Phase transitions of block-weighted planar maps

Arbeitsgemeinschaft Diskrete Mathematik
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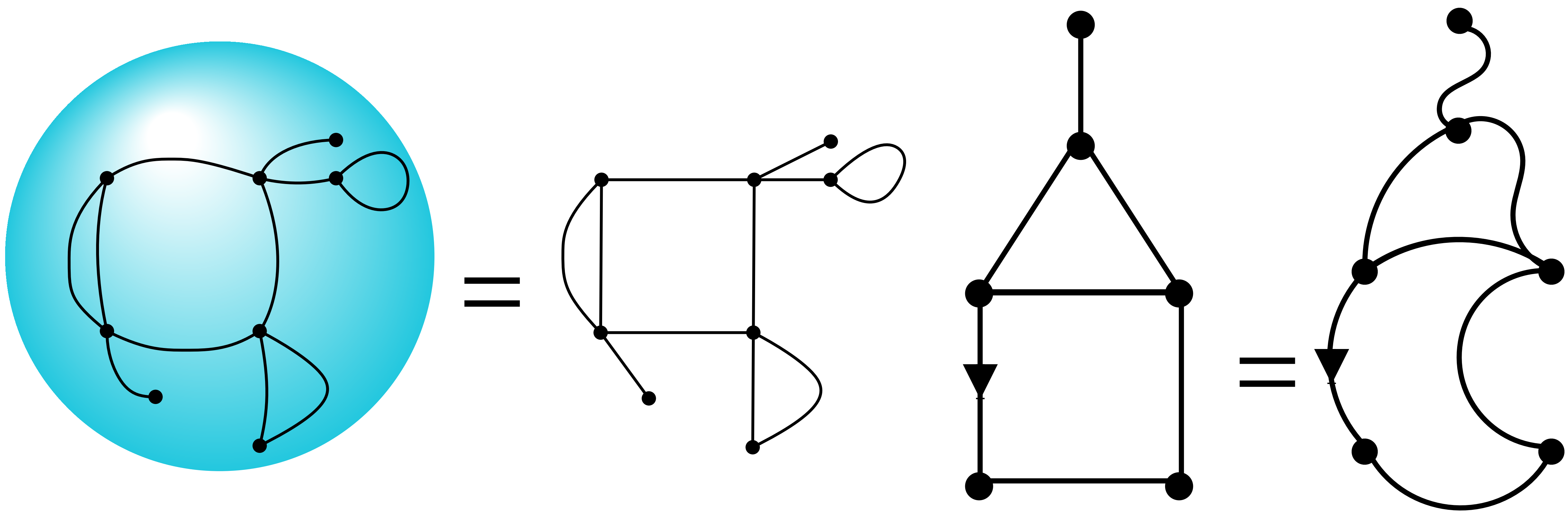
TU Wien

Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms



- Rooted planar map = map endowed with a marked oriented edge (represented by an arrow);
- Size $|\mathfrak{m}|$ = number of edges;
- Corner (does not exist for graphs !) = space between two consecutive edges around a vertex (trigonometric order).

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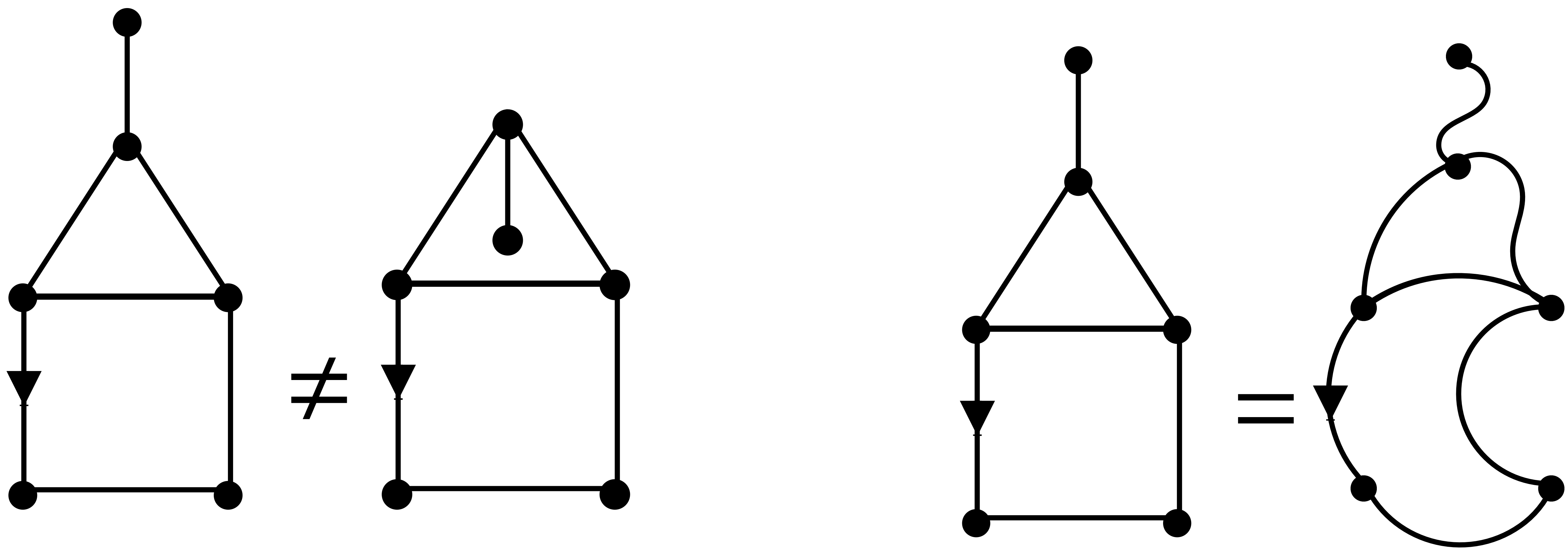

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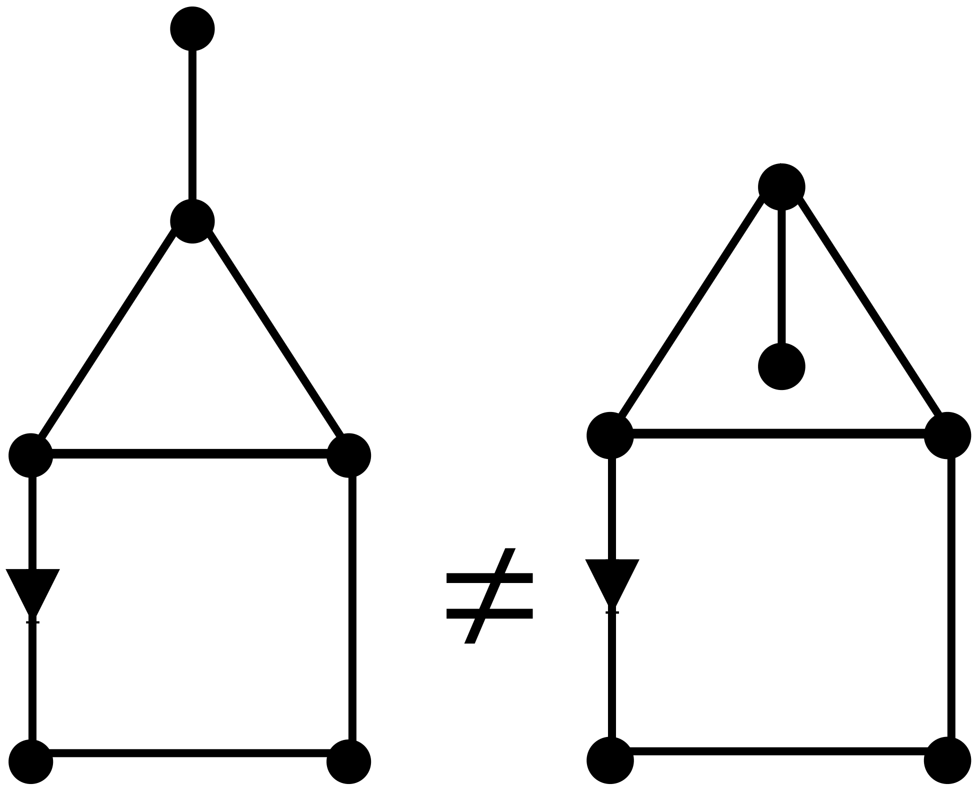

Planar map = planar graph + cyclic order on neighbours

- Rooted planar map = map endowed with a marked oriented edge (represented by an arrow);
- Size $|\mathfrak{m}|$ = number of edges;
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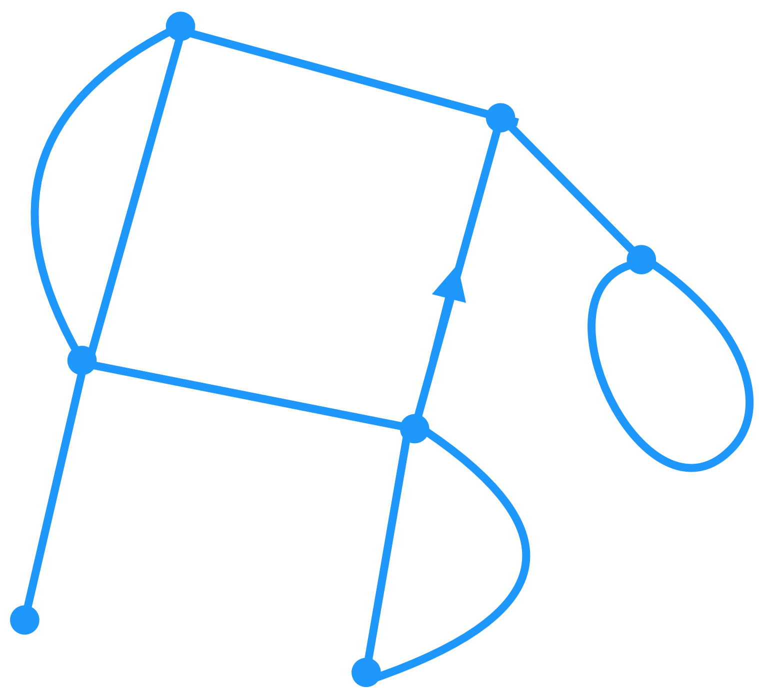
Universality results for planar maps

- Enumeration: $\kappa\rho^{-n}n^{-5/2}$ [Tutte 1963];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for quadrangulations [Le Gall 2013, Miermont 2013] and general maps [Bettinelli, Jacob, Miermont 2014];



Brownian Sphere $\mathcal{S}_e$

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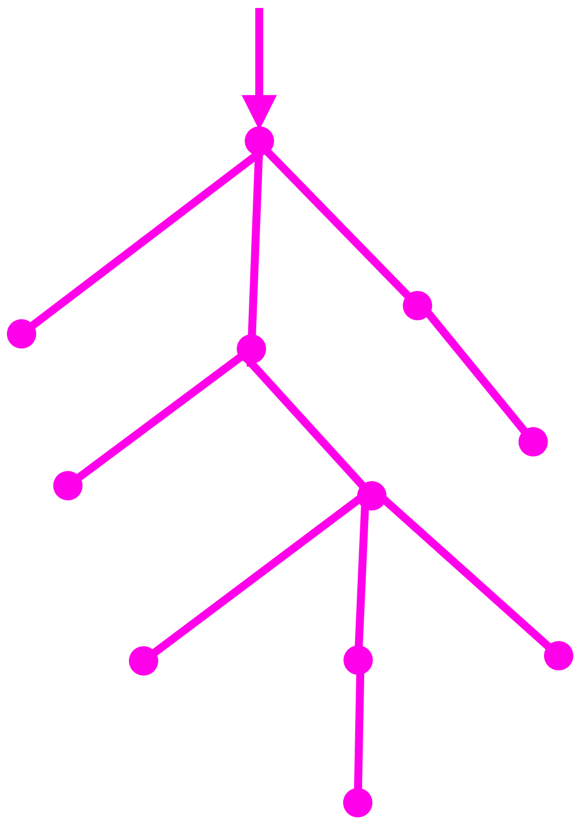
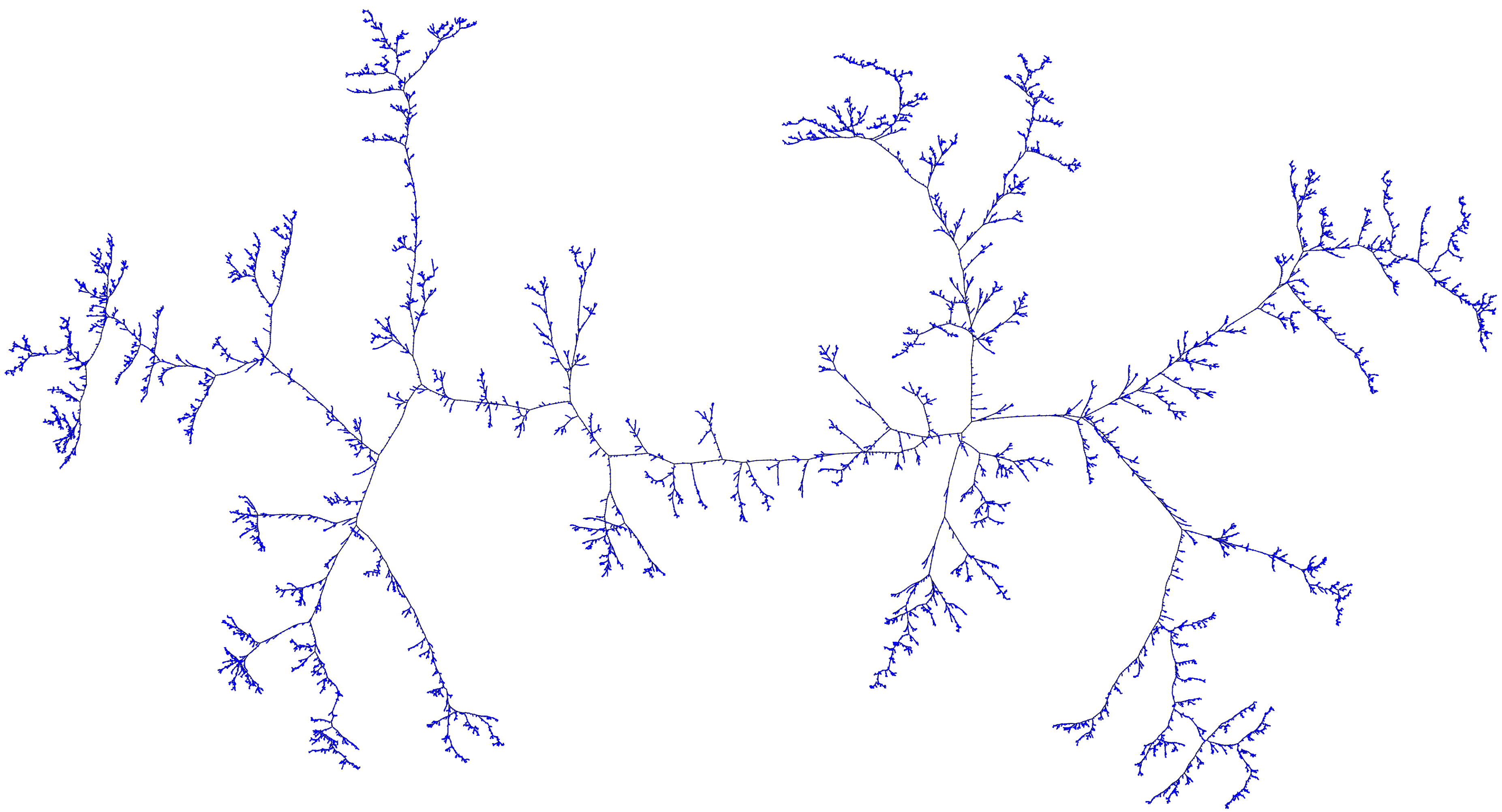
- Universality:
  - Same enumeration [Drmota, Noy, Yu 2020];
  - Same scaling limit, *e.g.* for triangulations & $2q$-angulations [Le Gall 2013], simple quadrangulations [Addario-Berry, Albenque 2017].

Universality results for plane trees

- Enumeration: $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];





Brownian Tree $\mathscr{T}_e$

Universality results for plane trees

- Enumeration: $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];

- Universality:
  - Same enumeration;
  - Same scaling limit, even for some classes of maps, *e.g.* outerplanar maps [Caraceni 2016], maps with a boundary of size >> $n^{1/2}$ [Bettinelli 2015].

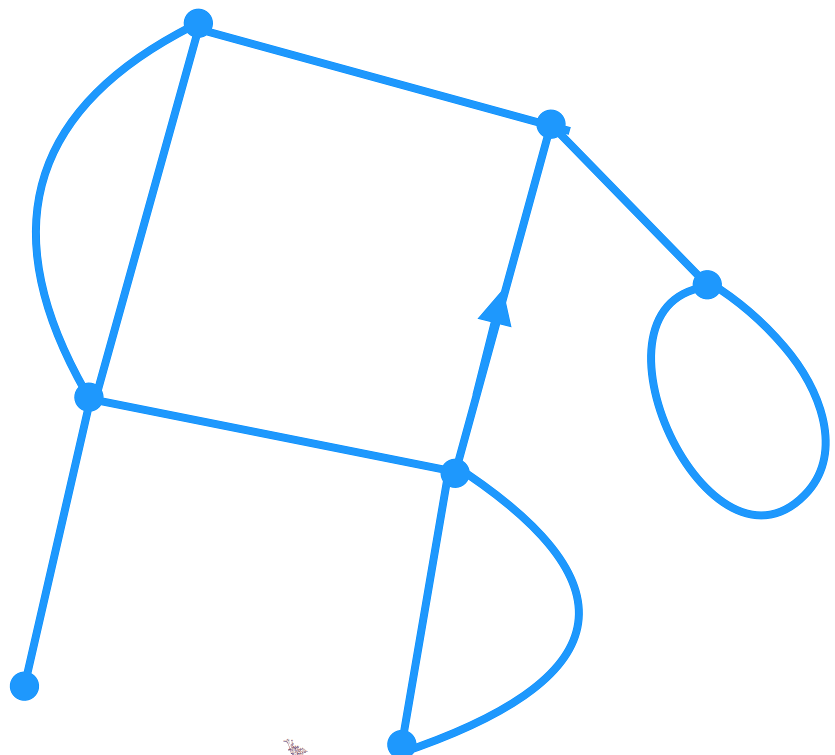
Models with (very) constrained boundaries

Motivation

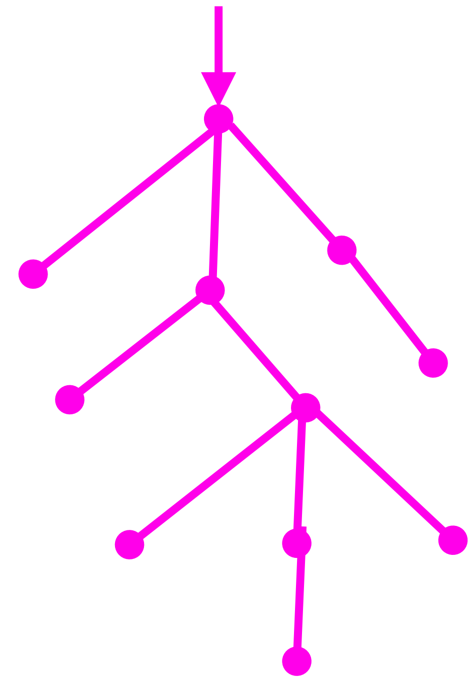
Inspired by [Bonzom Delepouve Rivasseau 2015].

Two rich situations with universality results:

Planar maps

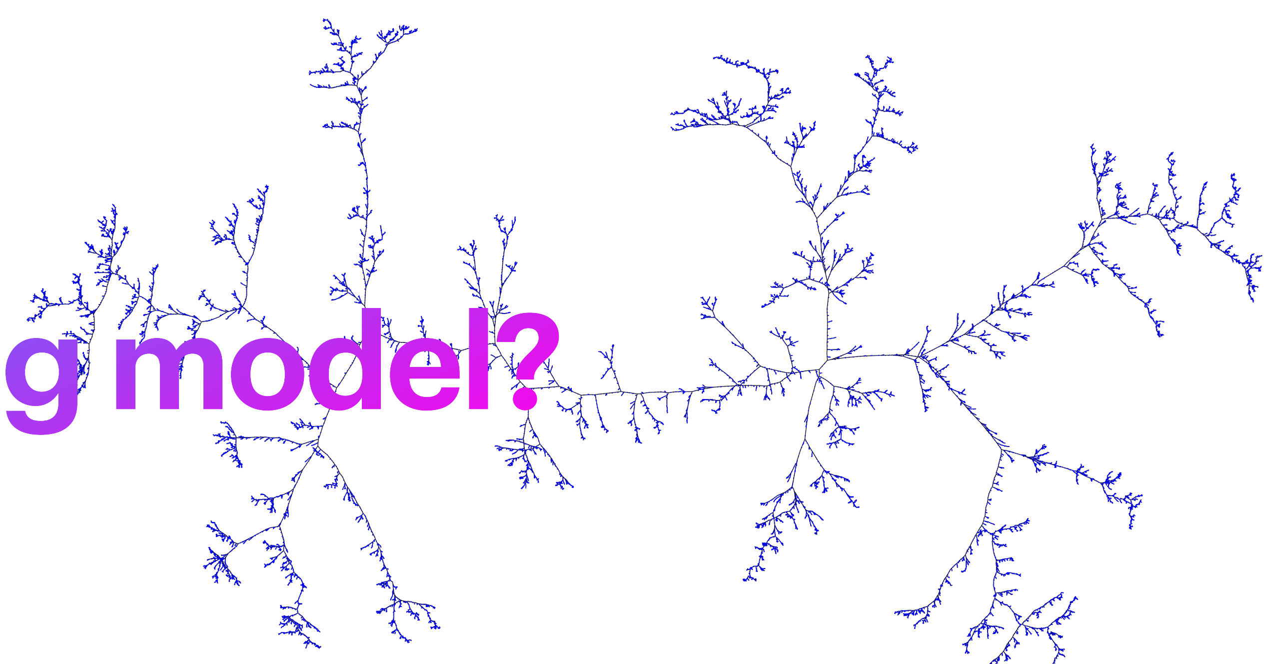


Plane trees



Brownian Sphere $\mathscr{S}_e$

Interpolating model?

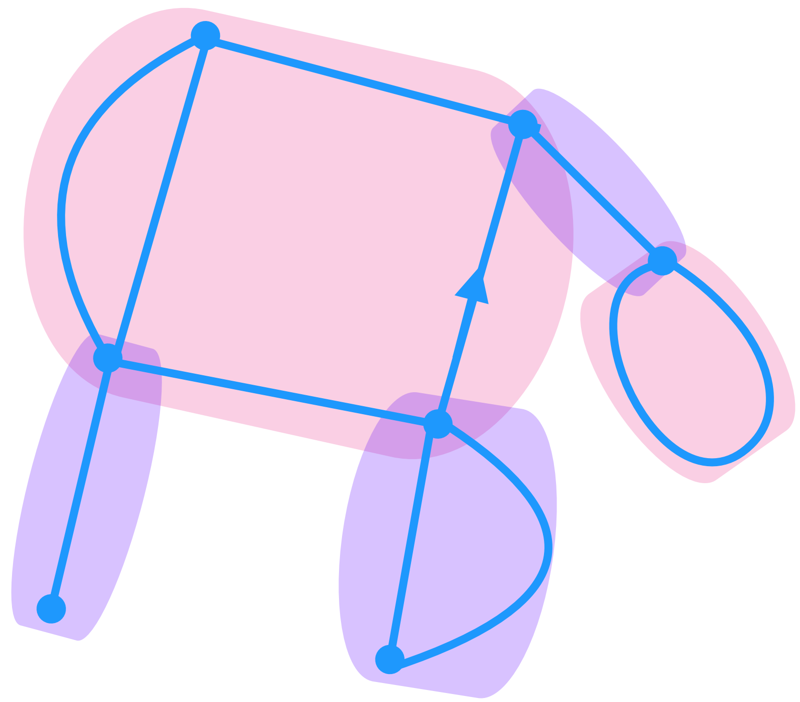


Brownian Tree $\mathscr{T}_e$

Model definition

2-connected = two vertices must be removed to disconnect.

Block = maximal (for inclusion) 2-connected submap.



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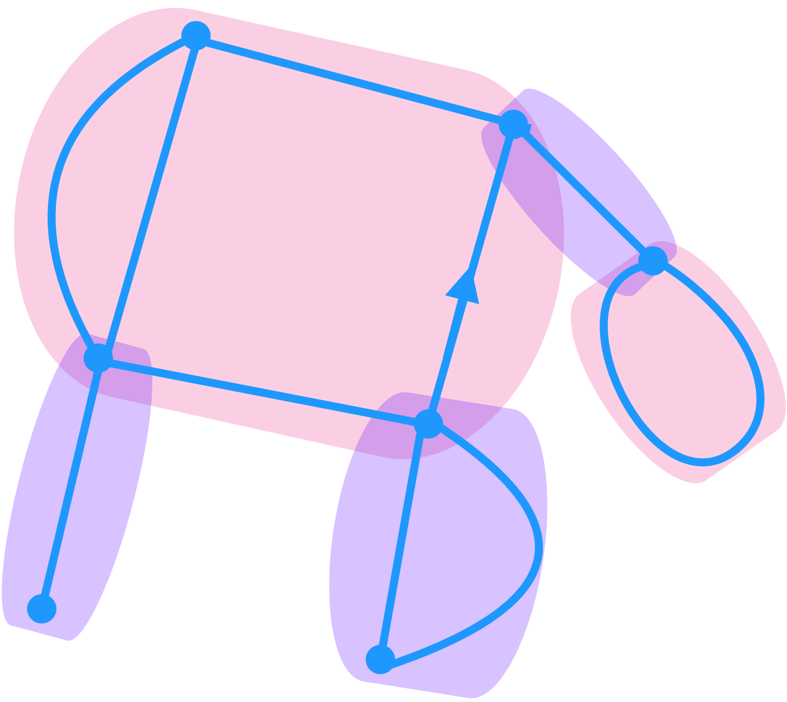

Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

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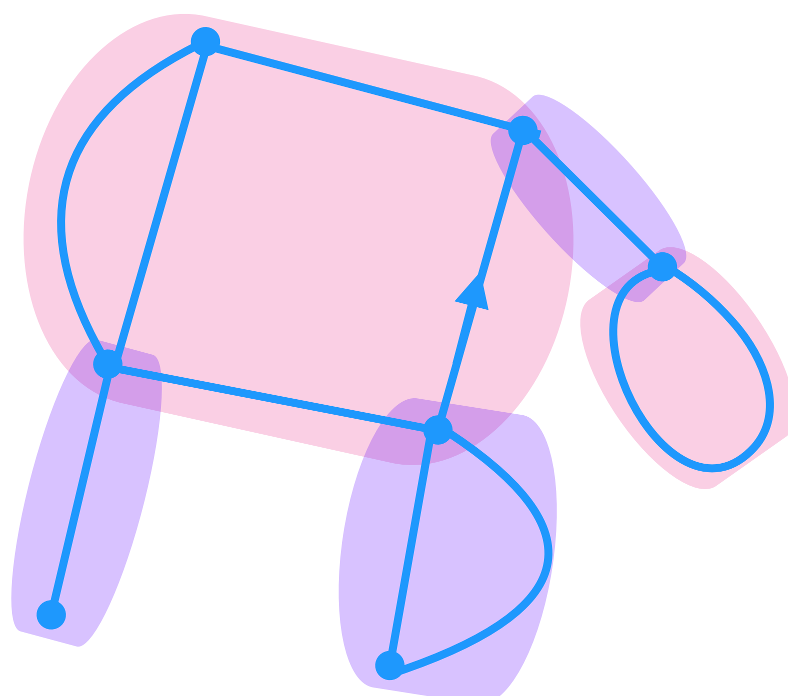
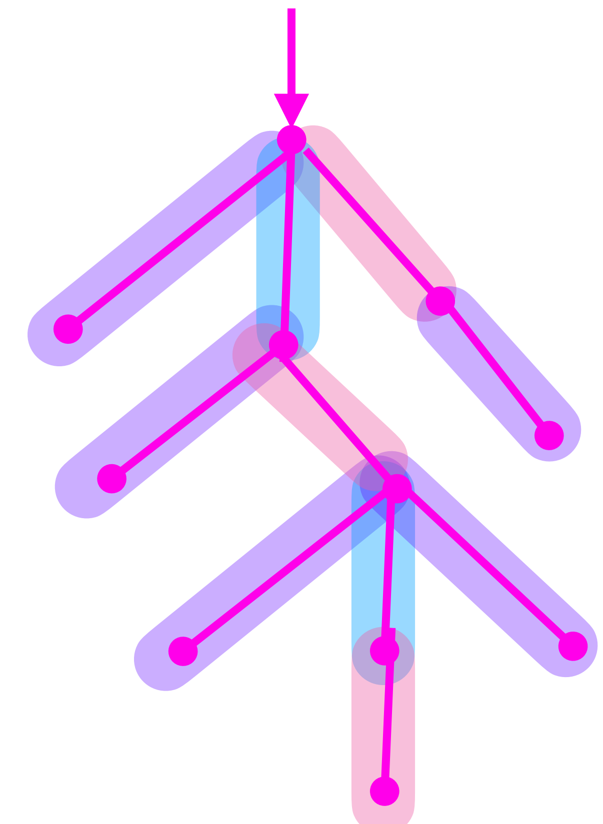




Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

Only small blocks.

Interpolating model using blocks!

Outline of the talk

Phase transitions of block-weighted planar maps

I. Model

II. Block tree of a map and its applications

Interlude. Quadrangulations

III. Scaling limits

IV. Extension to other families of maps

V. Extension to tree-rooted maps

VI. Perspectives

with William Fleurat

with Marie Albenque & Éric Fusy

I. Model

Model

Introduced by [Bonzom Delepouve Rivasseau 2015];

General setting in [Stufler 2020].

Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,

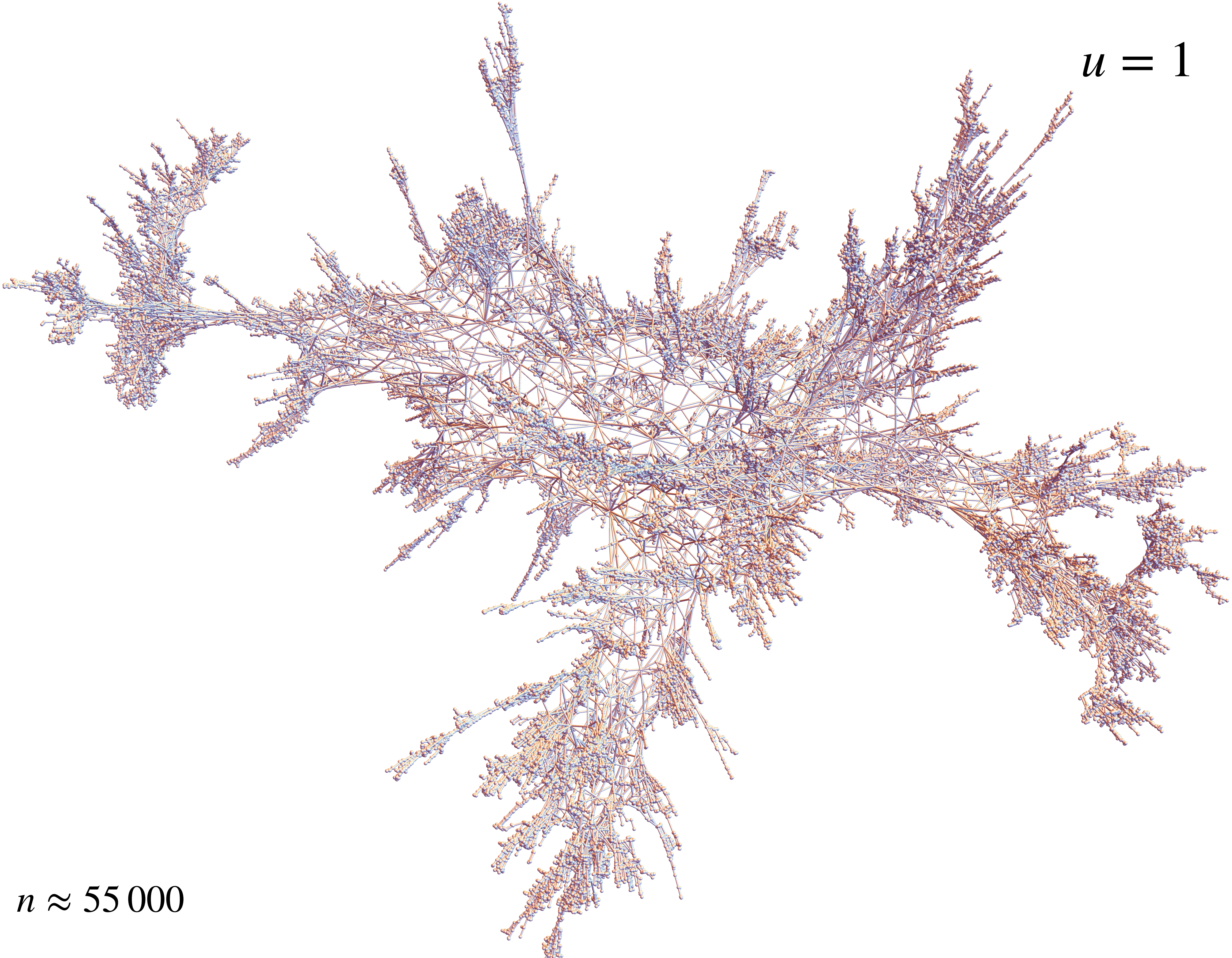
$\mathcal{M}_n$ = {maps of size n},

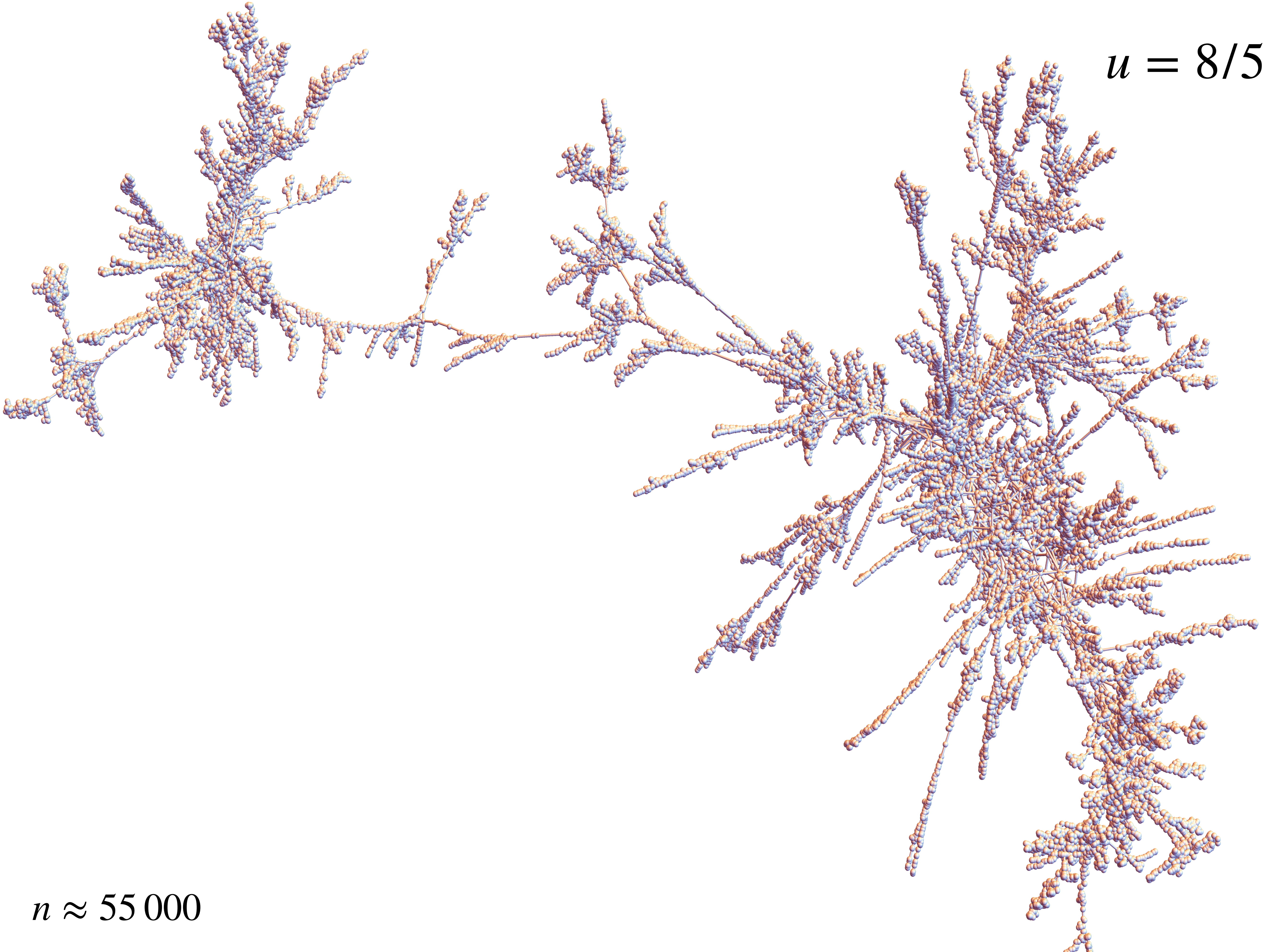
$\mathfrak{m} \in \mathcal{M}_n$,

$Z_{n,u}$ = normalisation.

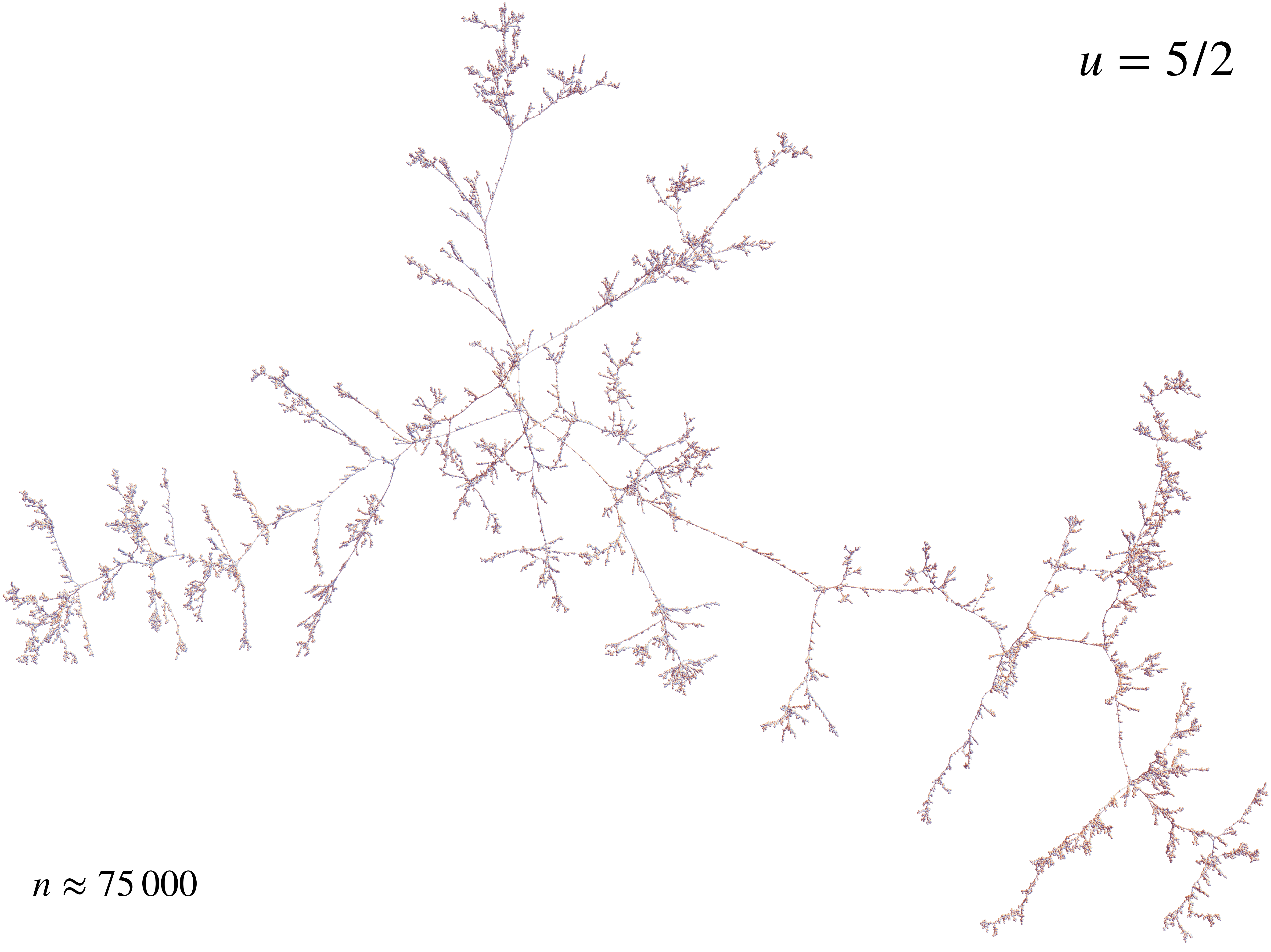
- $u = 1$: uniform distribution on maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected maps);
- $u \to \infty$: maximising the number of blocks (= trees!).

Given u, asymptotic behaviour when $n \to \infty$?


$u = 1$
$n \approx 55\,000$


$u = 8/5$
$n \approx 55\,000$


$u = 9/5$
$n \approx 80\,000$


$u = 5/2$
$n \approx 75\,000$


$u = 5$
$n \approx 50\,000$

Phase transition

Theorem [Fleurat, S. 24] Model exhibits a phase transition at $u = 9/5$. When $n \to \infty$:

- Subcritical phase $u < 9/5$: "general map phase" one huge block;
- Critical phase $u = 9/5$: a few large blocks;
- Supercritical phase $u > 9/5$: "tree phase" only small blocks.

We obtain explicit results on enumeration, size of blocks and scaling limits in each case.

→ *A phase transition in block-weighted random maps*
W. Fleurat & Z. S., Electronic Journal of Probability, 2024

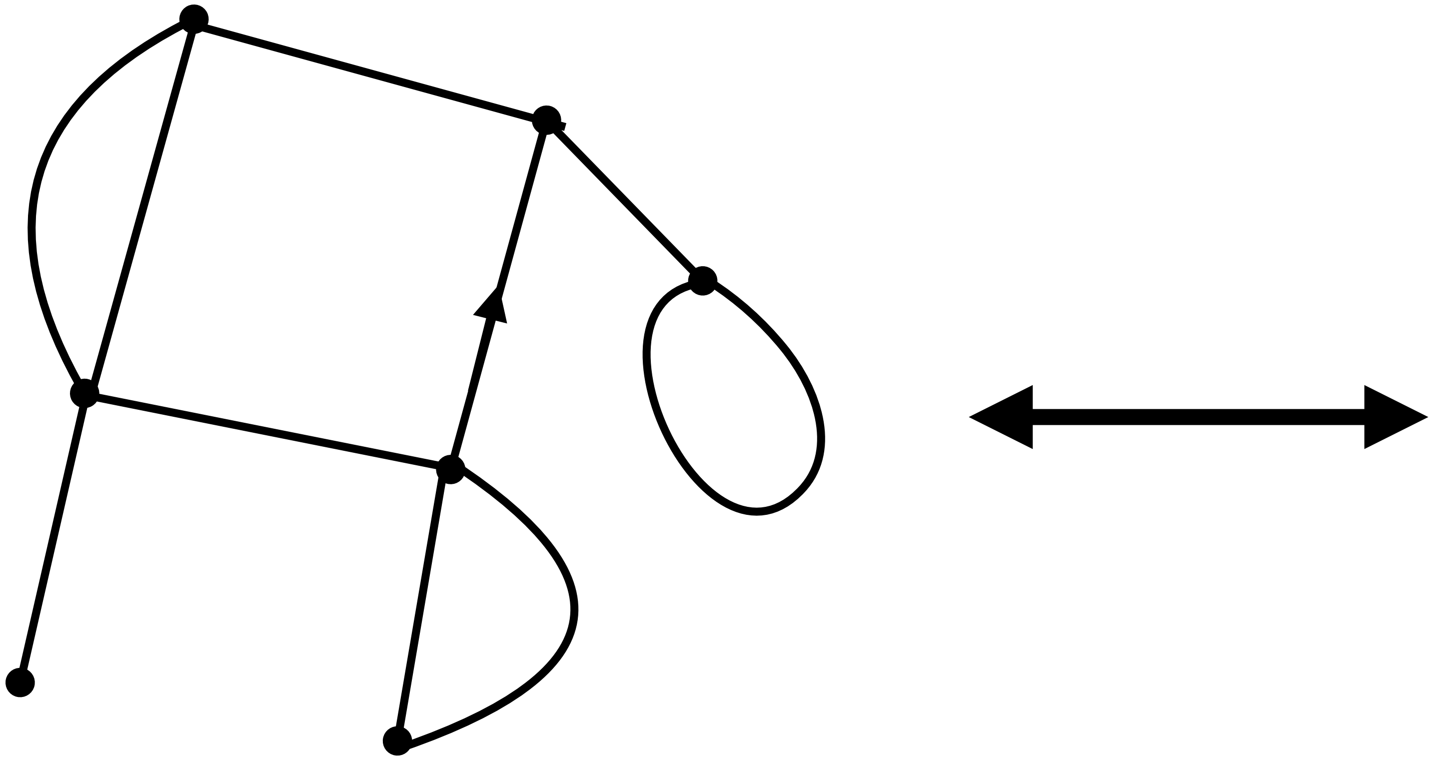
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration			
Size of - the largest block - the second one			
Scaling limit of M_n			

II. Block tree of a map and its applications

Decomposition of a map into blocks

Inspiration from [Tutte 1963]



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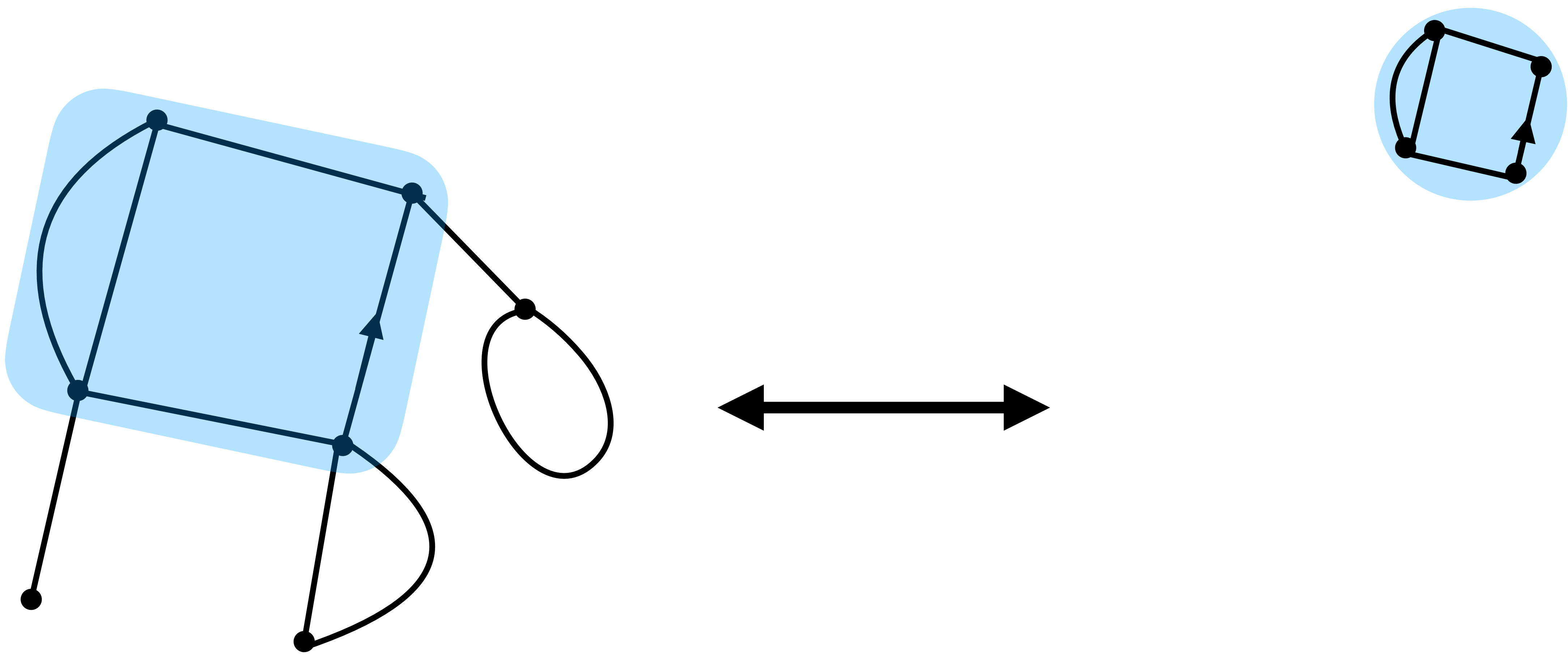

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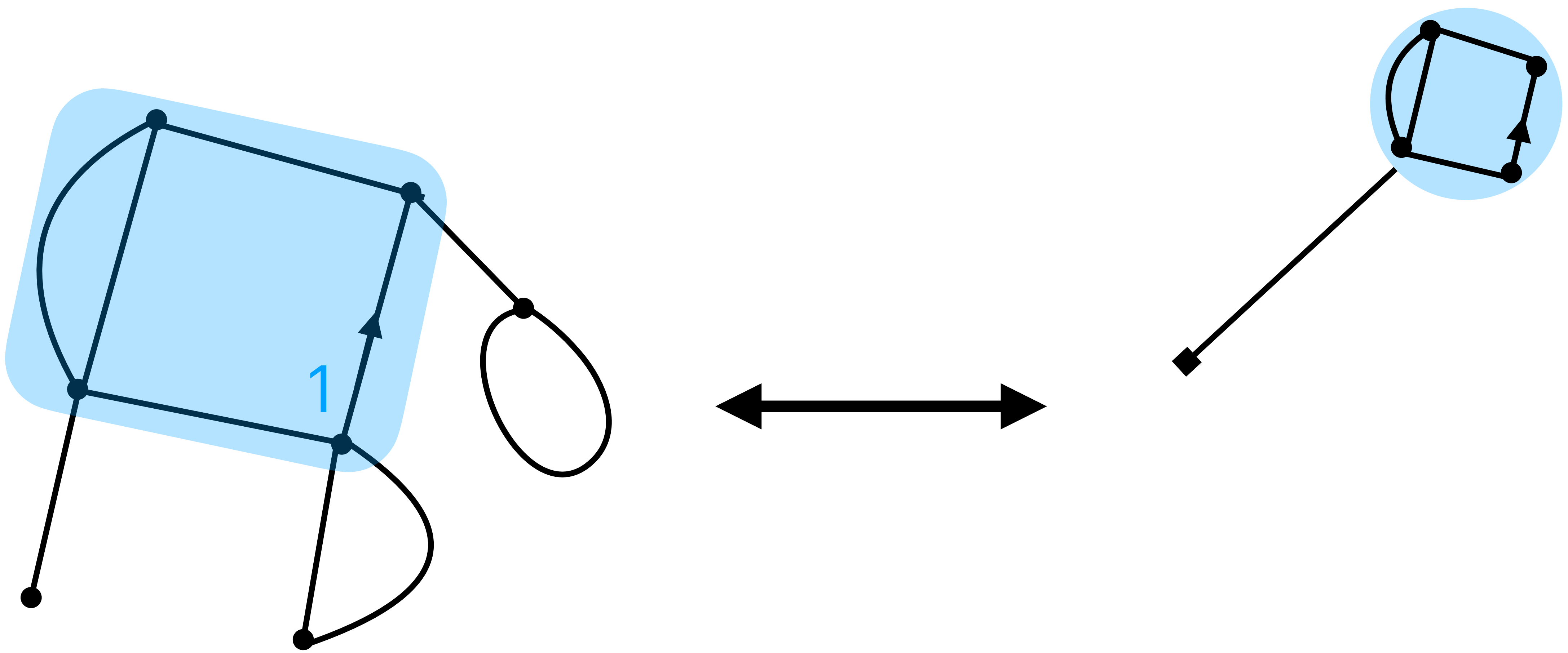

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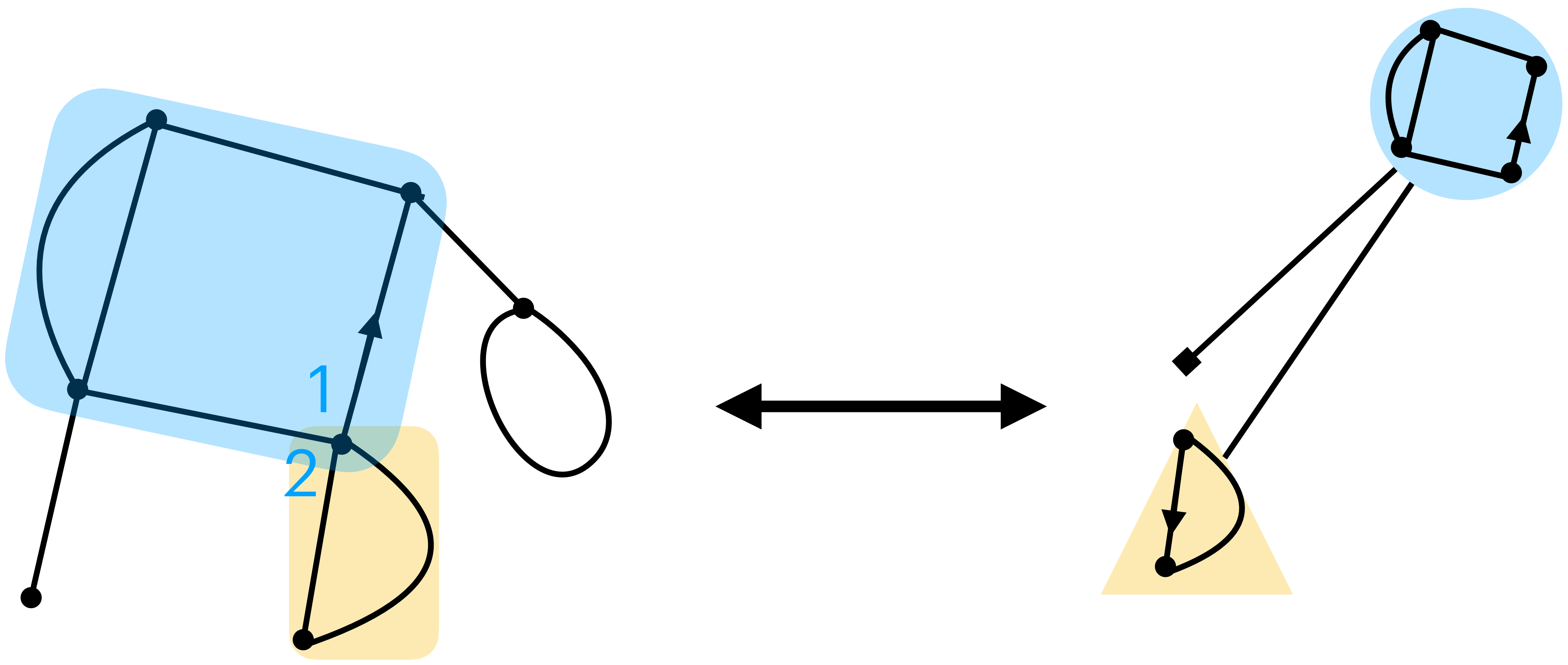

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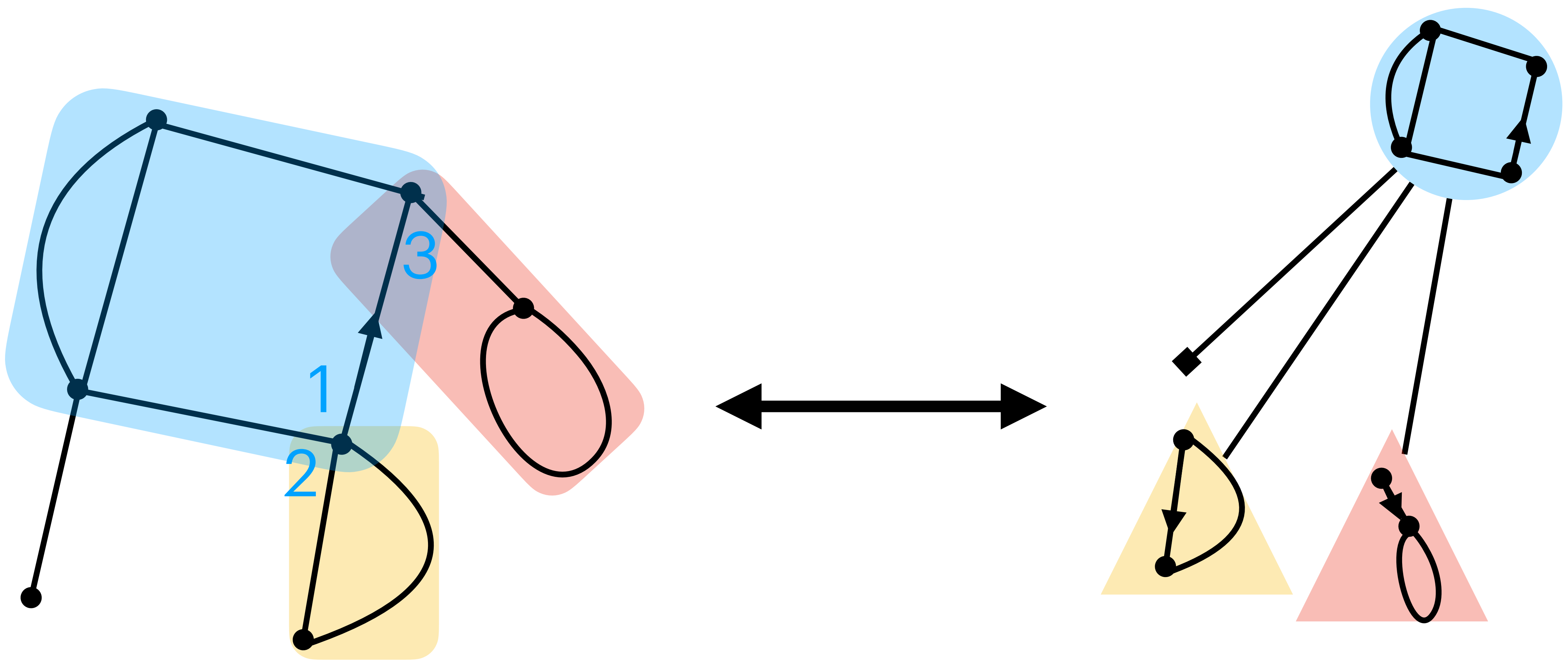

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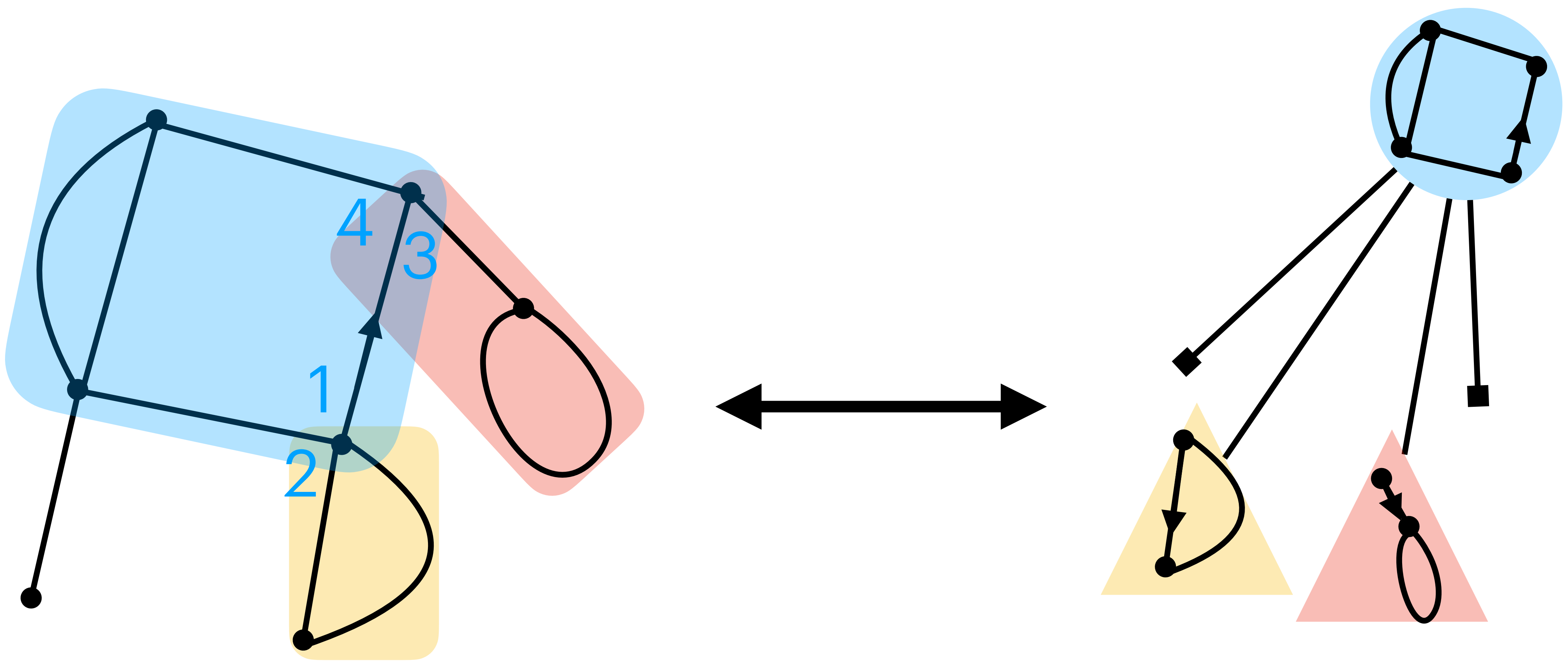

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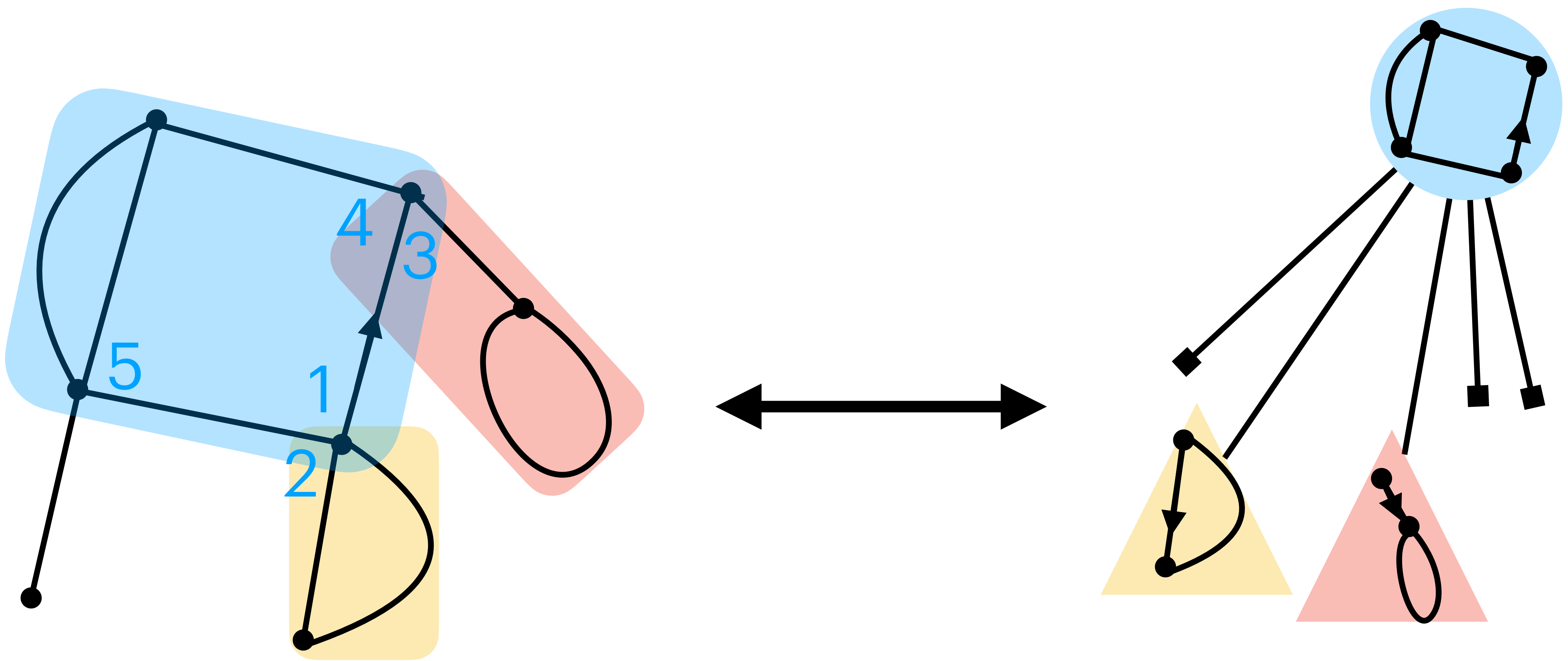

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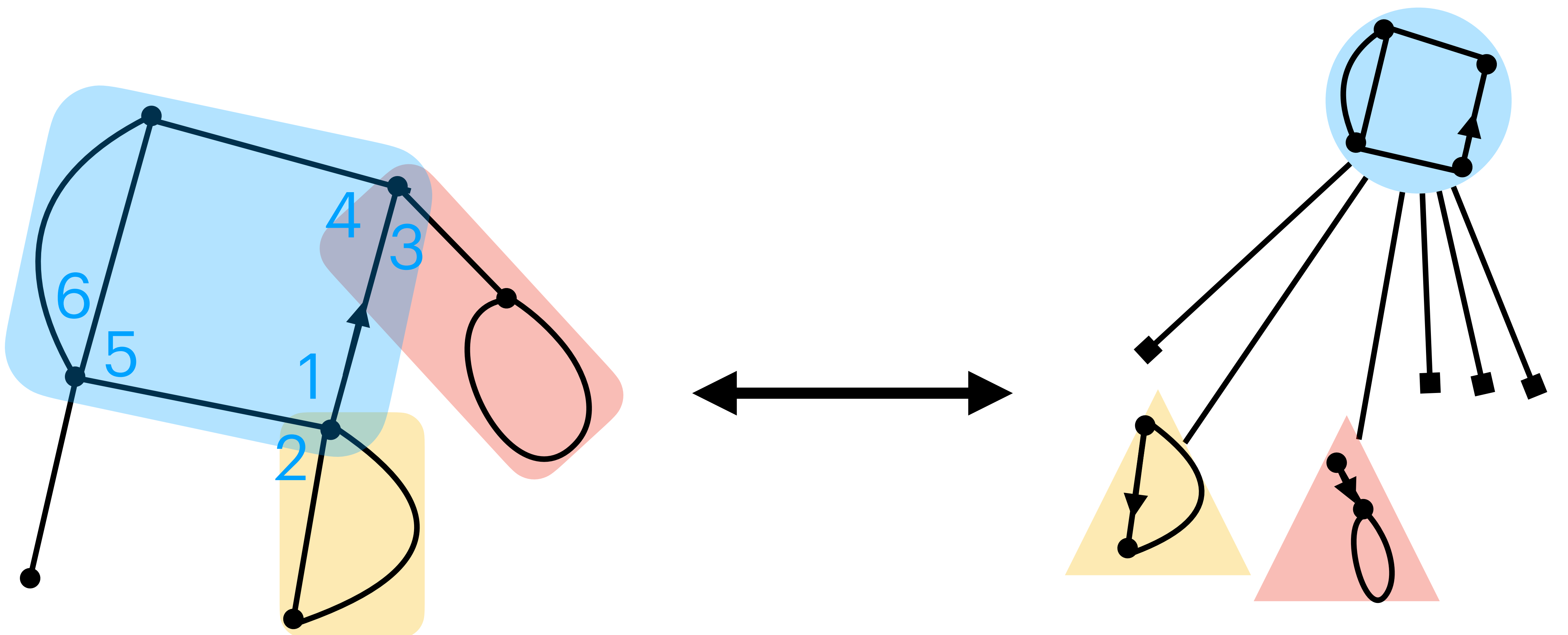

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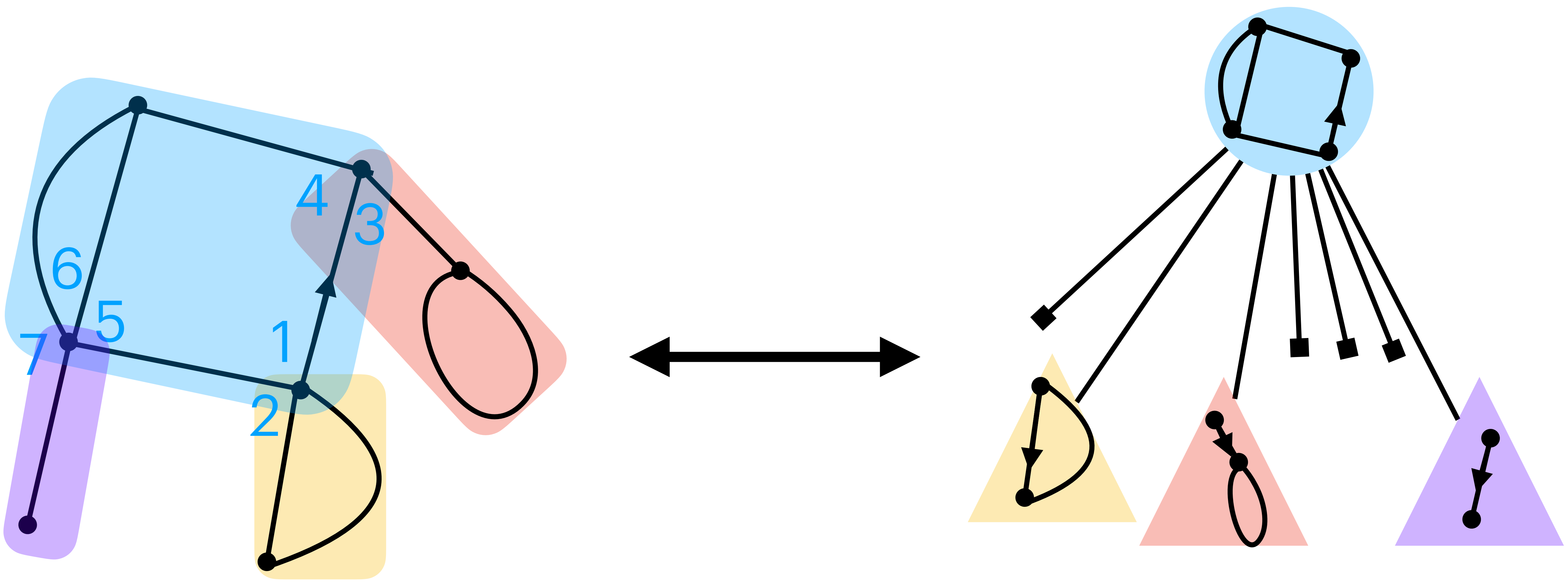

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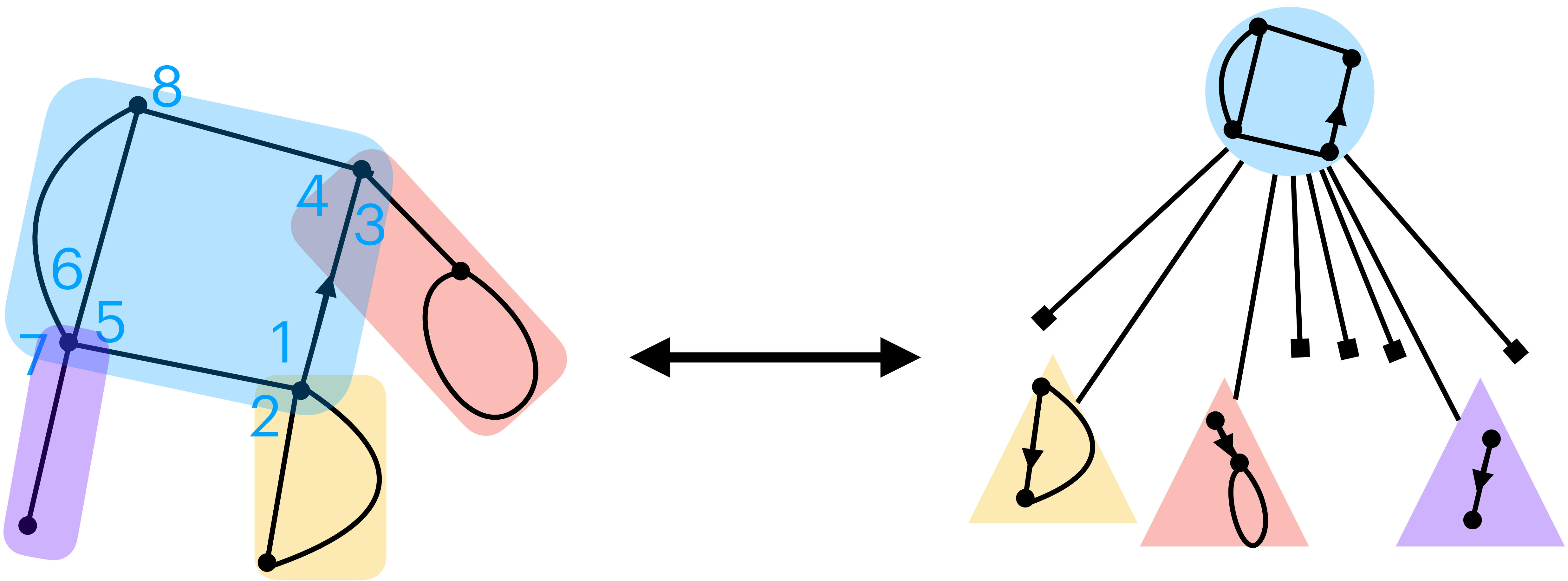

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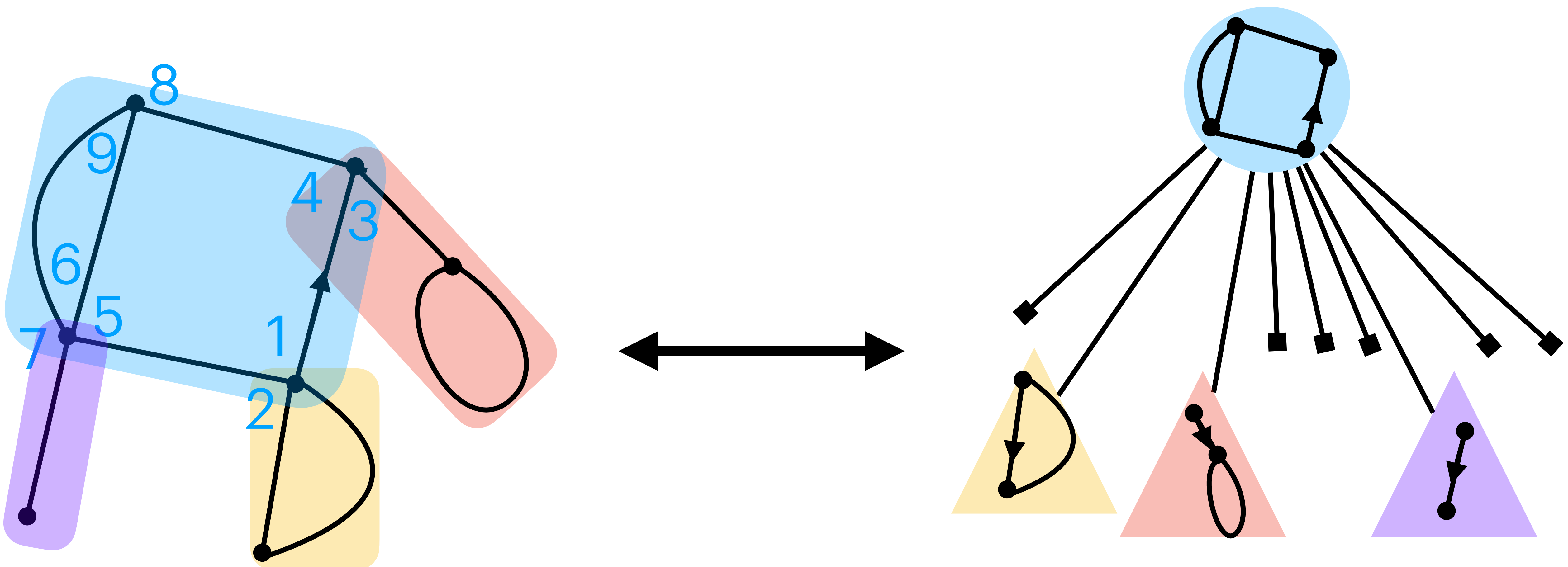

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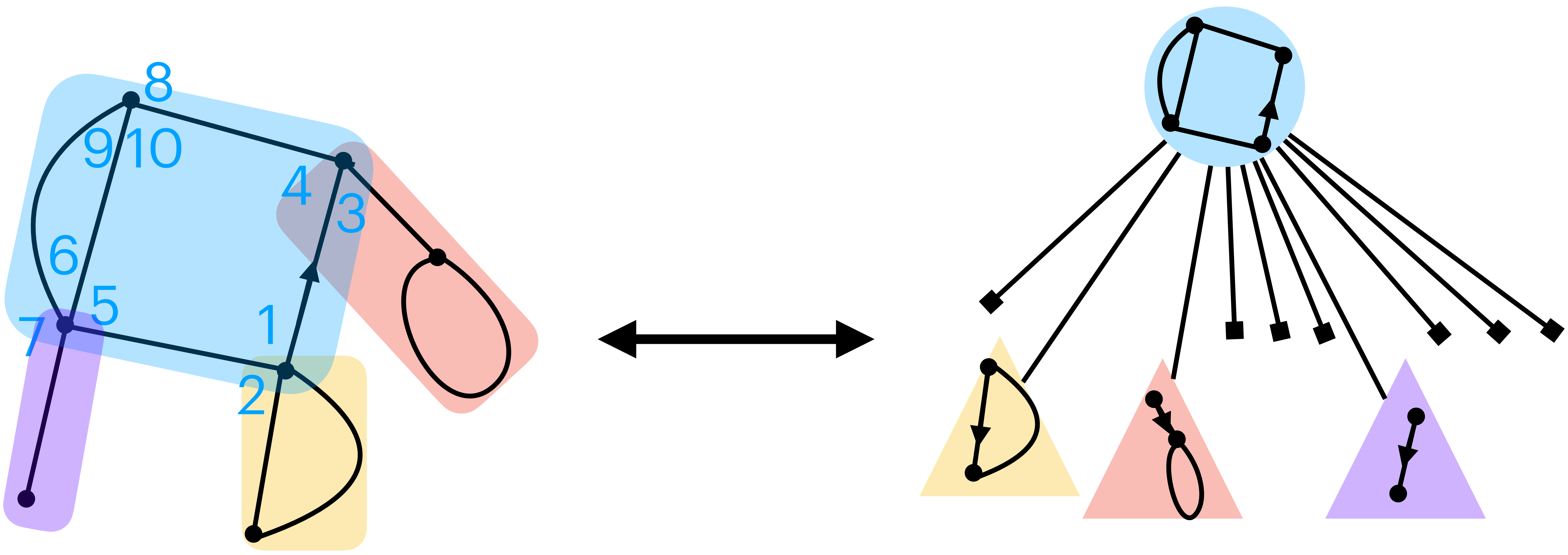

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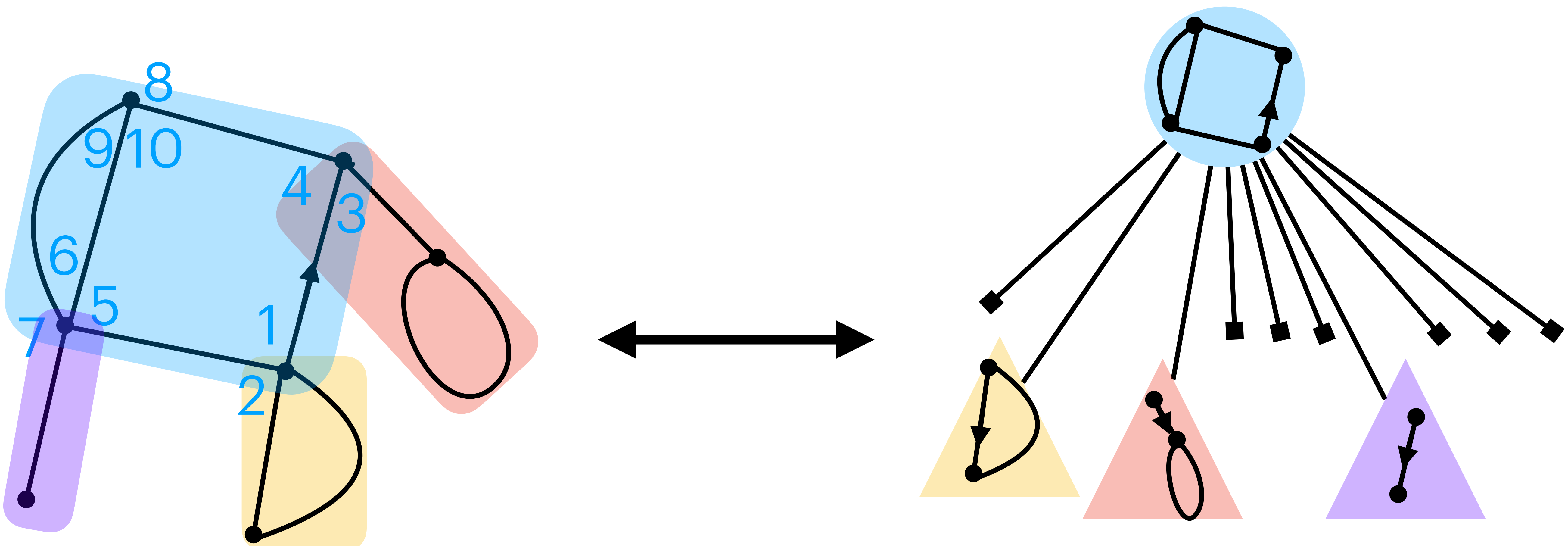


$$M(z) = B(zM^2(z))$$

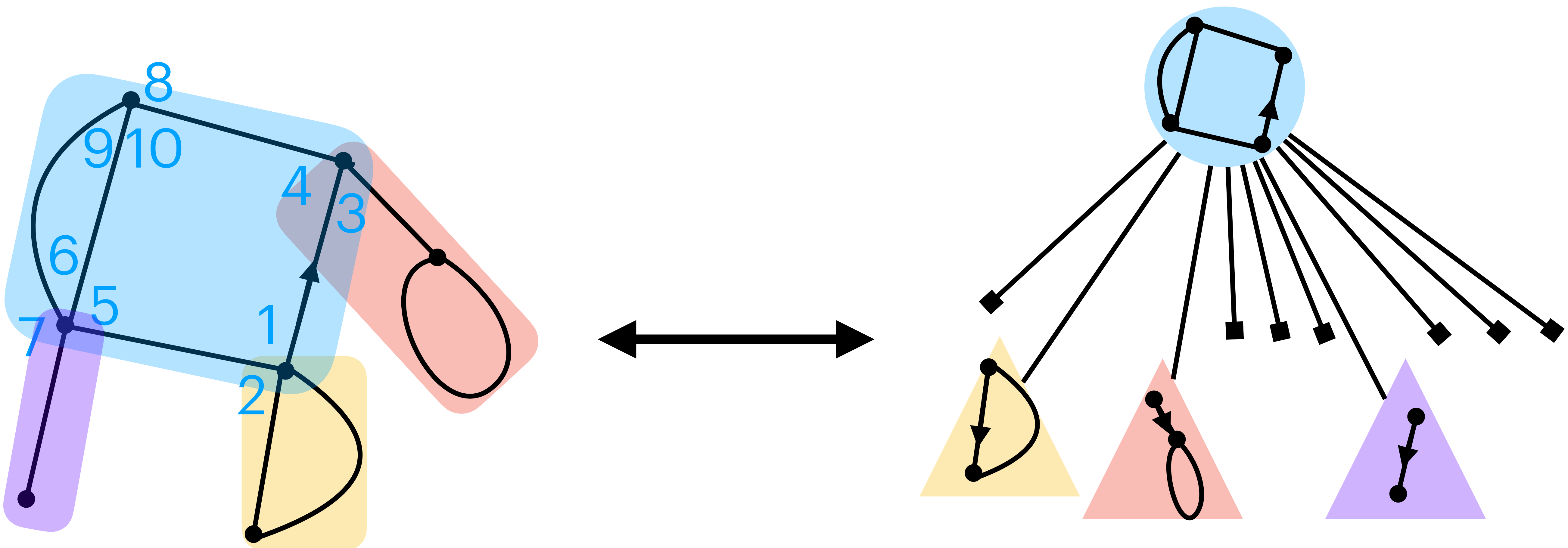
GS of maps

GS of 2-connected maps

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



$$M(z) = B(zM^2(z))$$

GS of maps

GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n}n^{-5/2}$	$\rho(u)^{-n}n^{-5/3}$	$\rho(u)^{-n}n^{-3/2}$
Size of - the largest block - the second one			
Scaling limit of M_n			

Decomposition of a map into blocks

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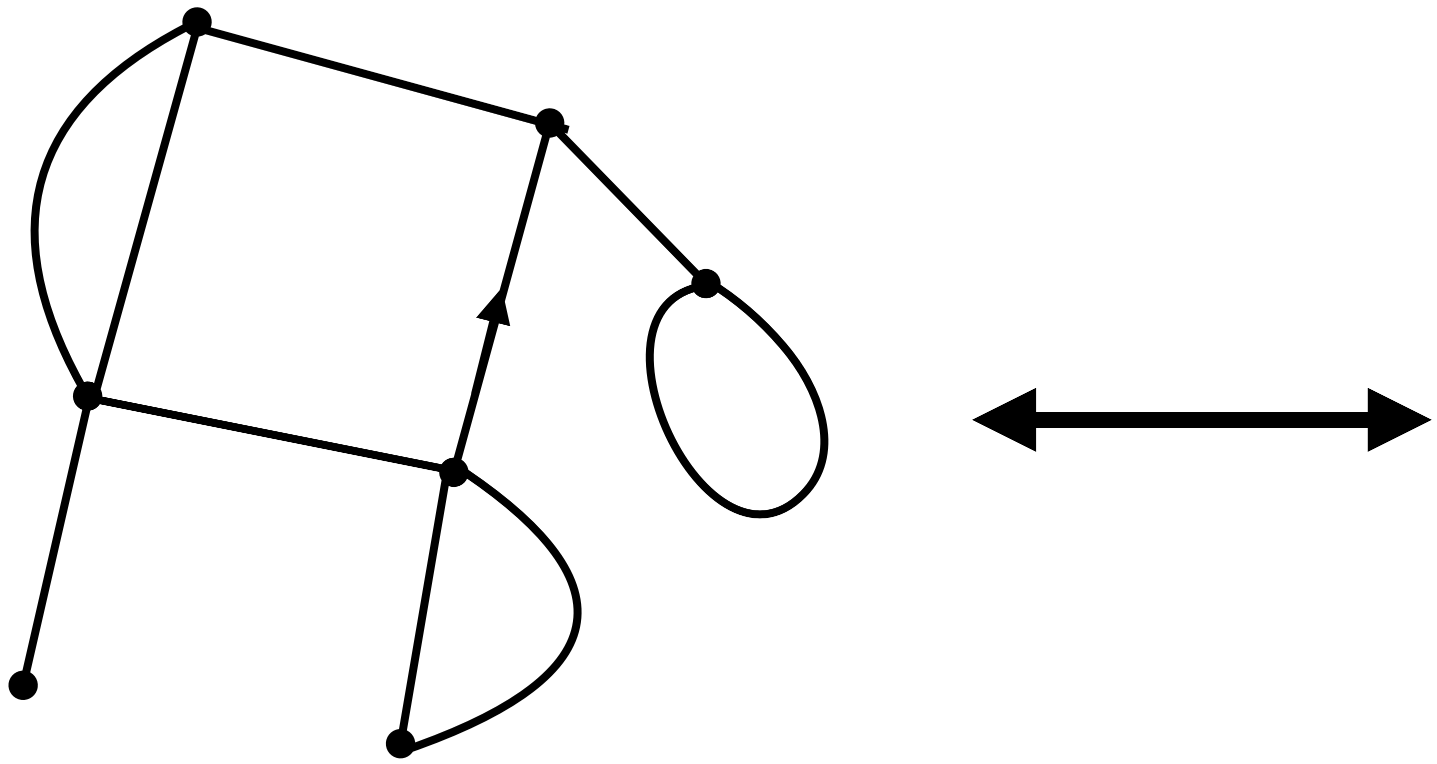

⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

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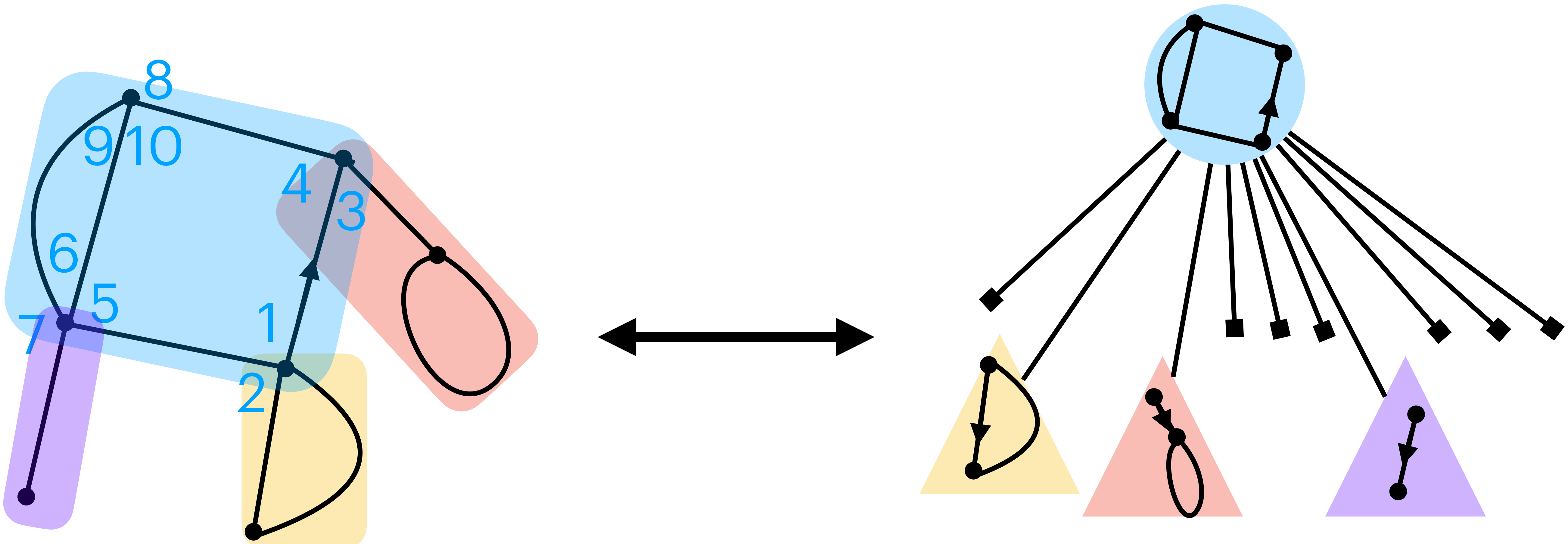

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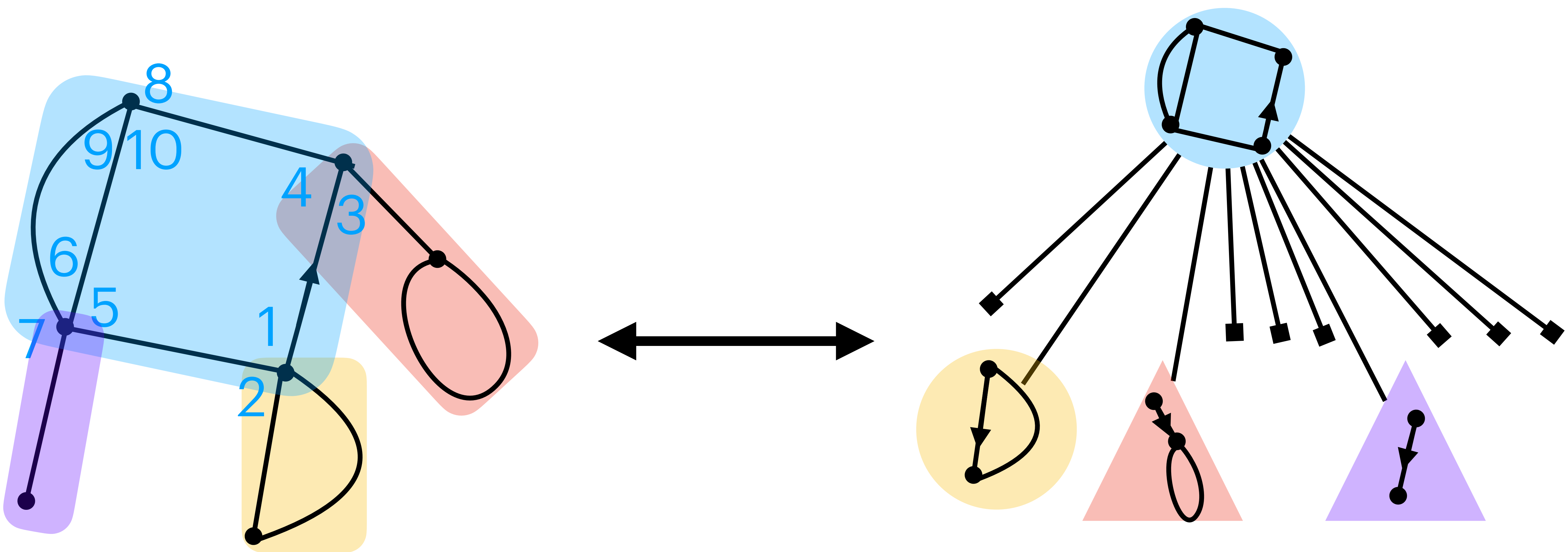

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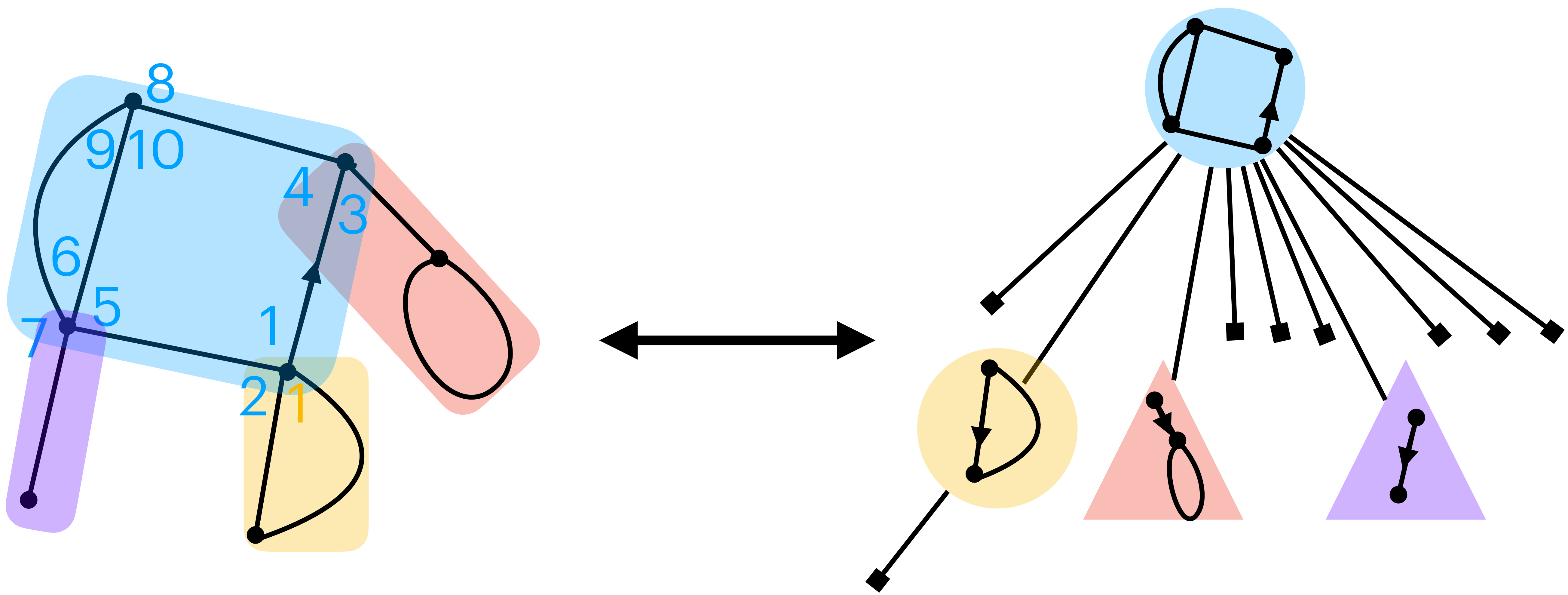

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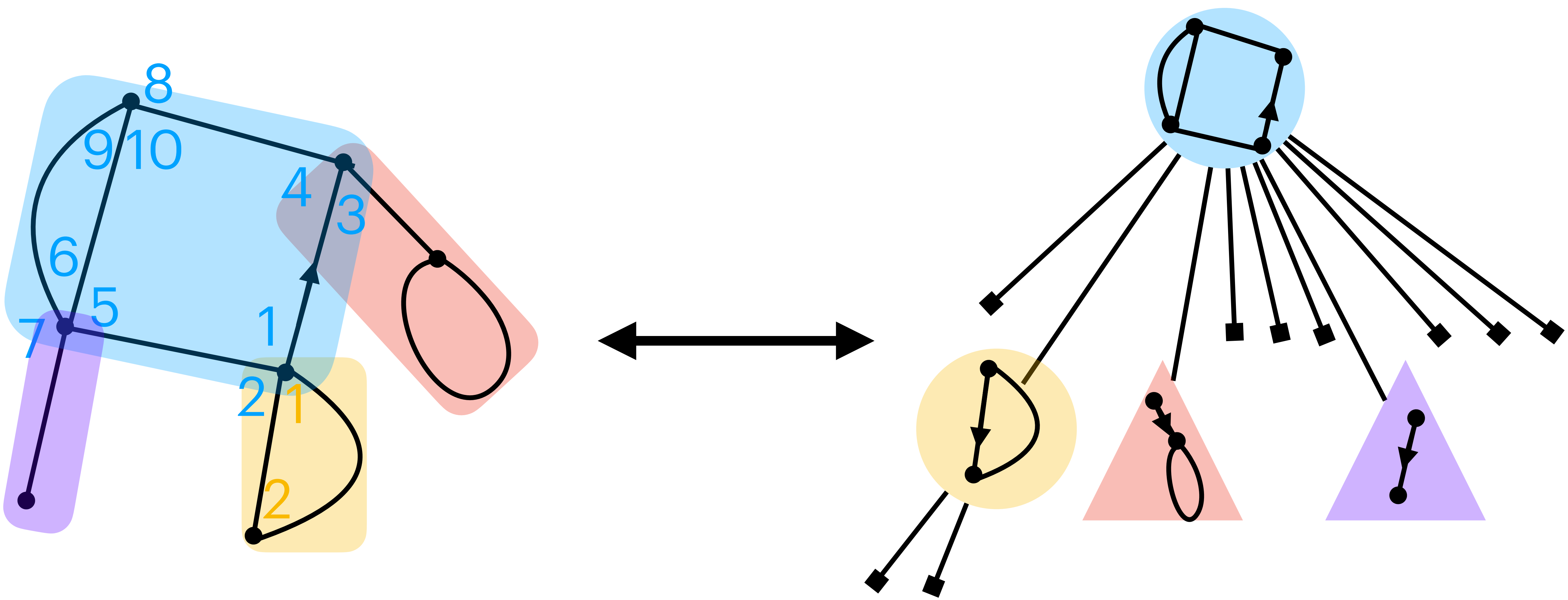

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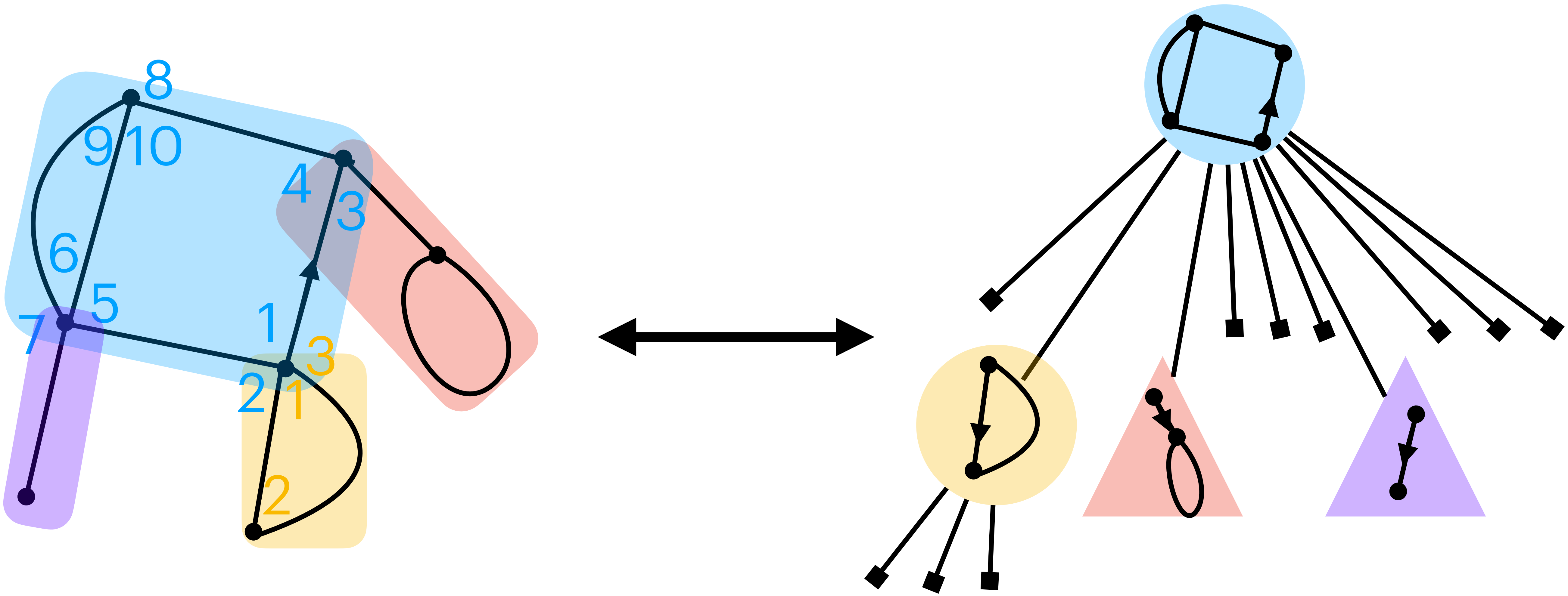

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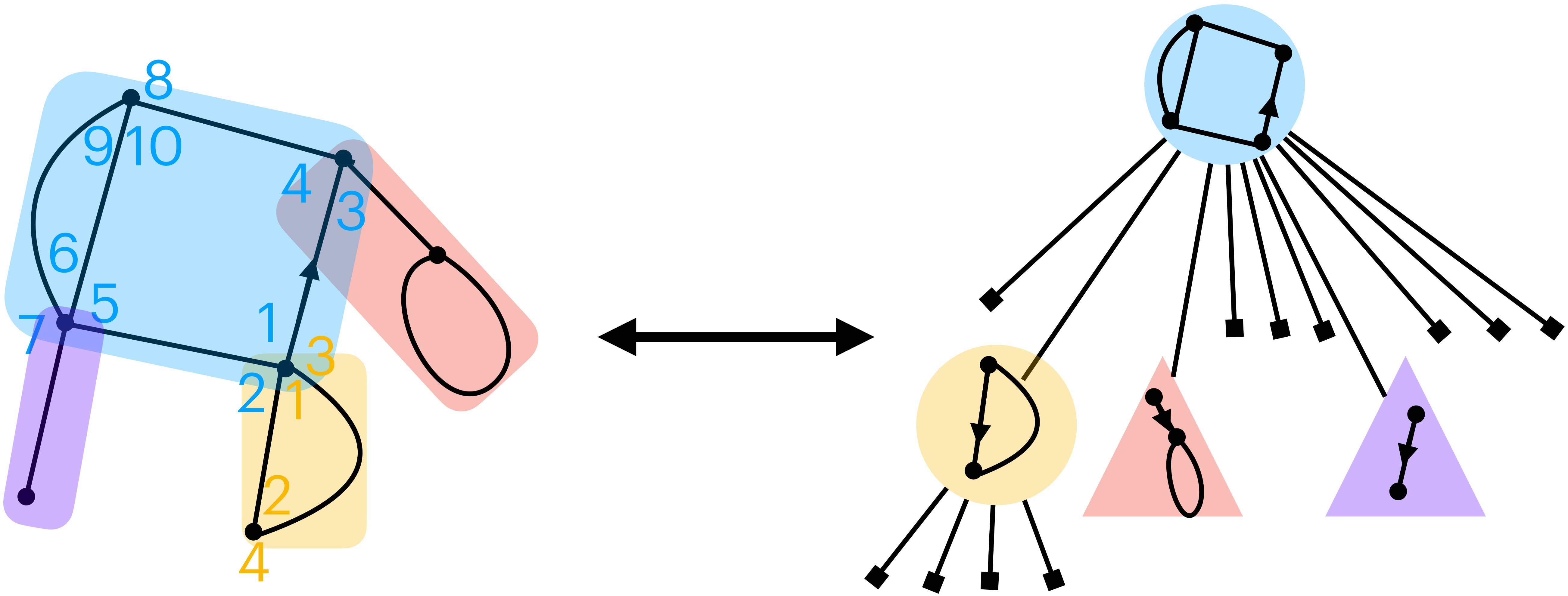

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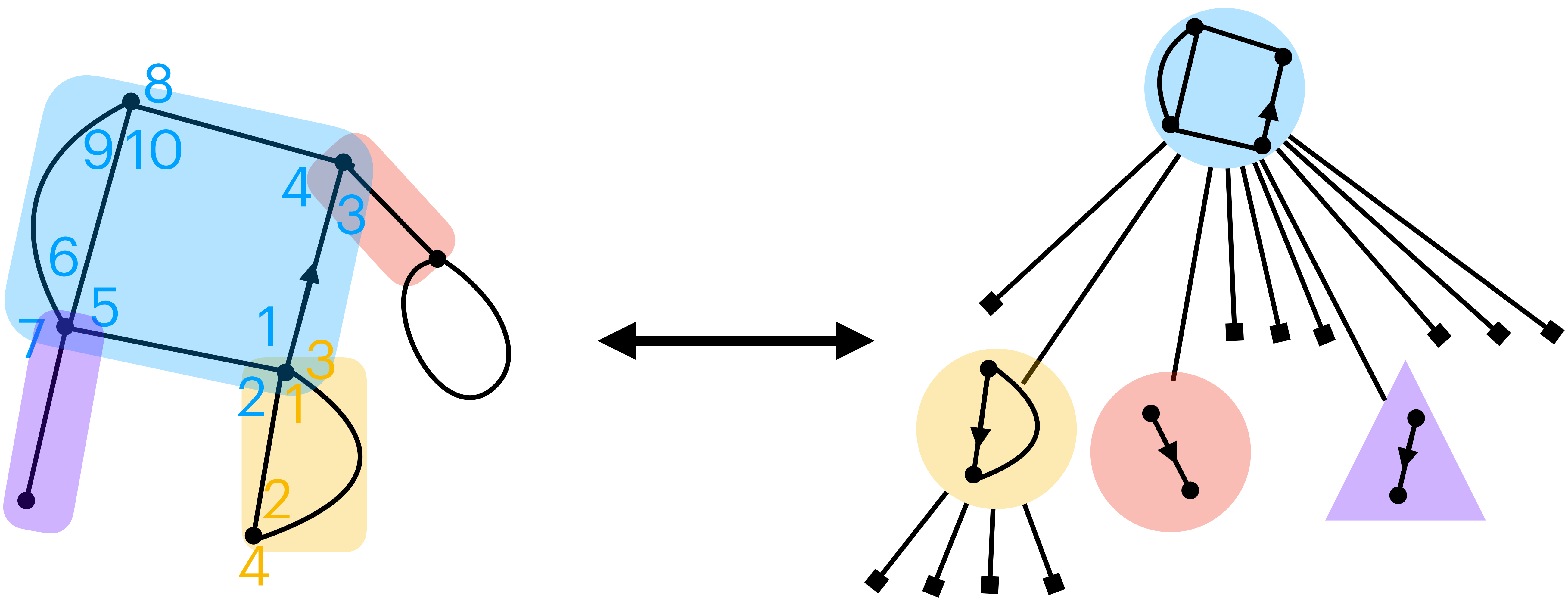

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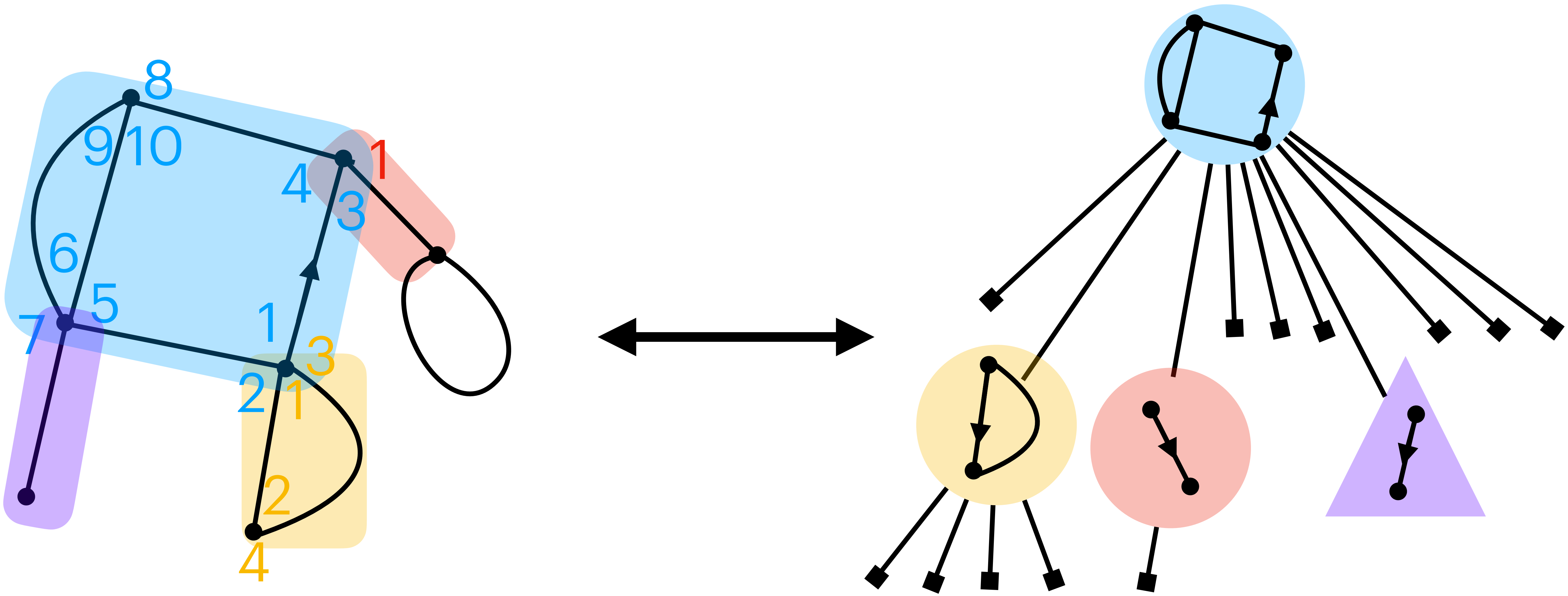

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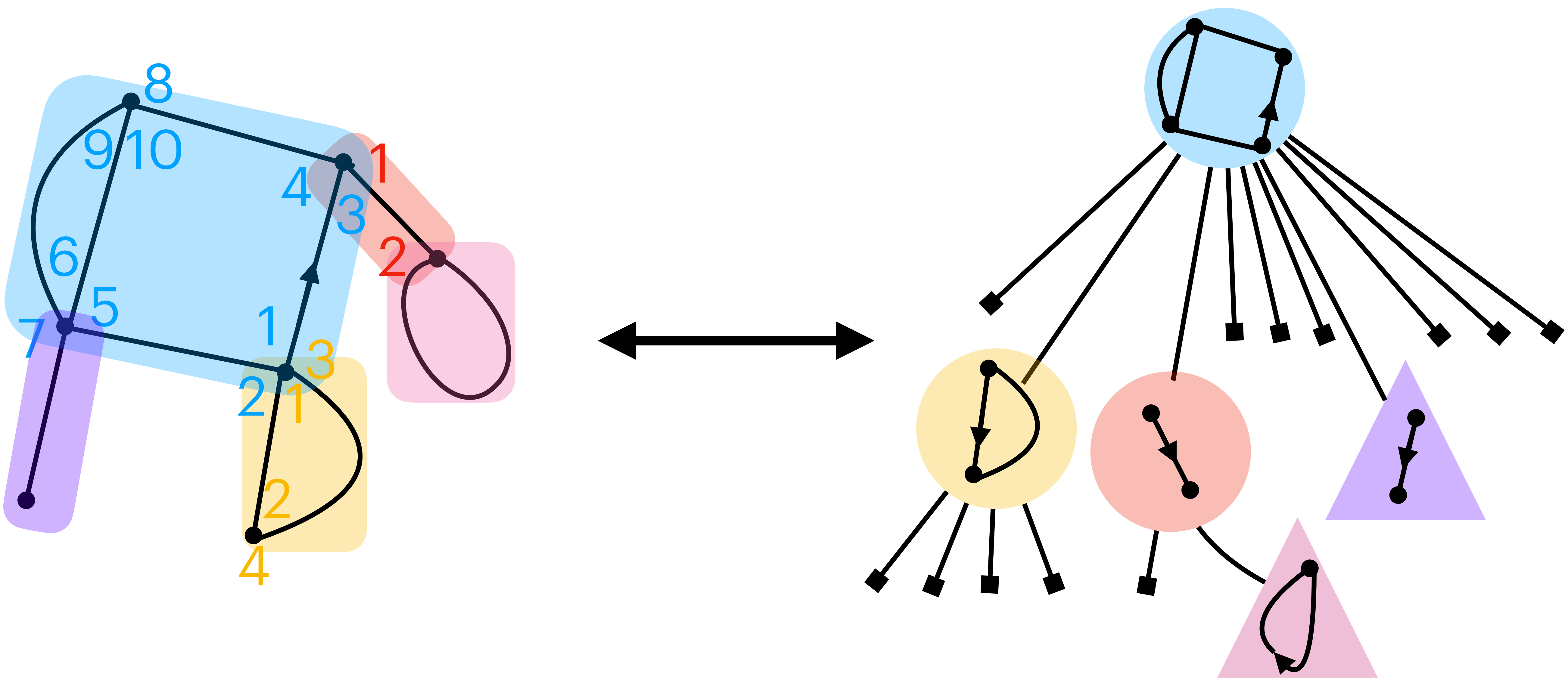

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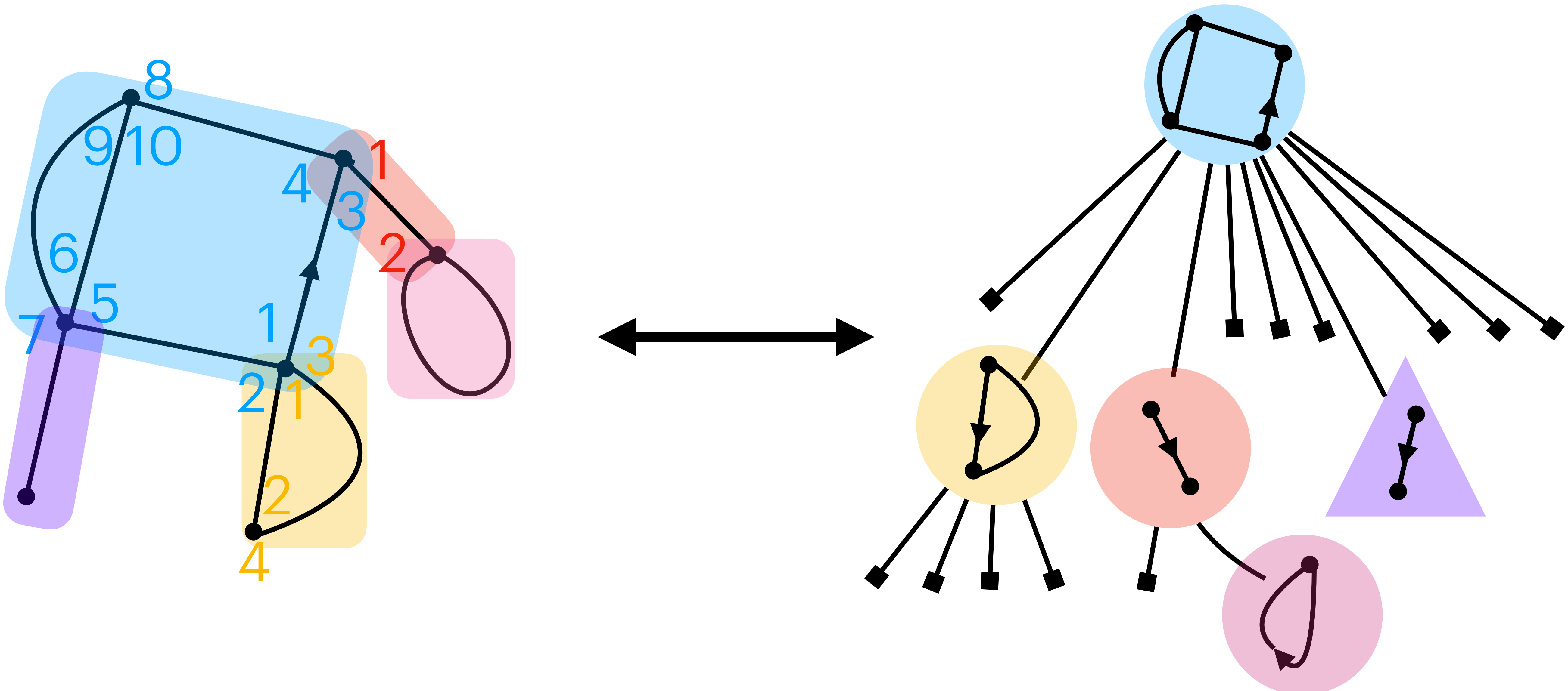

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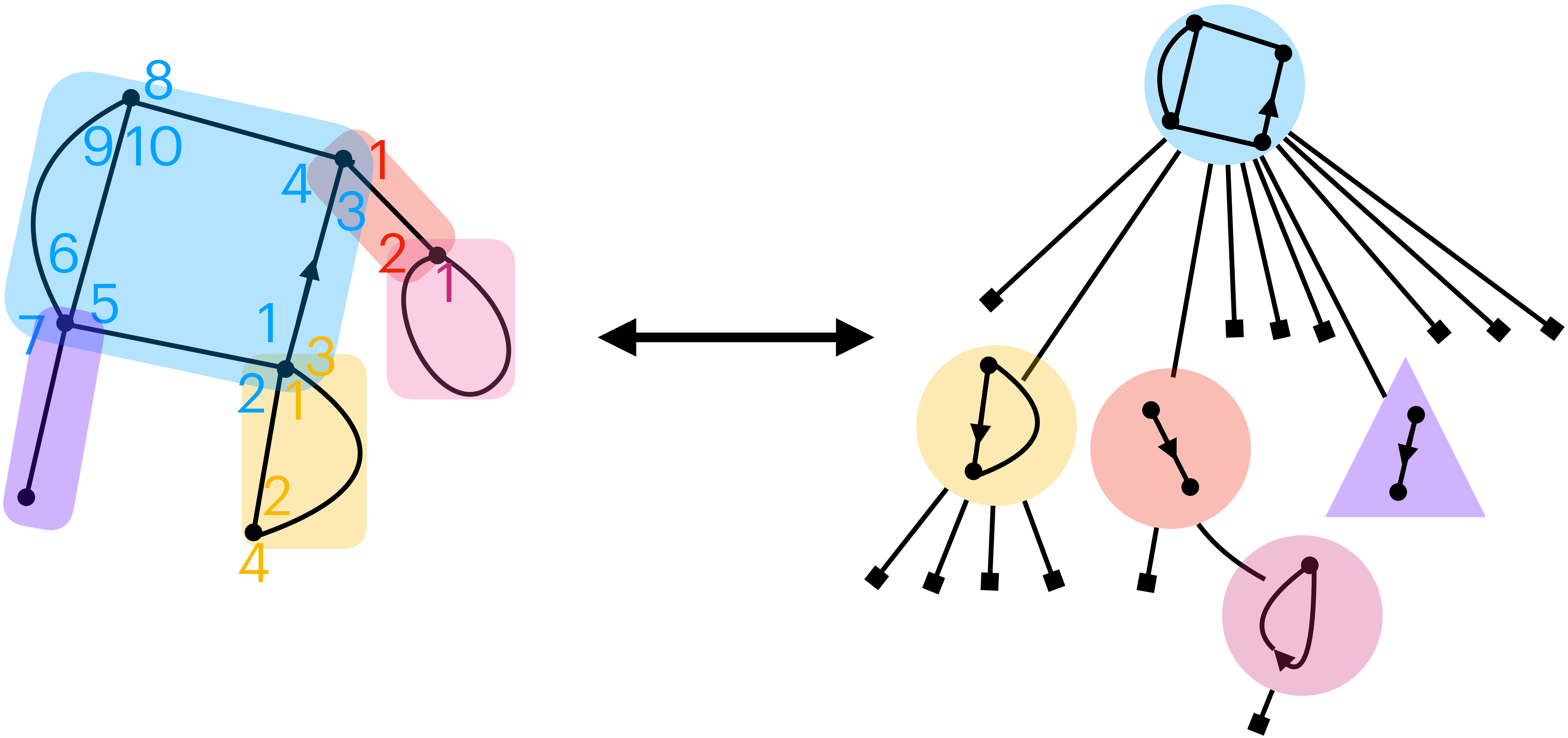

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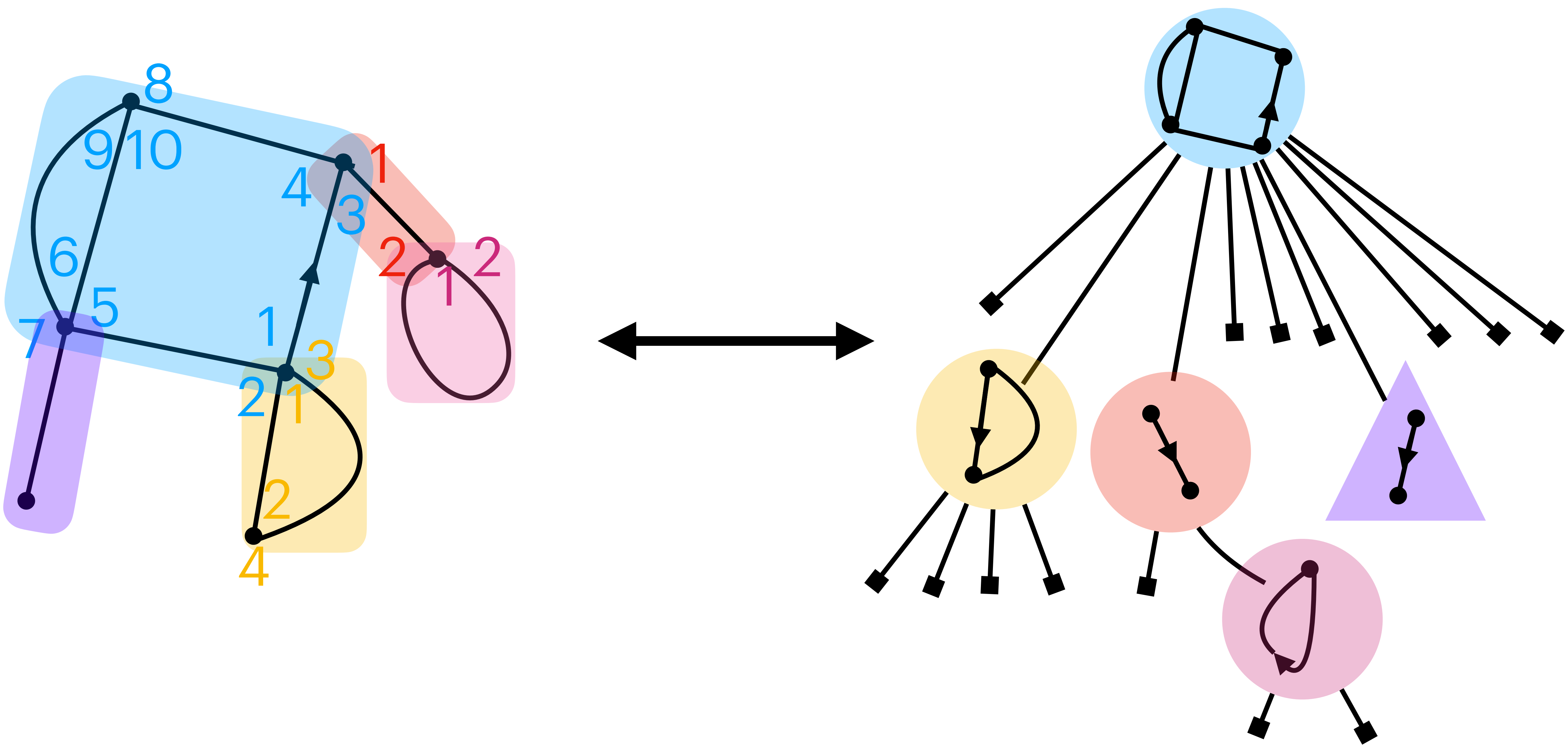

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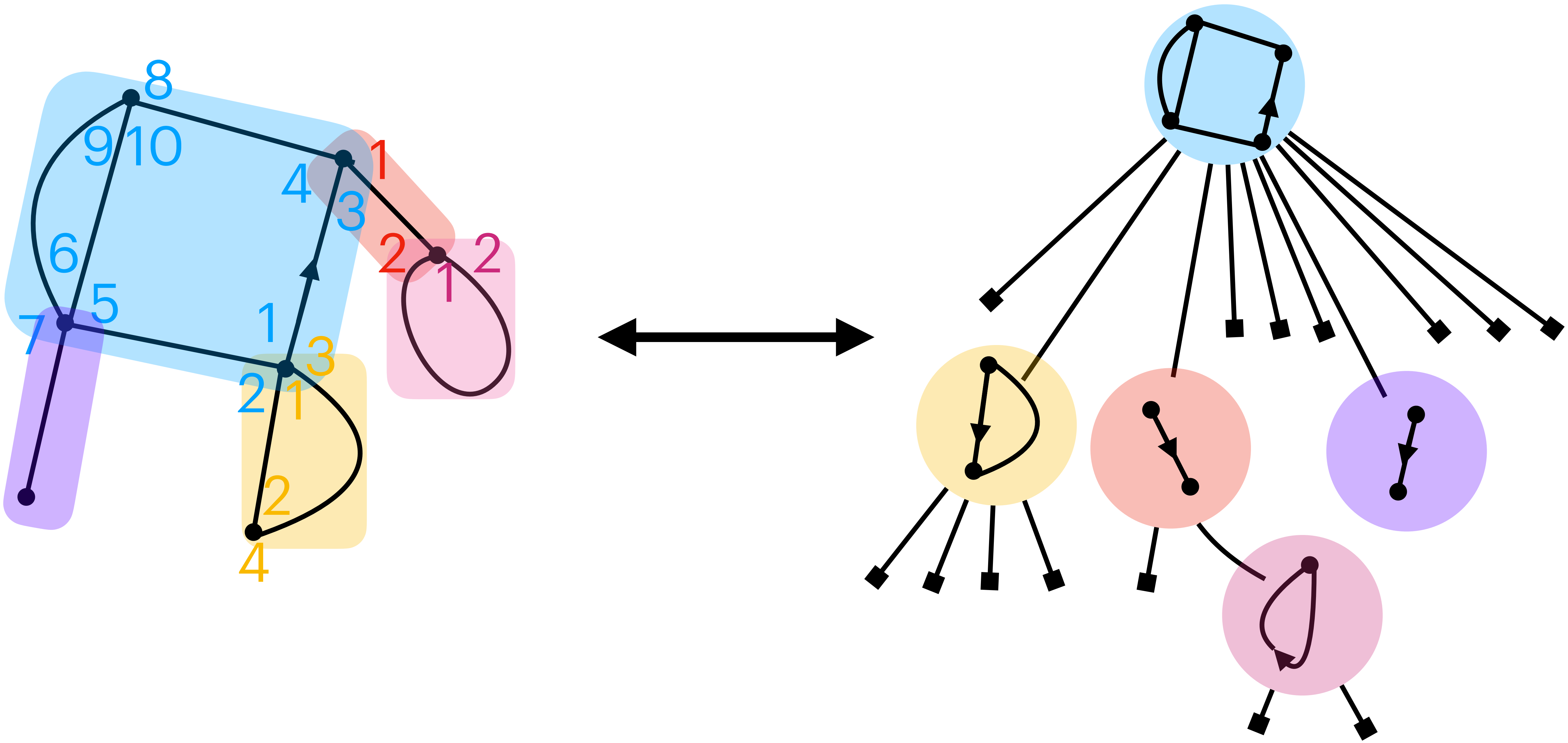

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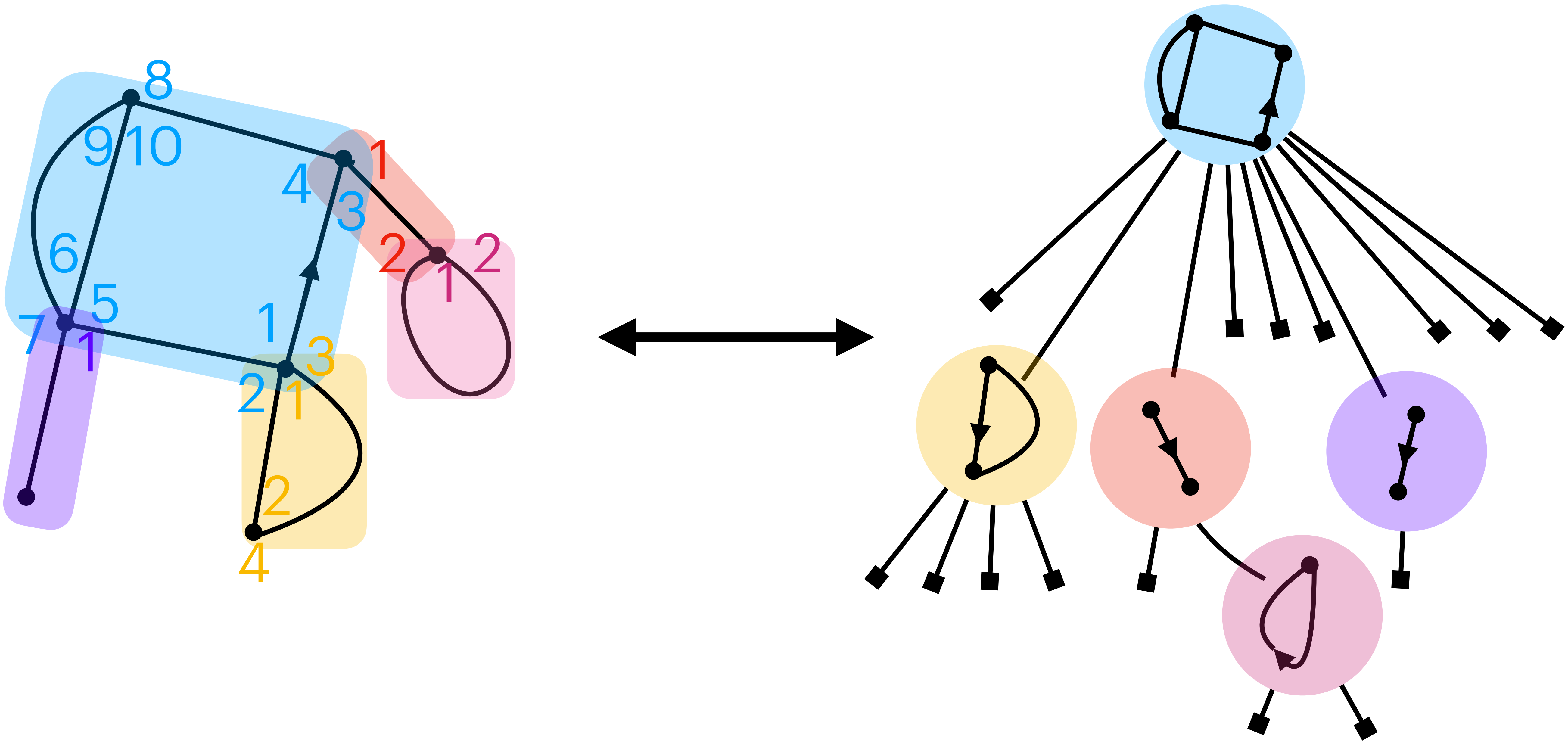

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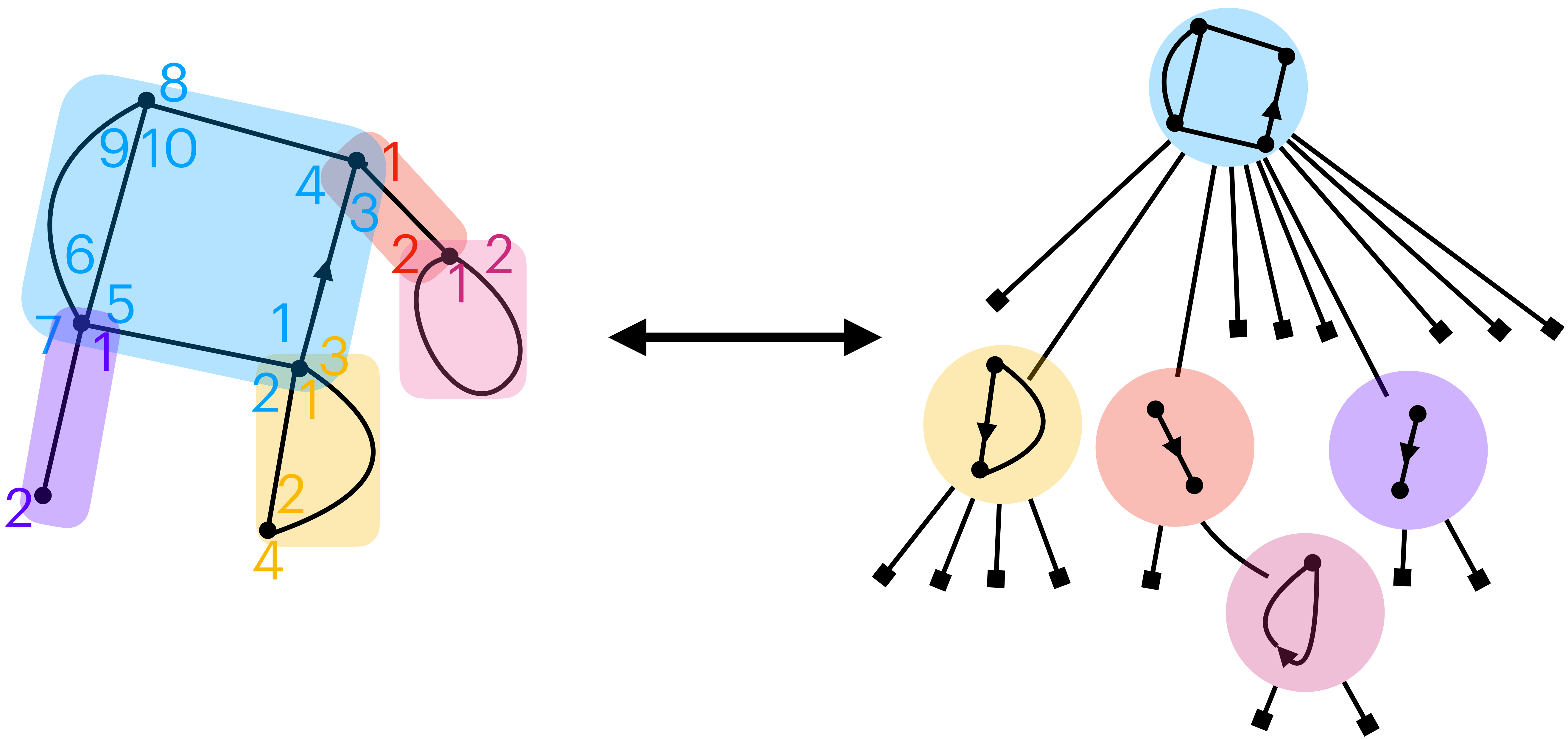

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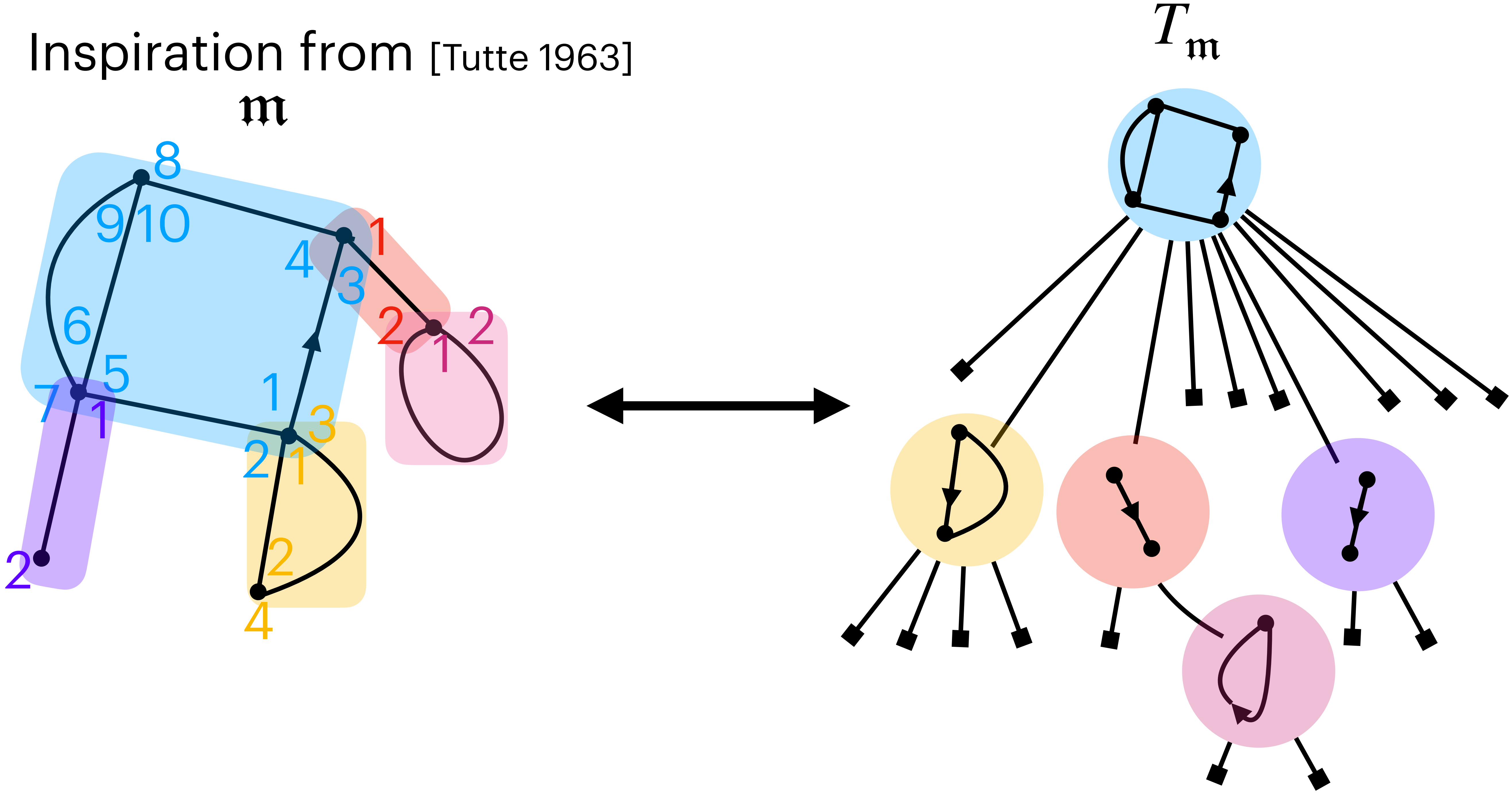

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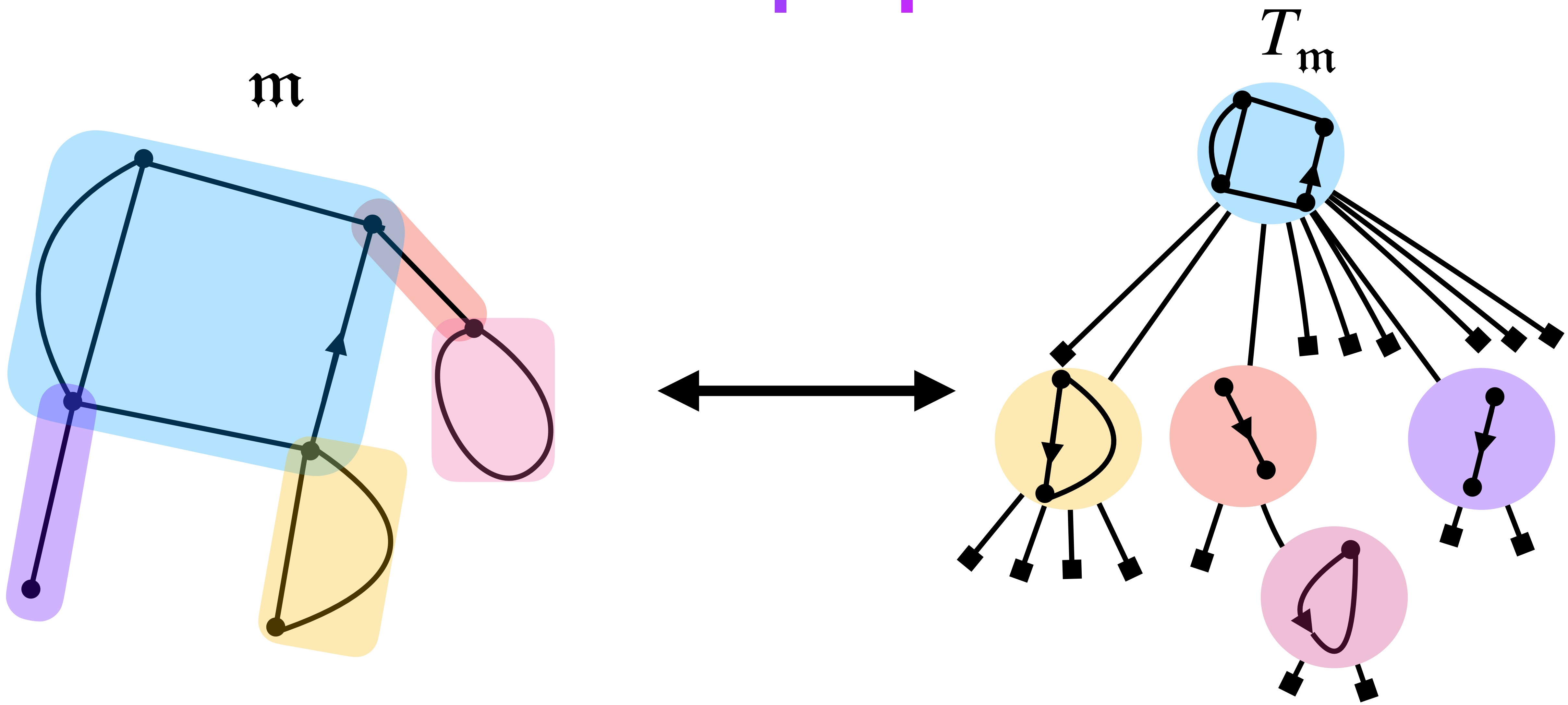
Block tree: properties



- $\mathfrak{m}$ is entirely determined by $T_{\mathfrak{m}}$ and $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ where $\mathfrak{b}_v$ is the block of $\mathfrak{m}$ represented by v in $T_{\mathfrak{m}}$;
- Internal node (with k children) of $T_{\mathfrak{m}} \leftrightarrow$ block of $\mathfrak{m}$ of size $k/2$.

T_{M_n} gives the block sizes of a random map M_n.

Block trees are BGW-trees

μ-Bienaymé-Galton-Watson (BGW) tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

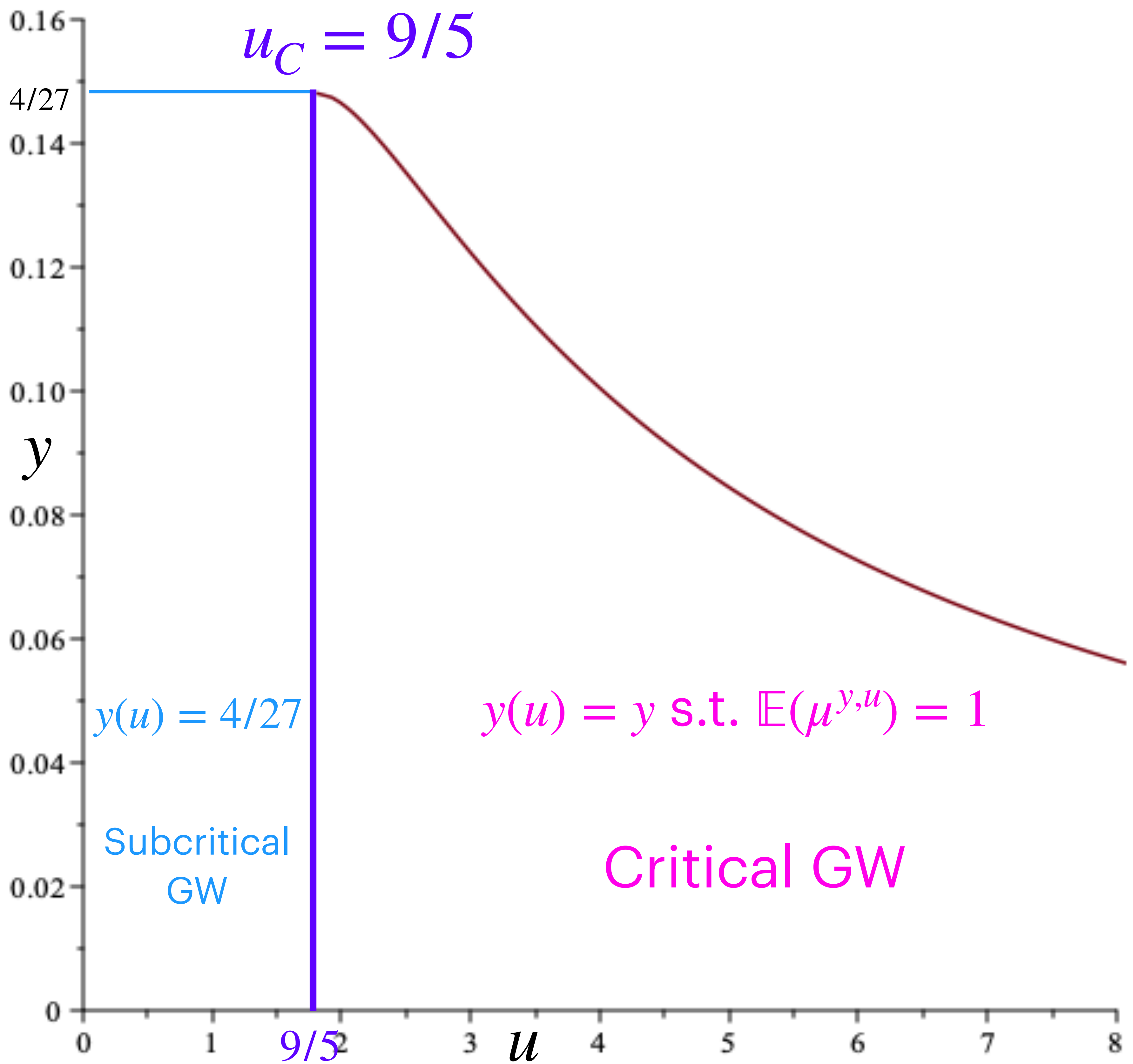
Theorem [Fleurat, S. 24]

$u > 0$

If $M_n \hookrightarrow \mathbb{P}_{n,u}$, then there exists an (explicit) $y = y(u)$ s.t. T_{M_n} has the law of a BGW tree of reproduction law $\mu^{y,u}$ conditioned to be of size $2n$, with

$$\mu^{y,u}(\{2k\}) = \frac{B_k y^k u^{\mathbf{1}_{k\neq 0}}}{uB(y) + 1 - u}.$$

Phase transition for $y(u)$



"Map regime" "Tree regime"

Largest blocks?

- Degrees of T_{M_n} give the block sizes of the map M_n;
- Largest degrees of a BGW tree are well-known [Janson 2012].

Rough intuition

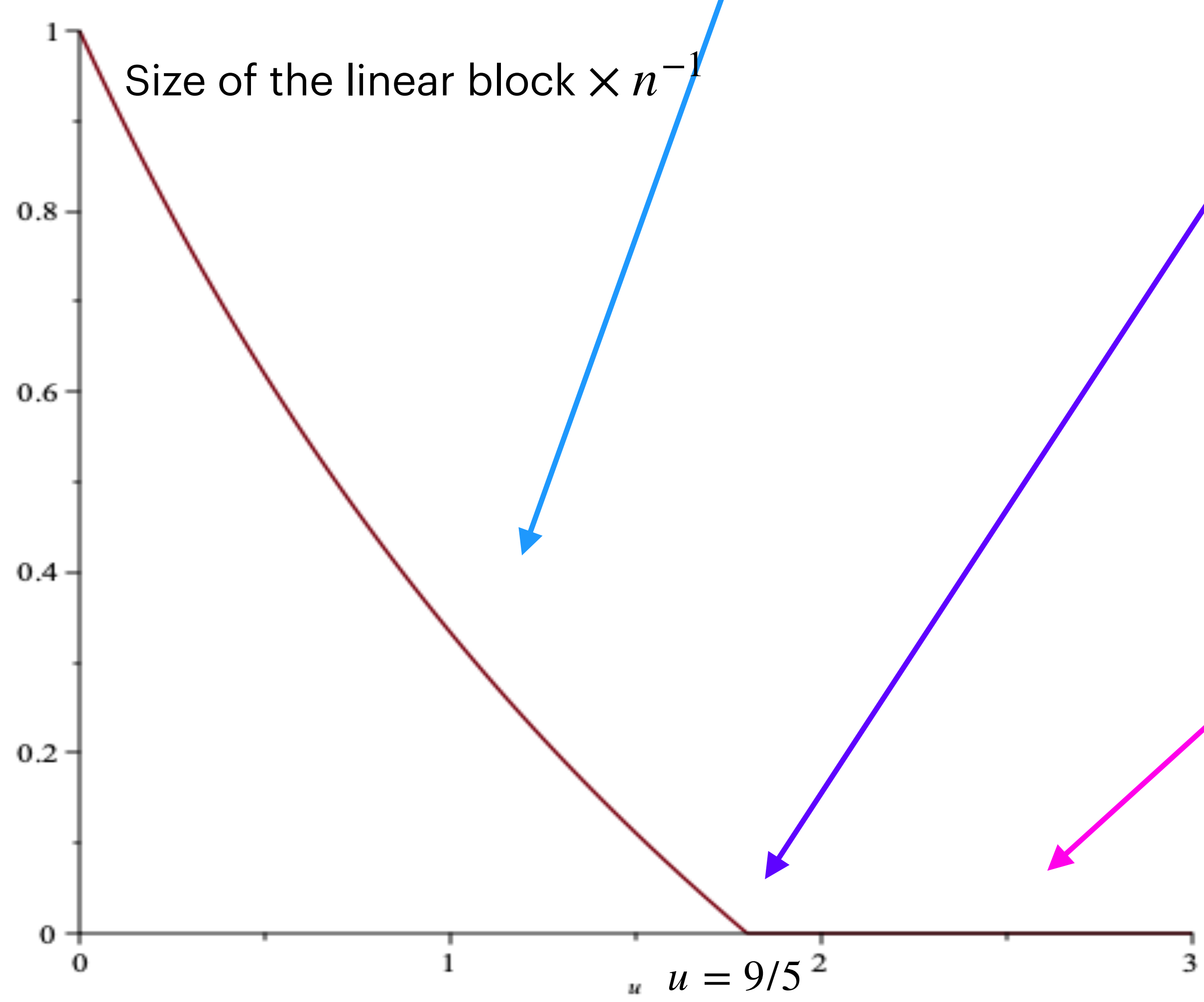
	$u < 9/5$	$u = 9/5$	$u > 9/5$
$\mu^{y(u),u}(\{2k\})$	$\sim c_u k^{-5/2}$		$\sim c_u \pi_u^k k^{-5/2}$
BGW tree	subcritical	critical	

Dichotomy between situations:

- Subcritical: condensation, cf [Jonsson Stefánsson 2011];
- Supercritical: behaves as maximum of independent variables.

Size $L_{n,k}$ of the k-th largest block

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
$L_{n,1}$	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
$L_{n,2}$	$\Theta(n^{2/3})$ [Stufler 2020]		



Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

Interlude: quadrangulations

Quadrangulations

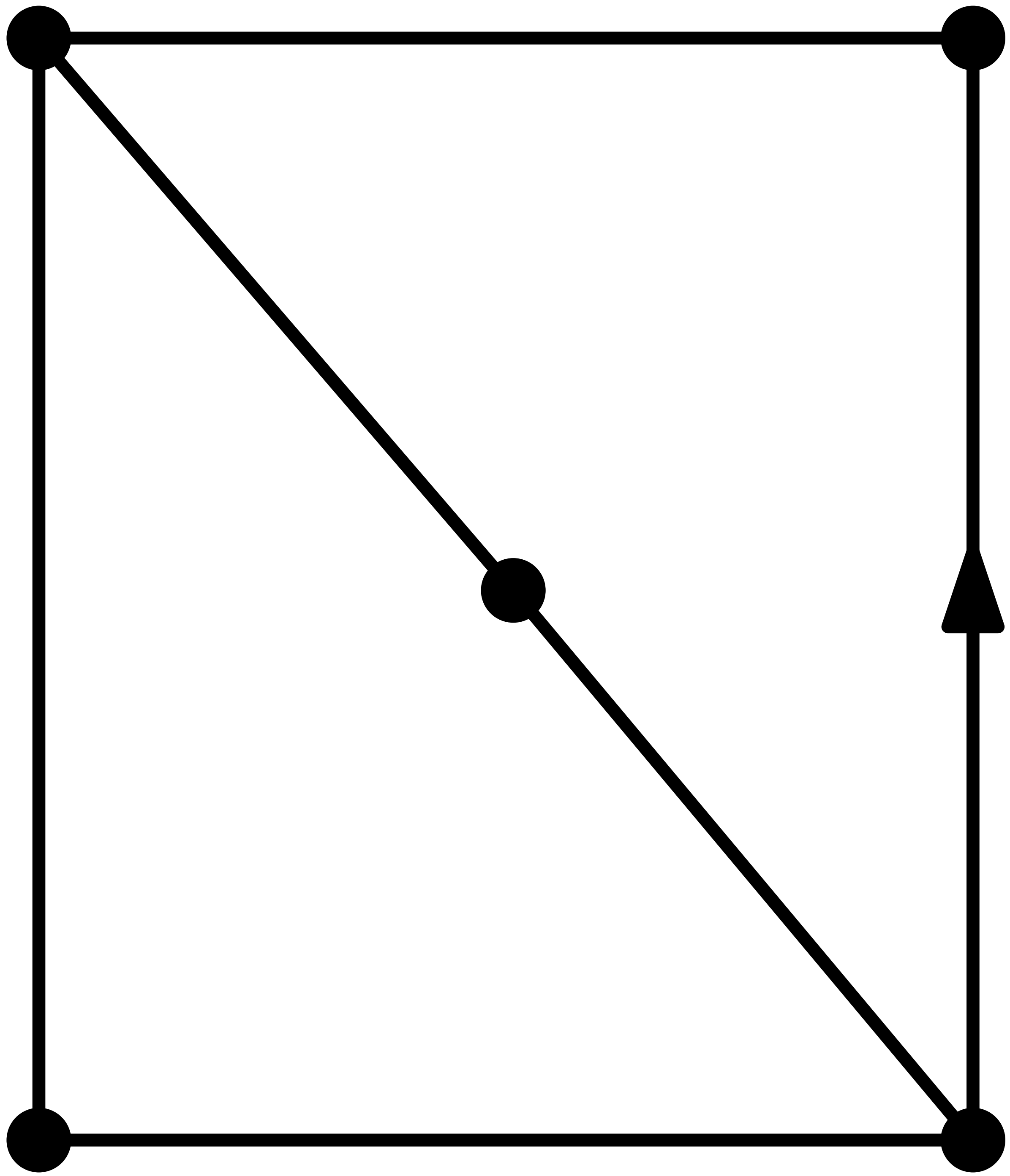
Def: map with all faces of degree 4.

Simple quadrangulation = no multiple edges.

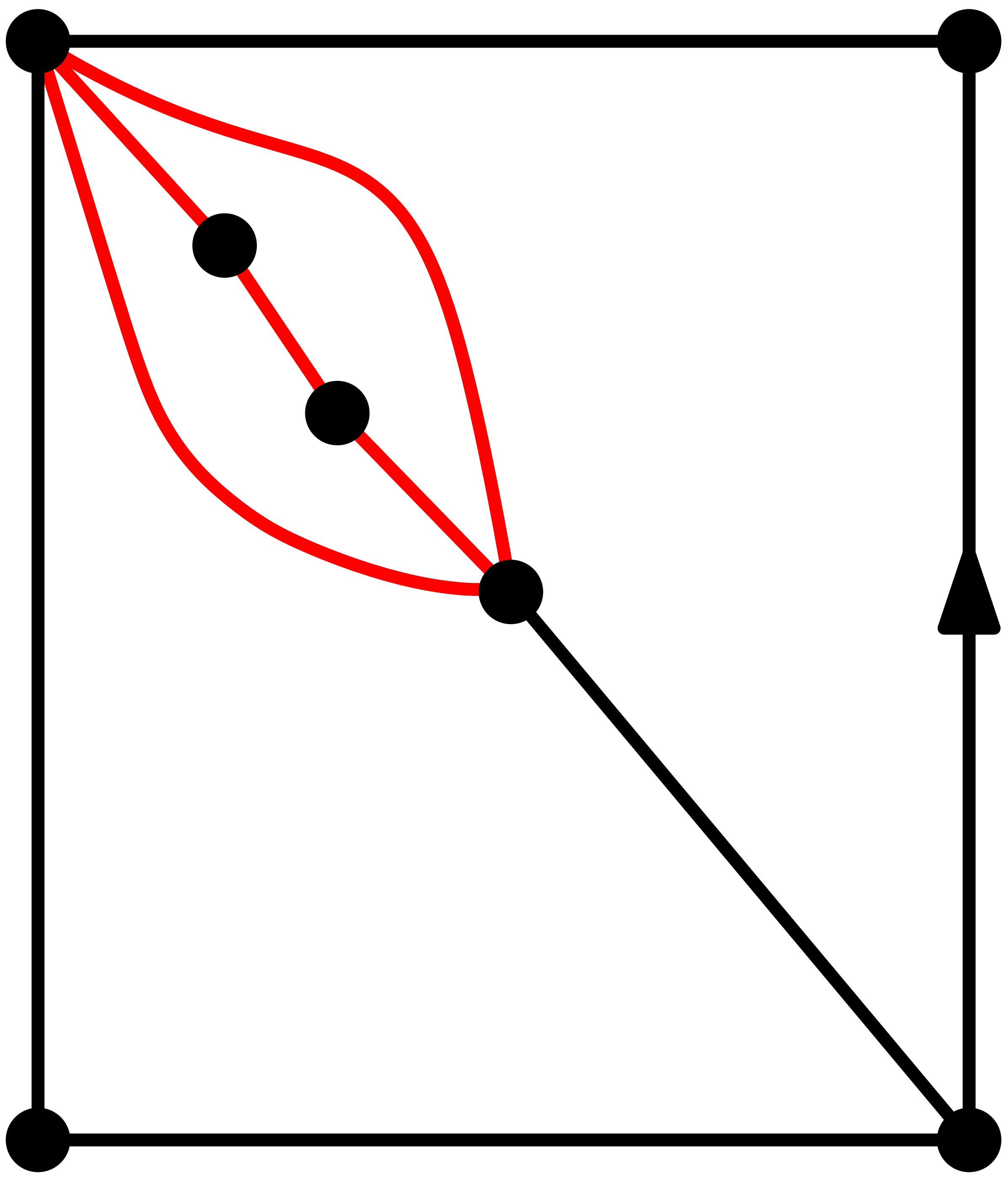
Size $|\mathbf{q}|$ = number of *faces*.

$$|V(\mathbf{q})| = |\mathbf{q}| + 2, \ |E(\mathbf{q})| = 2|\mathbf{q}|.$$

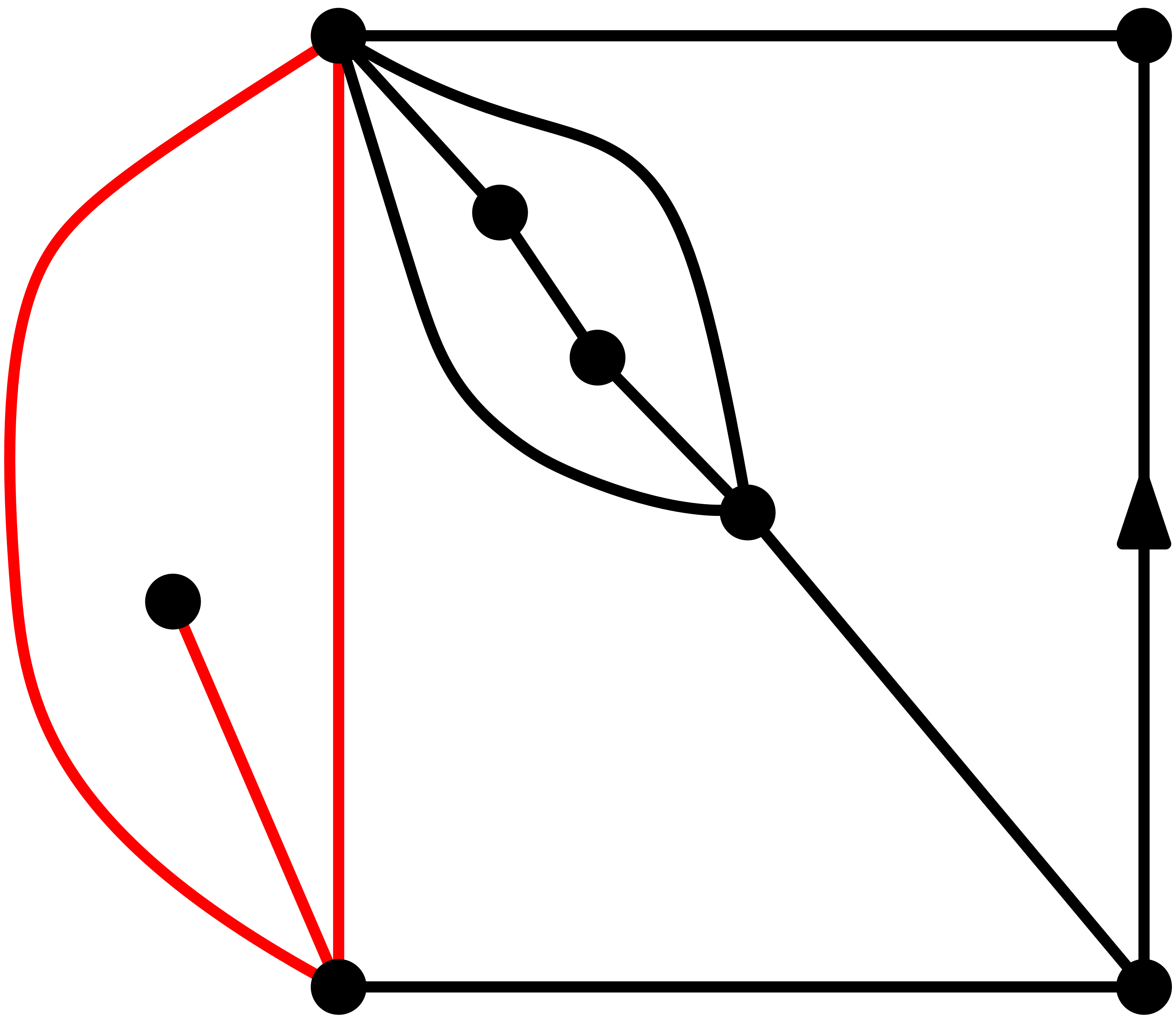
Construction of a quadrangulation from a simple core



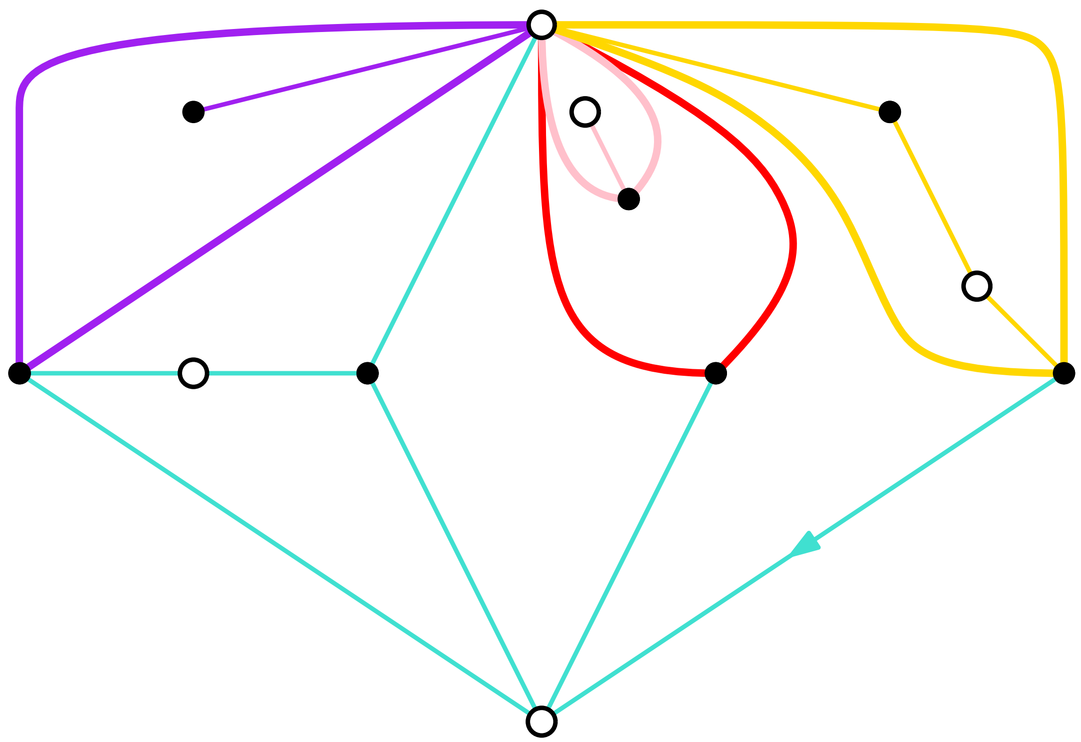
Construction of a quadrangulation from a simple core

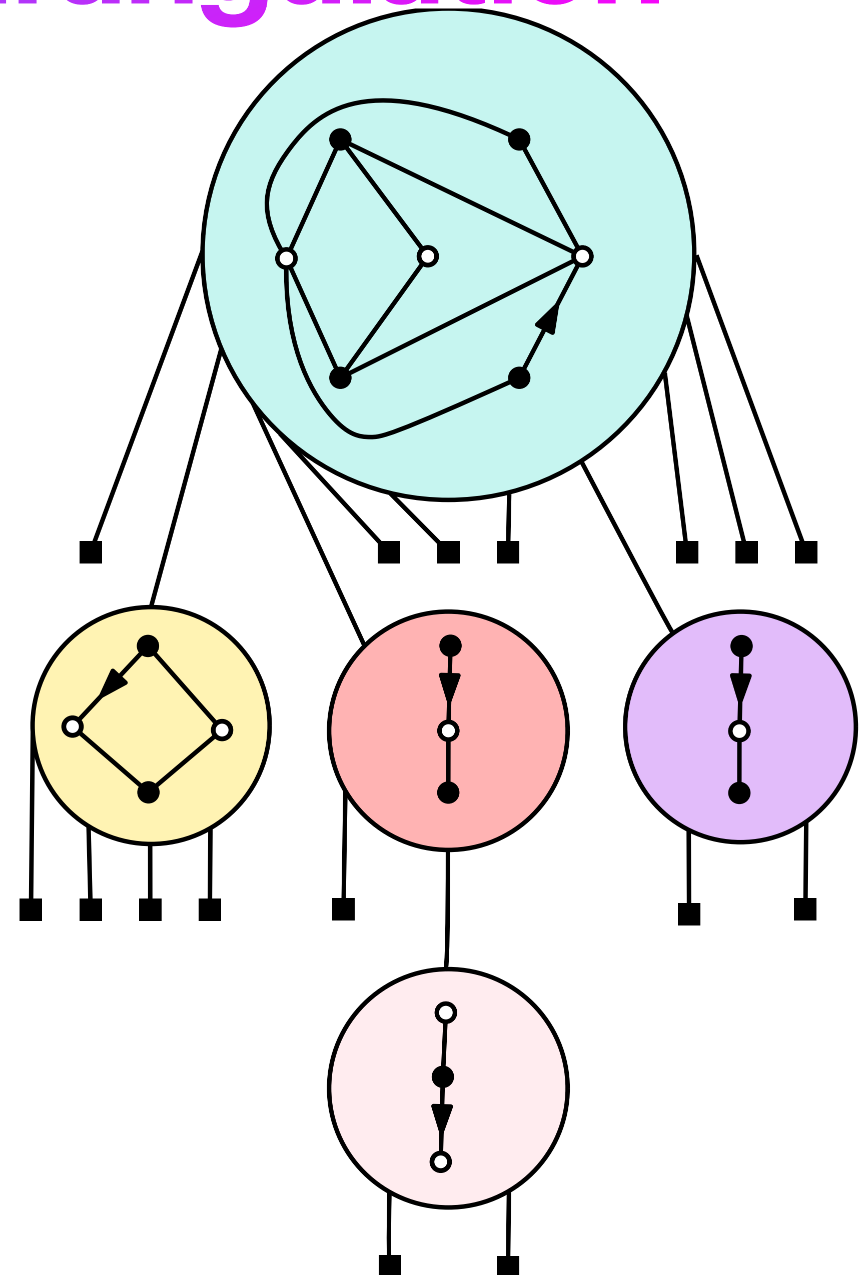


Construction of a quadrangulation from a simple core



Block tree for a quadrangulation



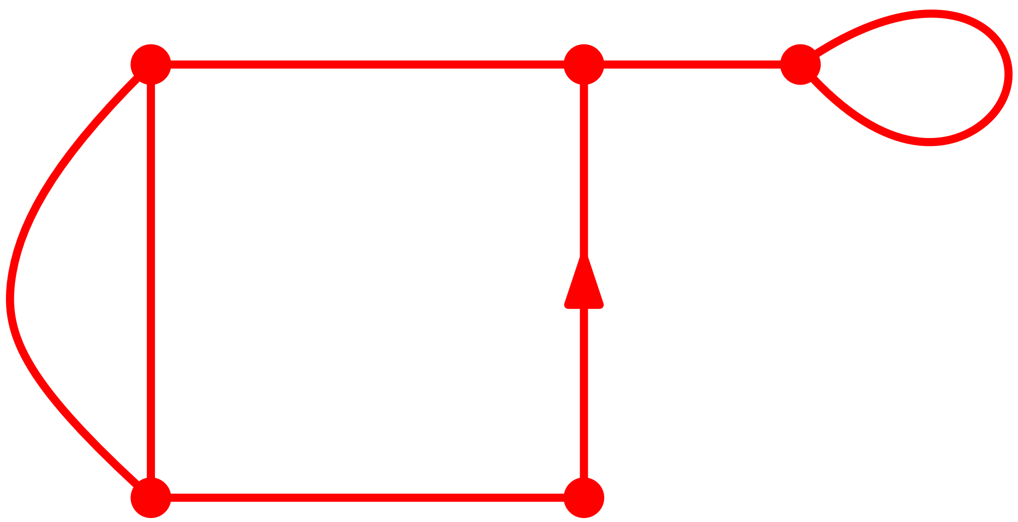


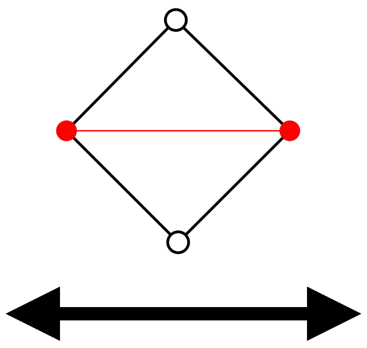
With a weight u on blocks: $Q(z,u) = uS(zQ^2(z,u)) + 1 - u$

Remember: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

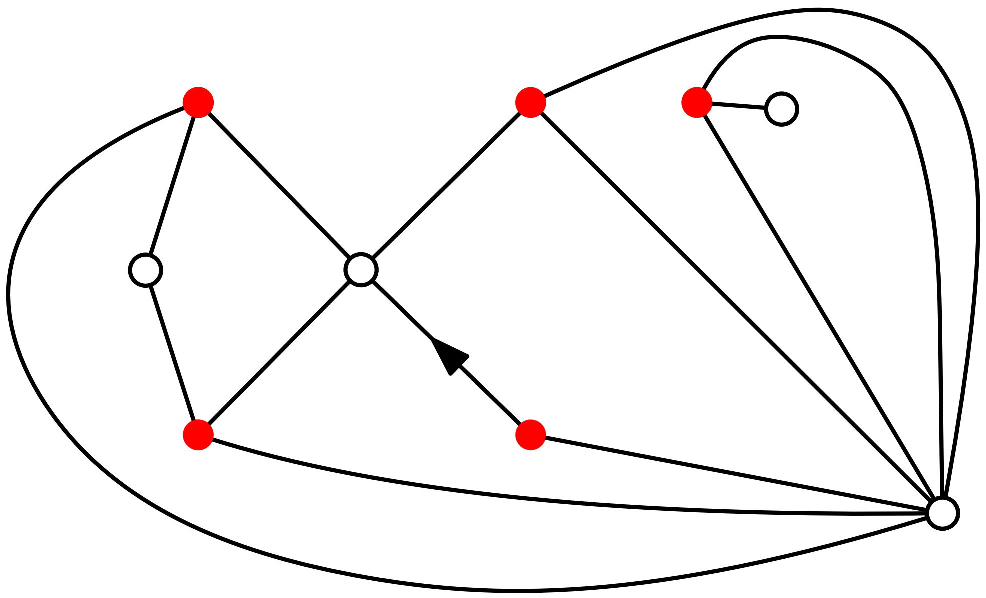
Tutte's bijection

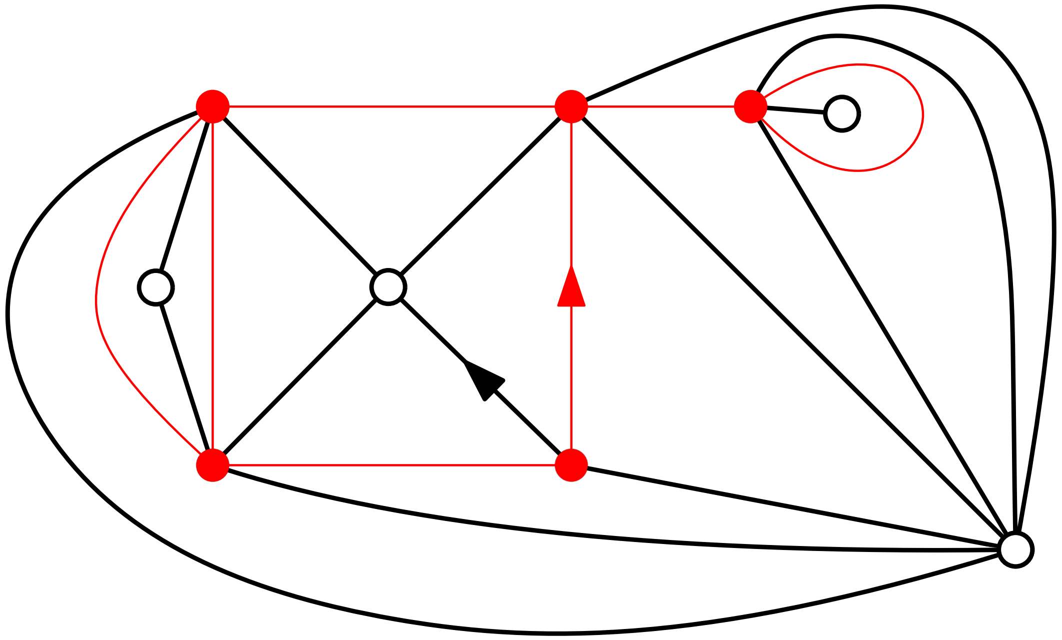
Map





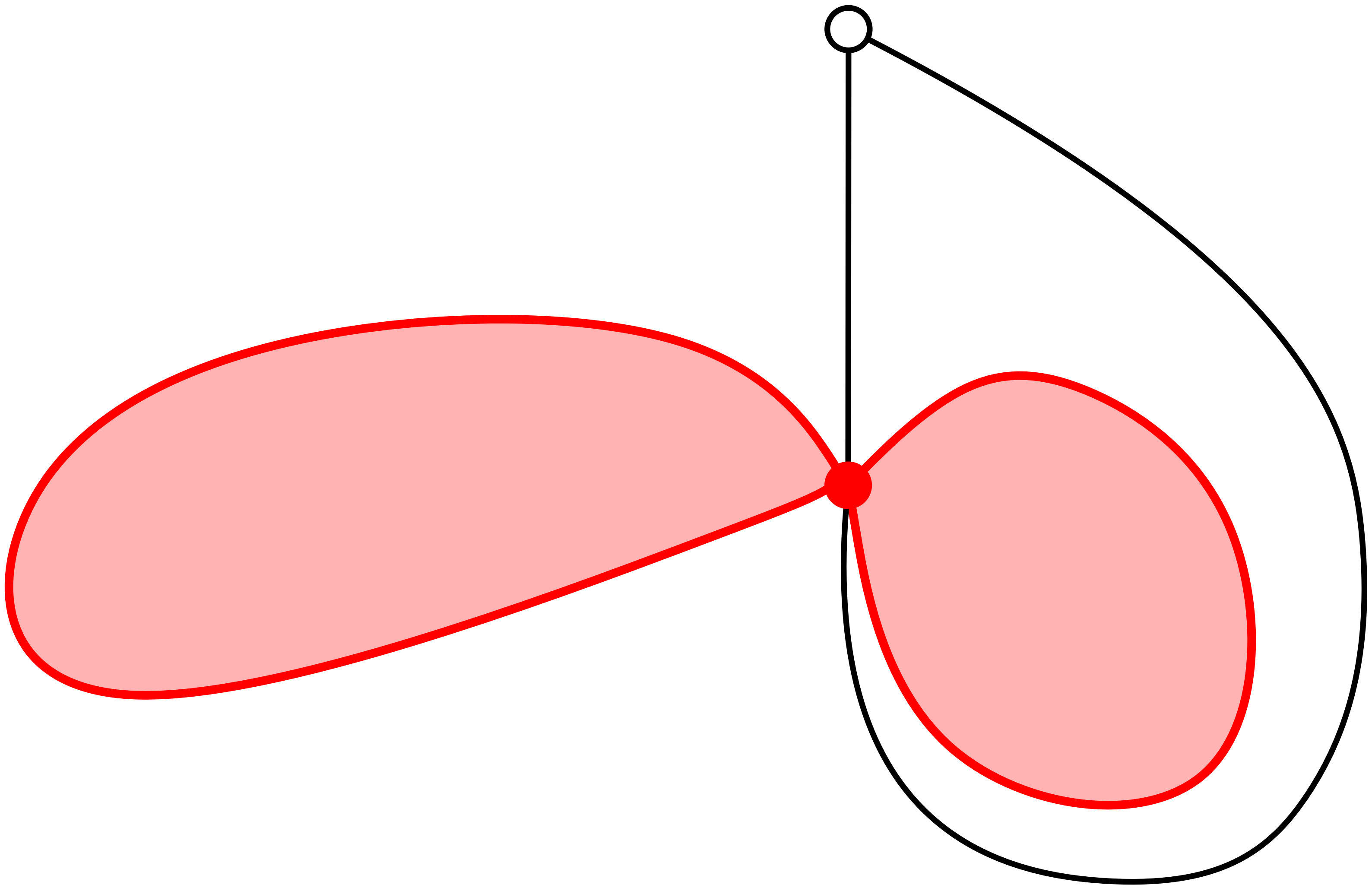
Quadrangulation





[Tutte 1963]

Tutte's bijection for 2-connected maps

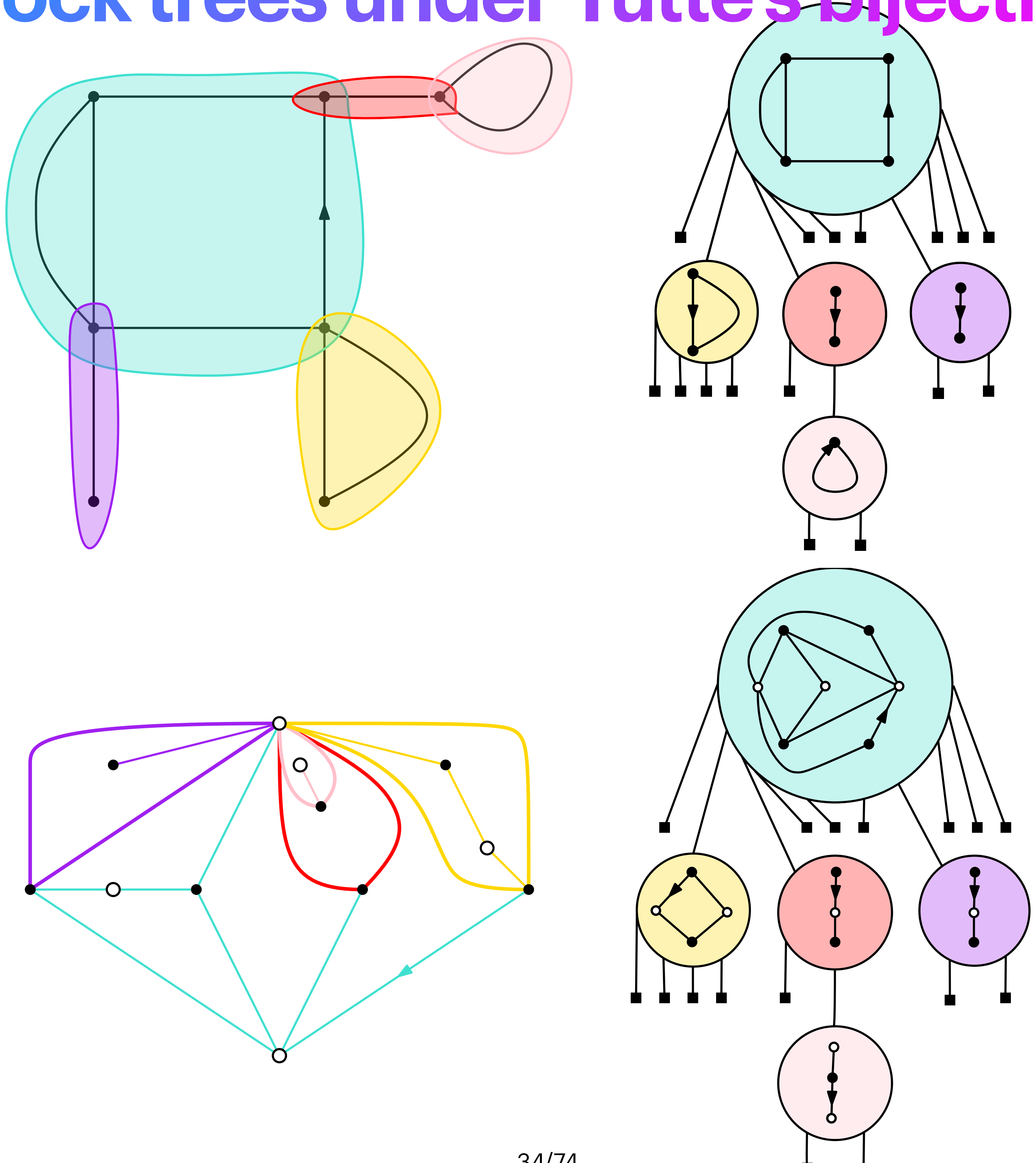


Cut vertex => multiple edge

2-connected maps <=> simple quadrangulations

[Brown 1965]

Block trees under Tutte's bijection



Implications on results

We choose: $\mathbb{P}_{n,u}(\mathfrak{q}) = \dfrac{u^{\#blocks(\mathfrak{q})}}{Z_{n,u}}$ where

$u > 0$,
$\mathcal{Q}_n$ = {quadrangulations of size n},
$\mathfrak{q} \in \mathcal{Q}_n$,
$Z_{n,u}$ = normalisation.

Results on the size of (2-connected) blocks can be transferred immediately for quadrangulations and their simple blocks.

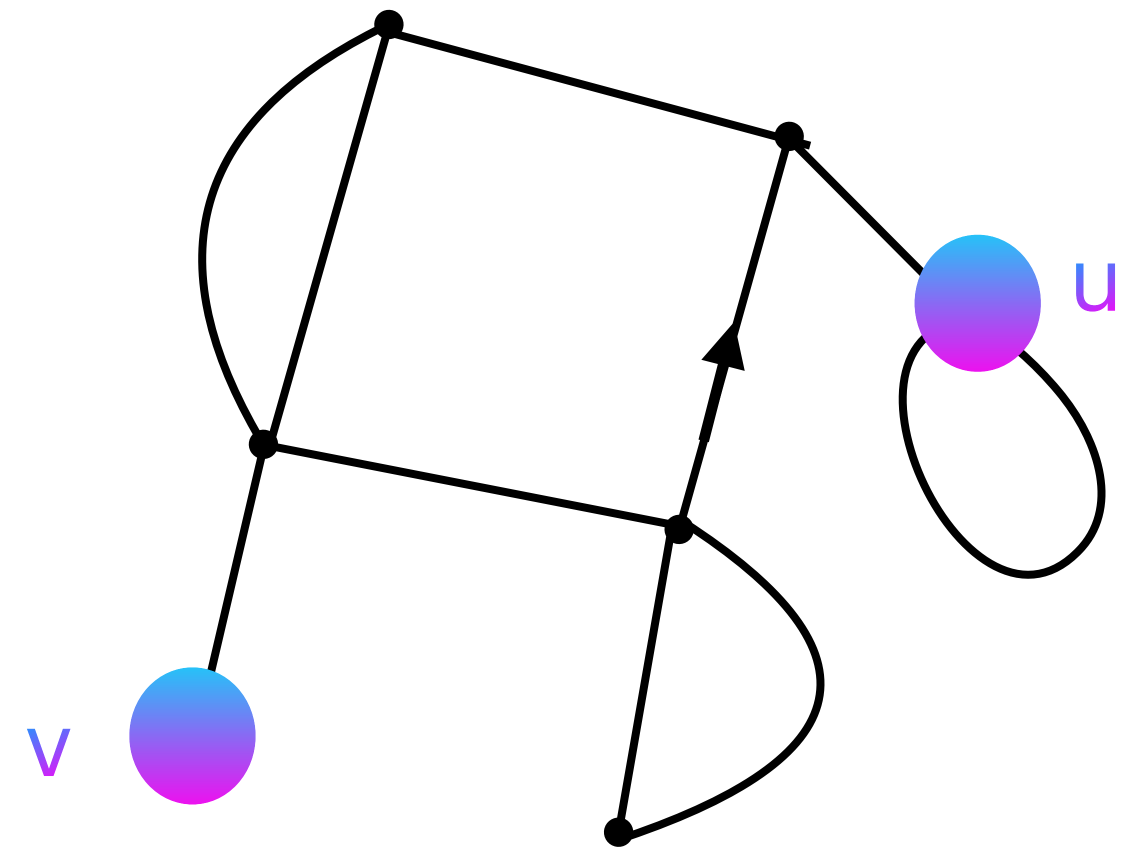
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016] for 2-c case	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

III. Scaling limits

Scaling limits

Convergence of the whole object considered as a (compact) metric space (with the graph distance), after renormalisation.



$d(u, v) = 4$

$M_n \hookrightarrow \mathbb{P}_{n,u}$ (map or quadrangulation)

What is the limit of the sequence of metric spaces $((M_n, d/n^{?}))_{n\in\mathbb{N}}$?

(Convergence for Gromov-Hausdorff(-Prokhorov) topology)

Scaling limit of supercritical and critical maps

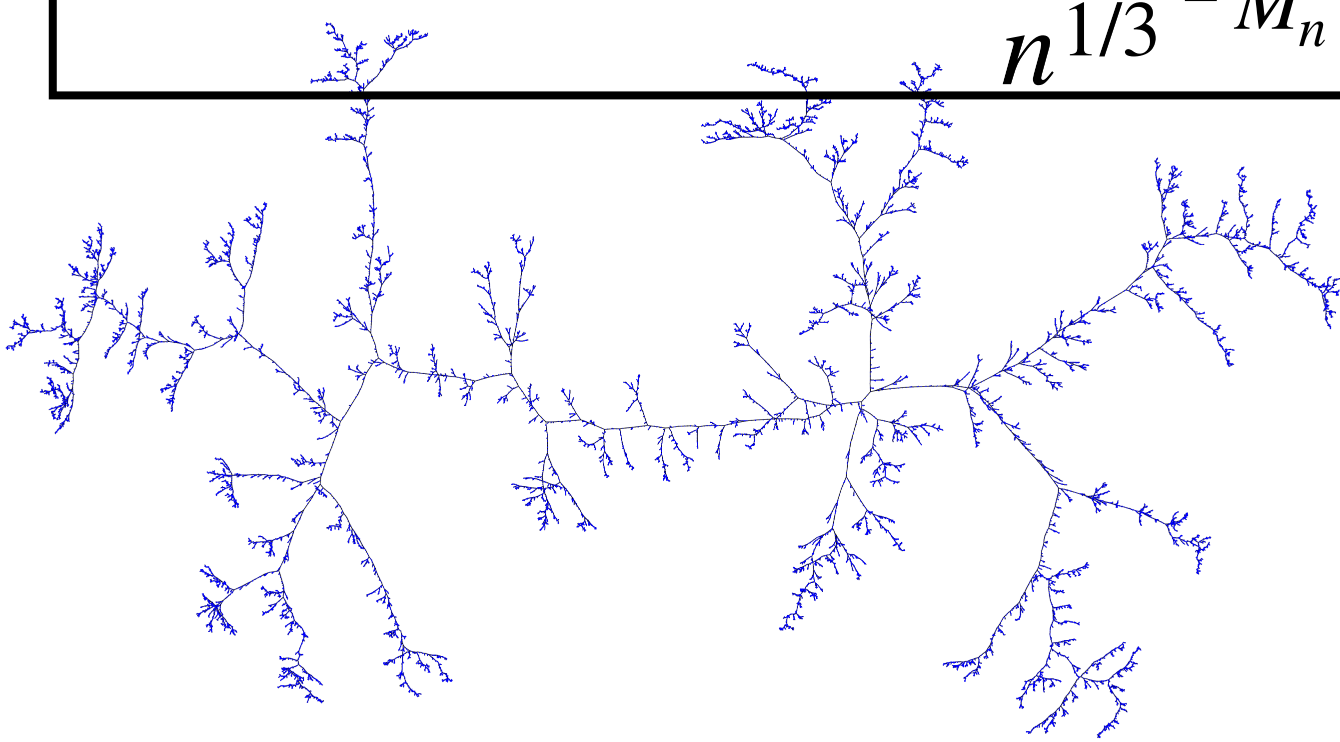
<u>Lemma</u> For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

- If $u > 9/5$,

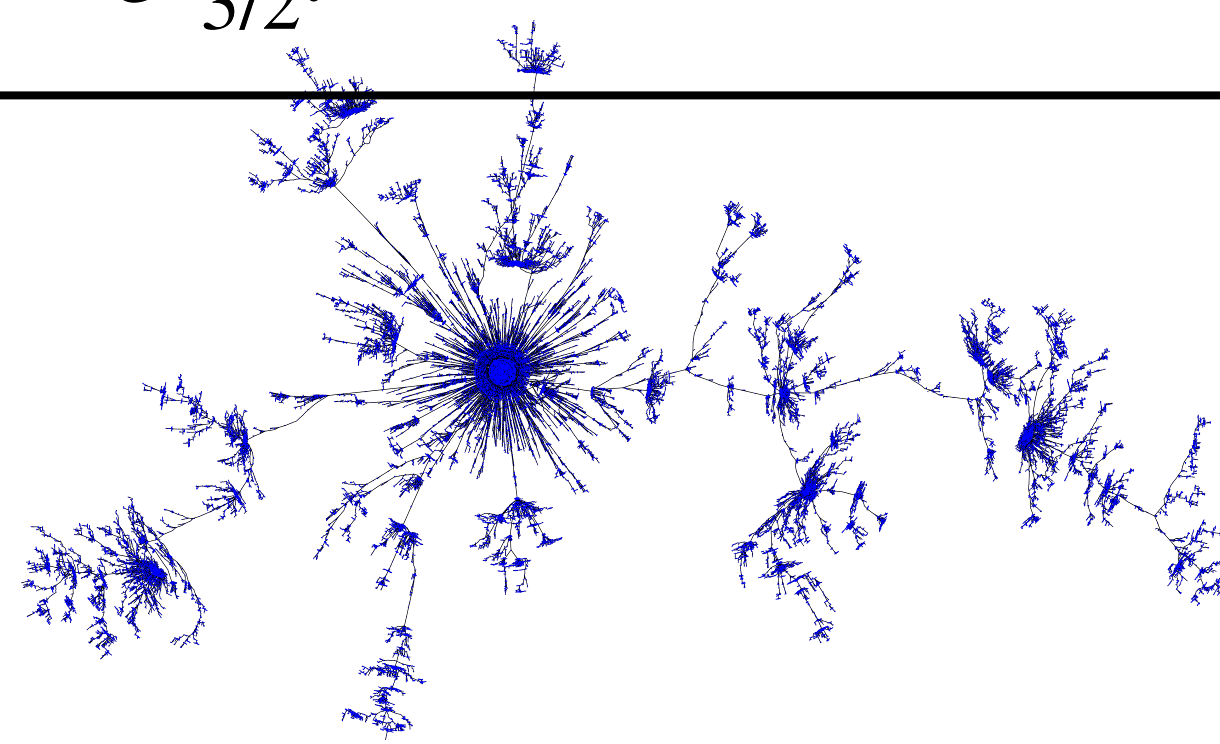
$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e.$$

- If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}.$$



Brownian Tree $\mathcal{T}_e$



Stable Tree $\mathcal{T}_{3/2}$

Scaling limit of supercritical and critical maps

Lemma For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

- If $u > 9/5$,

$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e.$$

- If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}.$$

Proof Known scaling limits of critical BGW trees

- with finite variance [Aldous 1993, Le Gall 2006];
- infinite variance and polynomial tails [Duquesne 2003].

Scaling limit of supercritical and critical maps

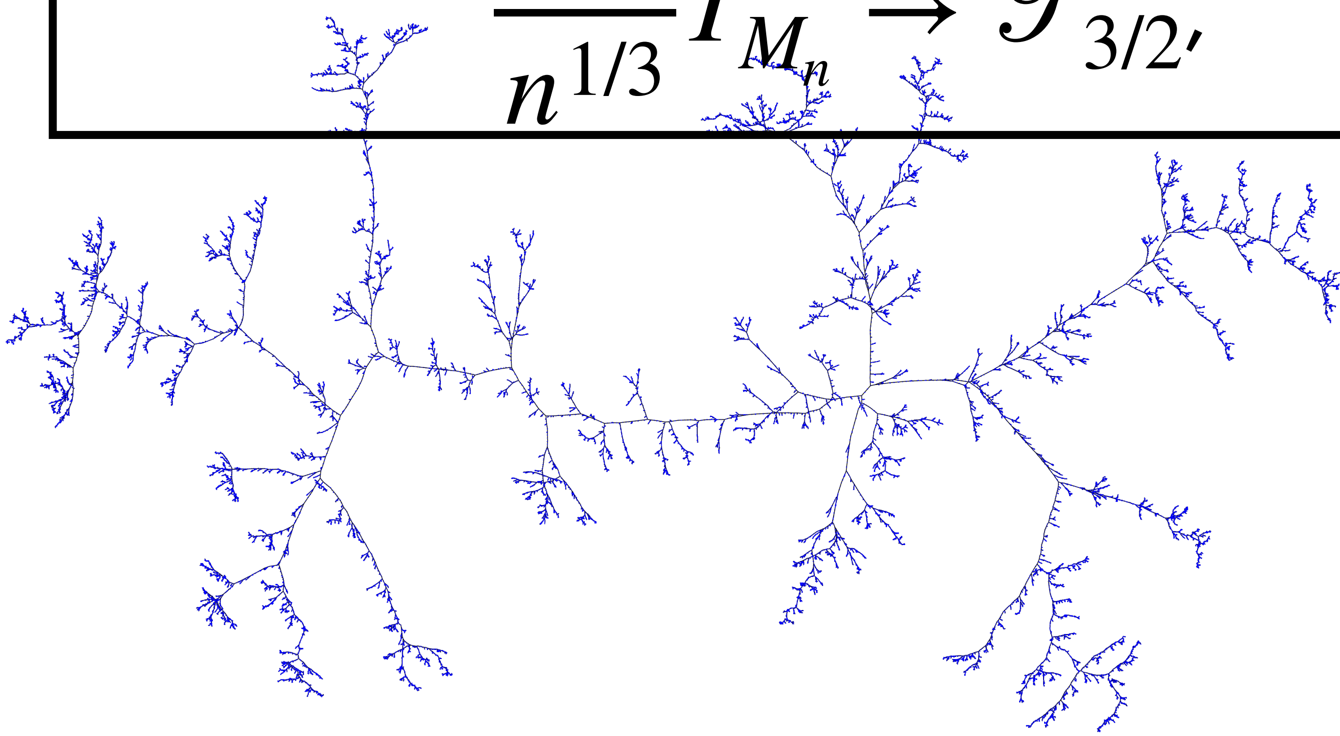
Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

- [Stufler 2020] If $u > 9/5$,

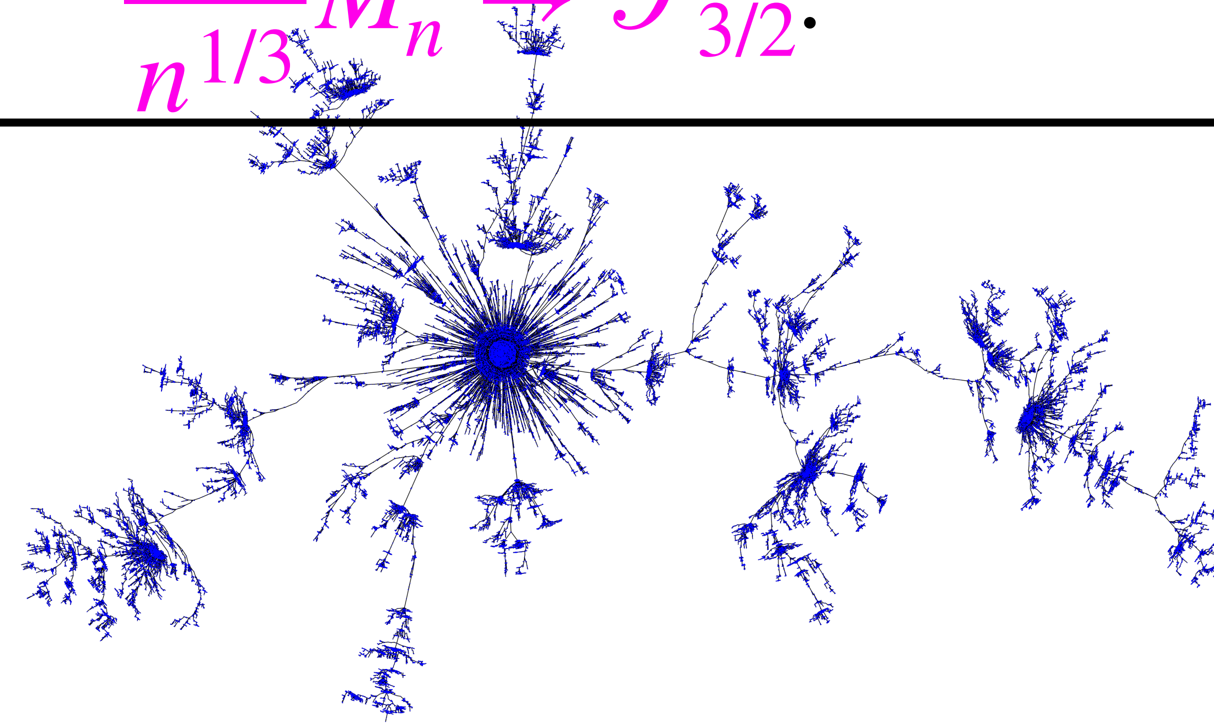
$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e, \qquad \frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e.$$

- [Fleurat, S. 24] If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}, \qquad \frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}.$$



Brownian Tree $\mathcal{T}_e$



Stable Tree $\mathcal{T}_{3/2}$

Scaling limit of supercritical and critical maps

Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

- [Stufler 2020] If $u > 9/5$,

$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e, \qquad \frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e.$$

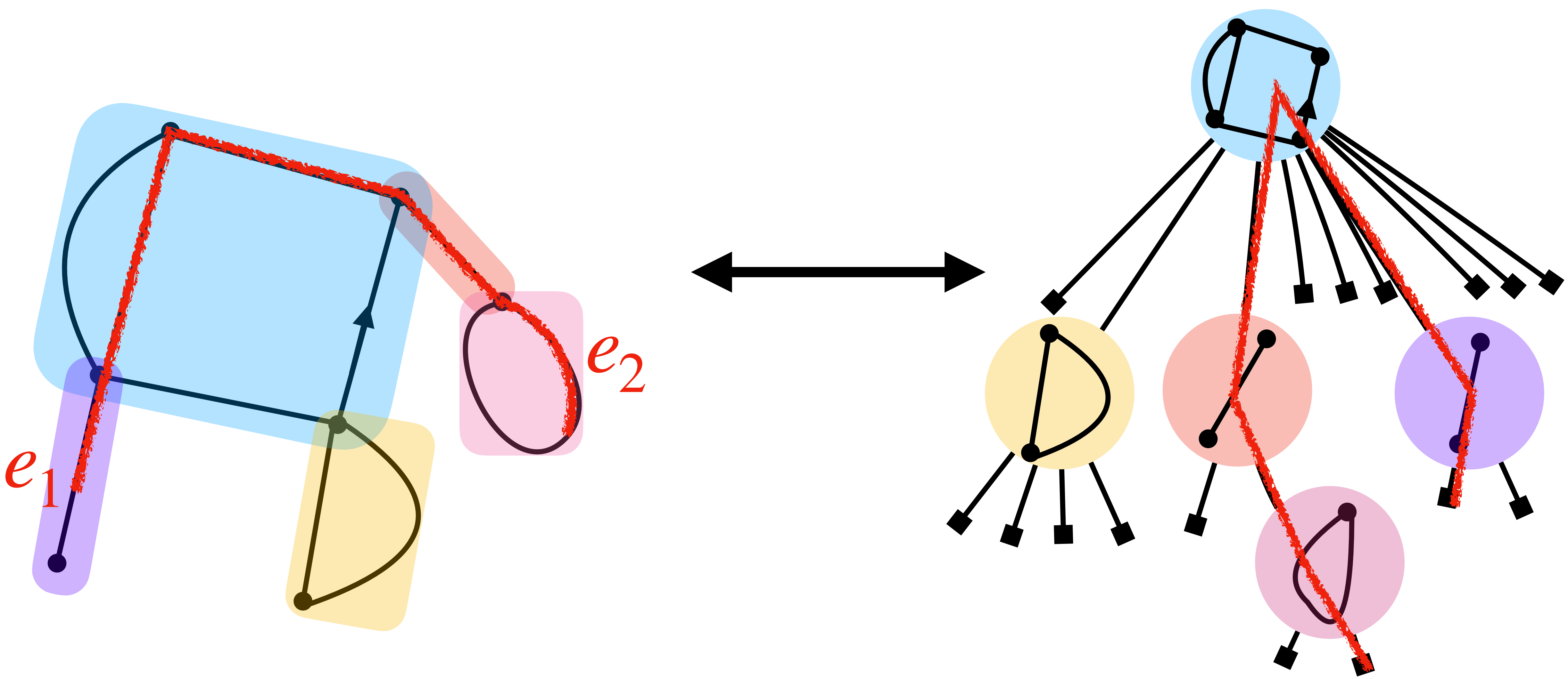
- [Fleurat, S. 24] If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}, \qquad \frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}.$$

Proof Distances in M_n behave like distances in T_{M_n}!

Supercritical and critical cases

Goal = show that distances in M_n behave like distances in T_{M_n}.

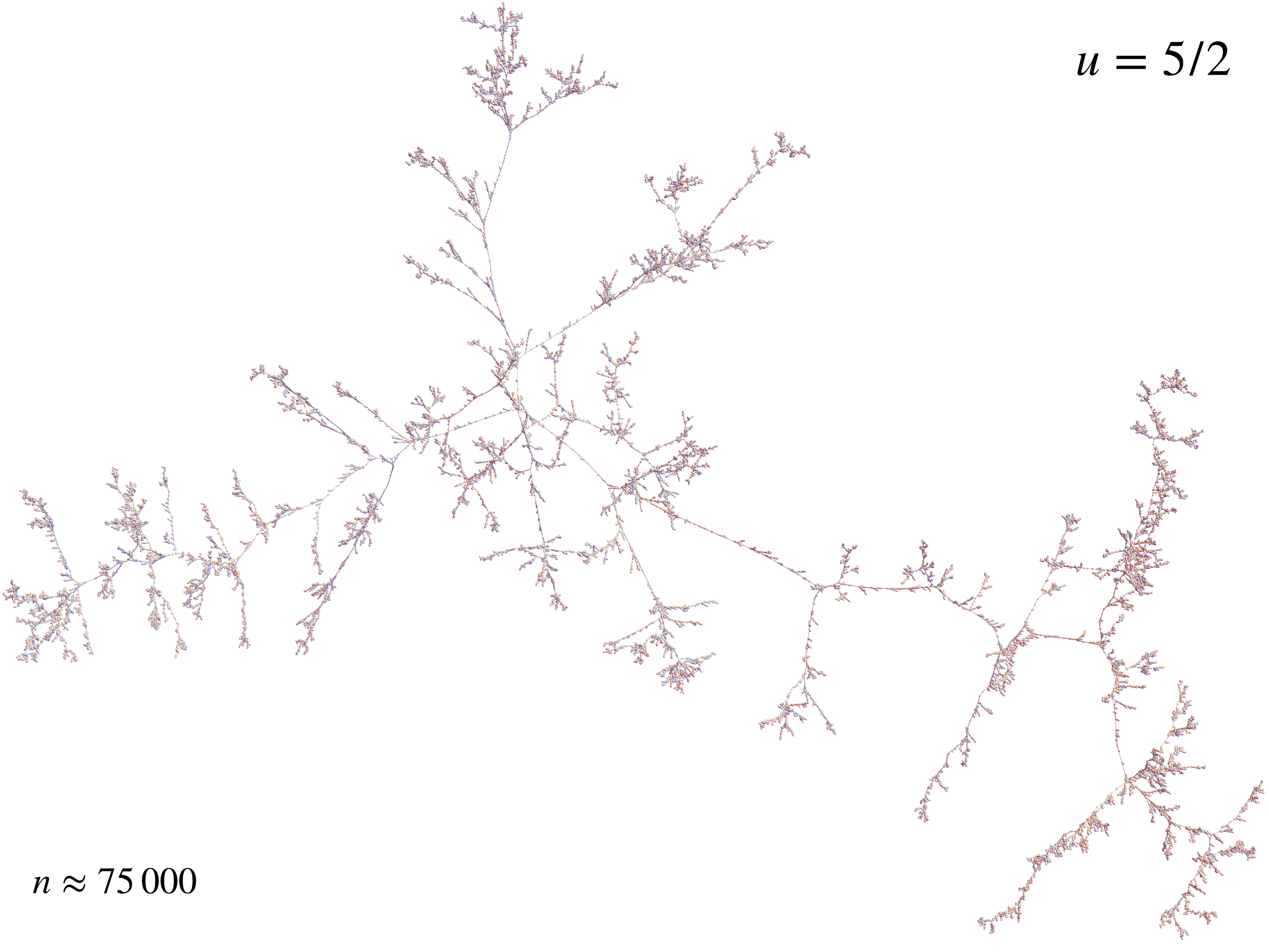


Let $\kappa = \mathbb{E}$("diameter" bipointed block). By a “law of large numbers”-type argument

$$d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2)\,.$$

Difficult for the critical case => large deviation estimates

$u = 9/5$
$n \approx 80\,000$


$u = 5/2$
$n \approx 75\,000$

$u = 5$
$n \approx 50\,000$

Scaling limits of subcritical maps

Theorem [Fleurat, S. 24] If $u < 9/5$, for $M_n \hookrightarrow \mathbb{P}_{n,u}$ and denoting $B(M_n)$ its largest block:

$$d_{GHP}\left(\frac{C_1(u)}{n^{1/4}}M_n, \frac{1}{n^{1/4}}B(M_n)\right) \to 0.$$

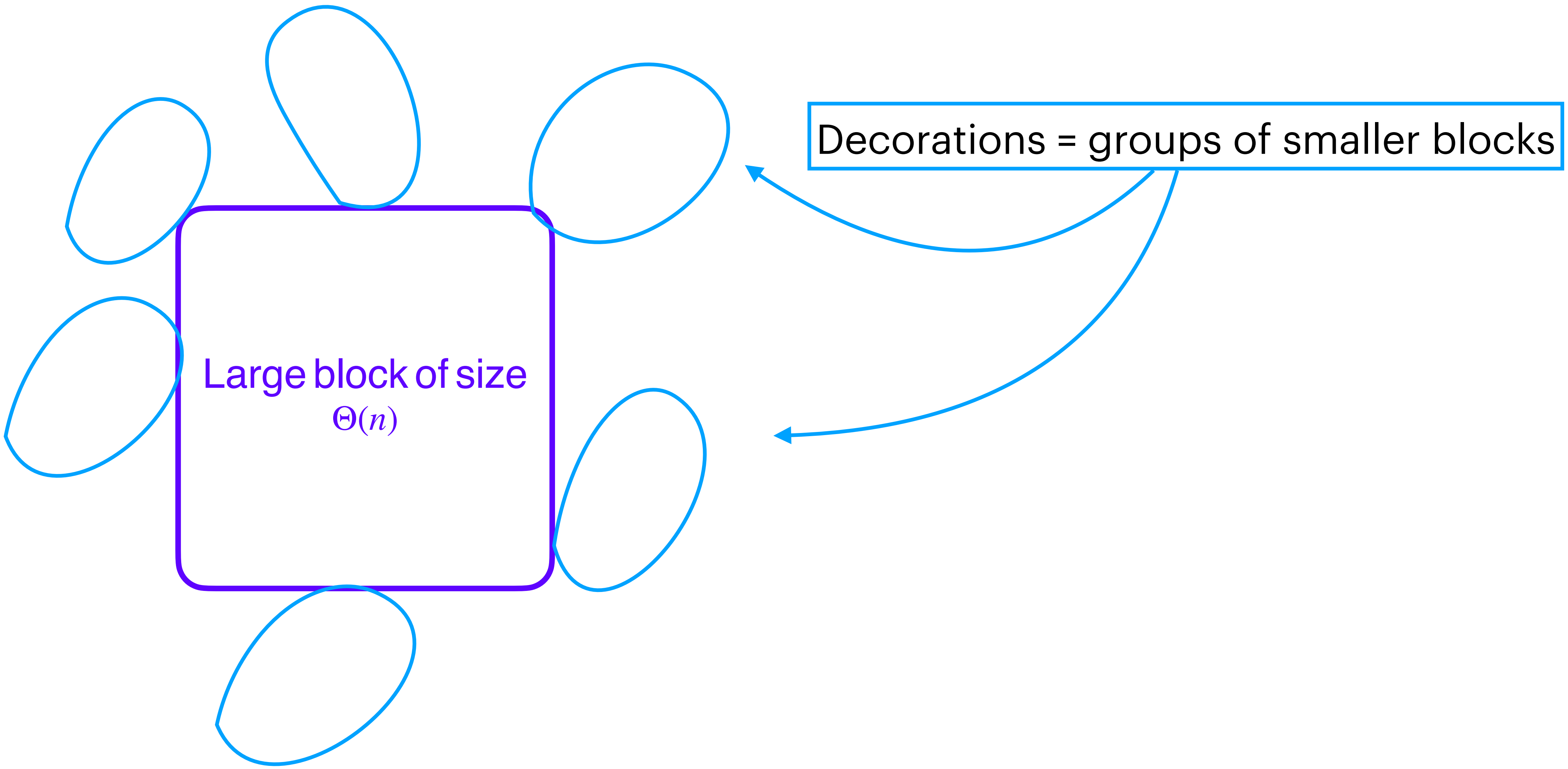
Brownian Sphere $\mathcal{S}_e$

So, if $cn^{-1/4}B_n \to \mathcal{S}_e$, then

$$\frac{C_1(u)}{cn^{1/4}}M_n \to \mathcal{S}_e.$$

See [Addario-Berry, Wen 2019] for a similar result and method.

Subcritical case



Diameter of a decoration ≤ blocks to cross × max diameter of blocks

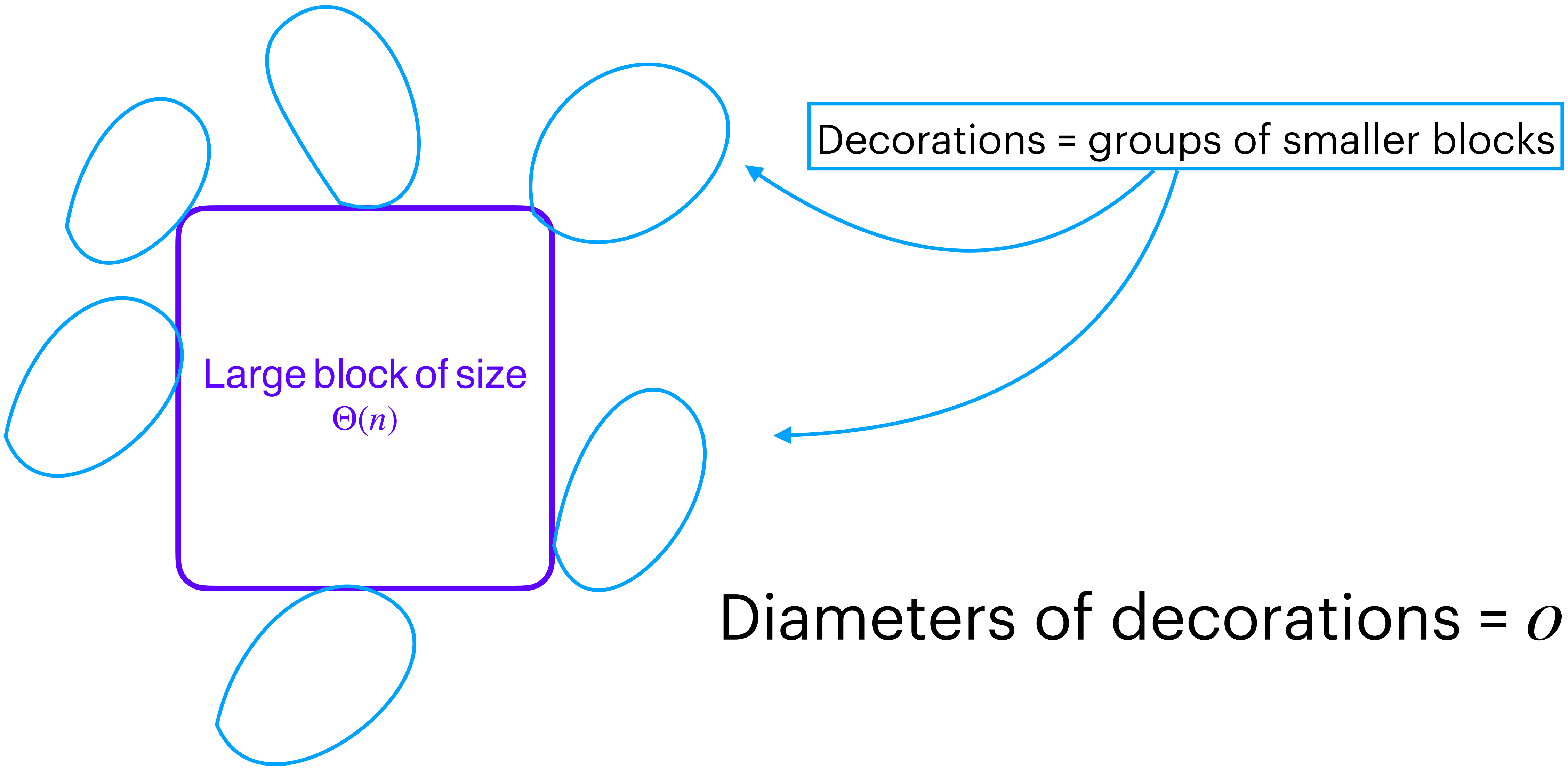
$$\leq diam(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = diam(T_{M_n}) \times O(n^{1/6+\delta})$$

$$= O(n^{1/6+2\delta}) = o(n^{1/4}) .$$

T_{M_n} is a subcritical BGW tree

[Chapuy Fusy Giménez Noy 2015]

Subcritical case



Diameters of decorations = $o(n^{1/4})$.

The scaling limit of M_n (rescaled by $n^{1/4}$) is the scaling limit of uniform blocks!

Scaling limits of subcritical maps

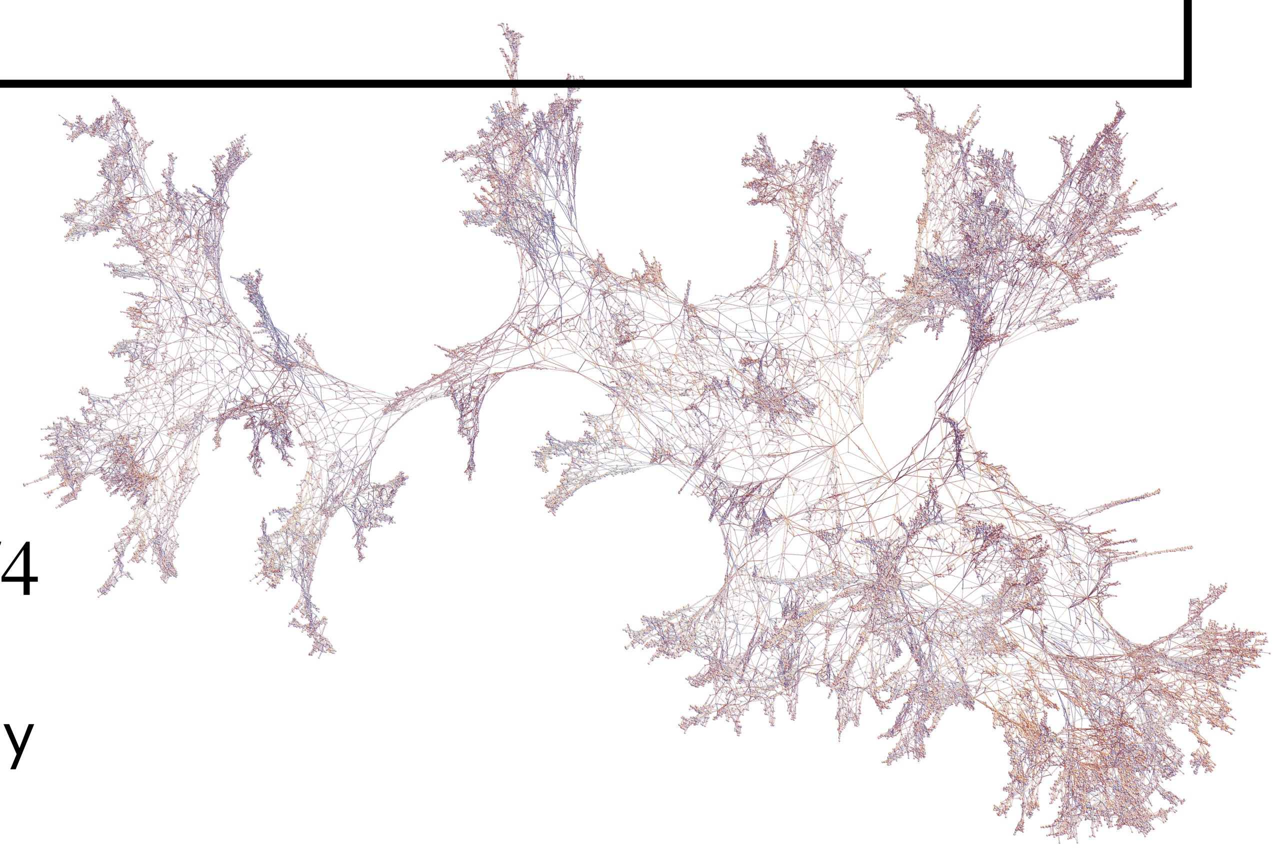
<u>Theorem</u> [Fleurat, S. 24] If $u < 9/5$, for $Q_n \hookrightarrow \mathbb{P}_{n,u}$ a quadrangulation:

$$\frac{C_1(u)}{n^{1/4}} Q_n \to \mathcal{S}_e.$$

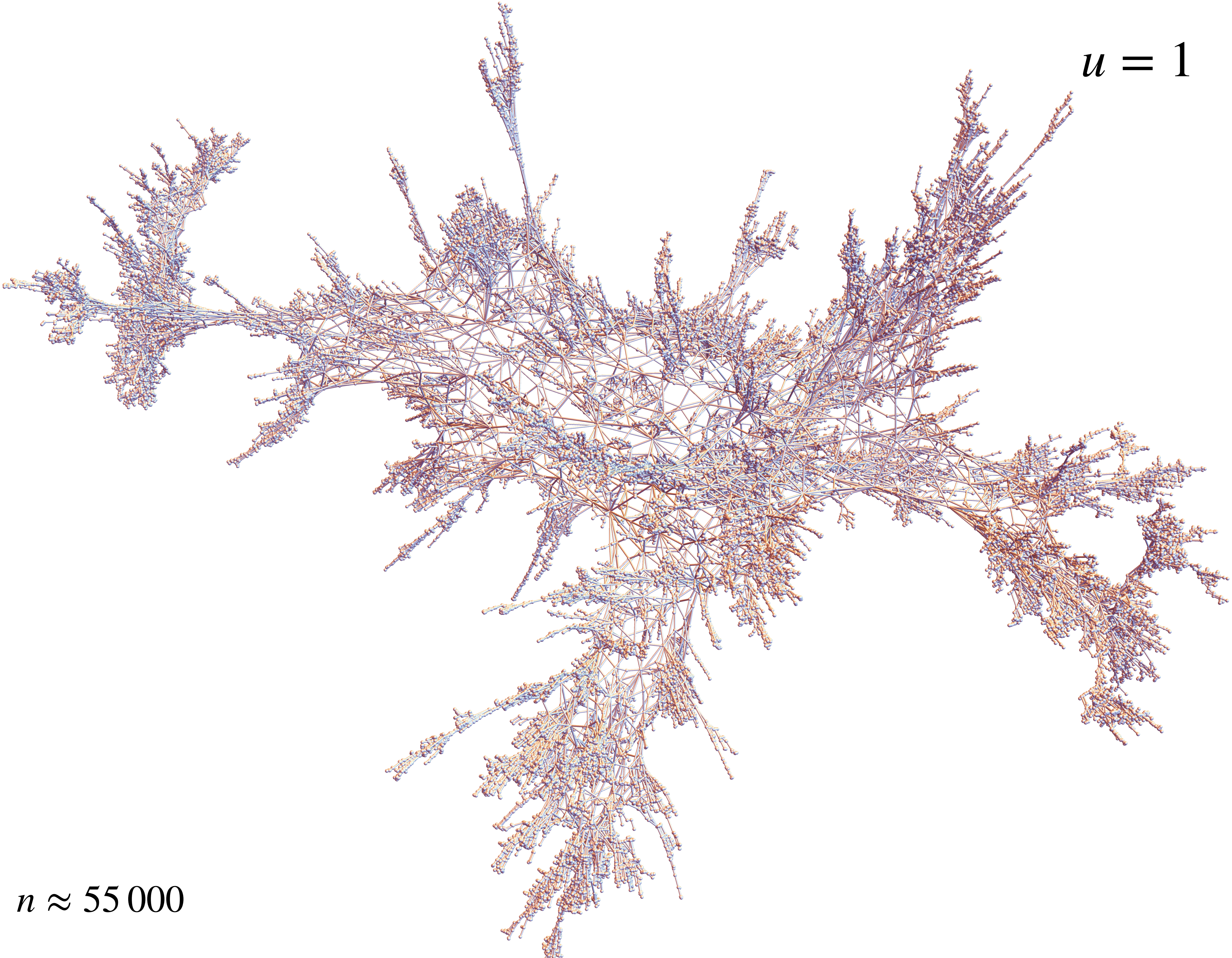
Moreover, Q_n and its simple core converge jointly to the same Brownian sphere.

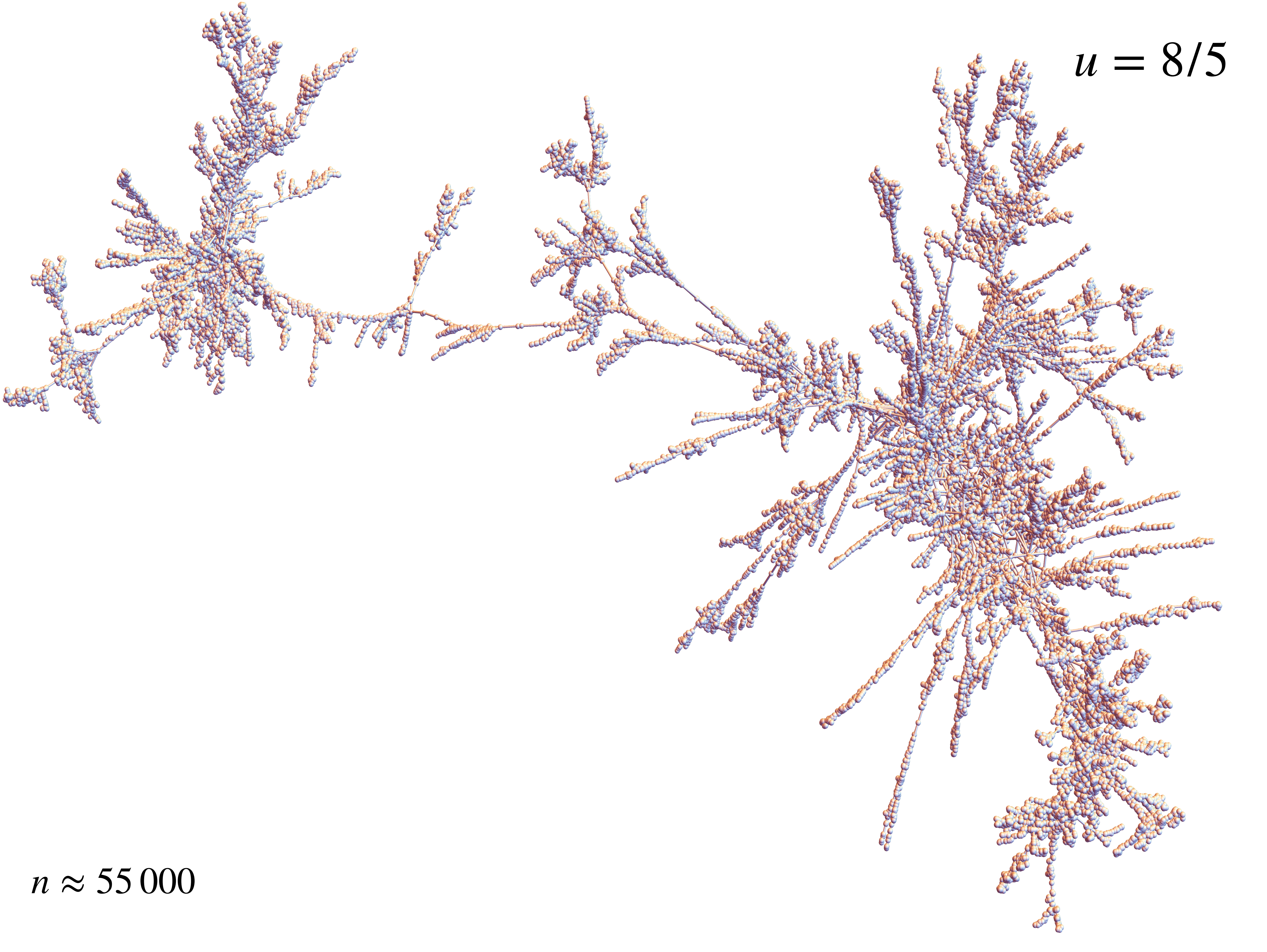
<u>Proof</u>

- Previous theorem;
- Scaling limit of uniform simple quad. rescaled by $n^{1/4}$ = Brownian sphere [Addario-Berry Albenque 2017].

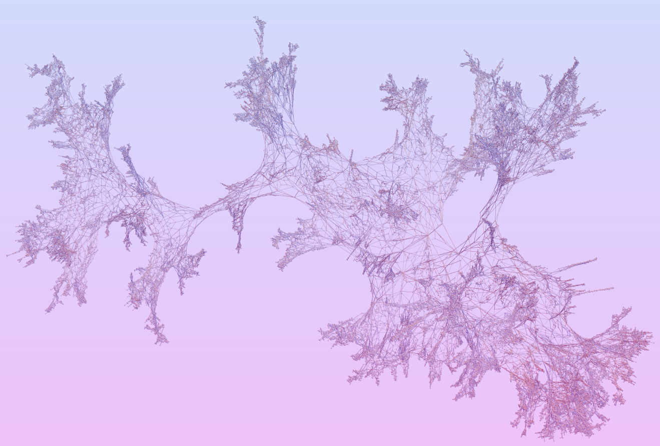
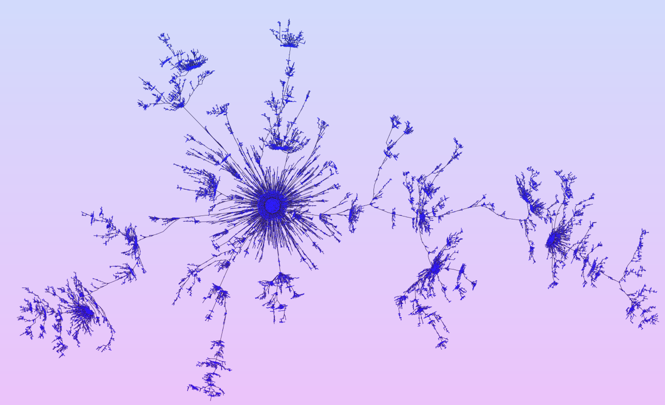
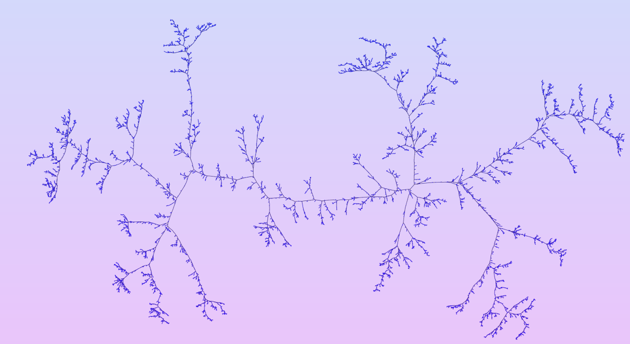


Brownian Sphere $\mathcal{S}_e$


$u = 1$
$n \approx 55\,000$


$u = 8/5$
$n \approx 55\,000$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n	$\frac{C_1(u)}{n^{1/4}} M_n \to \mathcal{S}_e$  Assuming the convergence of 2-connected maps towards the Brownian sphere	$\frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$ 	$\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e$ [Stufler 2020] 

IV. Extension to other families of maps

Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1+\mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$
loopless $M_2(z)$	simple $M_3(z)$	$z(1+M)$	–
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1+M)^2$	–
nonsep. $M_4(z)-z$	nonsep. simple $M_5(z)$	$z(1+M)$	–
nonsep. $M_4(z)/z-2$	3-connected $M_6(z)$	M	$z+2M^2/(1+M)$
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1+M)$	–
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1+M)^2$	–
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1+M)$	–
singular tri., $T_1(z)$	triang., $z+zT_2(z)$	$z(1+M)^3$	–
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1+M)^2$	–

Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$	u_C
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$	
loopless $M_2(z)$	simple $M_3(z)$	$z(1+M)$	–	81/17
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1+M)^2$	–	9/5
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1+M)$	–	135/7
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1+M)$	
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1+M)$	–	36/11
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$	
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1+M)^2$	–	52/27
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1+M)$	–	68/3
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1+M)^3$	–	16/7
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1+M)^2$	–	64/37

→ *Unified study of the phase transition for block-weighted random planar maps* Z. Salvy (Eurocomb'23)

Statement of the results

Theorem [S. 23] Model of the preceding table without coreless maps exhibits a phase transition at some explicit u_C.

When $n \to \infty$:

- Subcritical phase $u < u_C$: "general map phase" one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: "tree phase" only small blocks.

We obtain explicit results on enumeration and size of blocks in each case.

V. Extension to tree-rooted maps

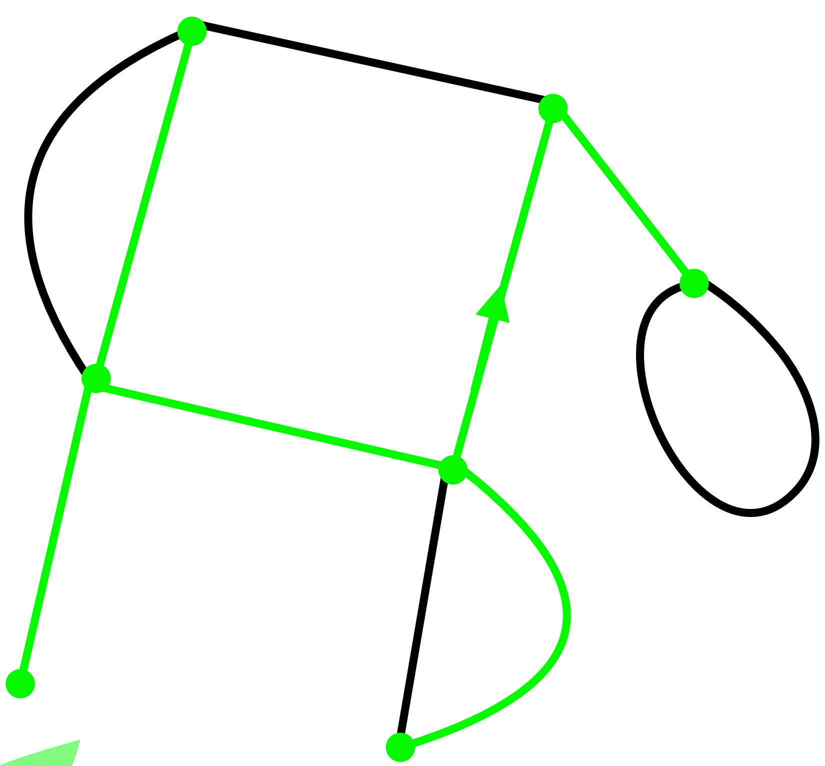
Decorated maps are interesting

Theoretical physics point of view:

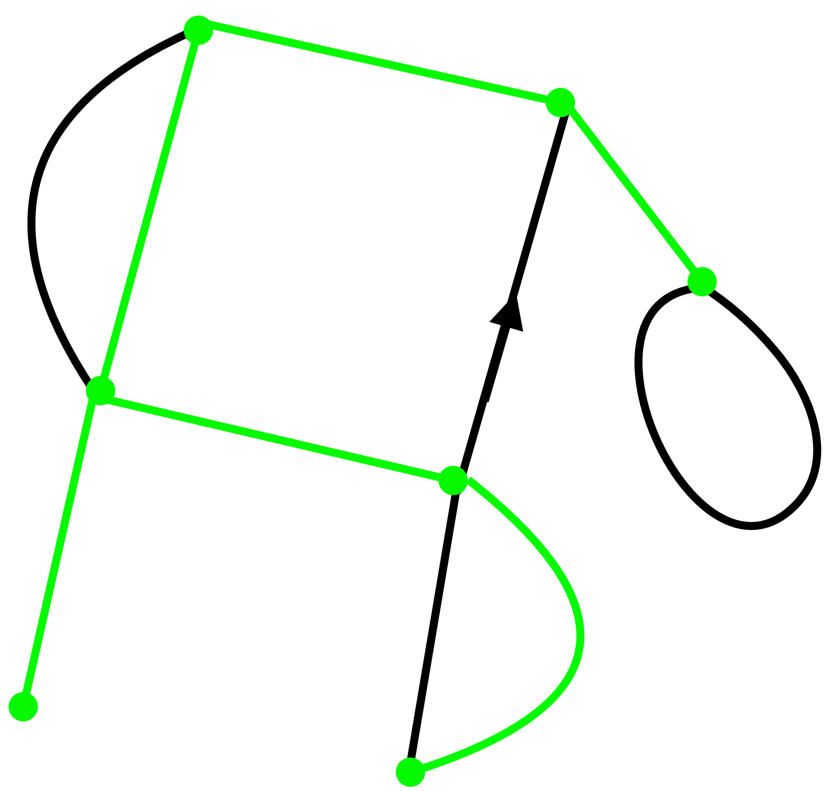
- Undecorated maps: “pure gravity” case (nothing happens on the surface);
- Decorated maps: things happen! new asymptotic behaviours! new universality classes! excitement!

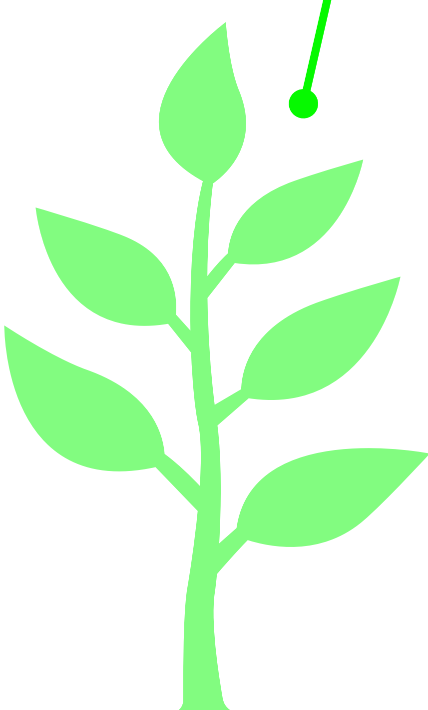
Tree-rooted maps

= (rooted planar) maps endowed with a spanning tree.

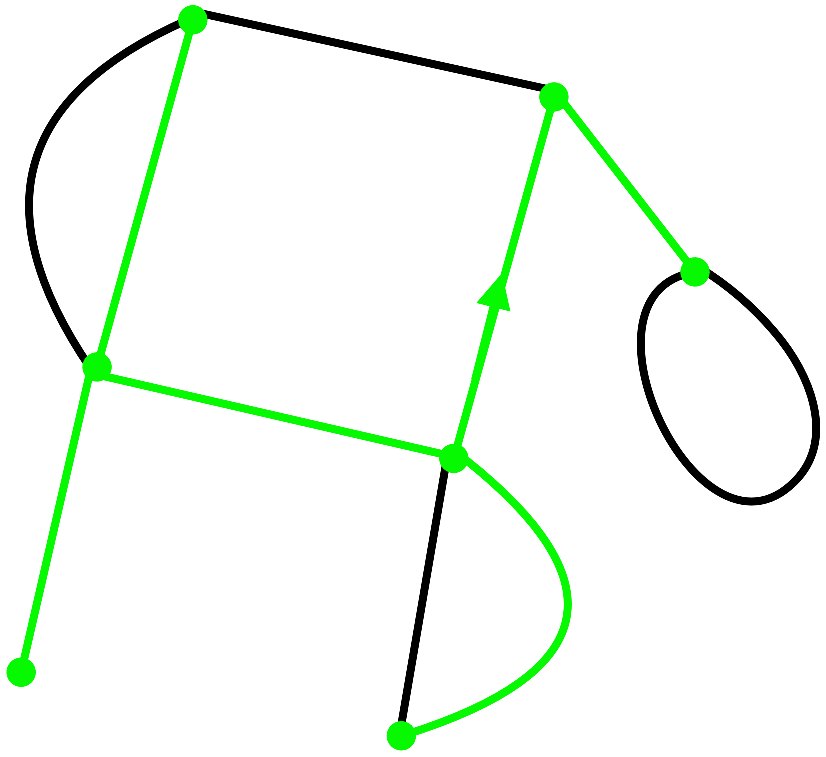


$\neq$





Tree-rooted maps



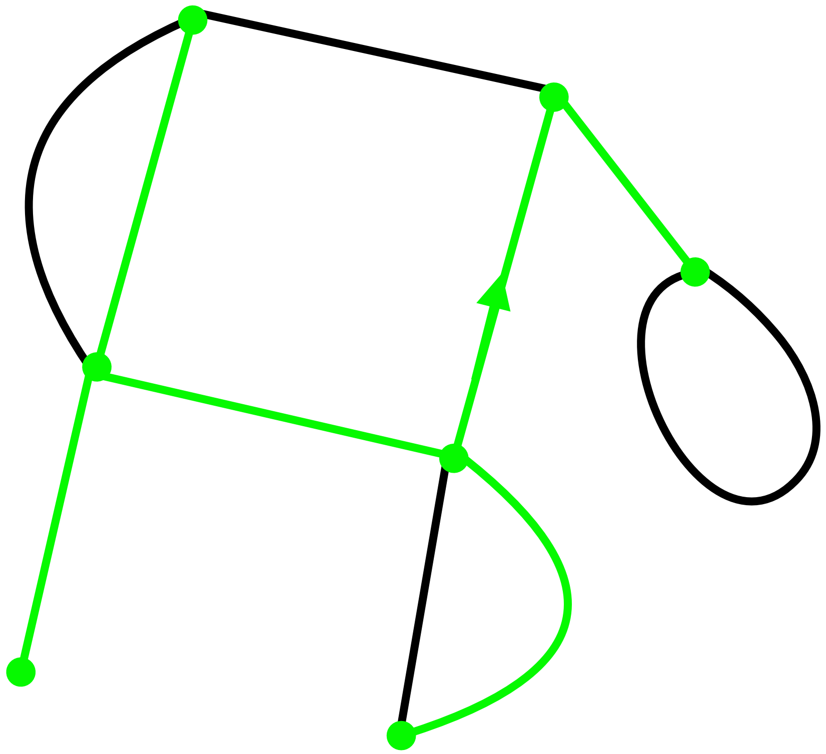
- Combinatorics well understood : Mullin's bijection;

$$[z^n]M(z) = \text{Cat}_n\text{Cat}_{n+1}$$

[Mullin 67]

- Geometry not so much.

Tree-rooted maps



- Combinatorics well understood : Mullin's bijection;

$$[z^n]M(z) = \mathrm{Cat}_n \mathrm{Cat}_{n+1}$$

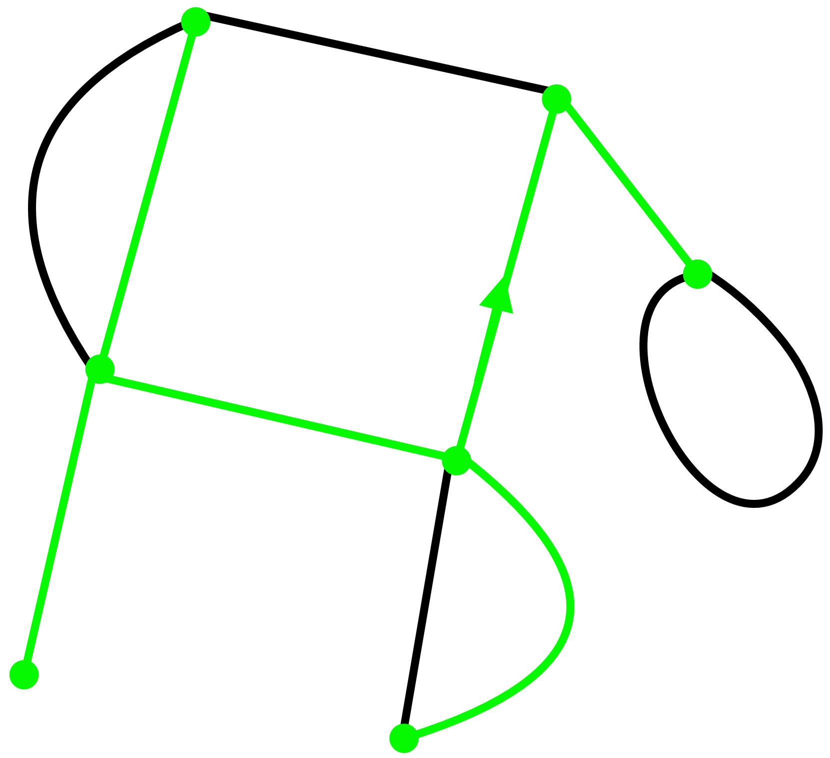
[Mullin 67]

- Geometry not so much.

We want a phase transition in tree-rooted maps.

=> Block-weighted tree-rooted maps.

Mullin's bijection



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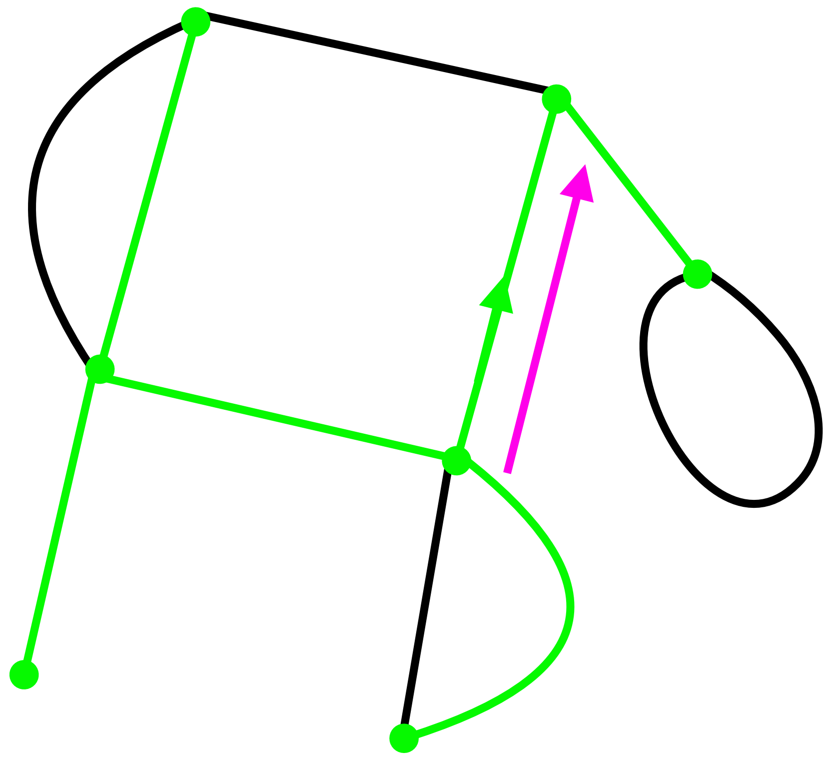

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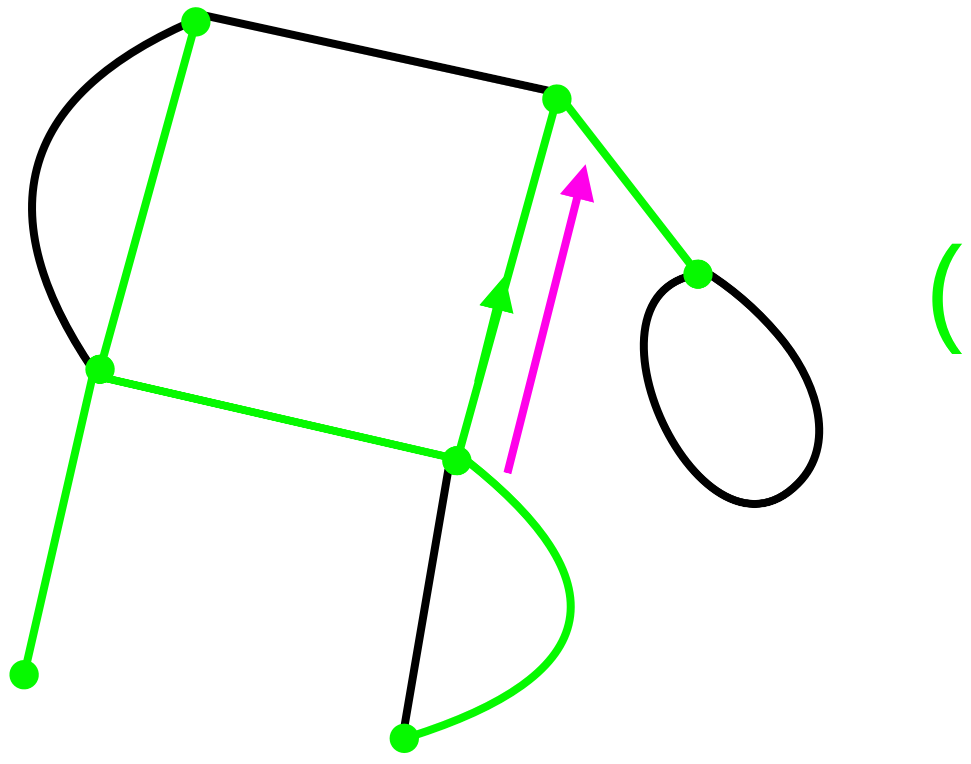

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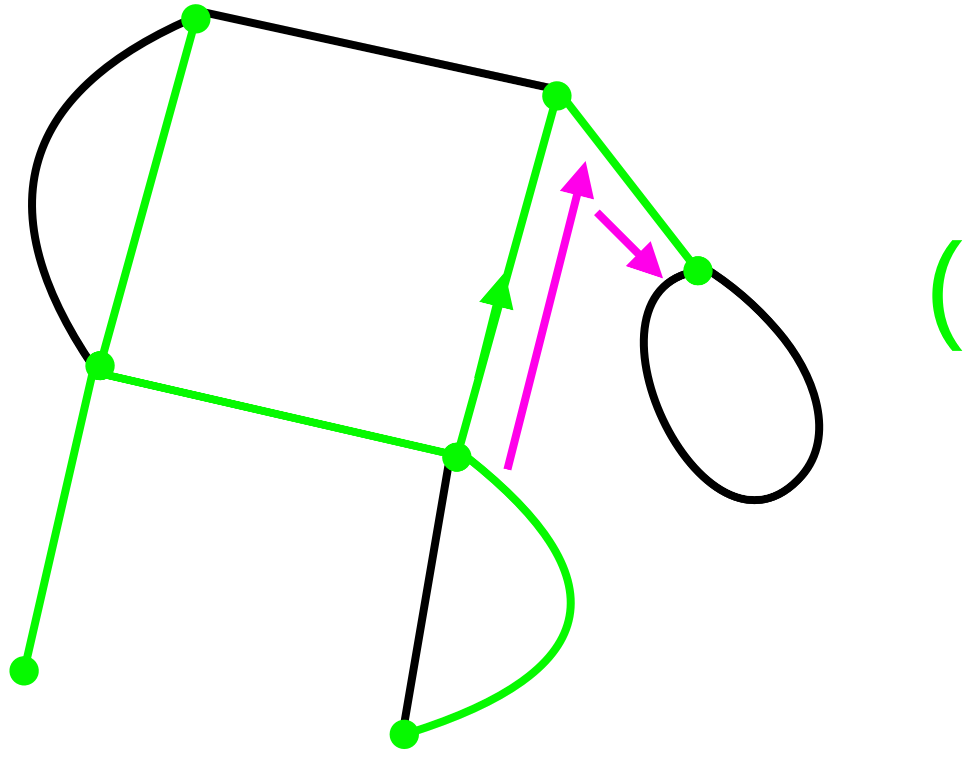

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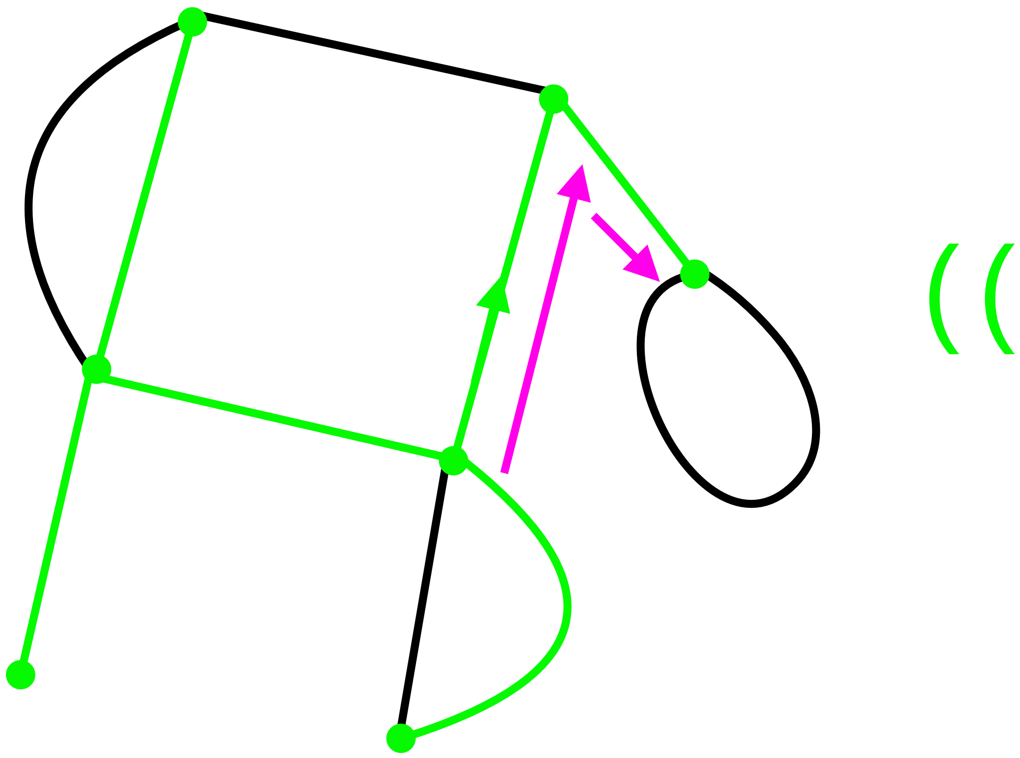

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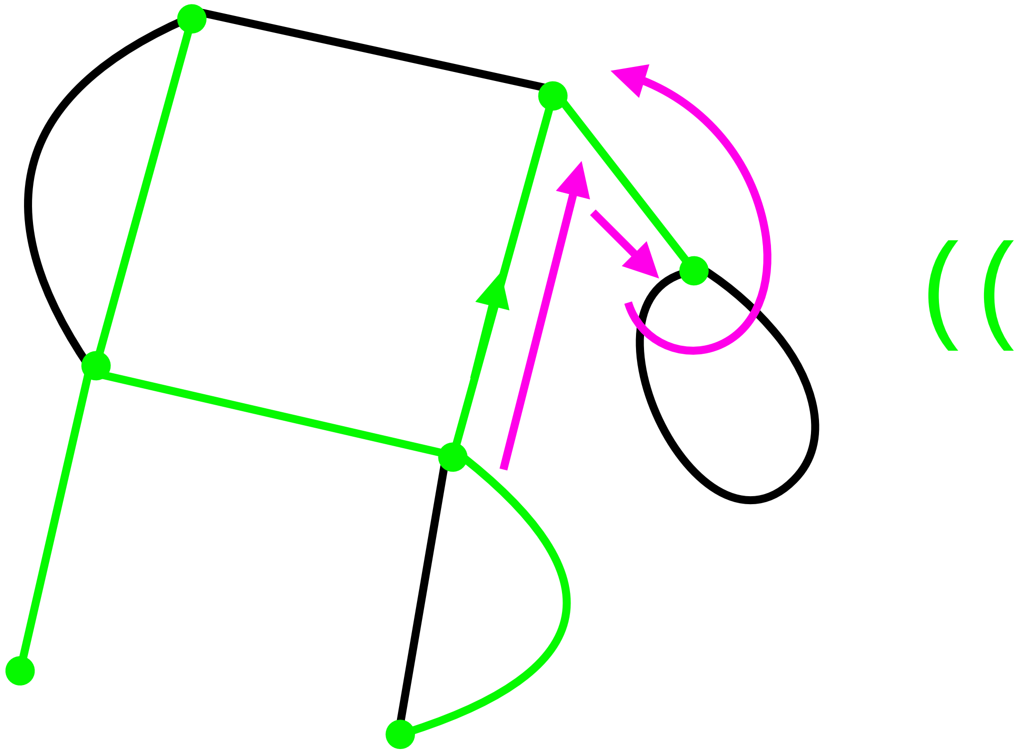

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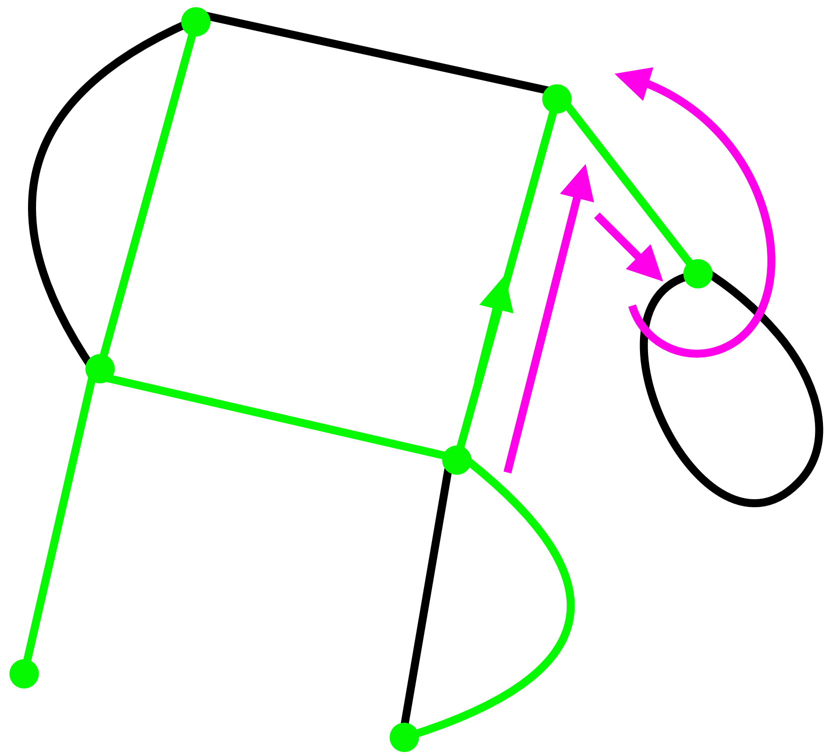

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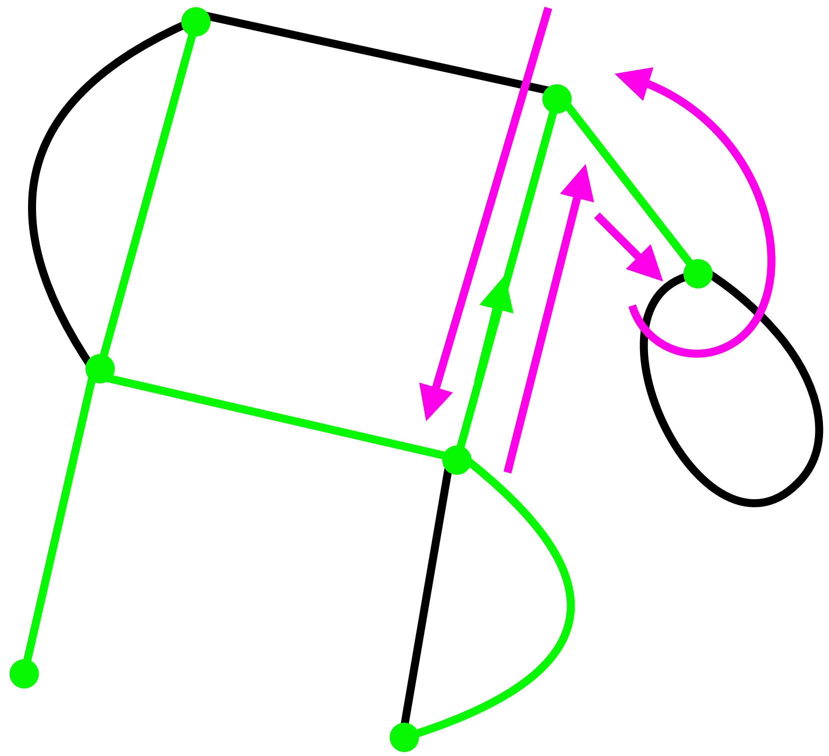

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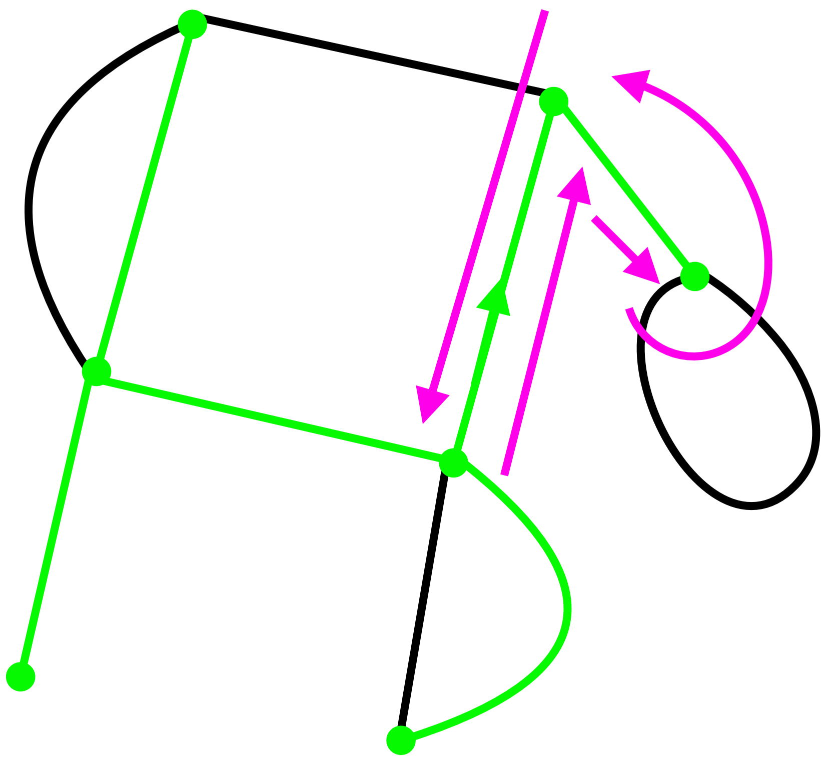

(([])[)

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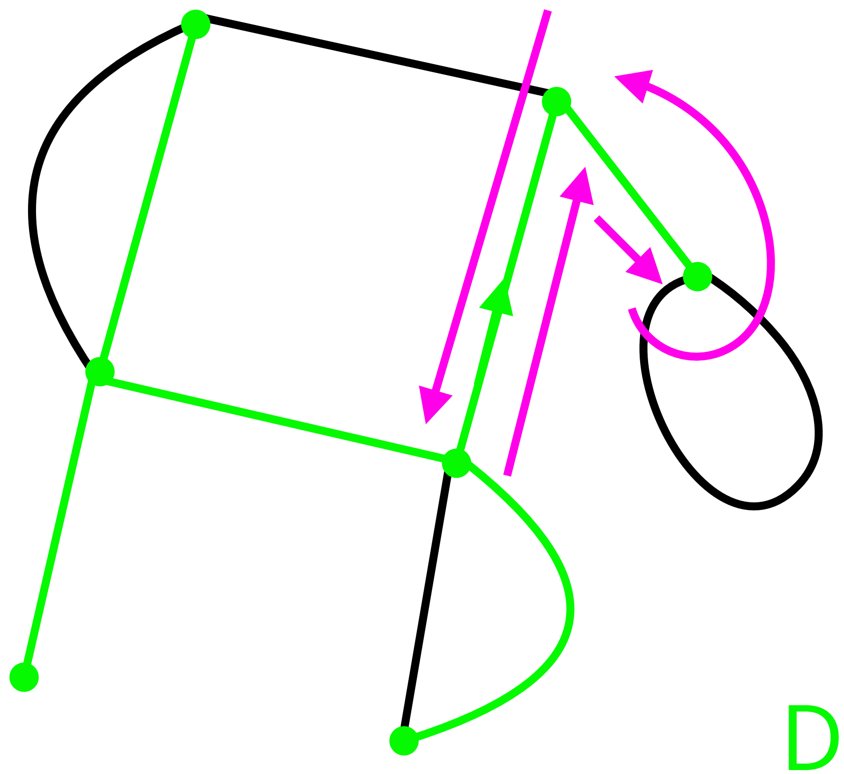

(([])[)((][)]())[(])

Dyck word of size $|\mathfrak{t}|$

Dyck word of size $|\mathfrak{m}| - |\mathfrak{t}|$

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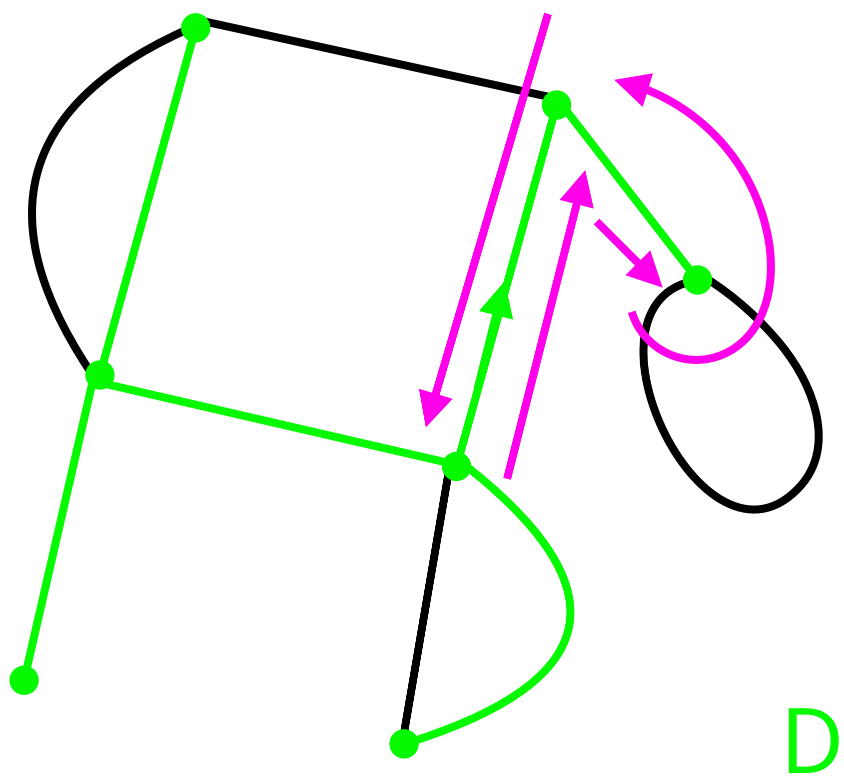

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$$[z^n]M(z) = \sum_{k=0}^{n} \binom{2n}{2k} \mathrm{Cat}_k \mathrm{Cat}_{n-k}$$

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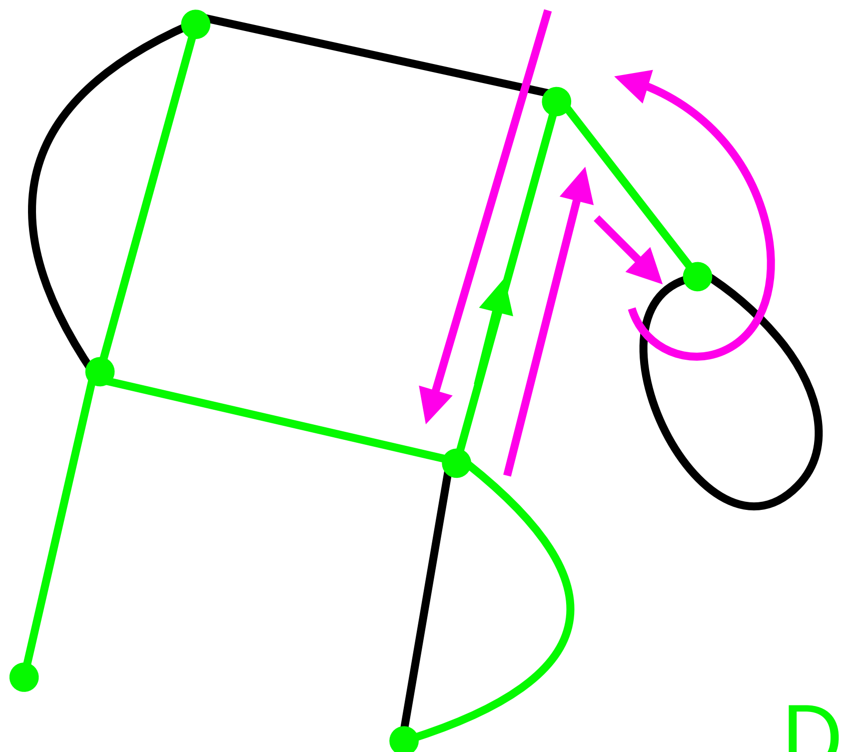

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Vandermonde identity

$$[z^n]M(z) = \mathrm{Cat}_n \mathrm{Cat}_{n+1}$$

[Mullin 67]

Model

Goal: parameter that affects the typical number of blocks.

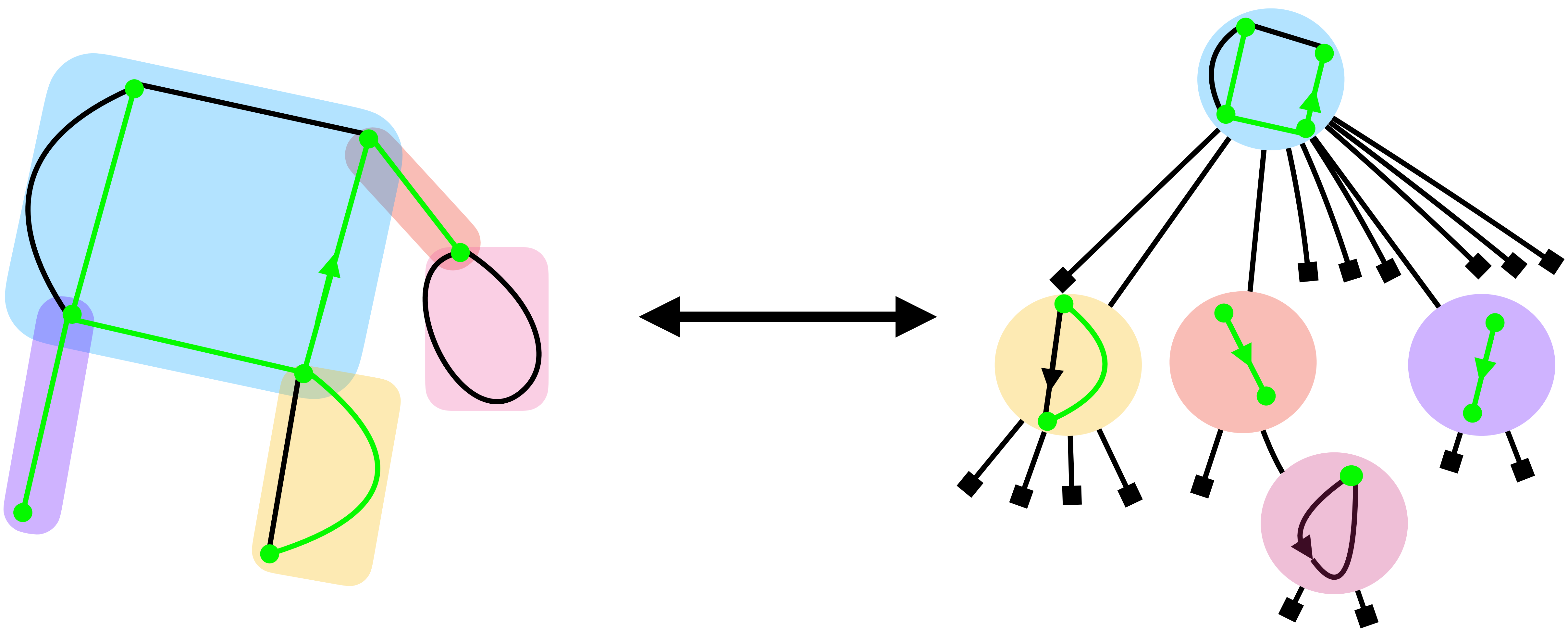
We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where $u > 0$, $\mathscr{M}_n$ = {tree-rooted maps of size n}, $\mathfrak{m} \in \mathscr{M}_n$, $Z_{n,u}$ = normalisation.

- $u = 1$: uniform distribution on tree-rooted maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected tree-rooted maps);
- $u \to \infty$: maximising the number of blocks (= tree-rooted trees!).

Given u, asymptotic behaviour when $n \to \infty$?

Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



$$M(z) = B(zM^2(z))$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

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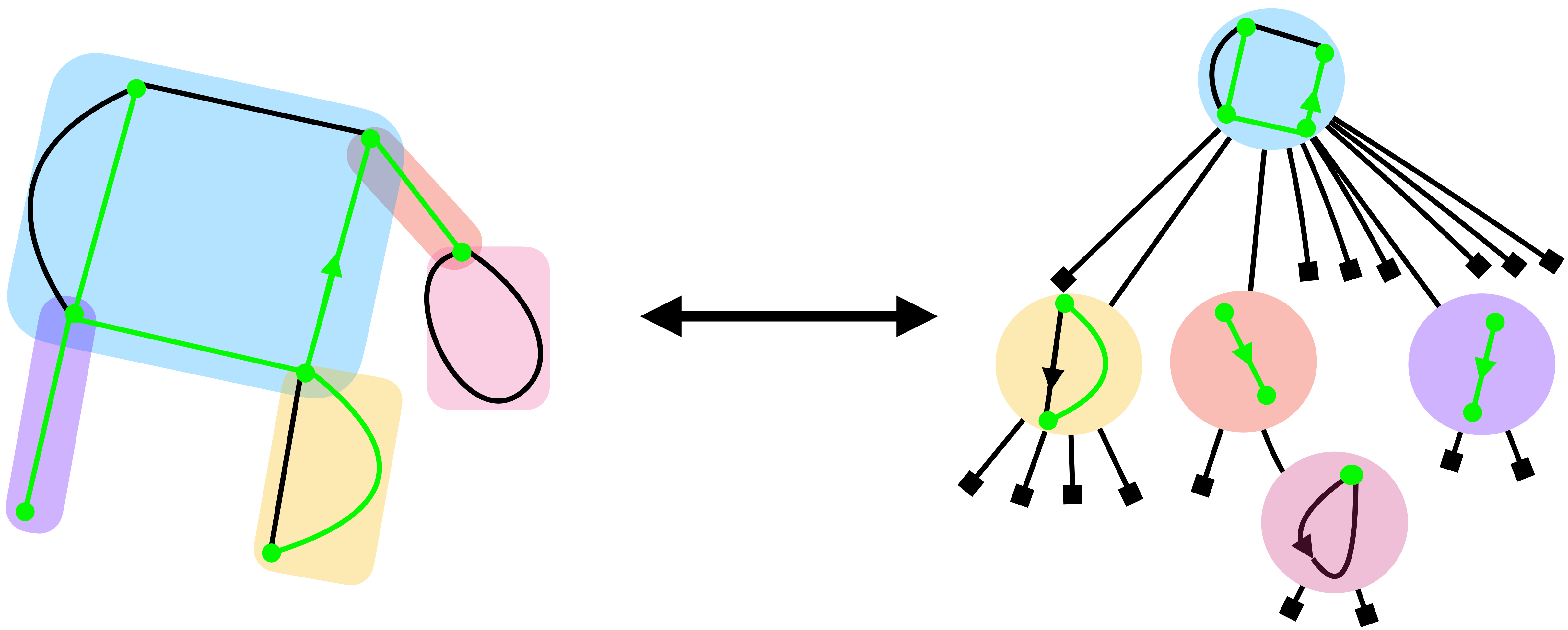

$$M(z,u) = uB(zM^2(z,u)) + 1 - u$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

So everything should be easy, right?

Tree-rooted maps are not so nice

$$M(z) = \sum_{n \geq 0} \mathsf{Cat}_n \mathsf{Cat}_{n+1} z^n \text{ so}$$

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- $M(\rho_M) = 8 - \frac{64}{3\pi} \simeq 1.2$ so M is not algebraic...

$$P\left(z, M(z)\right) = 0$$

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- Fortunately, it is still D-finite

$$P_0(z)\frac{\partial^2 M}{\partial z^2}(z) + P_1(z)\frac{\partial M}{\partial z}(z) + P_2(z)M(z) + P_3(z) = 0.$$

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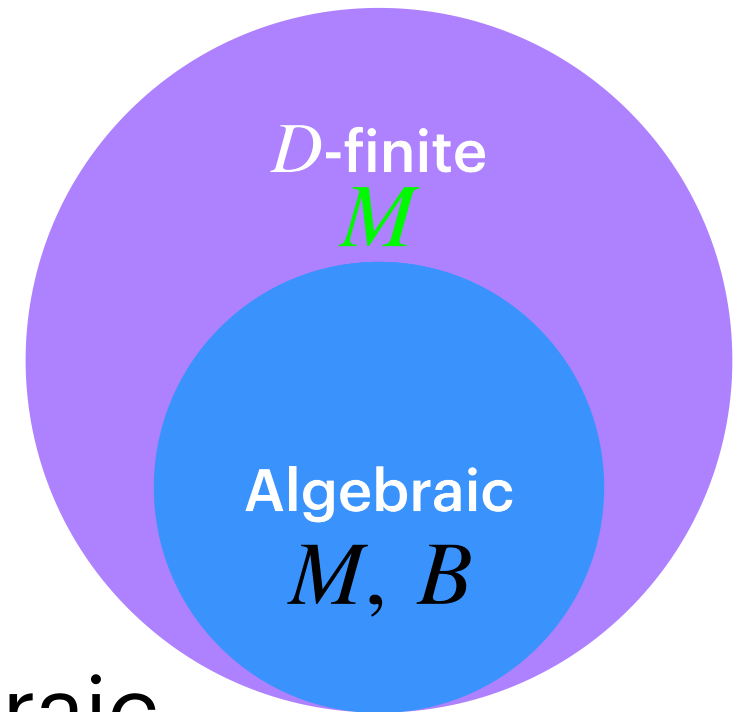

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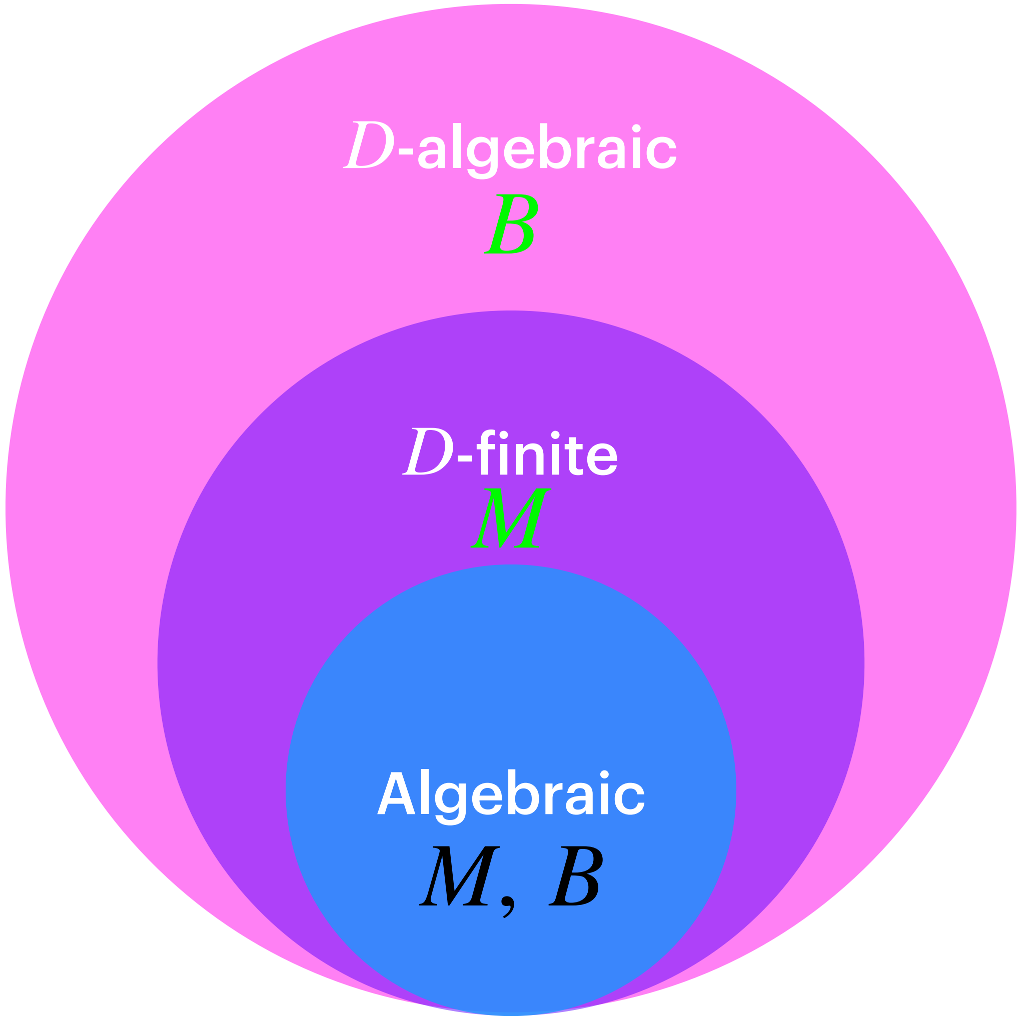

Enumeration of 2-connected tree-rooted maps

Using $M(z) = B(zM^2(z))$ and the properties of M, we show

Theorem [Albenque, Fusy, S. 24]

$$[y^n]B(y) \sim \frac{4(3\pi - 8)^3}{27\pi(4-\pi)^3} \times \rho_B^{-n} \times n^{-3}.$$

Phase transition

Theorem [Albenque, Fusy, S. 24] Model exhibits a phase transition at $u_C = \dfrac{9\pi(4-\pi)}{420\pi - 81\pi^2 - 512} \simeq 3.02$.

When $n \to \infty$:

- Subcritical phase $u < u_C$: "general tree-rooted map phase" one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: "tree phase" only small blocks.

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration			
Size of - the largest block - the second one			
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
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Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3\ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
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Scaling limit of M_n			

Ordered atoms of a Poisson Point Process

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
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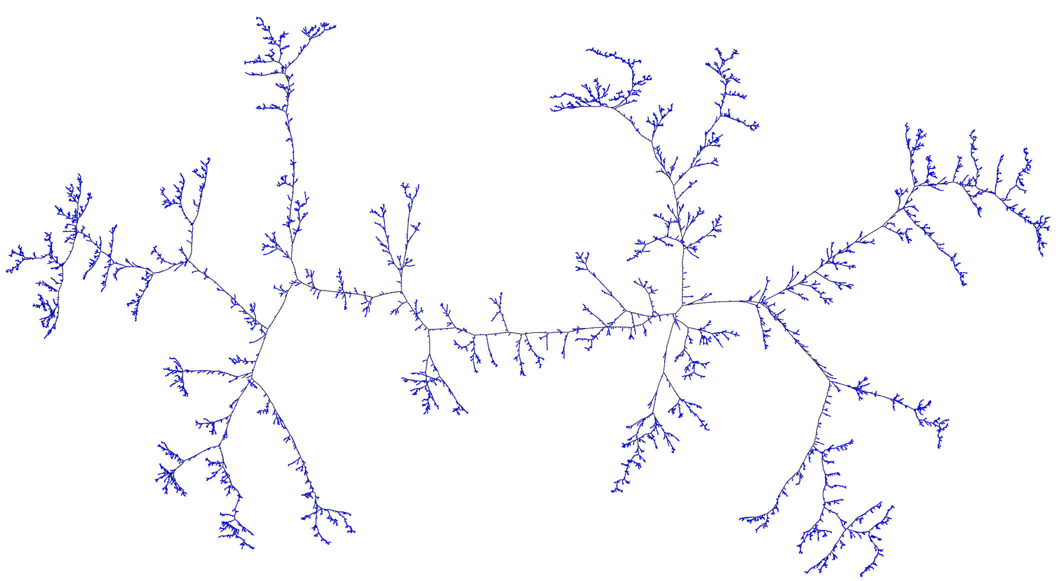
Scaling limit of M_n	?	$\frac{C_2 \ln(n)^{1/2}}{n^{1/2}} M_n \to \mathcal{T}_e$	[Stufler 2020] $\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e$



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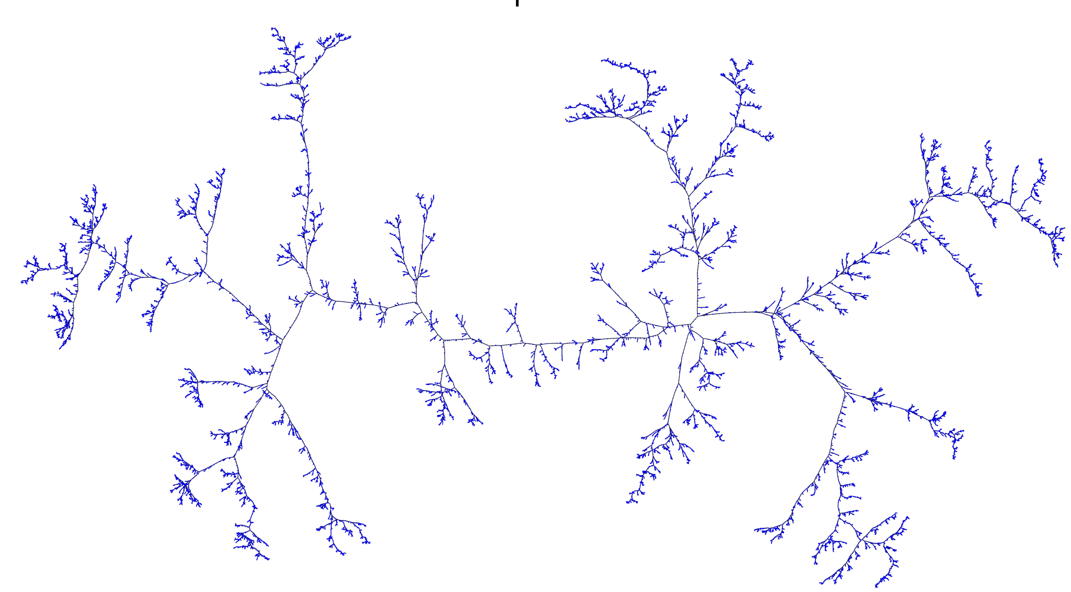

Interlude: tree-rooted quadrangulations

CANCELLED

Interlude: tree-rooted quadrangulations

$M(z) = Q(z)$ does not hold!

VI. Perspectives

Extensions to more involved decompositions

Block-weighted

- Maps into loopless blocks;
- 2-connected maps into 3-connected blocks...

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$
loopless $M_2(z)$	simple $M_3(z)$	$z(1+M)$	–
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1+M)^2$	–
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1+M)$	–
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1+M)$
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1+M)$	–
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1+M)^2$	–
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1+M)$	–
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1+M)^3$	–
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1+M)^2$	–

Extensions to more involved decompositions

Block-weighted

- Tree-rooted quadrangulations;
- Forested maps;
- Maps endowed with a Potts model / Ising model;
- 2-oriented quadrangulations (resp. 3-oriented triangulations) decomposed into irreducible blocks…

Critical window?

Phase transition very sharp => what if $u = 9/5 \pm \varepsilon(n)$?

- Block size results still hold if $u_n = 9/5 - \varepsilon(n)$, $\varepsilon^3 n \to \infty$;
- For $u_n = 9/5 + \varepsilon(n)$, this is the case as well: when $\varepsilon^3 n \to \infty$

$$L_{n,1} \sim 2.7648\, \varepsilon^{-2} \ln(\varepsilon^3 n)$$

(analogous to [Bollobás 1984]'s result for Erdős-Rényi graphs!);

- Results exist for scaling limits in ER graphs [Addario-Berry, Broutin, Goldschmidt 2010], open question in our case.

Is there a critical window? If so, what is its width?

Thank you!

▶ **Theorem 15.** *The random tree-rooted map* $\mathrm{M}_n^{(u)}$*, drawn according to* $\mathbb{P}_n^{(u)}$*, exhibits the following behaviours when* n *tends to infinity.*

Subcritical case. *For* $u < u_c$*, the largest bloc is macroscopic, and more precisely one has:*

$$\frac{\mathrm{LB}_1(\mathrm{M}_n^{(u)}) - (1 - E(u))n}{\sqrt{c(u) n \ln(n)}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1). \tag{23}$$

Furthermore, for any fixed $j \geq 2$*, it holds that* $\mathrm{LB}_j(\mathrm{M}_n^{(u)}) = \Theta_{\mathbb{P}}(n^{1/2})$ *and for* $x > 0$*:*

$$\mathbb{P}\left(\mathrm{LB}_j(\mathrm{M}_n^{(u)}) \leq x\sqrt{n}\right) \xrightarrow[n\to\infty]{} e^{-\lambda(x)} \sum_{p=0}^{j-2} \frac{\lambda(x)^p}{p!}, \qquad \text{where } \lambda(x) := \frac{c(u)}{2x^2}. \tag{24}$$

Critical case. *For* $u = u_C$*, for any fixed* $j \geq 1$*, it holds that* $\mathrm{LB}_j(\mathrm{M}_n^{(u)}) = \Theta_{\mathbb{P}}(n^{1/2})$*. More precisely, up to a shift of indices, the sizes of the blocks exhibit a similar behavior as the sizes of non-macroscopic blocks in the subcritical regime, namely, for* $x > 0$*:*

$$\mathbb{P}\left(\mathrm{LB}_j(\mathrm{M}_n^{(u)}) \leq x\sqrt{n}\right) \xrightarrow[n\to\infty]{} e^{-\lambda(x)} \sum_{p=0}^{j-1} \frac{\lambda(x)^p}{p!}, \qquad \text{where } \lambda(x) := \frac{c(u_C)}{2x^2}. \tag{25}$$

Supercritical case. *For* $u > u_C$*, for all fixed* $j \geq 1$*, it holds as* $n \to \infty$ *that*

$$\mathrm{LB}_j(\mathrm{M}_n^{(u)}) = \frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3\ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O_{\mathbb{P}}(1).$$

▶ **Remark 16.** One can get a local limit theorem for $\mathrm{LB}_1(\mathrm{M}_n^{(u)})$ in the subcritical case as in [24] (up to the technicality that nodes of the block tree have only even numbers of children). Furthermore, one can state a joint limit law for the sizes $\mathrm{LB}_j(\mathrm{M}_n^{(u)})$. For any fixed $r \geq 1$,

$$\left(\frac{c(u)}{2} \left(\frac{\mathrm{LB}_j(\mathrm{M}_n^{(u)})}{\sqrt{n}} \right)^{-2}, \; 2 \leq j \leq r+1 \right) \xrightarrow[n\to\infty]{(d)} (A_1, \ldots, A_r),$$

where the A_i are the decreasingly ordered atoms of a Poisson Point Process of rate 1 on $\mathbb{R}_+$. The same joint limit law holds at u_C (with j from 1 to r).