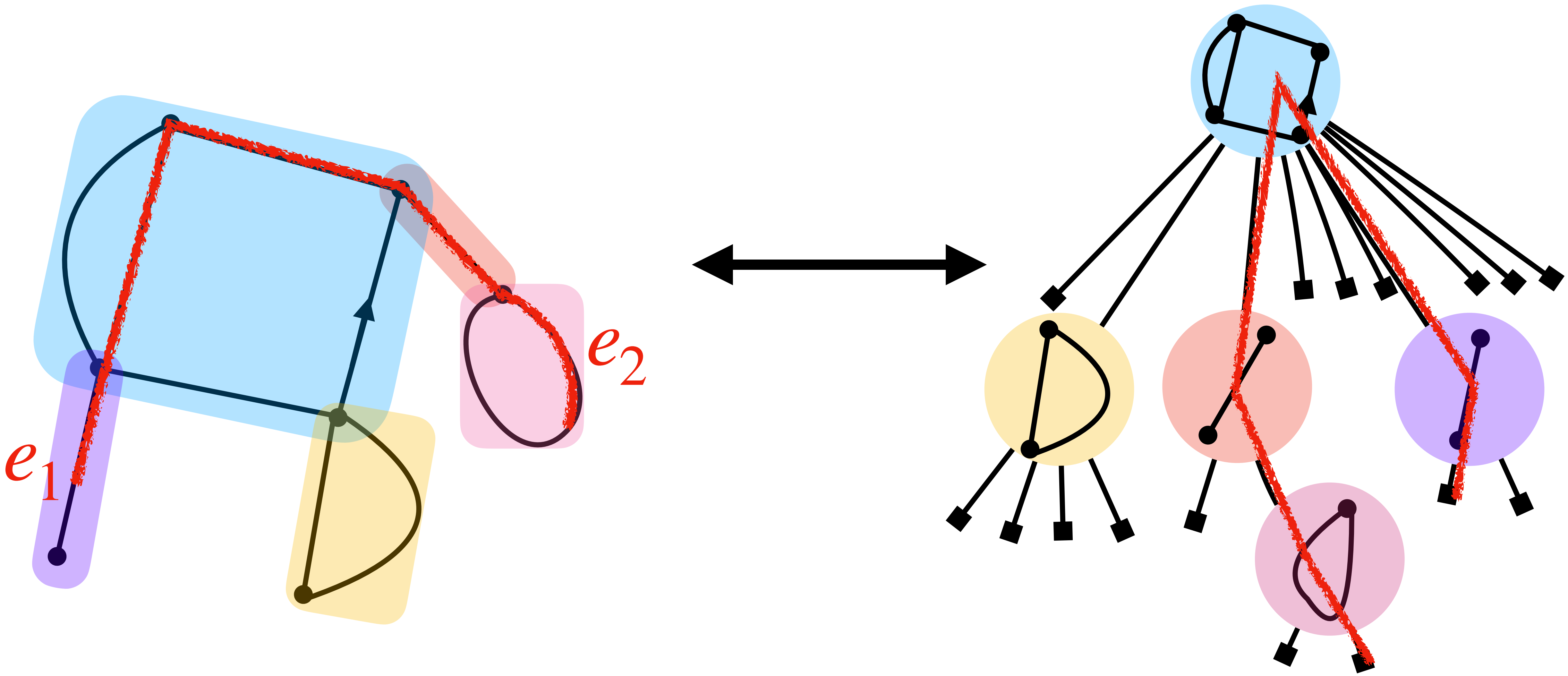
$$\frac{c}{n^{1/3}} T_{M_n} \xrightarrow{(d)} \mathcal{T}_{3/2} \qquad \frac{C}{n^{1/3}} M_n \xrightarrow{(d)} \mathcal{T}_{3/2}$$

Proof Distances in M_n behave like distances in T_{M_n}!

Supercritical and critical cases

Goal = show that distances in M_n behave like distances in T_{M_n}.



Let $\kappa = \mathbb{E}$("diameter" bipointed block). By a "law of large numbers"-type argument

$$d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2)\,.$$

Difficult for the critical case => large deviation estimates

Supercritical and critical cases

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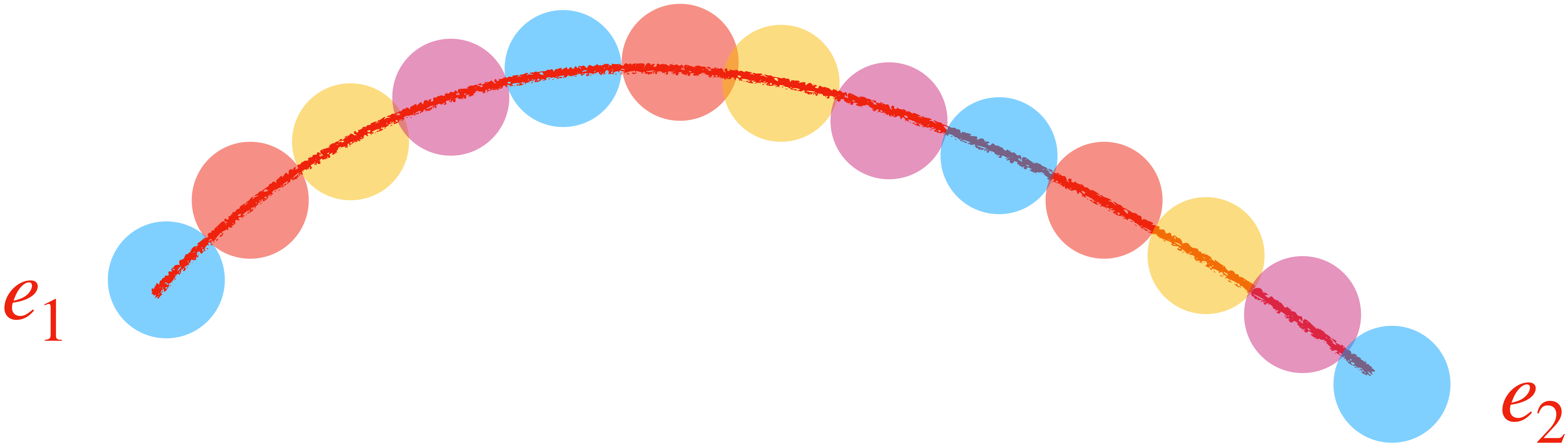

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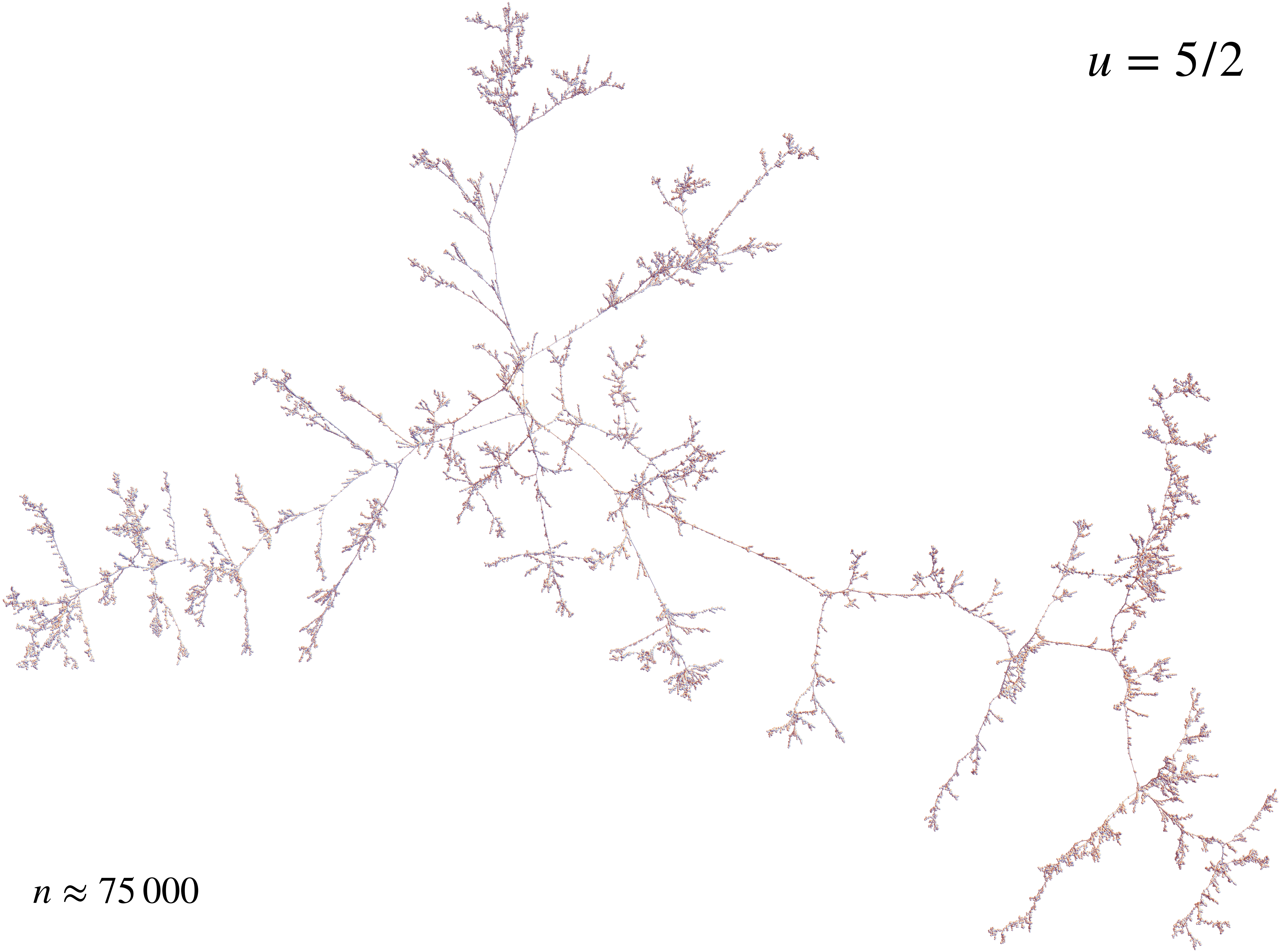
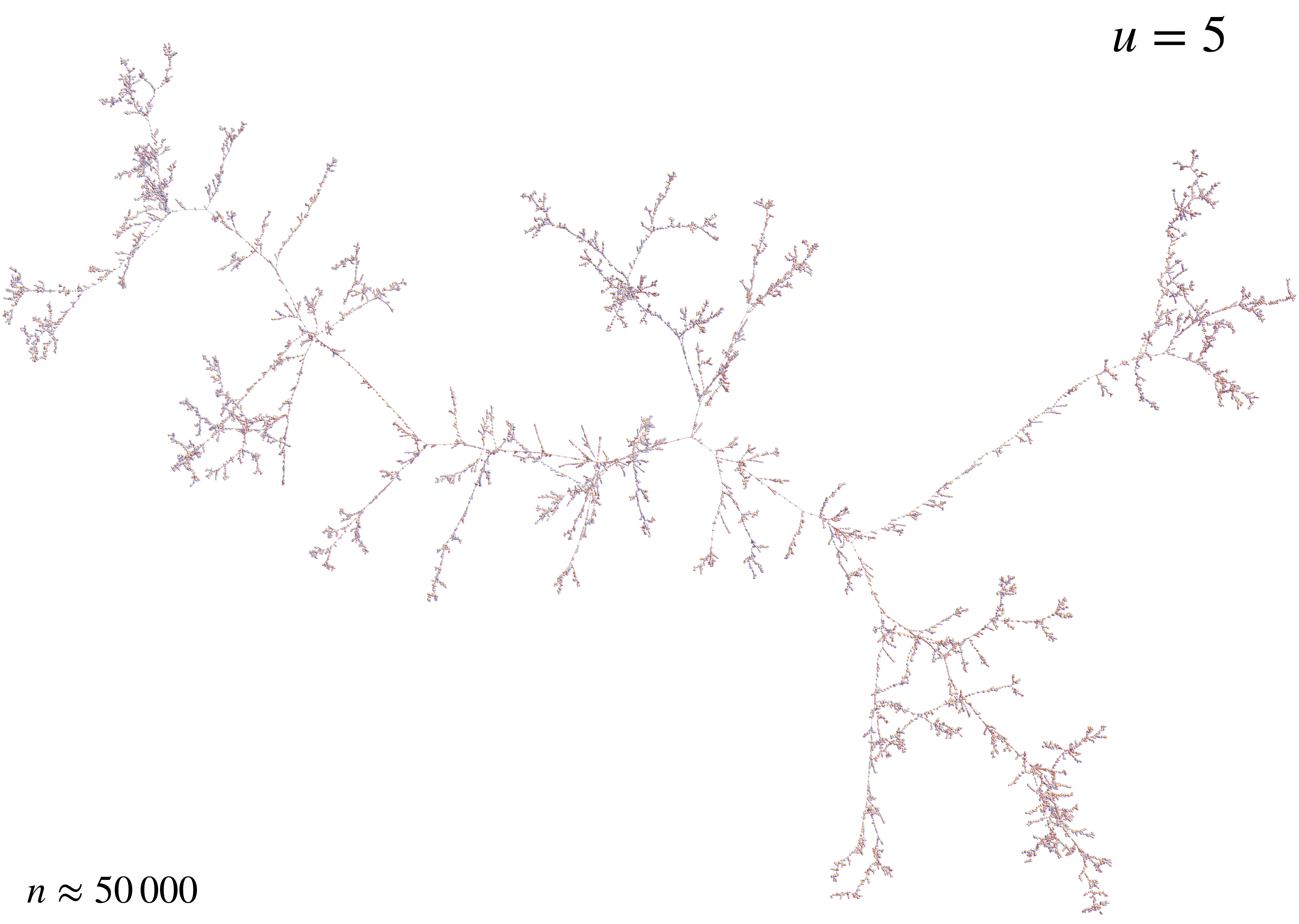
Difficult for the critical case => large deviation estimates


$u = 9/5$
$n \approx 80\,000$


$u = 5/2$
$n \approx 75\,000$


$u = 5$
$n \approx 50\,000$

Scaling limits of subcritical maps

Theorem [Fleurat, S. 24] If $u < 9/5$, for $M_n \hookrightarrow \mathbb{P}_{n,u}$ and denoting $B(M_n)$ its largest block:

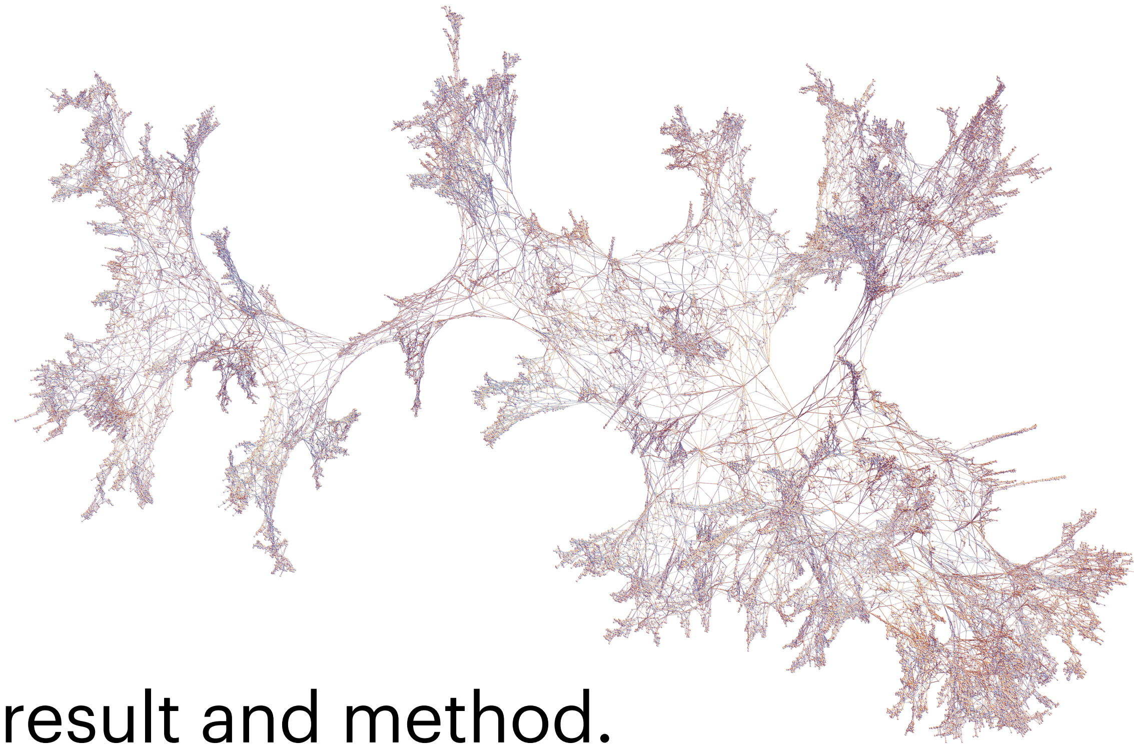
$$d_{GHP}\left(\frac{C(u)}{n^{1/4}}M_n, \frac{1}{n^{1/4}}B(M_n)\right) \xrightarrow{(d)} 0$$

So, if $cn^{-1/4}B_n \xrightarrow{(d)} \mathcal{S}_e$ then

$$\frac{C(u)}{cn^{1/4}}M_n \xrightarrow{(d)} \mathcal{S}_e$$

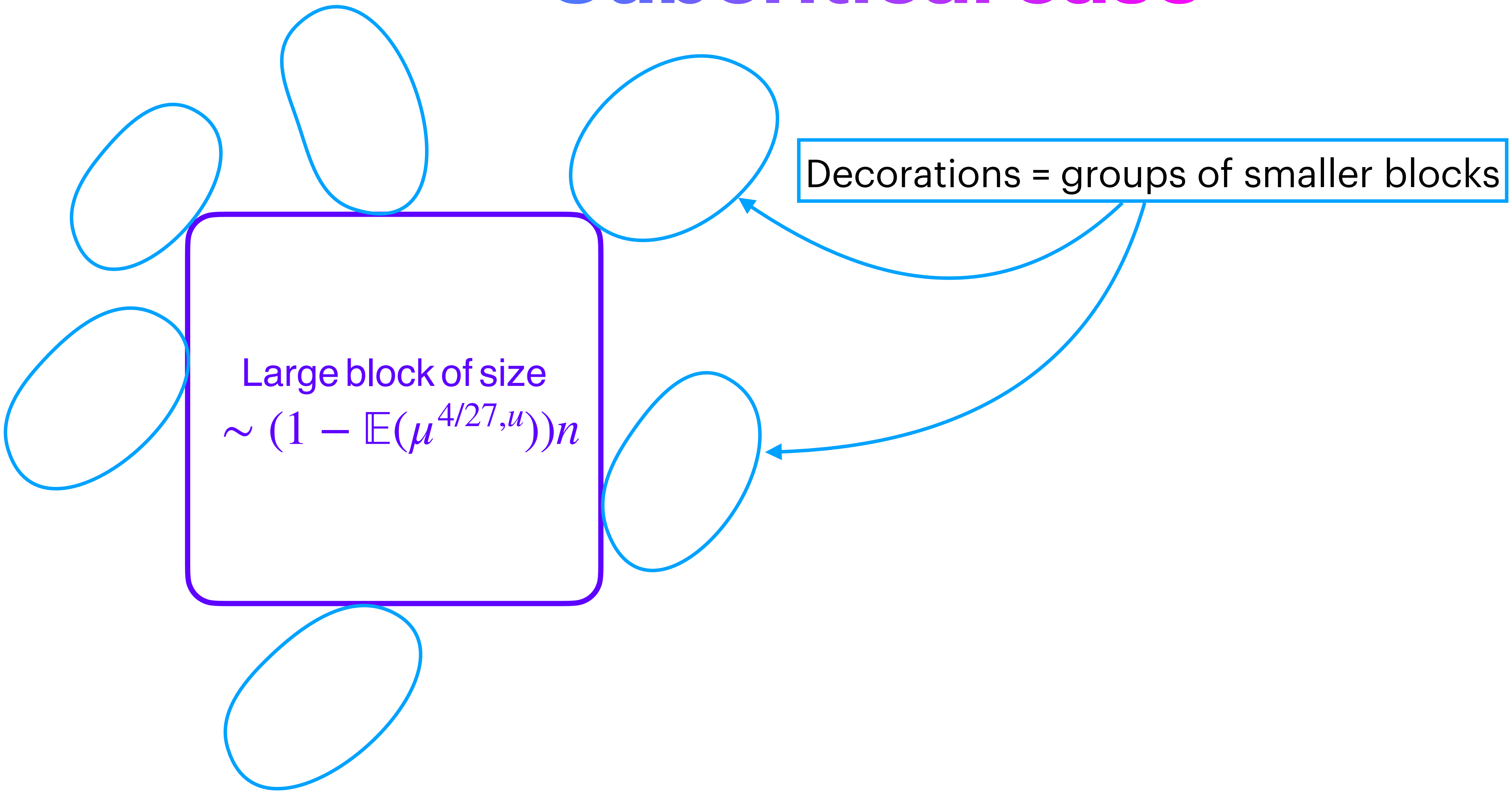
which is assumed for 2-c maps.

Brownian Sphere $\mathcal{S}_e$



See [Addario-Berry, Wen 2019] for a similar result and method.

Subcritical case



Diameter of a decoration ≤ blocks to cross × max diameter of blocks

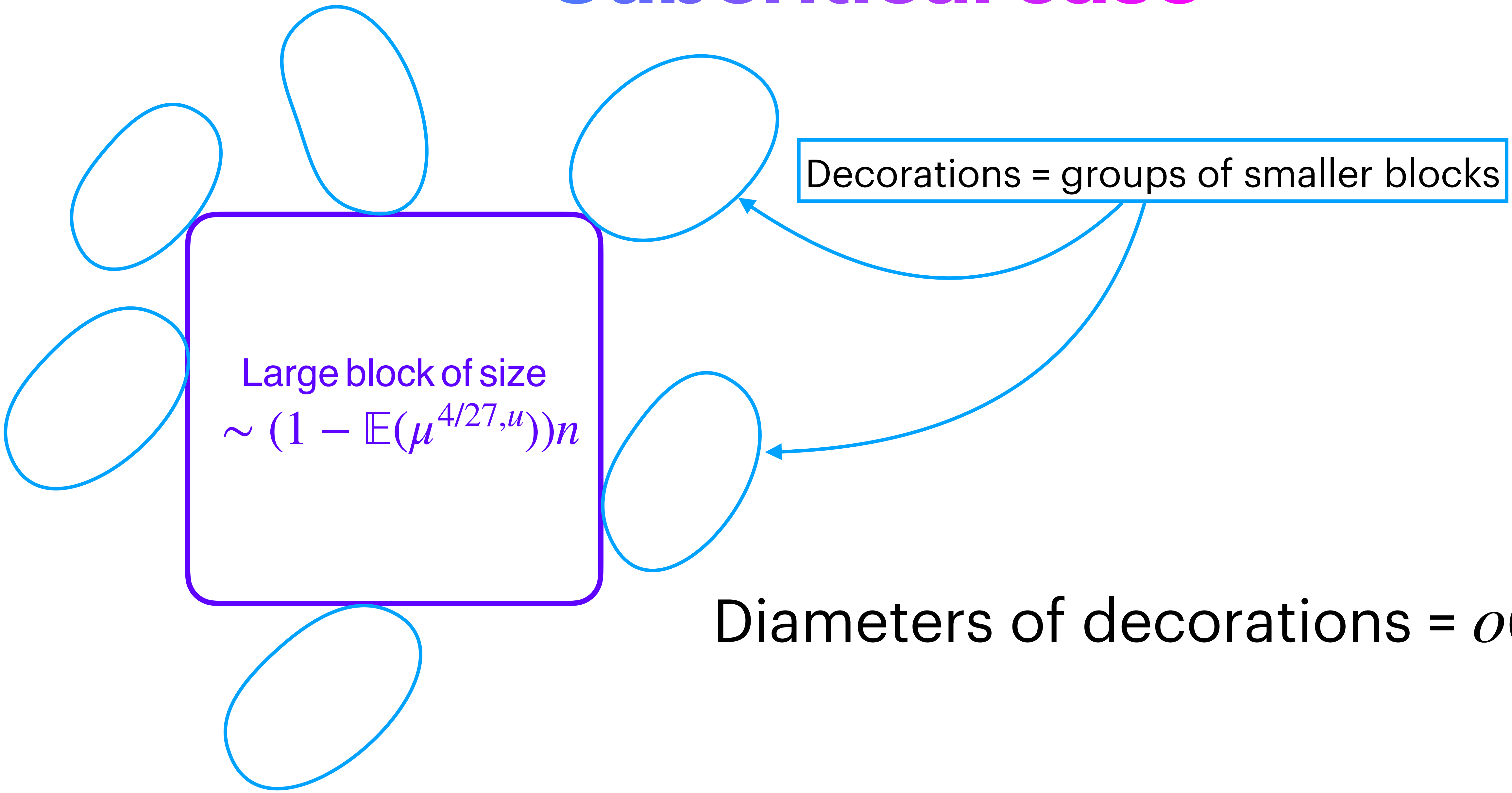
$$\leq diam(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = diam(T_{M_n}) \times O(n^{1/6+\delta})$$

$$= O(n^{1/6+2\delta}) = o(n^{1/4}) \, .$$

T_{M_n} is a subcritical BGW tree

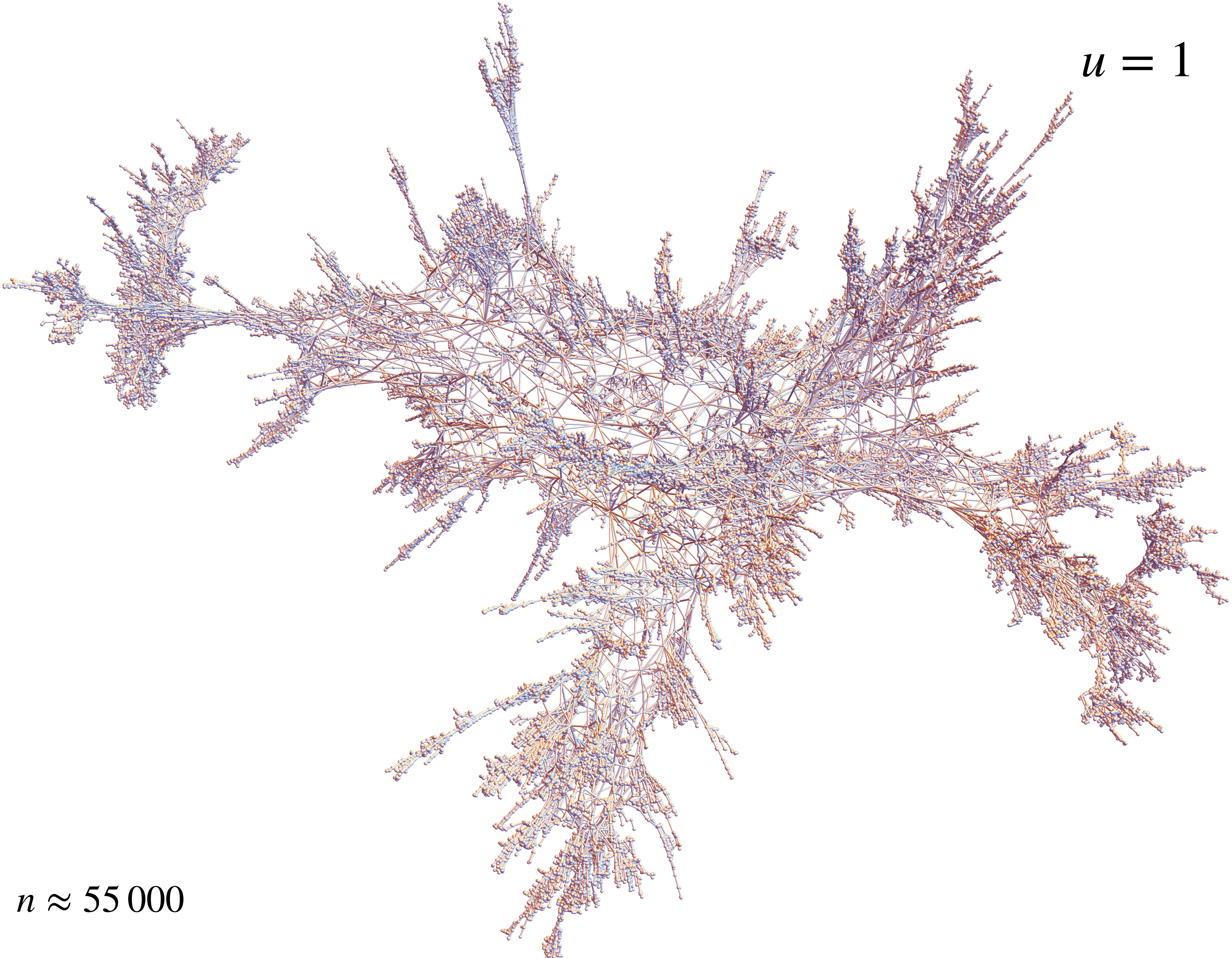
[Chapuy Fusy Giménez Noy 2015]

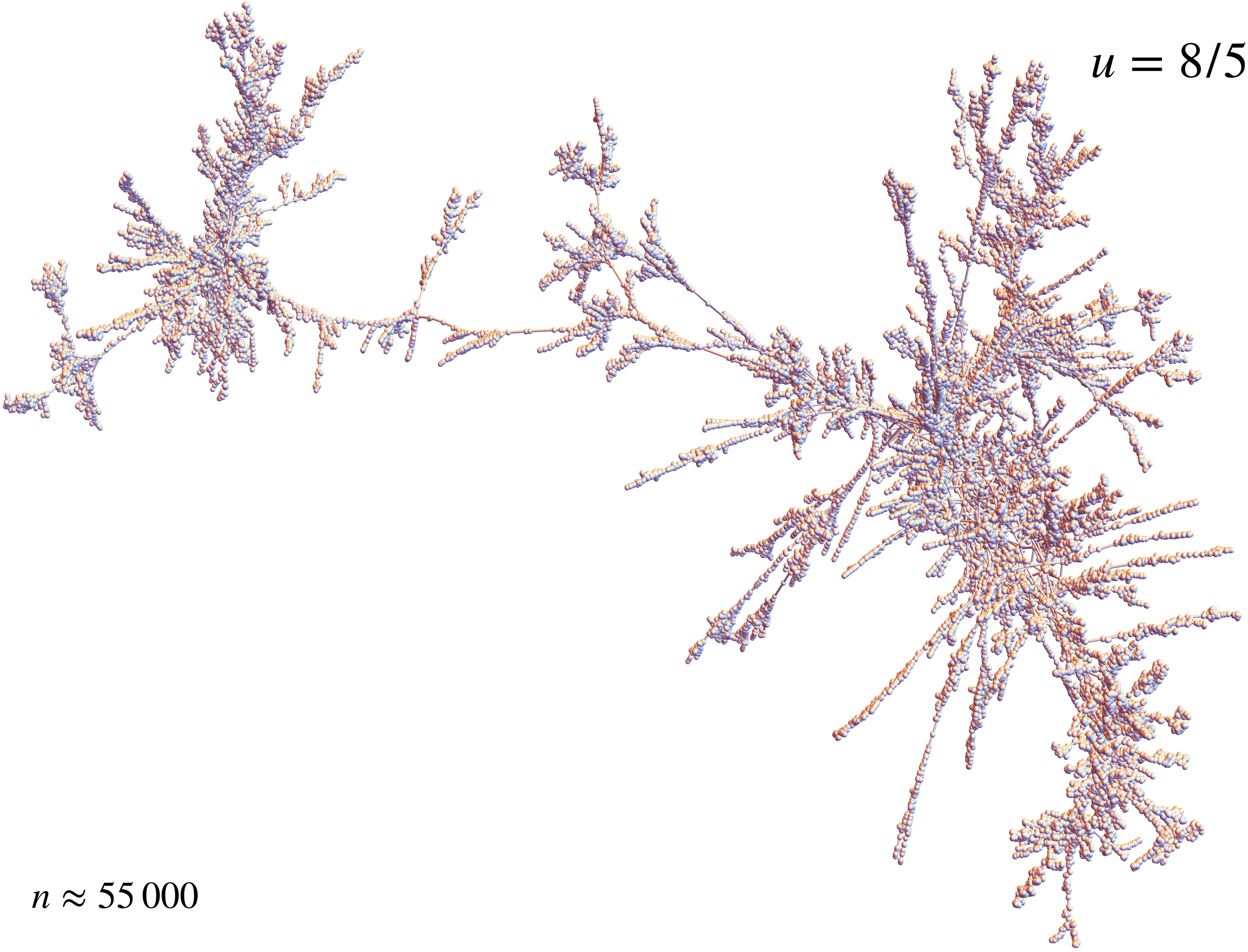
Subcritical case



Diameters of decorations = $o(n^{1/4})$.

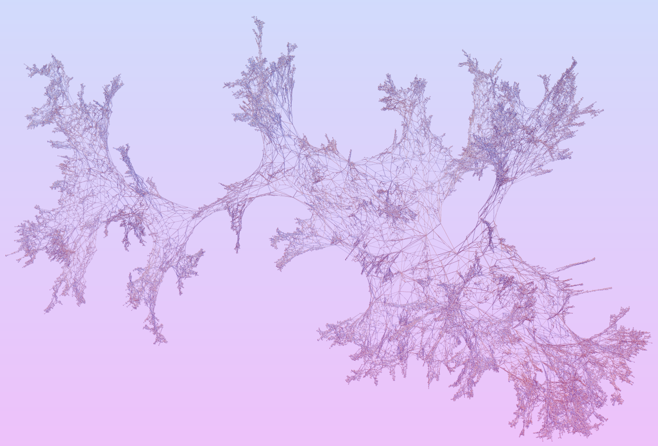
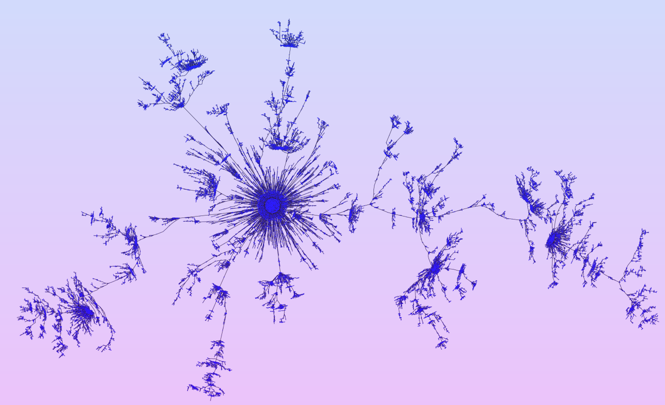
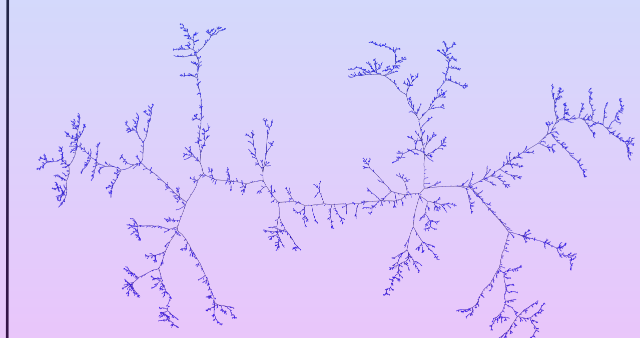
The scaling limit of M_n (rescaled by $n^{1/4}$) is the scaling limit of uniform blocks!


$u = 1$
$n \approx 55\,000$



$u = 8/5$
$n \approx 55\,000$

Results

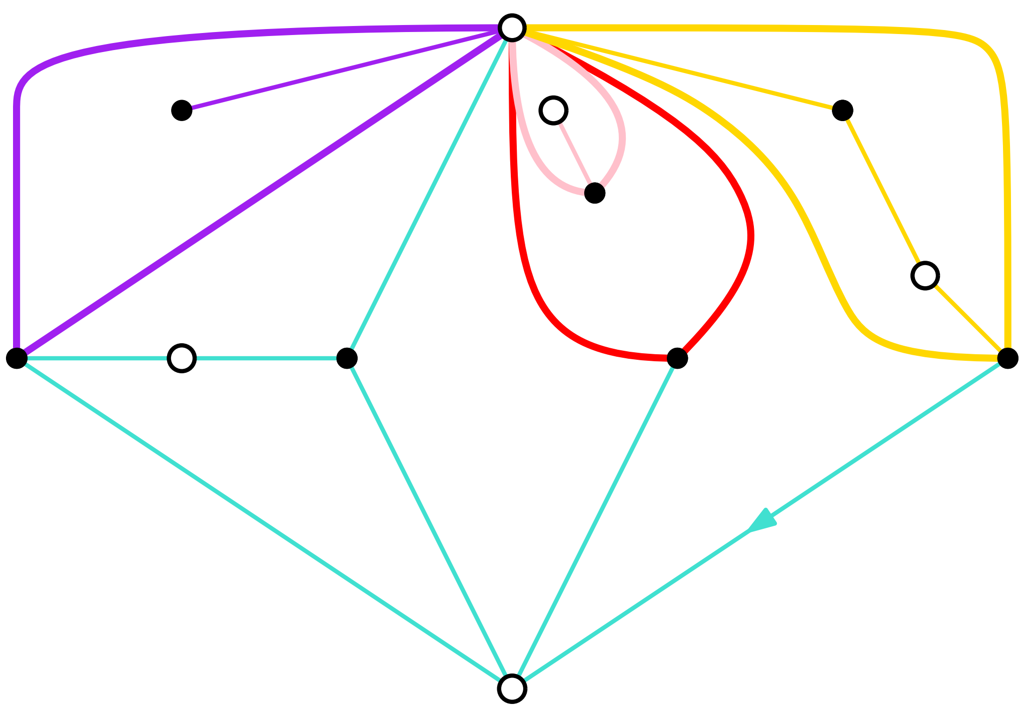
For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\dfrac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \dfrac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n	$\dfrac{C(u)}{n^{1/4}} M_n \xrightarrow{(d)} \mathcal{S}_e$  Assuming the convergence of 2-connected maps towards the Brownian sphere	$\dfrac{C}{n^{1/3}} M_n \xrightarrow{(d)} \mathcal{T}_{3/2}$ 	$\dfrac{C(u)}{n^{1/2}} M_n \xrightarrow{(d)} \mathcal{T}_2$ [Stufler 2020] 

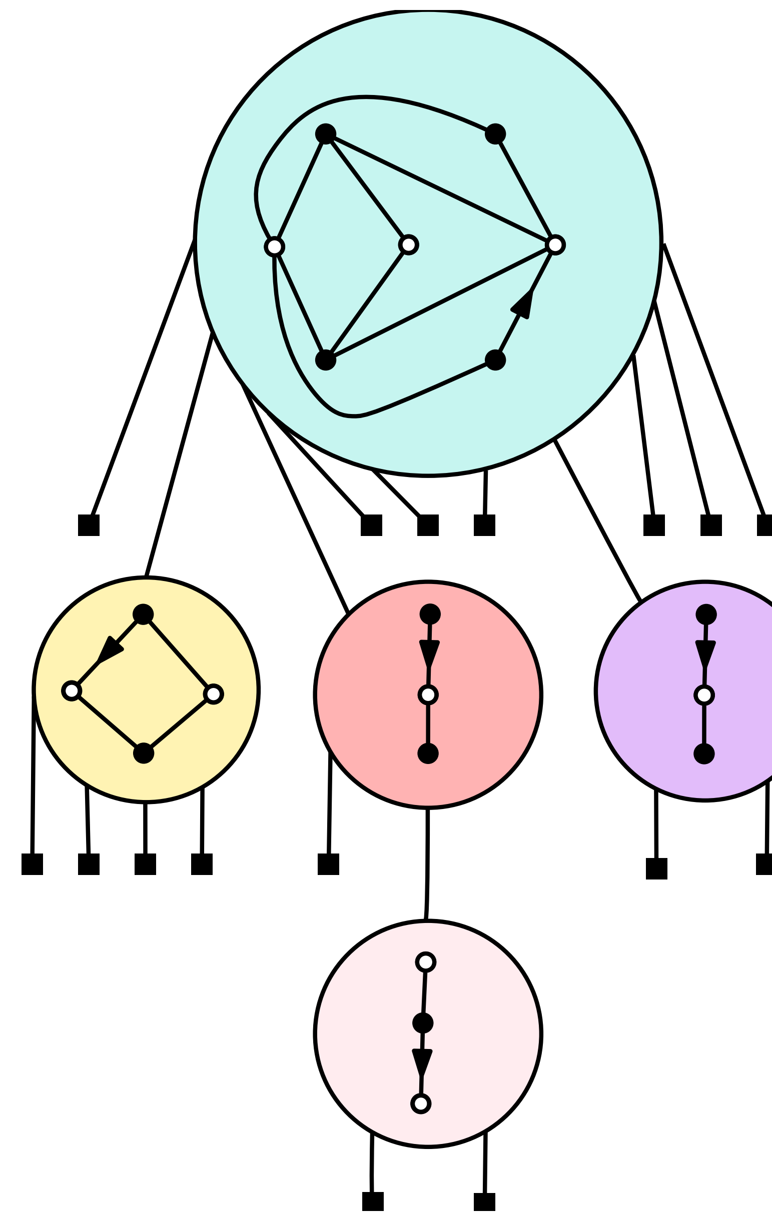
Interlude. Quadrangulations

Block-weighted quadrangulations

Quadrangulation = map with all faces of degree 4.

Simple quadrangulation = no multiple edges.





=> Same enumeration (Tutte's bijection), metric properties better known

Scaling limits of subcritical maps

Theorem [Fleurat, S. 24] If $u < 9/5$, for $Q_n \hookrightarrow \mathbb{P}_{n,u}$ a quadrangulation and denoting $B(Q_n)$ its largest block

$$\frac{C(u)}{n^{1/4}} \left(Q_n, B(Q_n)\right) \xrightarrow{(d)} \left(\mathcal{S}_e, \mathcal{S}_e\right)$$

Proof

- Previous theorem;
- Scaling limit of uniform simple quad. rescaled by $n^{1/4}$ = Brownian sphere [Addario-Berry Albenque 2017].

Brownian Sphere $\mathcal{S}_e$

IV. Extension to other families of maps

Extension to other models

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$	u_C
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1 - z(1+M))^2$	$z(1+M)^2$	
loopless $M_2(z)$	simple $M_3(z)$	$z(1+M)$	–	81/17
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1+M)^2$	–	9/5
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1+M)$	–	135/7
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1+M)$	
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1+M)$	–	36/11
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1 - z(1+M))^2$	$z(1+M)^2$	
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1+M)^2$	–	52/27
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1+M)$	–	68/3
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1+M)^3$	–	16/7
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1+M)^2$	–	64/37

[Banderier, Flajolet, Schaeffer, Soria 2001]

Statement of the results

Theorem [S. 23] Each model of the preceding table without coreless maps exhibits a phase transition at some explicit u_C.

When $n \to \infty$:

- Subcritical phase $u < u_C$: "general map phase" one macroscopic block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: "tree phase" only small blocks.

We obtain explicit results on enumeration and limit laws for the size of the largest blocks in each case.

V. Extension to tree-rooted maps

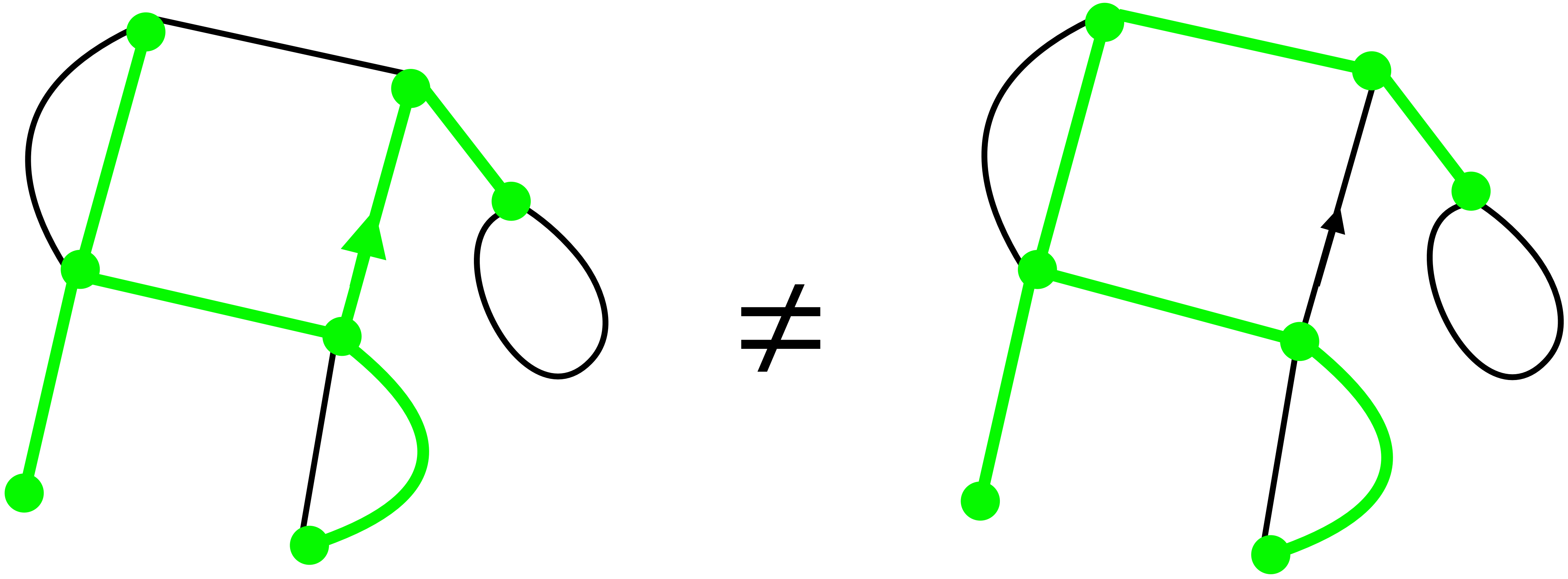
Escaping universality: decorated maps

Theoretical physics point of view:

- Undecorated maps: “pure gravity” case;
- Decorated maps: enables to study models in the presence of matter => new asymptotic behaviours & new universality classes!

Tree-rooted maps

= (rooted planar) maps endowed with a spanning tree.



Tree-rooted maps

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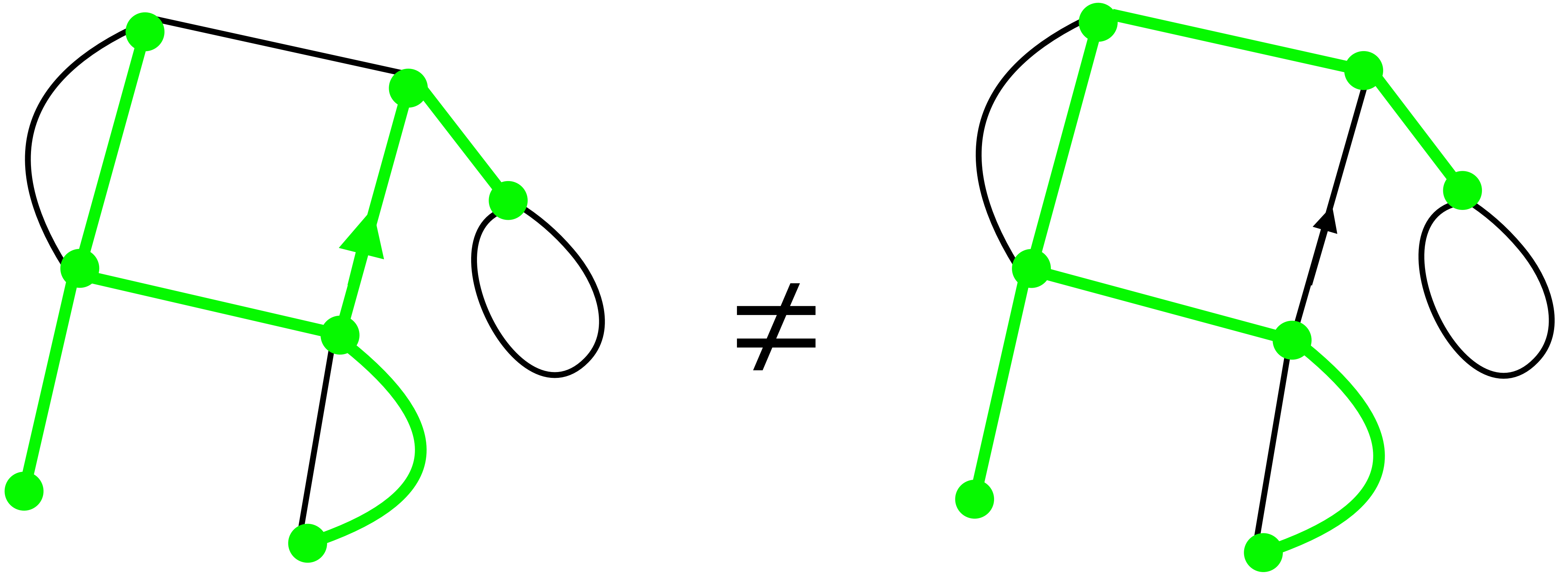


Combinatorics well understood: Mullin's bijection

$$[z^n]M(z) = \mathrm{Cat}_n \mathrm{Cat}_{n+1}$$

[Mullin 1967; Bernardi 2006]

Model

Phase transition for tree-rooted maps => block-weighting

Fix $u > 0$, define

$$\mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$$

$\mathfrak{m} \in \{$tree-rooted maps of size $n\}$,

$Z_{n,u}$ = normalisation.

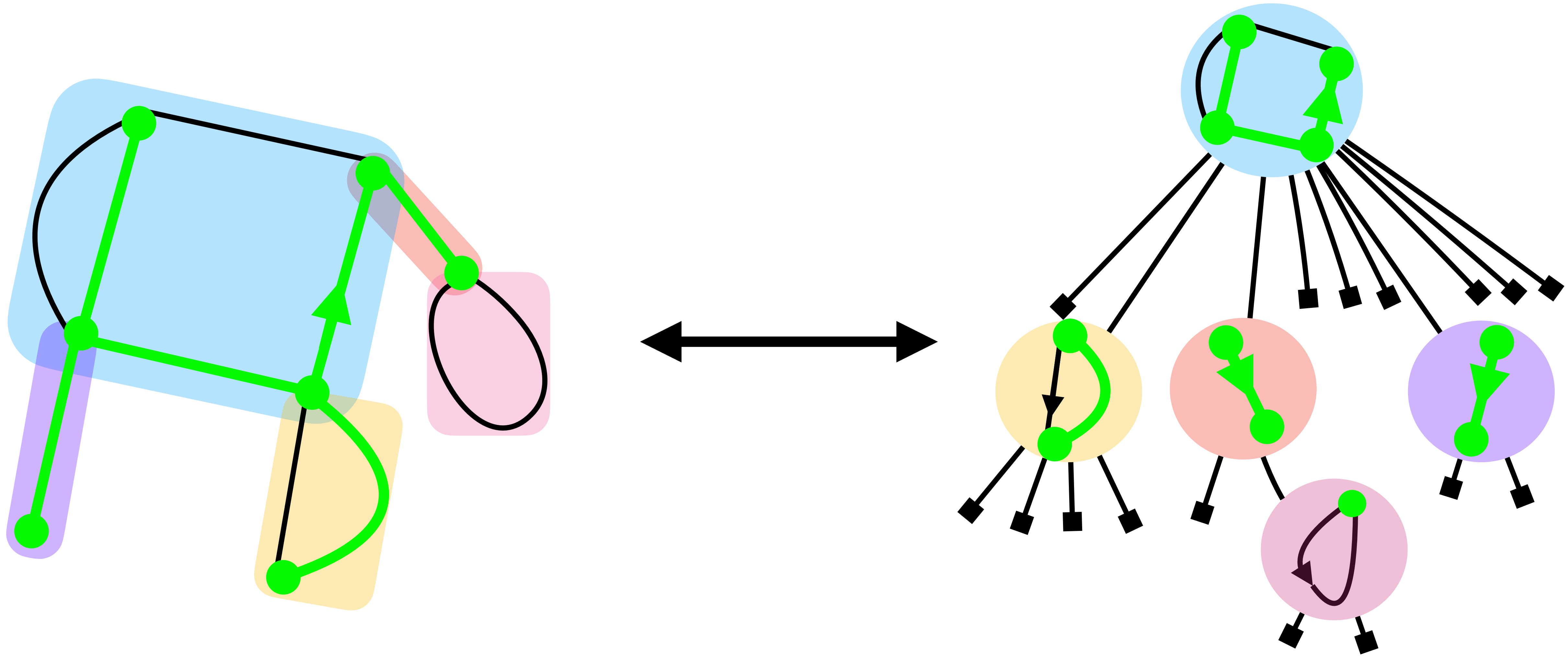
- $u = 1$: uniform distribution on tree-rooted maps of size n;
- $u \to 0$: 2-connected tree-rooted maps;
- $u \to \infty$: tree-rooted trees = trees!

Given u, asymptotic behaviour when $n \to \infty$?

[Stufler 2020]

Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



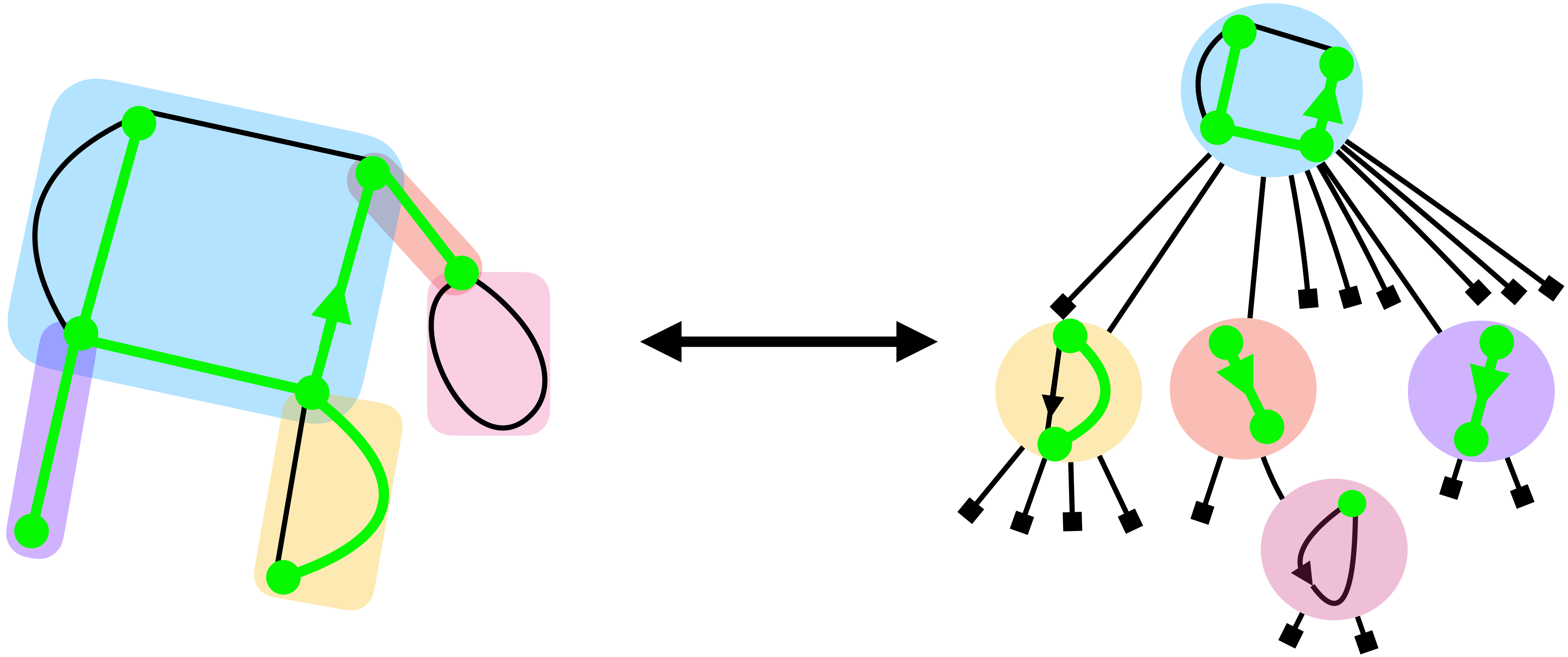
$$M(z) = B(zM^2(z))$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



$$M(z,u) = uB(zM^2(z,u)) + 1 - u$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

So everything should be easy, right?

Tree-rooted maps are not so easy

$$M(z) = \sum_{n \geq 0} \mathrm{Cat}_n \mathrm{Cat}_{n+1} z^n \text{ so}$$

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- $[z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3}$ so M is not algebraic…

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$$P\left(z, M(z)\right) = 0$$

- $[z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3}$ so M is not algebraic…
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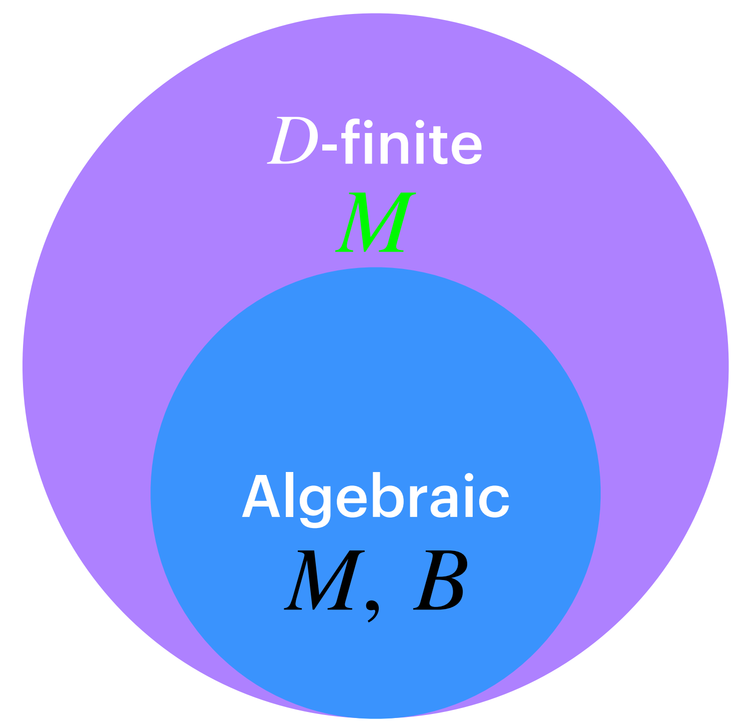
- M is still D-finite

$$P_0(z)\frac{\partial^2 M}{\partial z^2}(z) + P_1(z)\frac{\partial M}{\partial z}(z) + P_2(z)M(z) + P_3(z) = 0$$



Tree-rooted maps are not so easy

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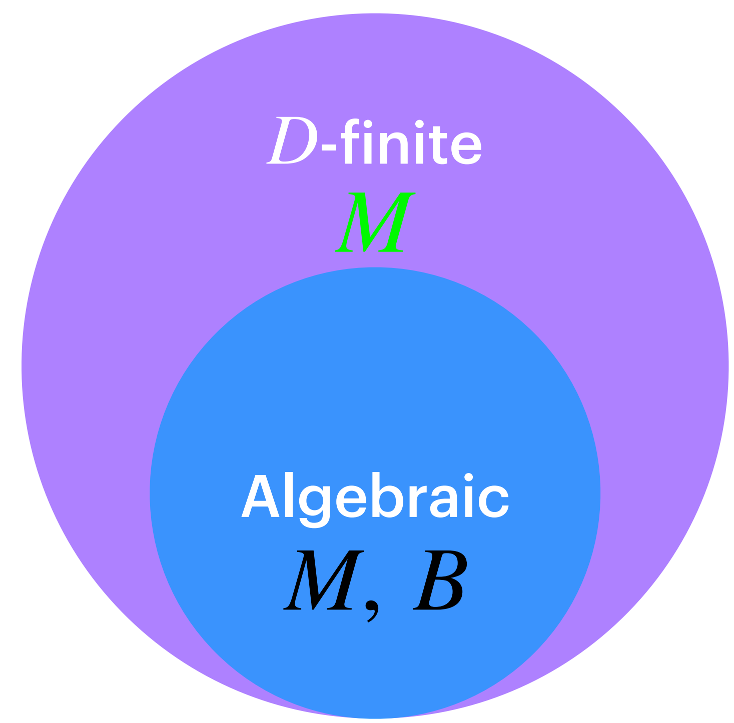

$$z^2(1-16z)\frac{\partial^2 M}{\partial z^2}(z) + 4z(1-12z)\frac{\partial M}{\partial z}(z) + 2(1-6z)M(z) - 2 = 0$$

2-connected tree-rooted maps are tricky

Using $M(z) = B(zM^2(z))$ and the properties of M, we show

2-connected tree-rooted maps are tricky

Using $M(z) = B(zM^2(z))$ and the properties of M, we show

- $\rho_B = \rho_M M^2(\rho_M) = \dfrac{4(3\pi - 8)^2}{9\pi^2} \approx 0.091$

is not algebraic so B is not D-finite

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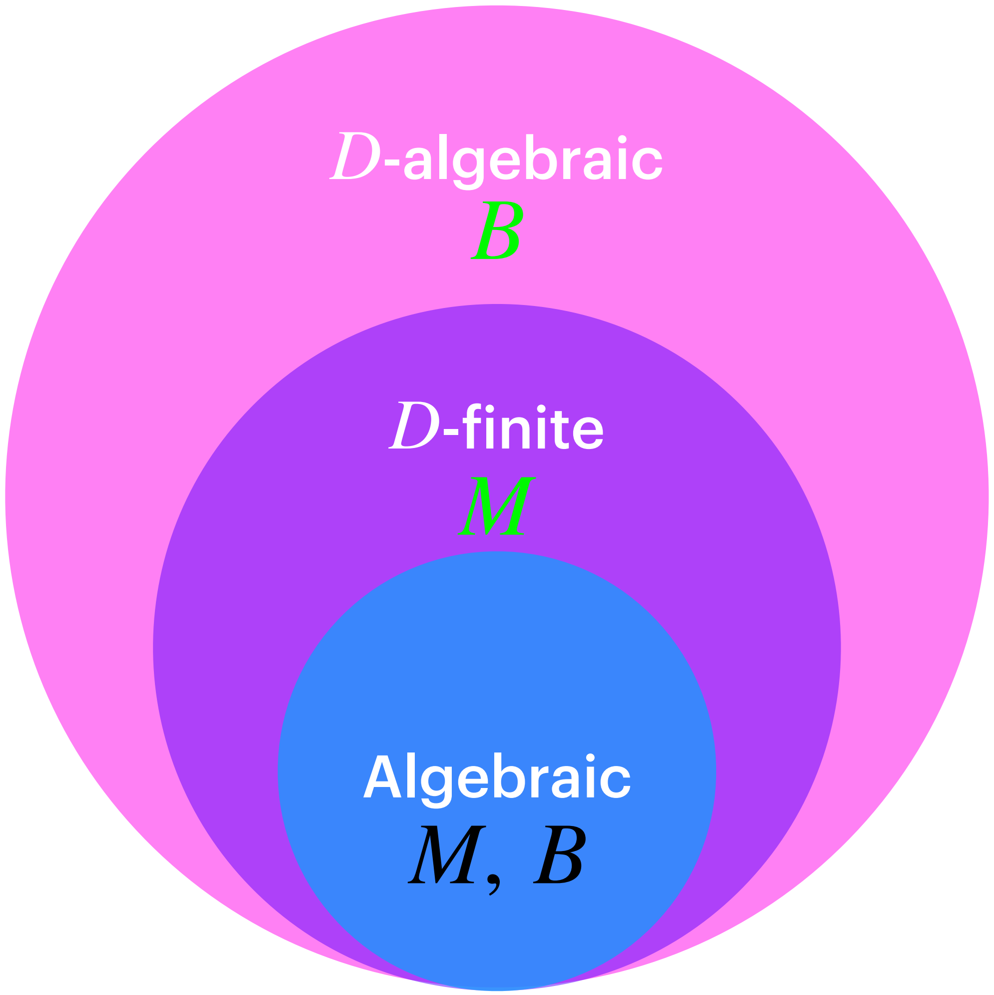

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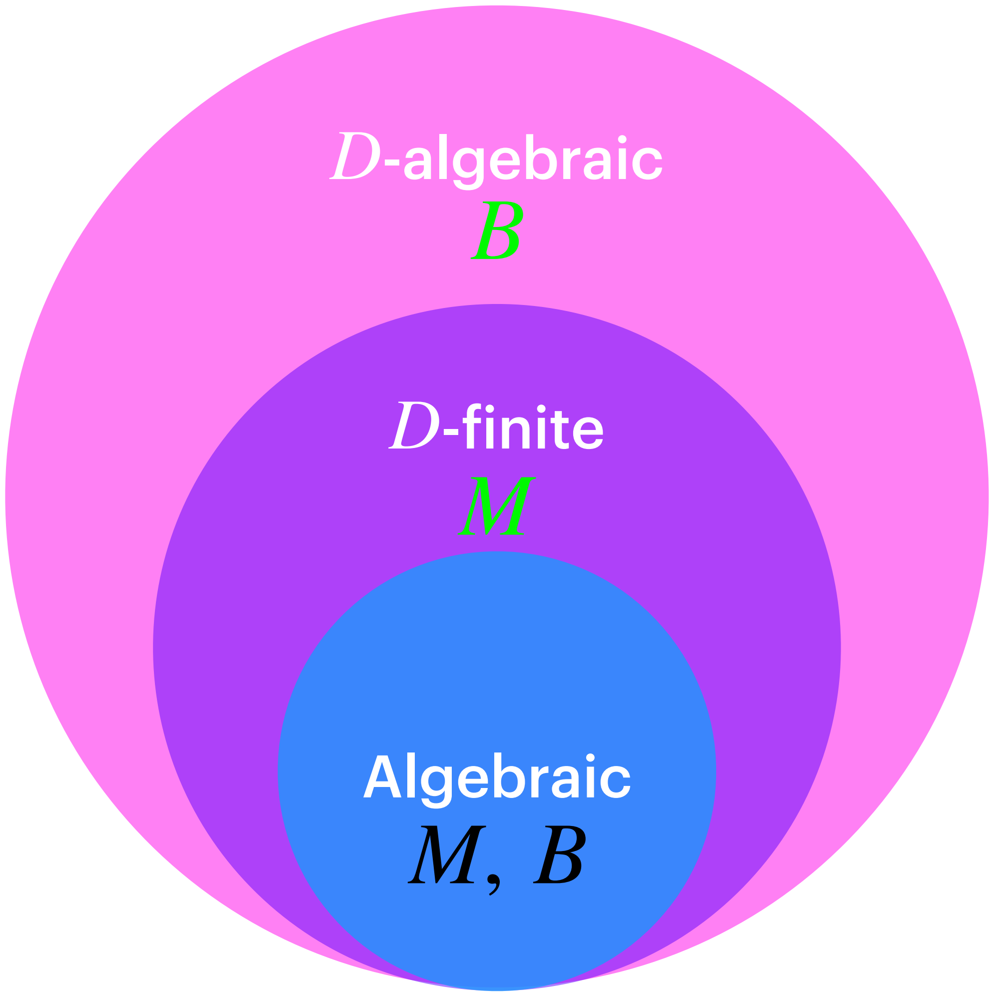

- B is D-algebraic

$$P\left(\frac{\partial^2 B}{\partial y^2}(y), \frac{\partial B}{\partial y}(y), B(y), y\right) = 0.$$

2-connected tree-rooted maps are tricky

Using $M(z) = B(zM^2(z))$ and the properties of M, we show

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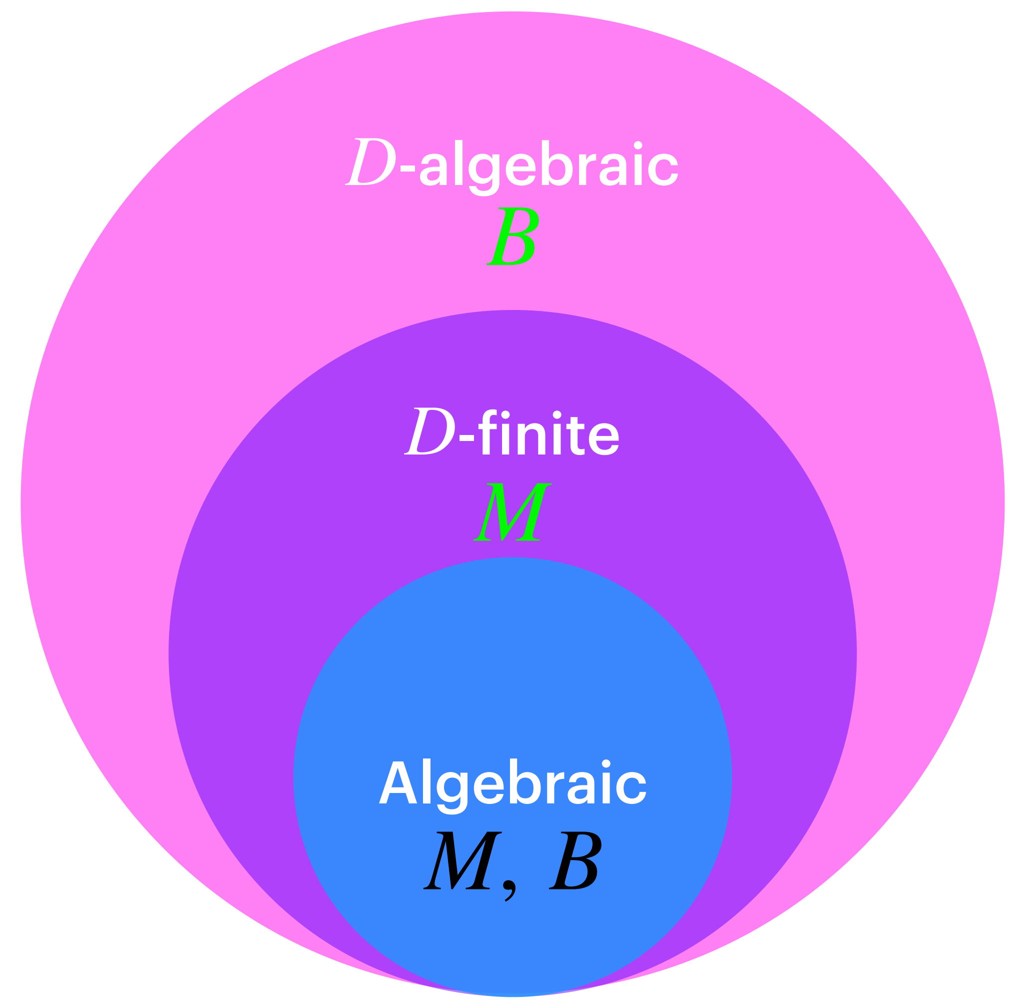

$\simeq 0.22$

Theorem [Albenque, Fusy, S. 24]

$$[y^n]B(y) \sim \frac{4(3\pi - 8)^3}{27\pi(4-\pi)^3} \times \rho_B^{-n} \times n^{-3}.$$

Phase transition

Theorem [Albenque, Fusy, S. 24] Model exhibits a phase transition at $u_C = \dfrac{9\pi(4-\pi)}{420\pi - 81\pi^2 - 512} \simeq 3.02.$

When $n \to \infty$:

- Subcritical phase $u < u_C$: "general tree-rooted map phase" one macroscopic block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: "tree phase" only small blocks.

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration			
Size of - the largest block - the second one			
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one			
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3\ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
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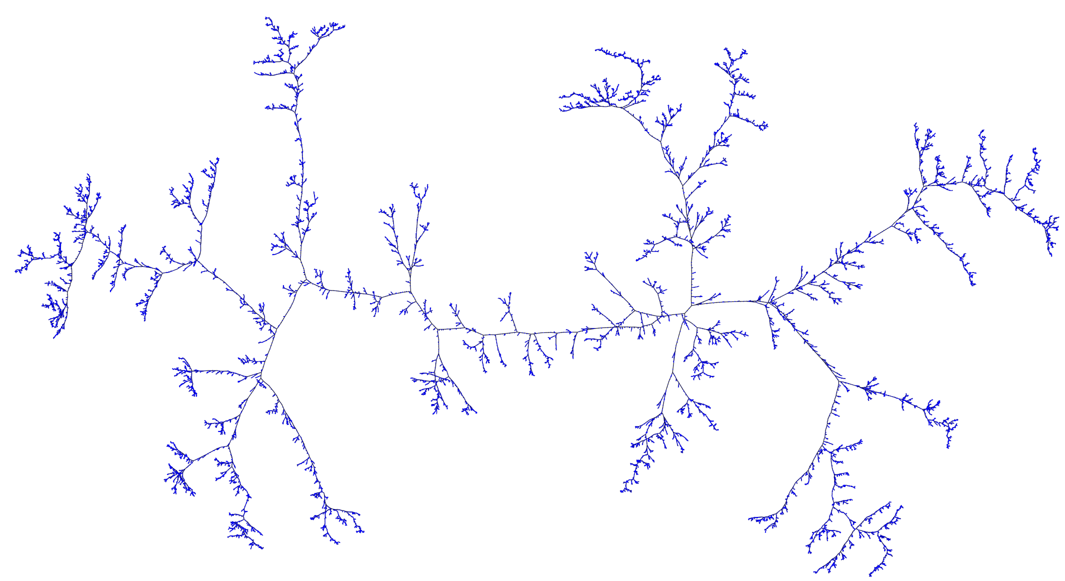
Ordered atoms of a Poisson Point Process

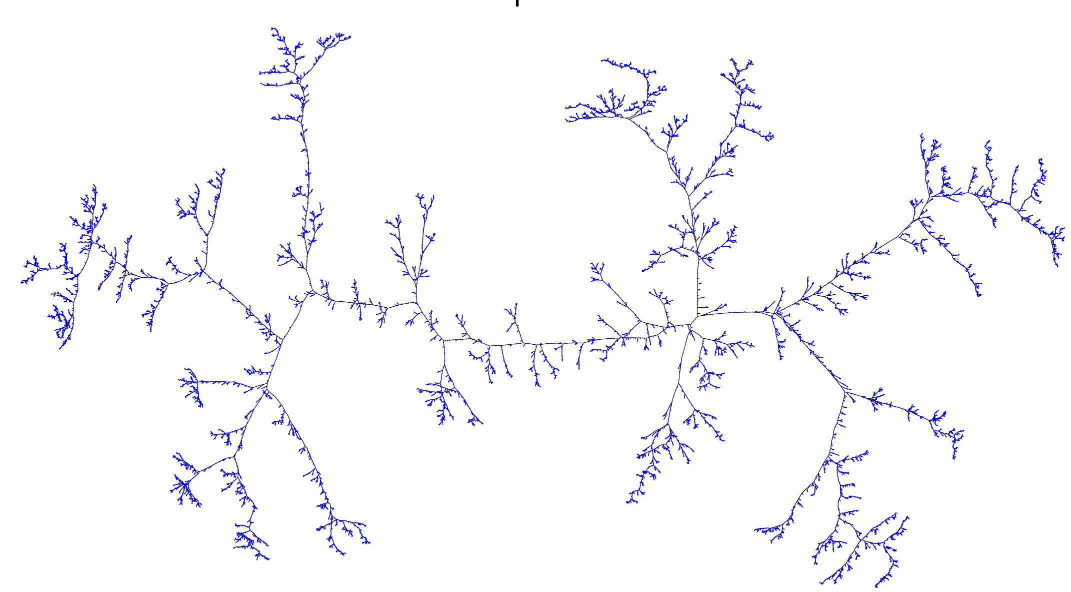
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3\ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
Scaling limit of M_n	?	$\frac{C \ln(n)^{1/2}}{n^{1/2}} M_n \to \mathcal{T}_e$	[Stufler 2020] $\frac{C(u)}{n^{1/2}} M_n \to \mathcal{T}_e$



Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3\ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
Scaling limit of M_n	?	$\frac{C \ln(n)^{1/2}}{n^{1/2}} M_n \to \mathcal{T}_e$ 	[Stufler 2020] $\frac{C(u)}{n^{1/2}} M_n \to \mathcal{T}_e$

VI. Perspectives

Extensions to more involved decompositions

Block-weighted

- Maps into loopless blocks;
- (Bipartite) maps into (bipartite) bridgeless blocks;
- 2-connected maps into 3-connected blocks;
- Simple quadrangulations into irreducible blocks...

Require new methods, same results expected

Extensions to more involved decompositions

Block-weighted

- Tree-rooted quadrangulations;
- Forested maps;
- Maps endowed with a Ising model or a Potts model;
- 2-oriented quadrangulations decomposed into irreducible blocks;
- 3-oriented triangulations decomposed into irreducible blocks...

Critical window?

Phase transition very sharp => what if $u = 9/5 \pm \varepsilon(n)$?

Block size results still hold

- if $u_n = 9/5 - \varepsilon(n)$, $\varepsilon^3 n \to \infty$;
- If $u_n = 9/5 + \varepsilon(n)$, $\varepsilon^3 n \to \infty$:

$$L_{n,1} \sim 2.7648\, \varepsilon^{-2} \ln(\varepsilon^3 n)$$

(analogous to [Łuczak 1990]'s result for Erdős-Rényi graphs!);

Results exist for scaling limits in ER graphs [Addario-Berry, Broutin, Goldschmidt 2010].

Is there a critical window? If so, what is its width?

Thank you!