Bijections for planar maps

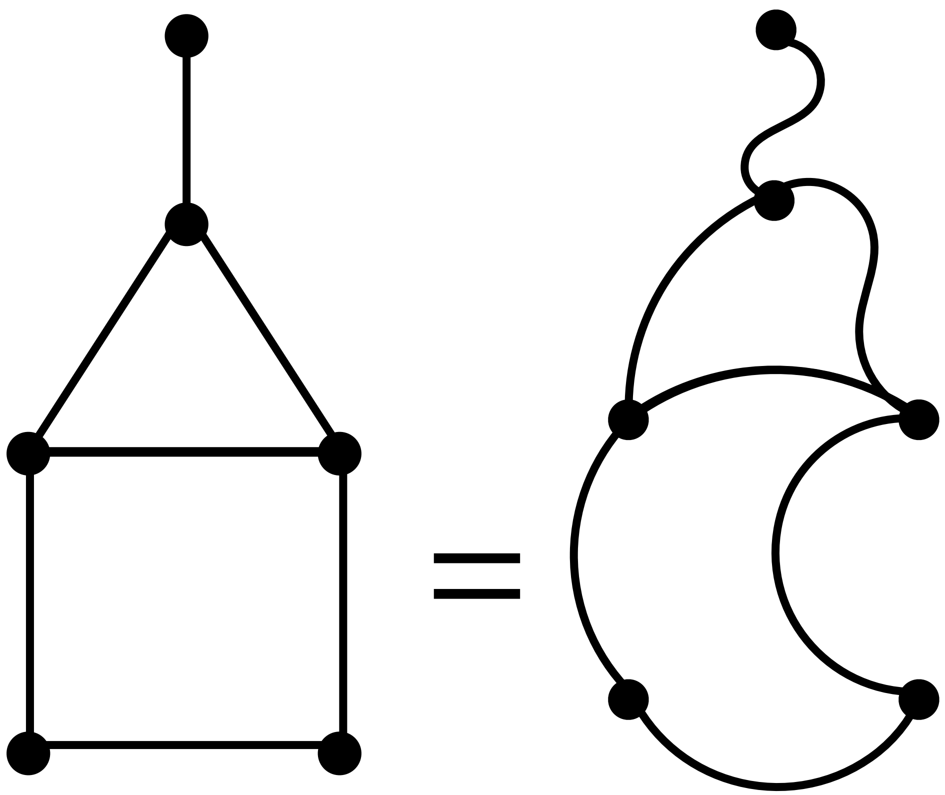
SFB reading group
31 October 2024

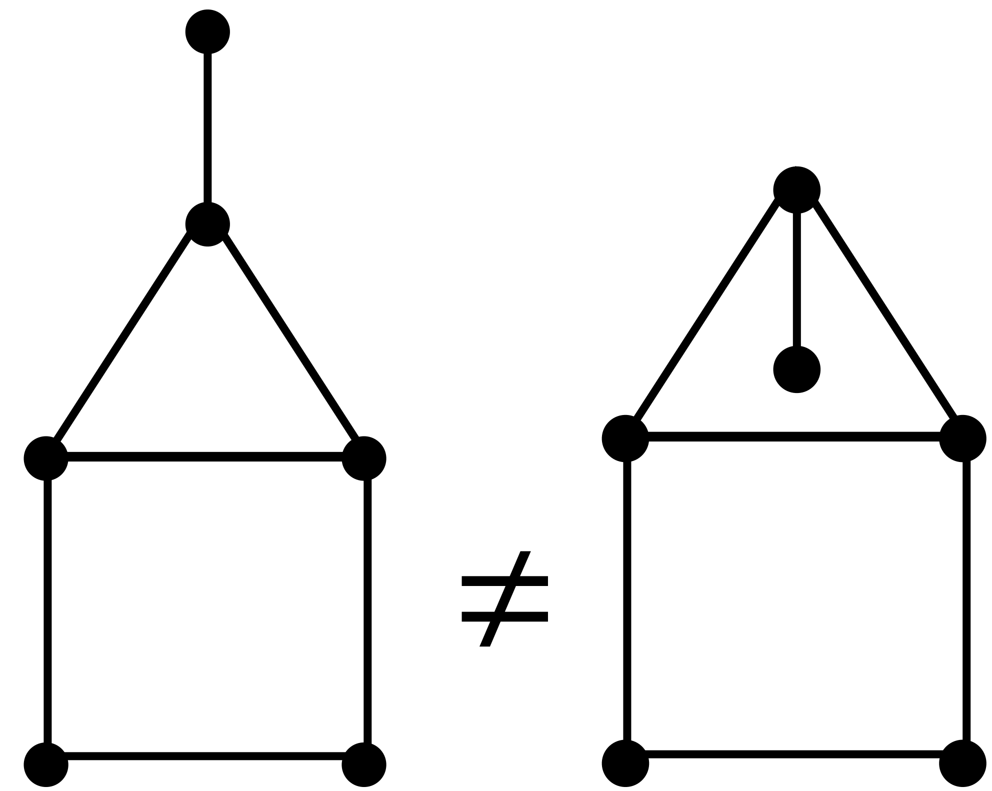
Zéphyr Salvy (he/they)

TU Wien, Austria

Planar maps

Planar map = embedding on the sphere of a connected planar graph, considered up to homeomorphisms



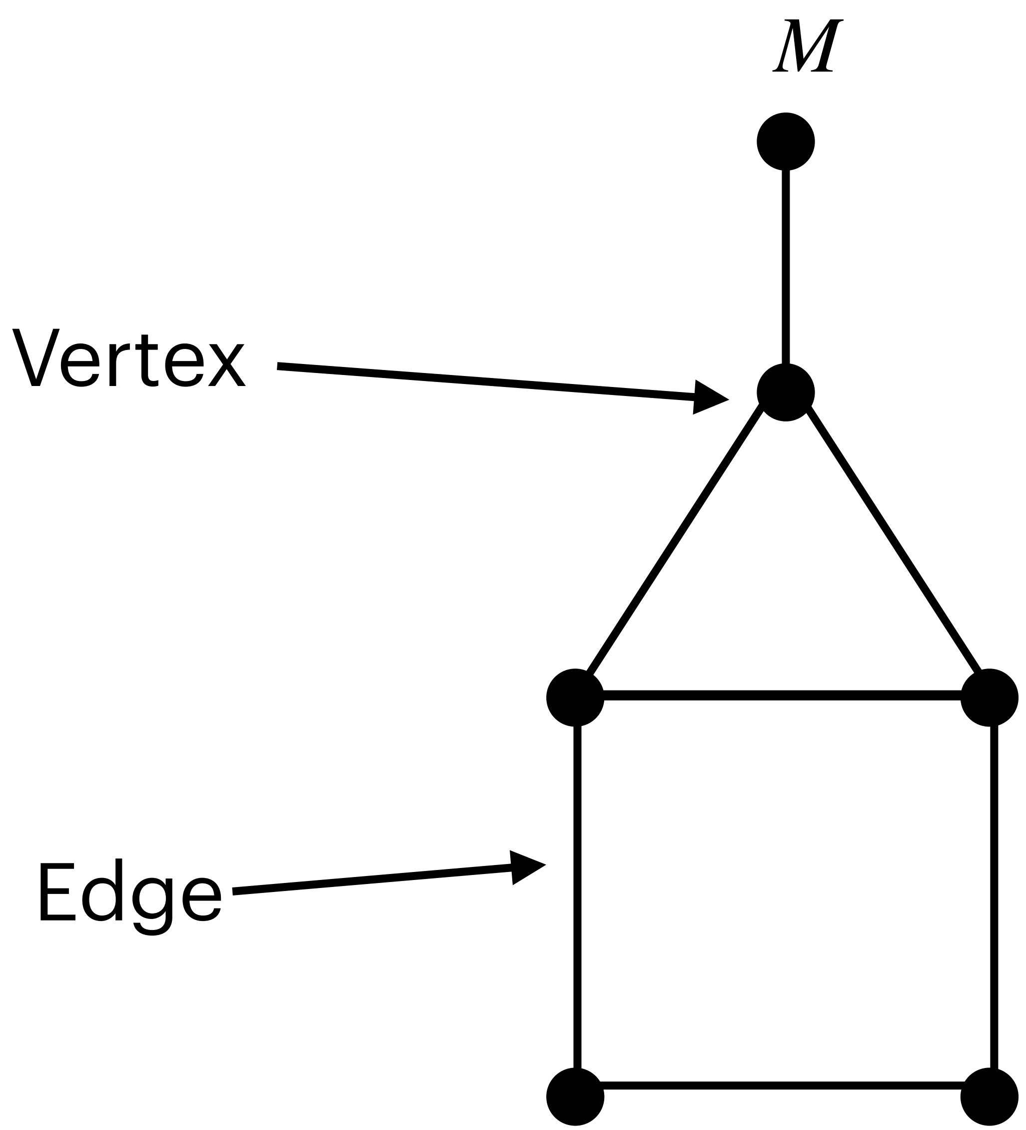


Planar map = planar graph + cyclic order on neighbours

Very interesting objects for computer science, mathematics & physics.

Vocabulary for maps

Planar map = planar graph + cyclic order on neighbours



Exist for graphs

Vocabulary for maps

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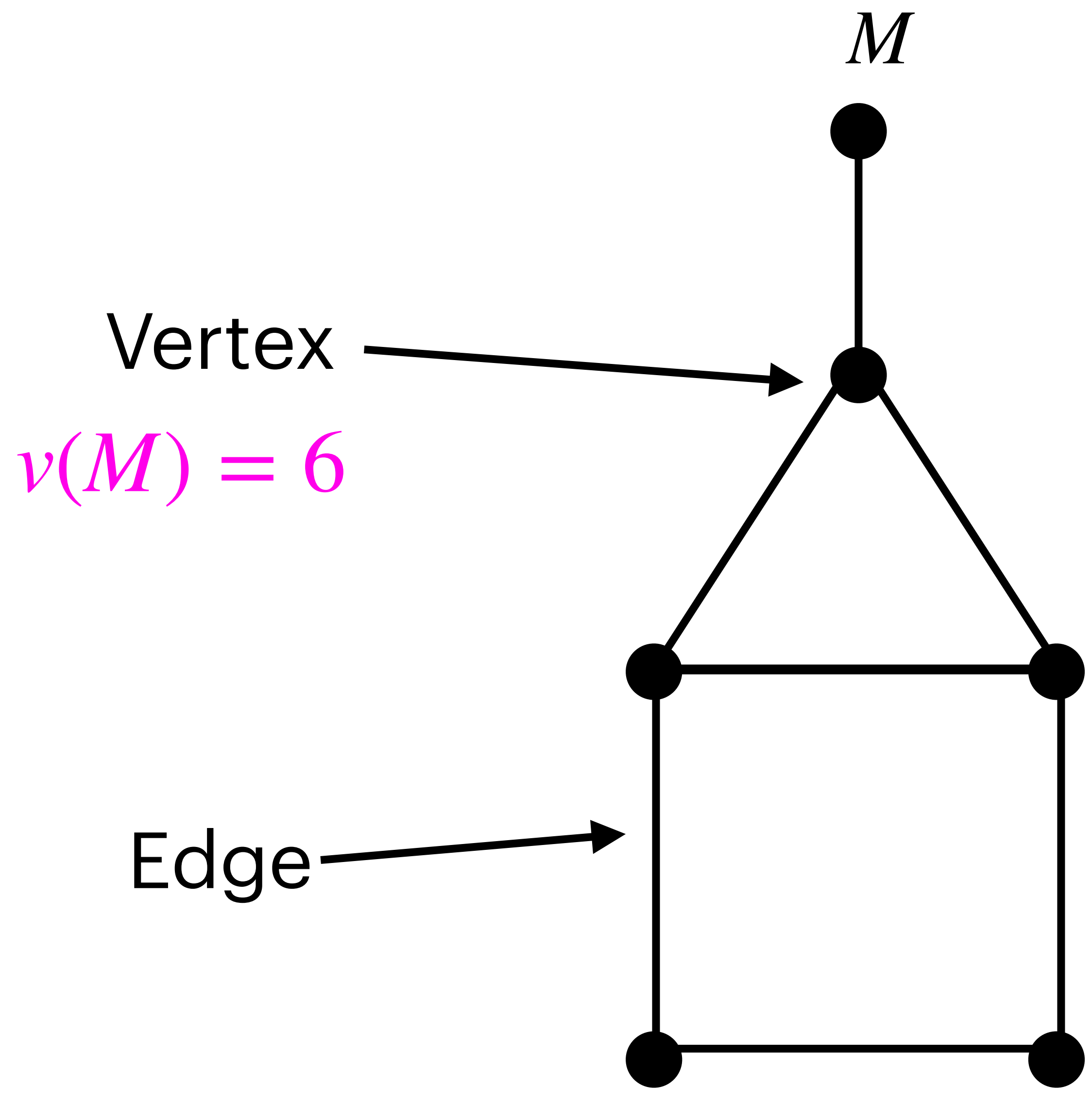

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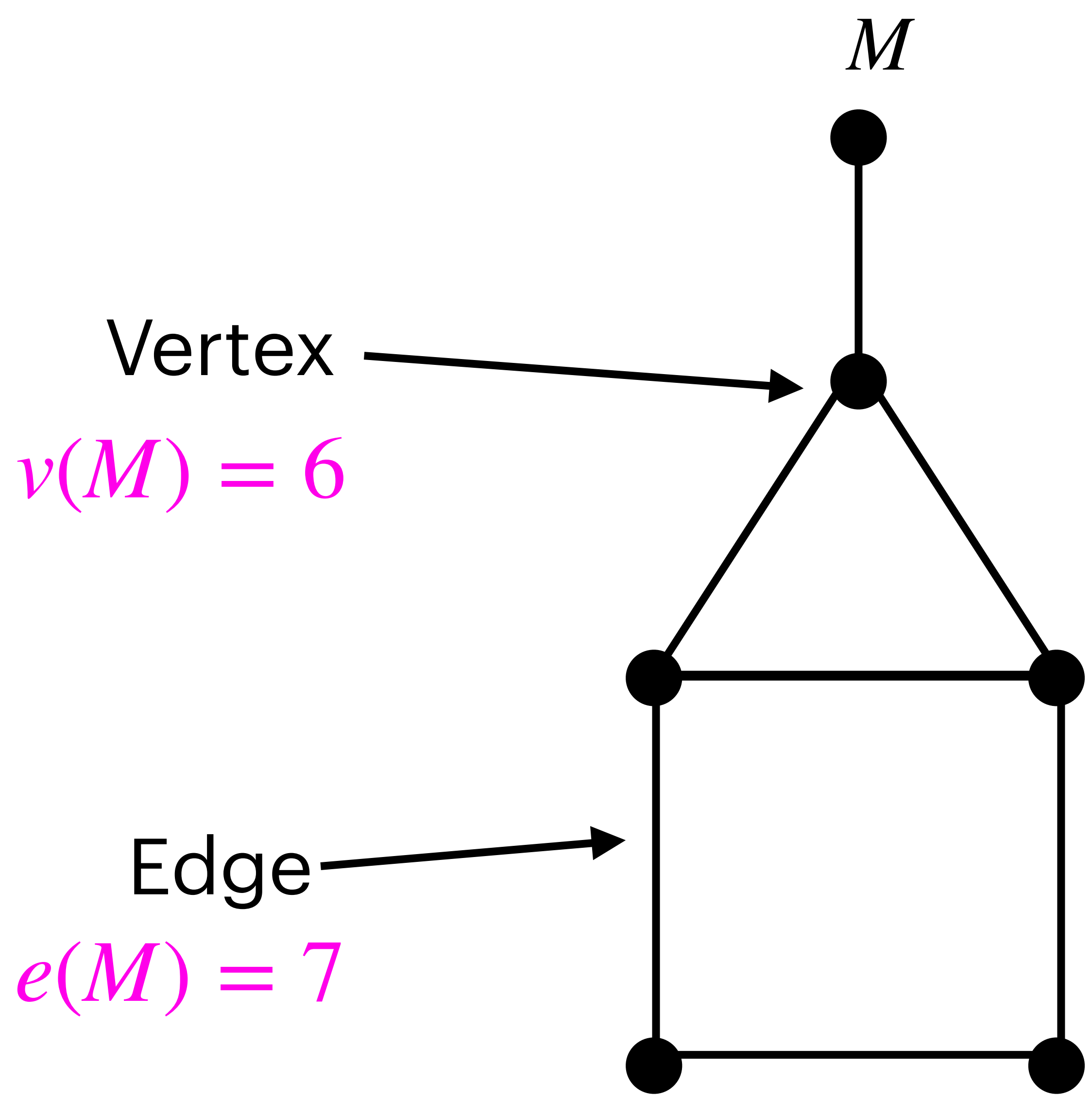

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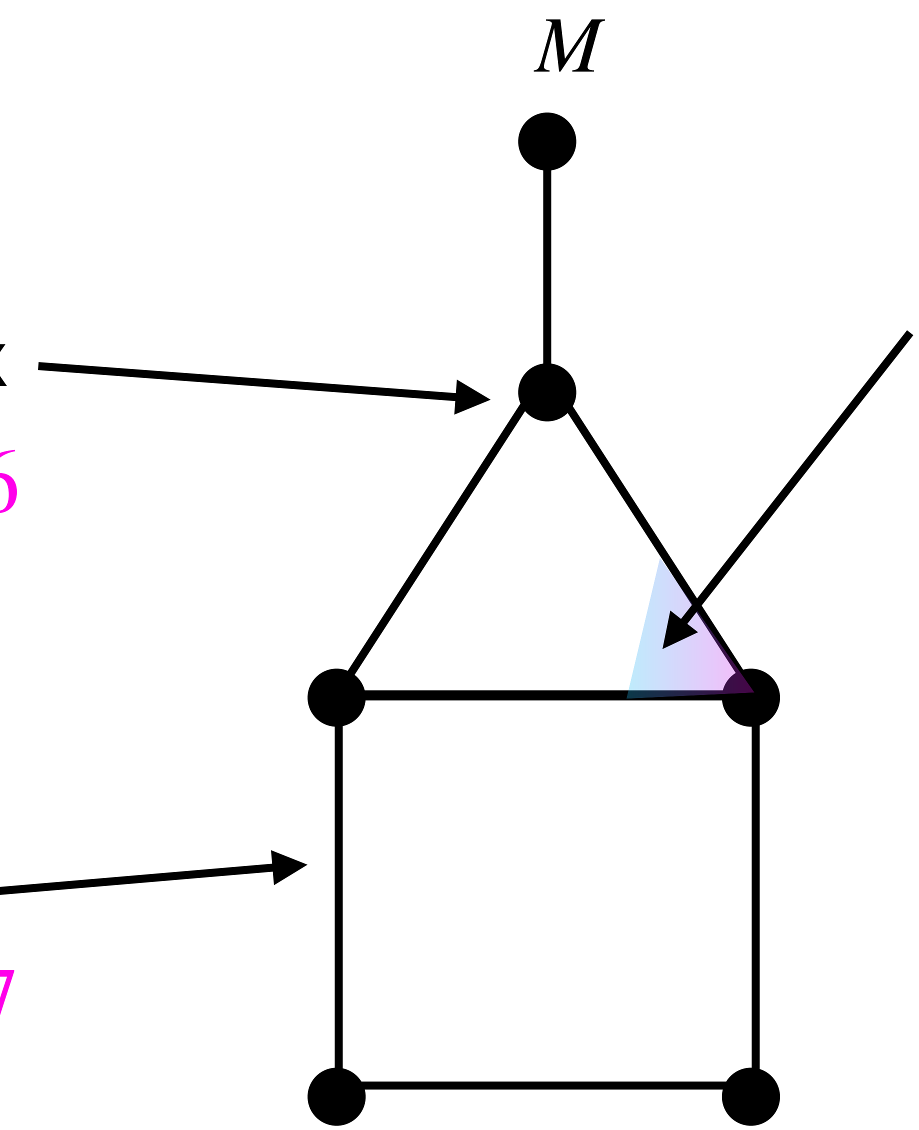

Vertex

$v(M) = 6$

Edge

$e(M) = 7$

Corner = space between two consecutive edges around a vertex (trigonometric order).

Exist for graphs

Do not exist for graphs

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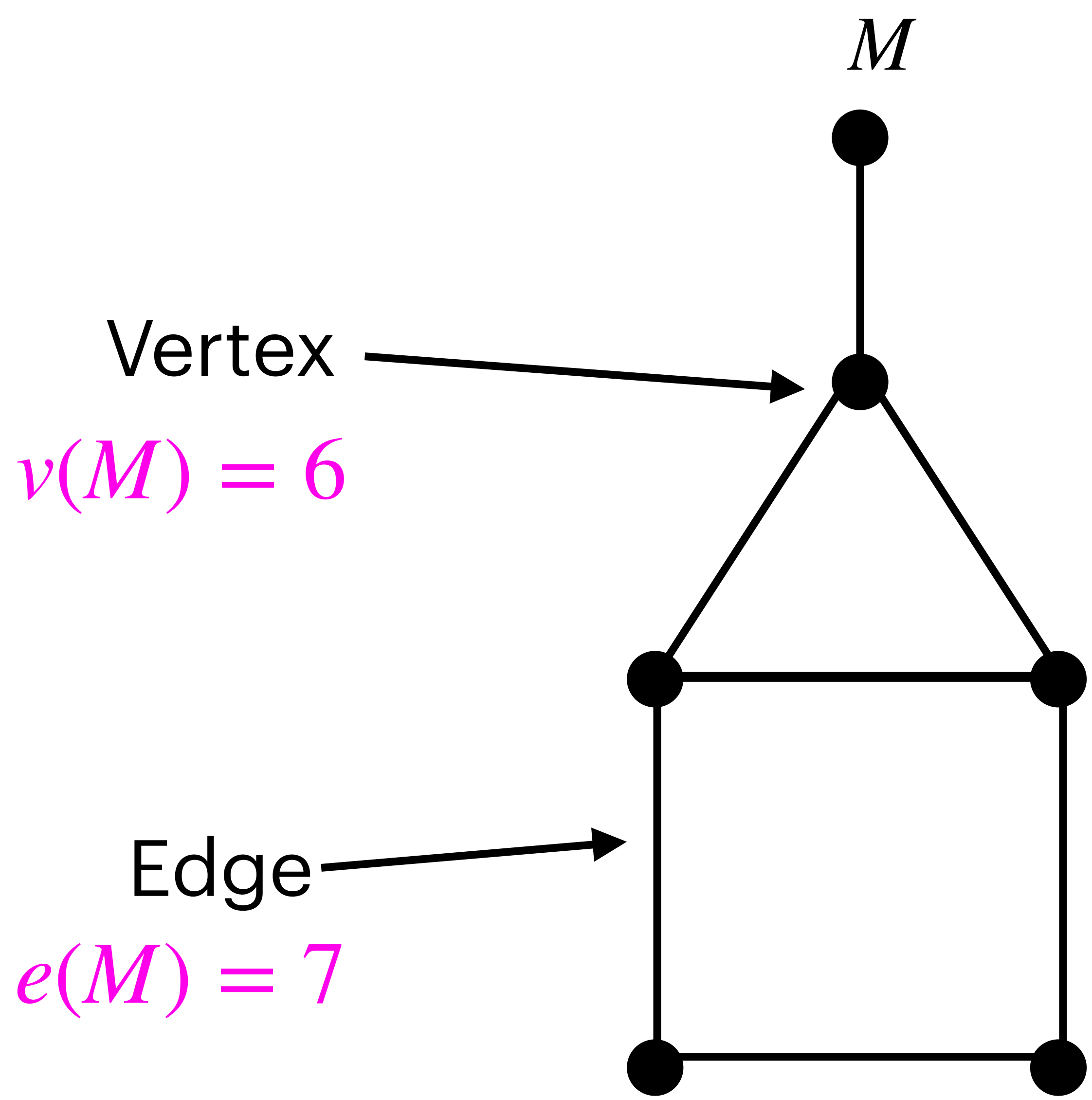

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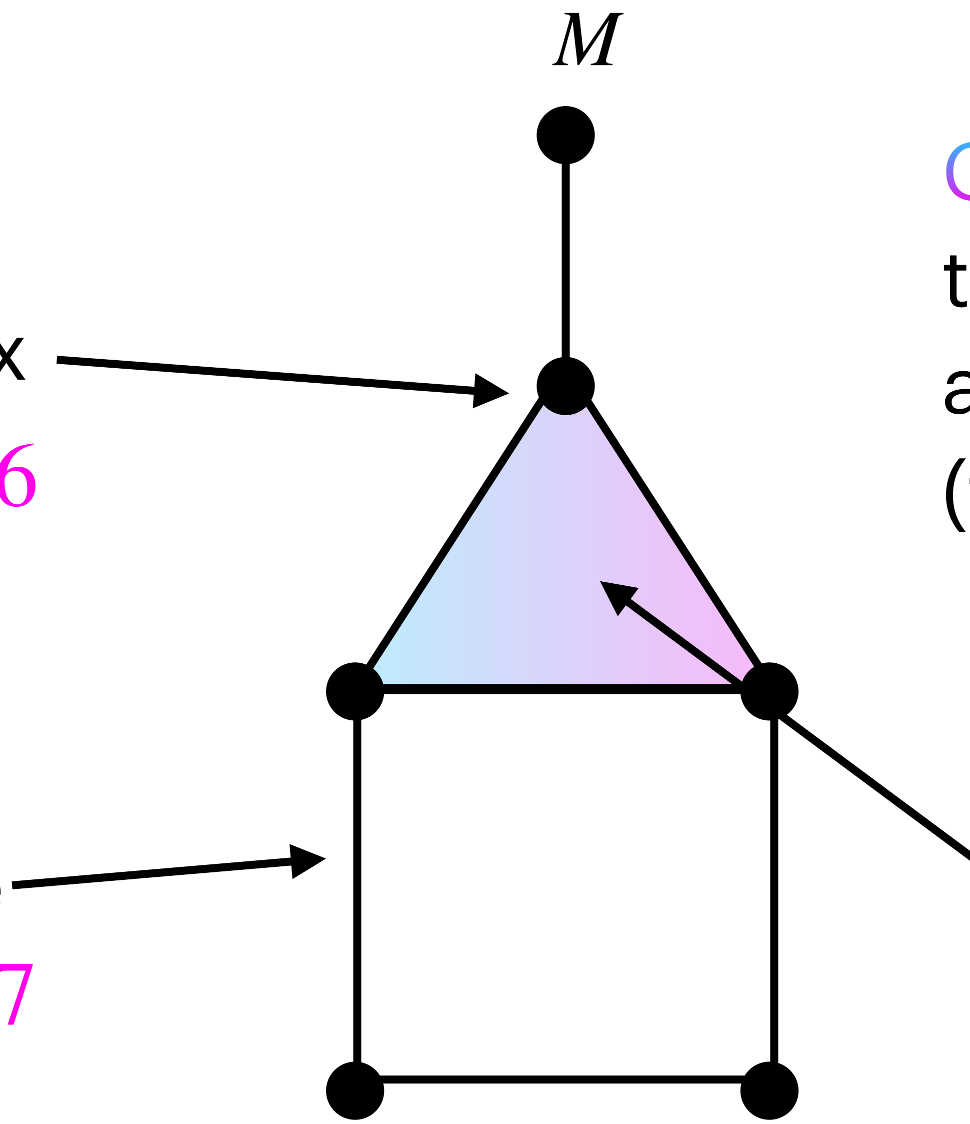

Vertex

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Edge

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Exist for graphs

Corner = space between two consecutive edges around a vertex (trigonometric order).

Face = connected components of the space without the graph

$f(M) = 3$

Do not exist for graphs

Vocabulary for maps

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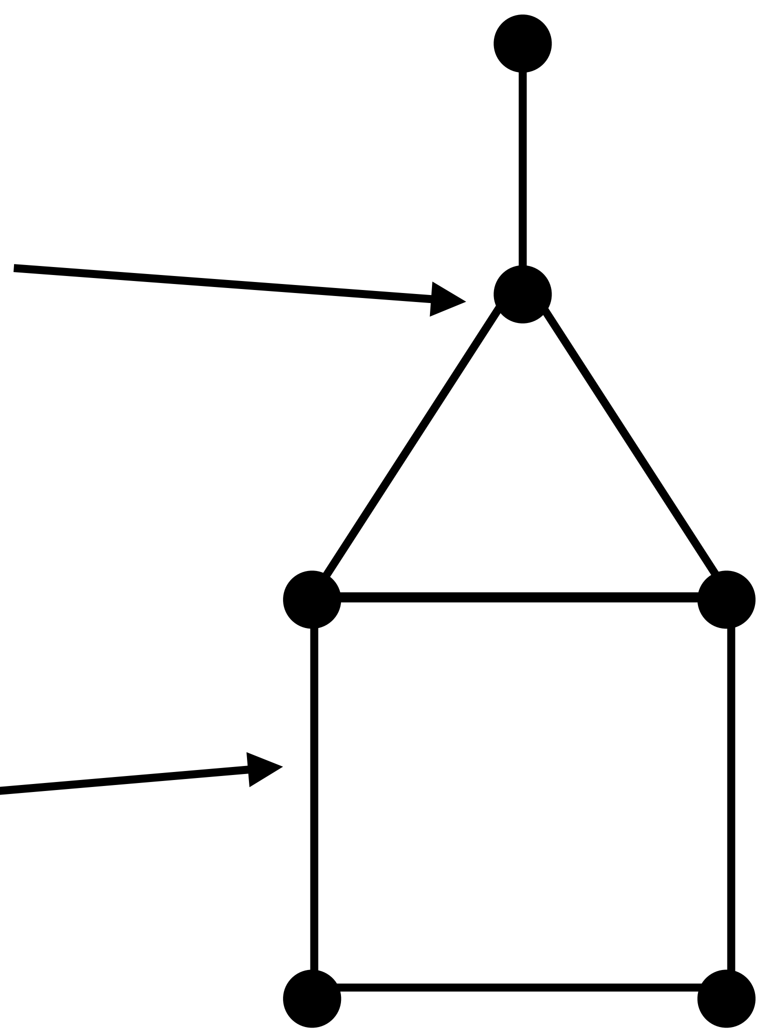
M



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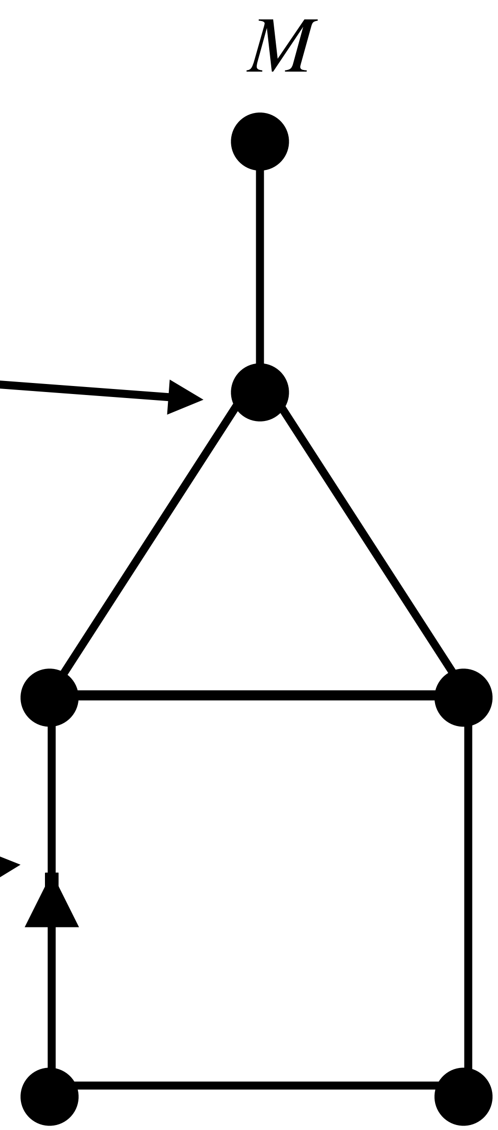

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Do not exist for graphs

Rooted planar map = map endowed with a marked oriented edge (represented by an arrow).

Outline of the lecture

Bijections for planar maps

I. Duality construction

II. Tutte's bijection

III. Cori—Vauquelin—Schaeffer's bijection

→ *Handbook of Enumerative Combinatorics, Chapter "Planar Maps", G. Schaeffer (2015)*

IV. Bouttier—Di Francesco—Guitter's bijection

→ *"Planar maps as labelled mobiles", J. Bouttier, P. Di Francesco and E. Guitter (2004)*

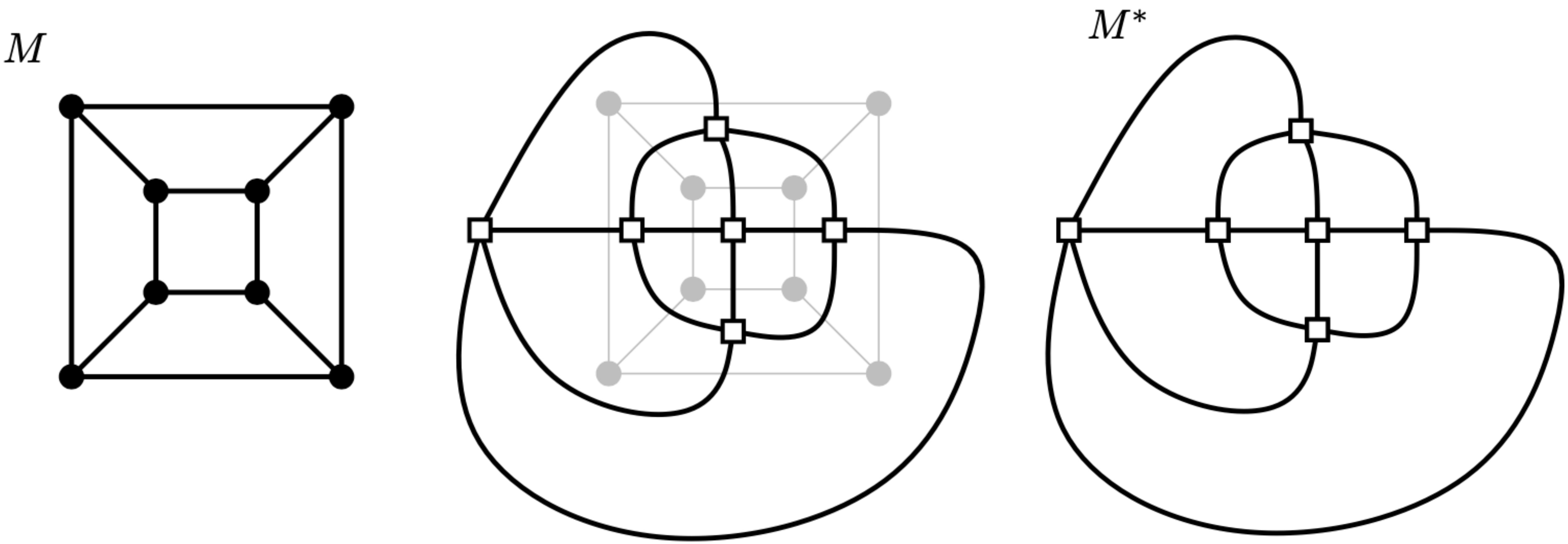
V. Conclusion

Other source: M. Albenque's MPRI course.

I. Duality construction

Duality

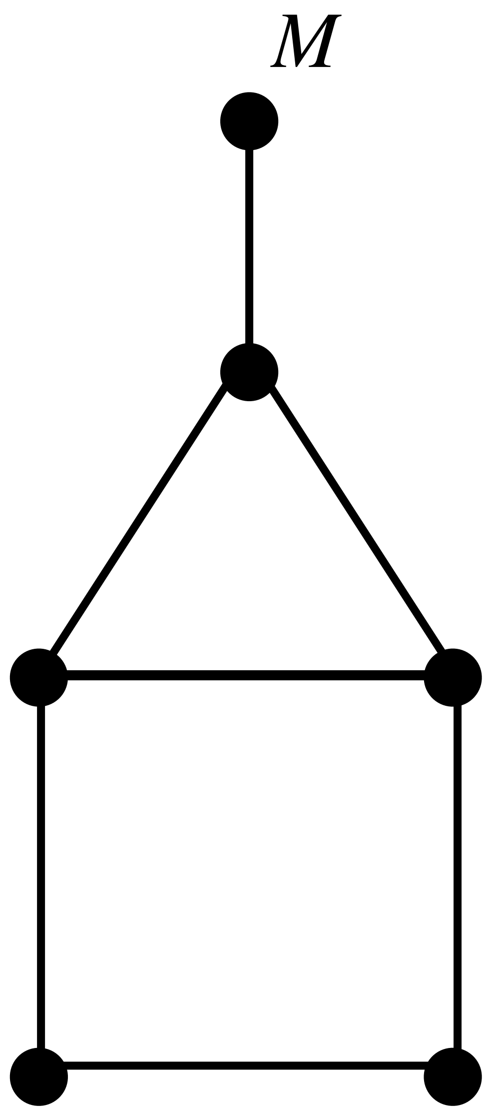
[Schaeffer 15, Figure 1.3]



[Schaeffer 15, Theorem 1] Duality is an involution on the set of planar maps. It preserves the number of edges, and exchanges the numbers of vertices and faces:

$$M^{**} = M,\ e(M^*) = e(M), \text{ and } v(M^*) = f(M).$$

Euler's formula for planar maps



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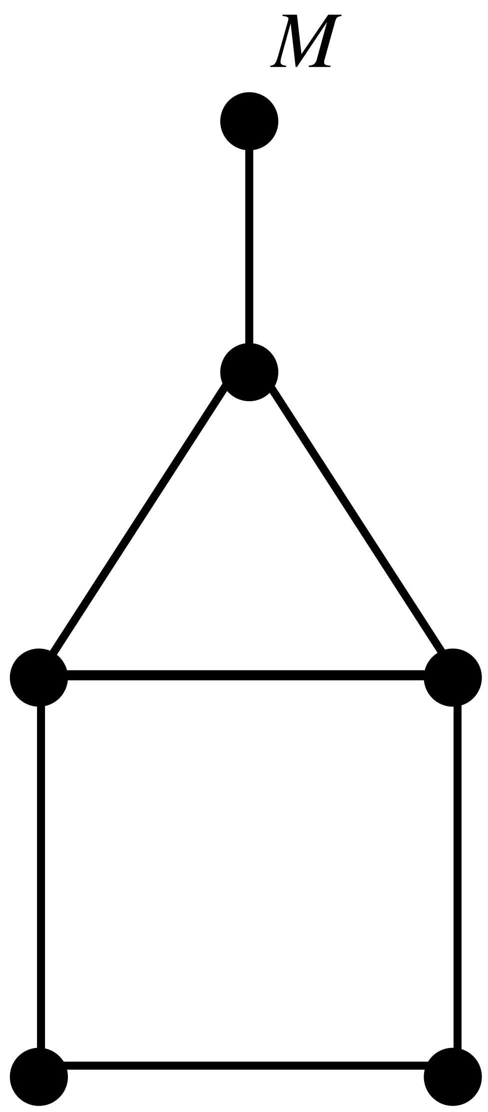


$v(M) = 6$

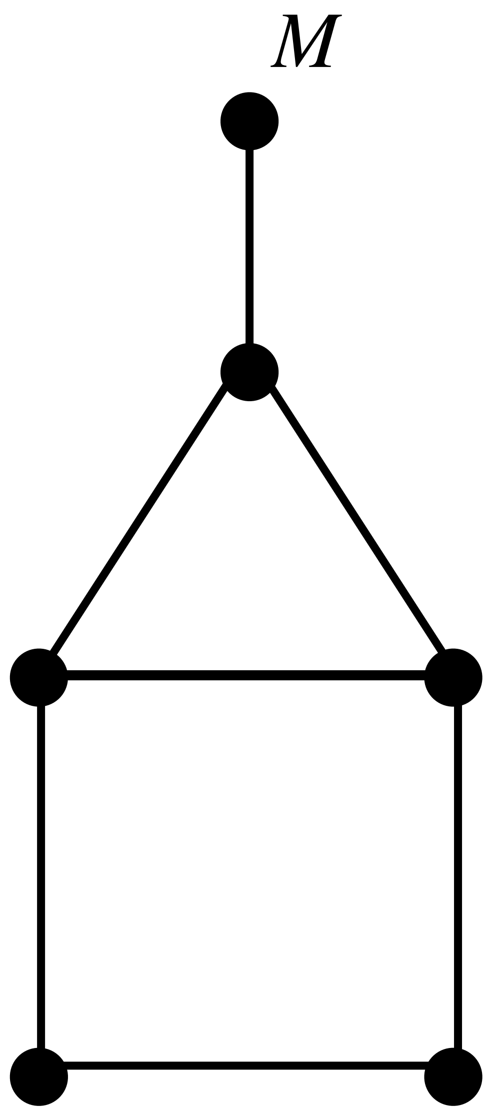
$f(M) = 3$

$e(M) = 7$

For all planar maps M, it holds that

$$v(M) + f(M) = e(M) + 2.$$

Euler's formula for planar maps



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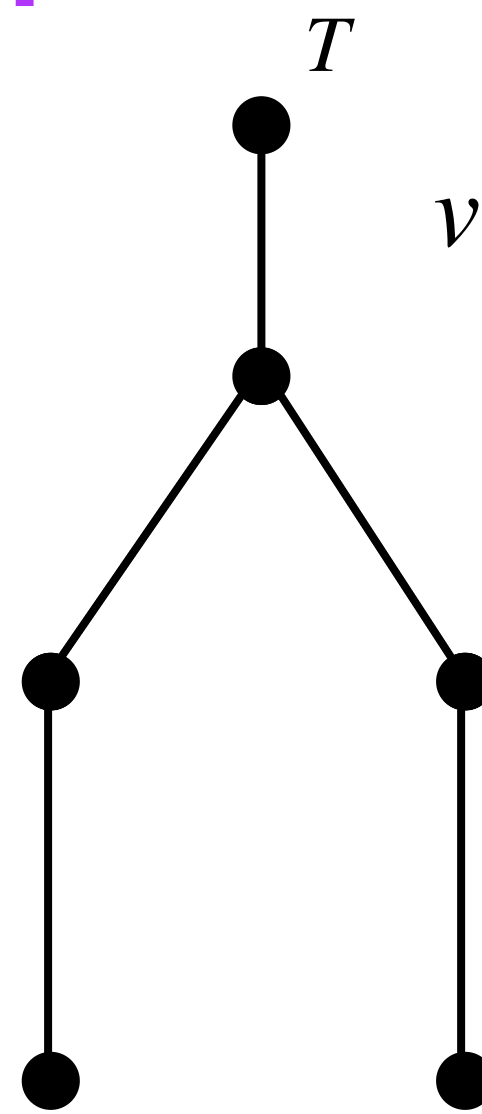

$v(T) = e(T) + 1$

$f(T) = 1$

For all planar maps M, it holds that

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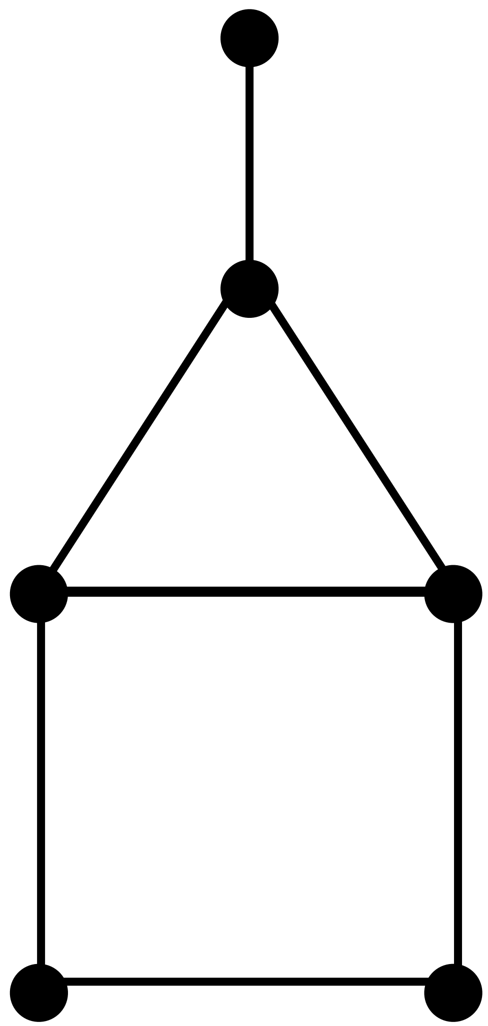
→ True for trees

Euler's formula for planar maps

M



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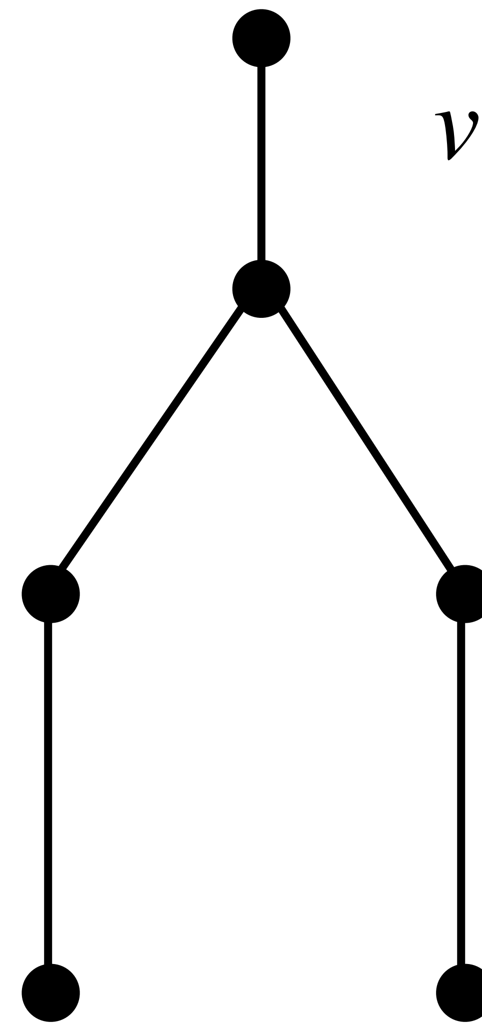
T



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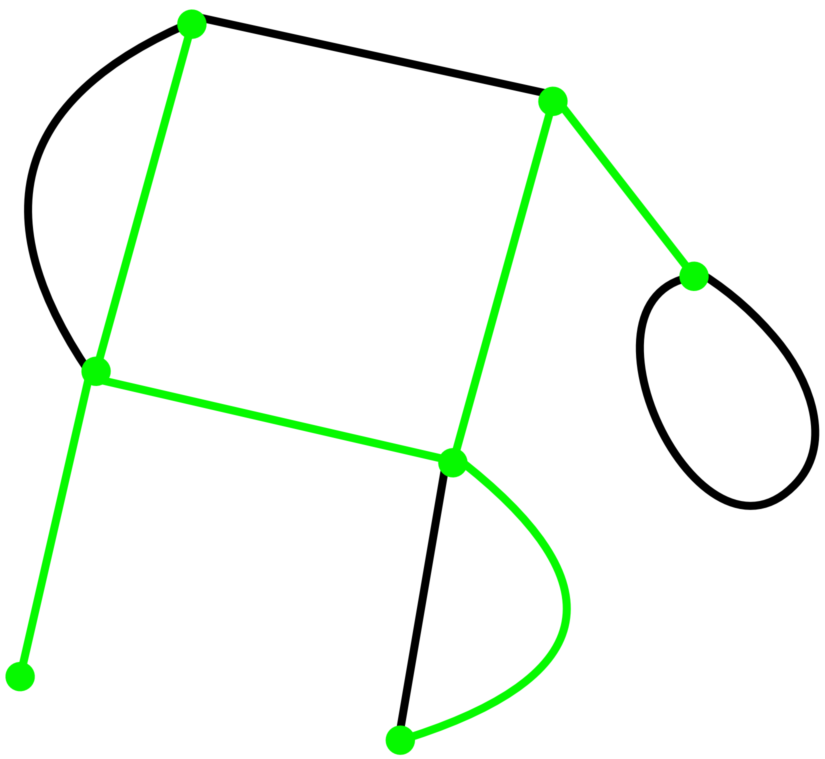
For all planar maps M, it holds that

$$v(M) + f(M) = e(M) + 2.$$

→ Interpretation via duality

Interpretation of Euler's formula (1/2)

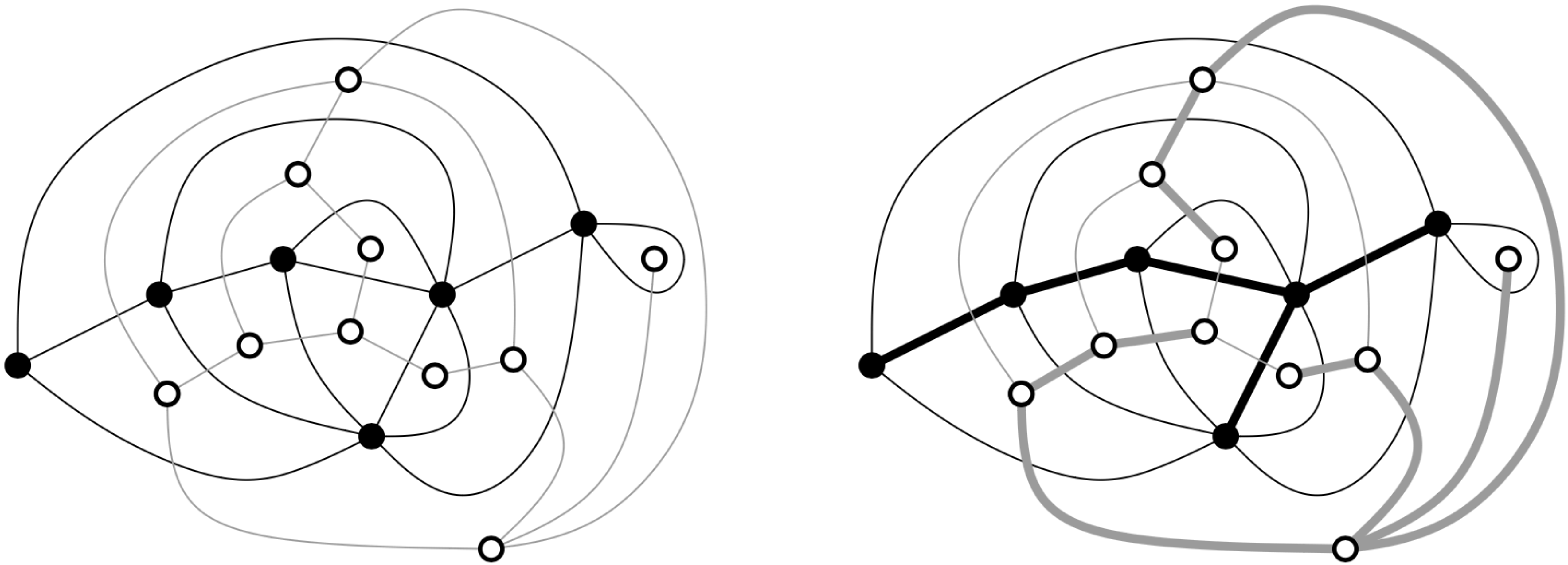
Spanning tree = tree of edges that covers all vertices.



Interpretation of Euler's formula (1/2)

Spanning tree = tree of edges that covers all vertices.

[Schaeffer 15, Theorem 2] The "dual of a spanning tree" is a spanning tree of the dual.



[Schaeffer 15, Figure 1.4]

Interpretation of Euler's formula (2/2)

For all planar maps M, it holds that

$$|V(M)| + |F(M)| = |E(M)| + 2.$$

$|E(M)|$ edges of a map =

- $|V(M)| - 1$ edges of a spanning tree;
- $|F(M)| - 1$ edges of the dual spanning tree.

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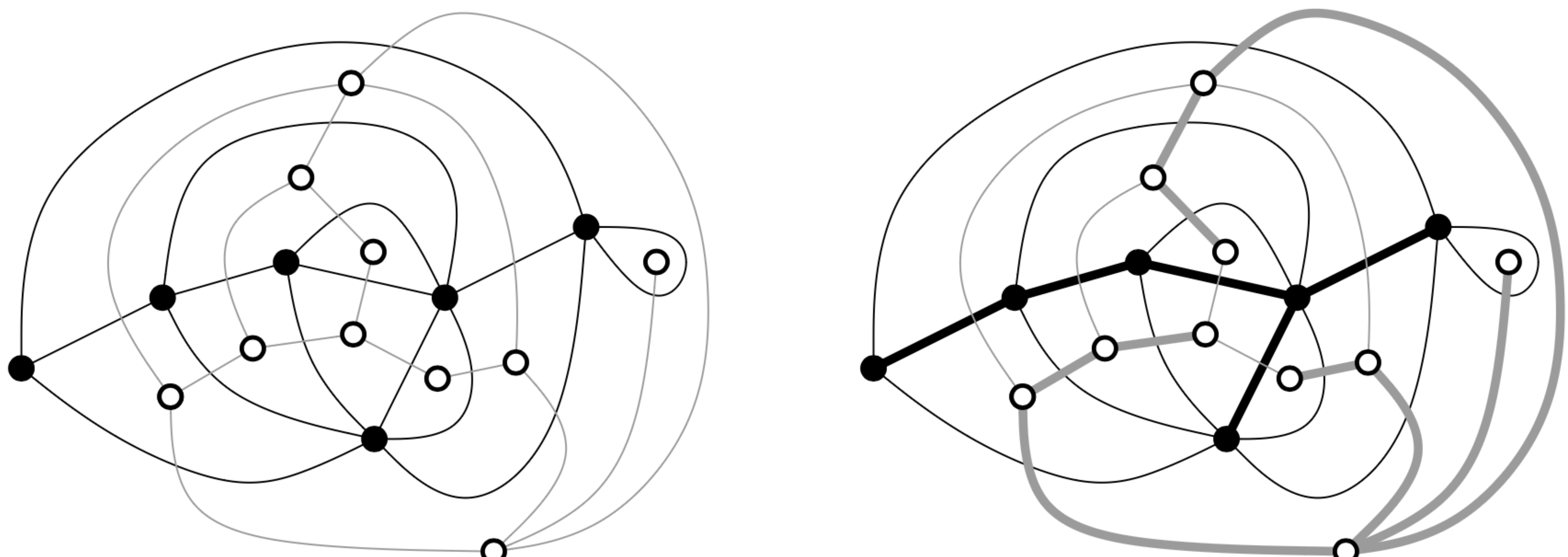

[Schaeffer 15, Figure 1.4]

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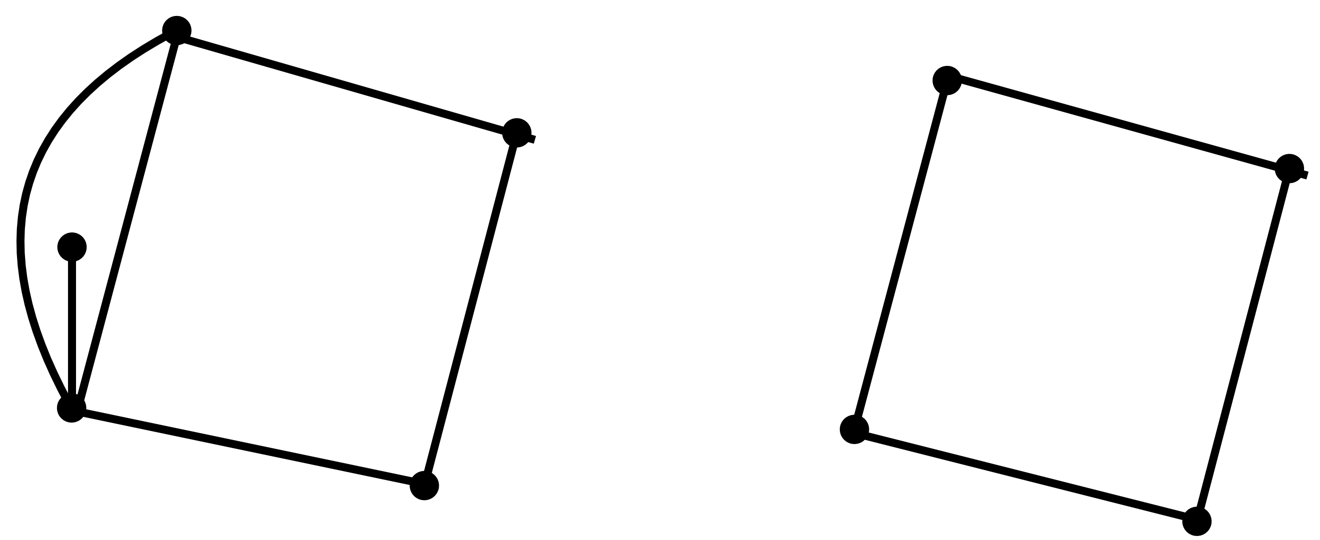
II. Tutte's bijection

Quadrangulations

Quadrangulation Q = map with all faces of degree 4.



Therefore,

$$e(Q) = 2f(Q)\,.$$

So, by Euler's formula,

$$v(Q) = f(Q) + 2.$$

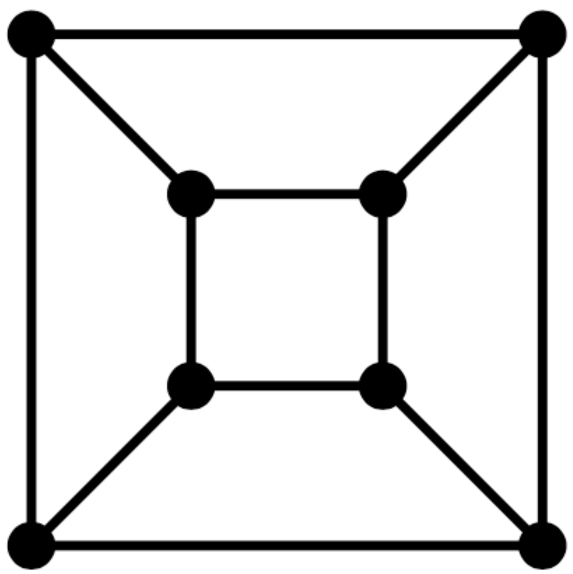
Tutte's bijection (1/2)

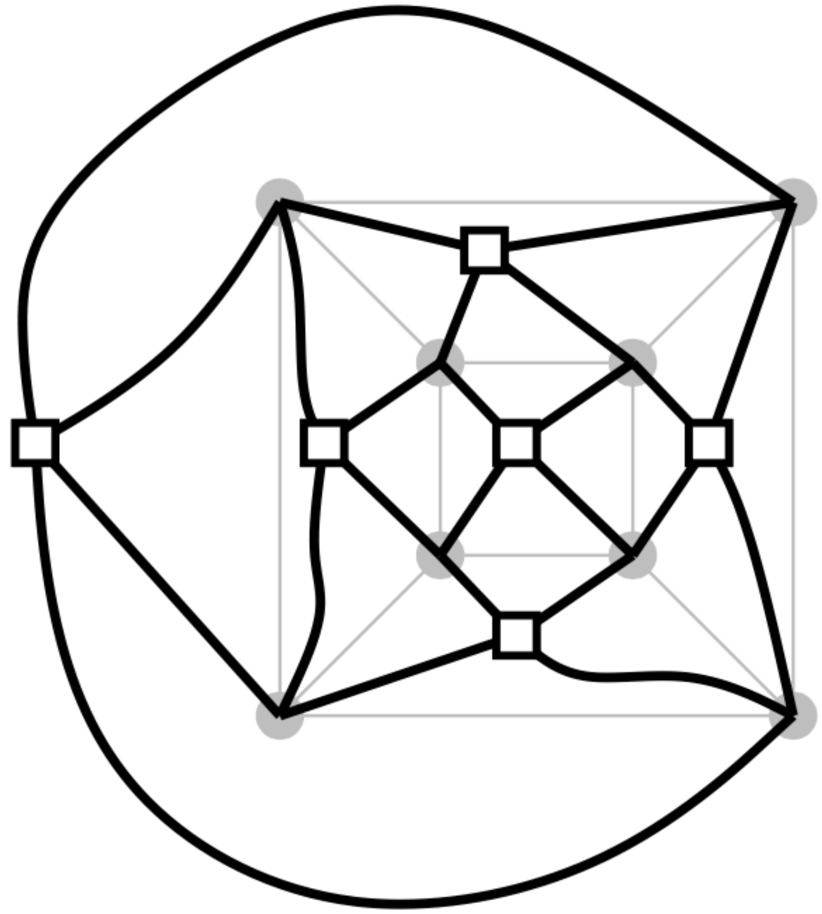
[Tutte 1

[Schaeffer 15, Figure 1.3]

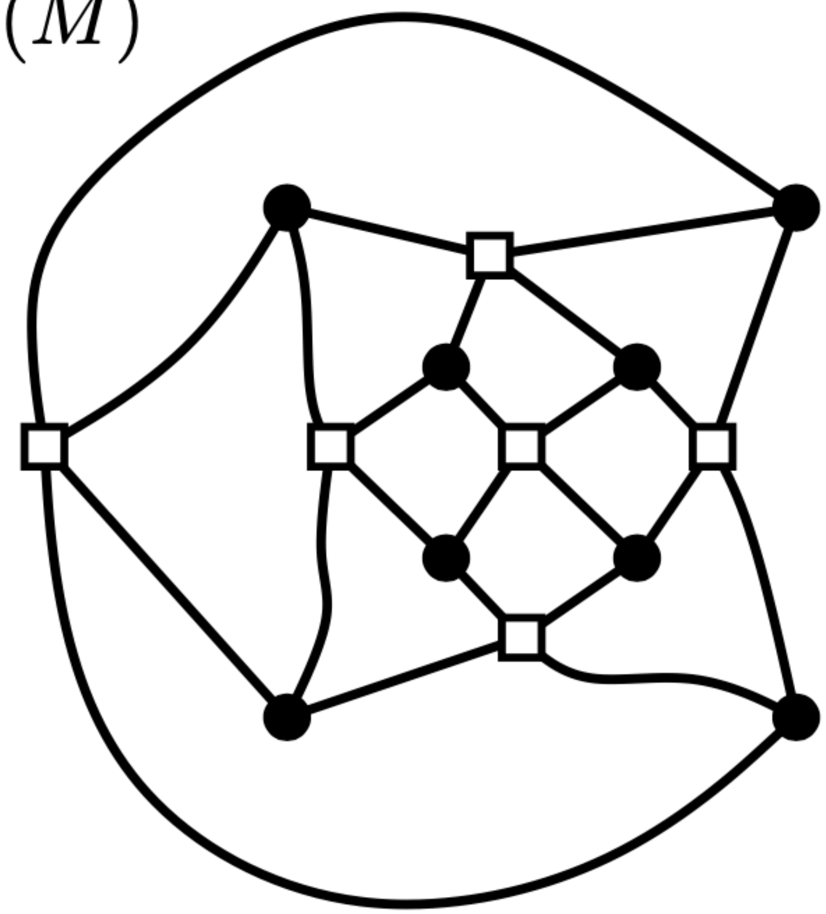
"incidence map"

M

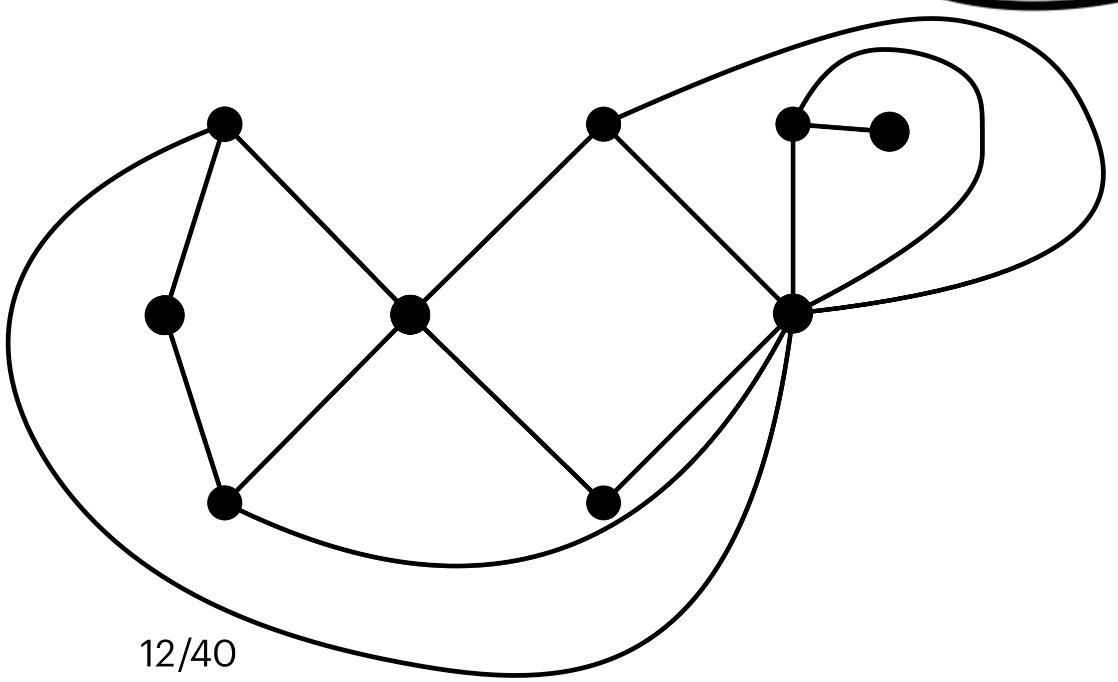




$Q(M)$



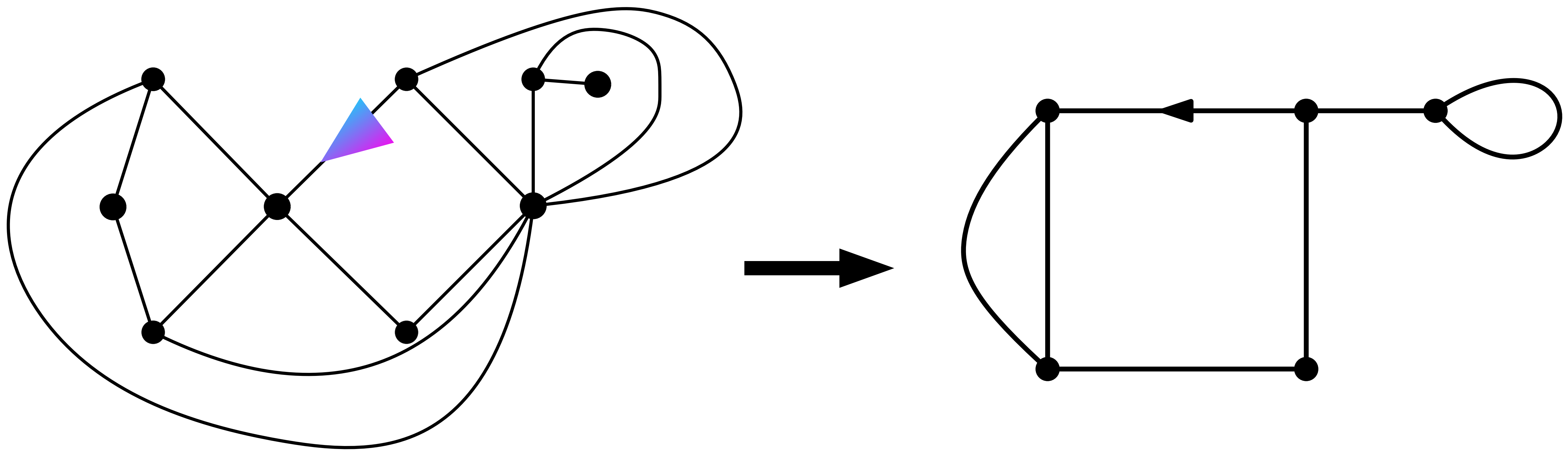
What is the inverse of



?

Tutte's bijection (2/2)

Inverse construction:



Theorem [Tutte 1963] Tutte's bijection sends rooted maps with n edges to rooted quadrangulations with n faces.

III. Cori—Vauquelin—Schaeffer's bijection (CVS)

CVS bijection

Bijection between quadrangulations and decorated trees.
→ Trees are easier to study than maps!

CVS bijection

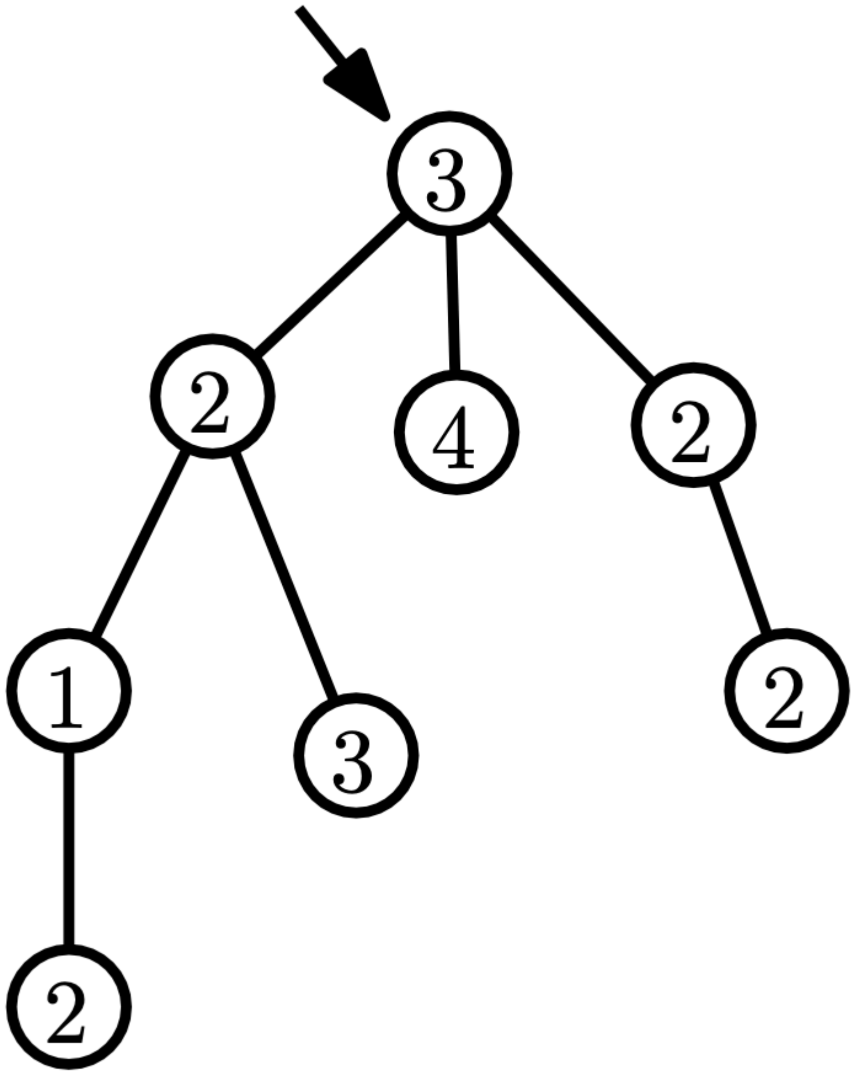
Bijection between quadrangulations and decorated trees.
→ Trees are easier to study than maps!

[Schaeffer 15, Theorem 10 and Corollary 7] The CVS bijection sends rooted quadrangulations with n faces and a marked vertex to rooted well-labelled trees with n edges and an additional global label in $\{+1, -1\}$.

Well-labelled trees

(Rooted) well-labelled tree = decorated rooted plane tree where

- Each vertex carries a positive integer label,
- There is a vertex of label 1;
- Along each edge, the difference in labels is at most 1.



[Schaeffer 15, Figure 1.18]

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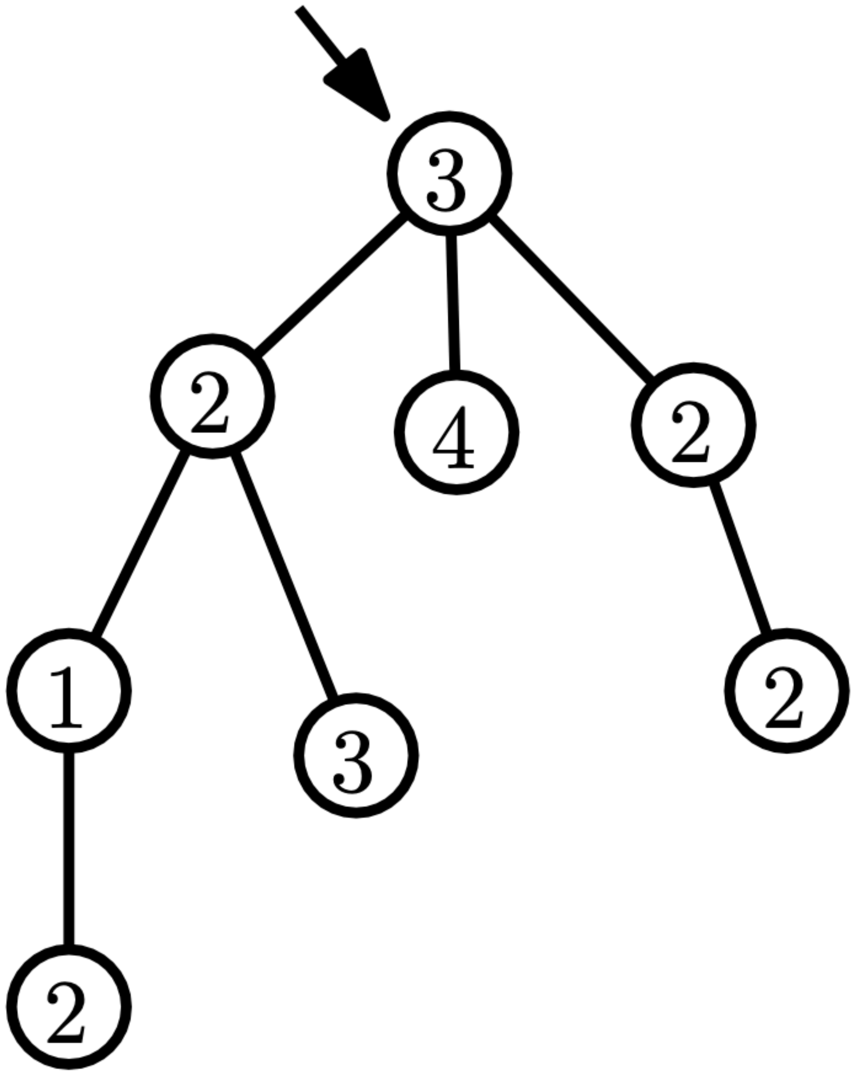

How many are there?

[Schaeffer 15, Figure 1.18]

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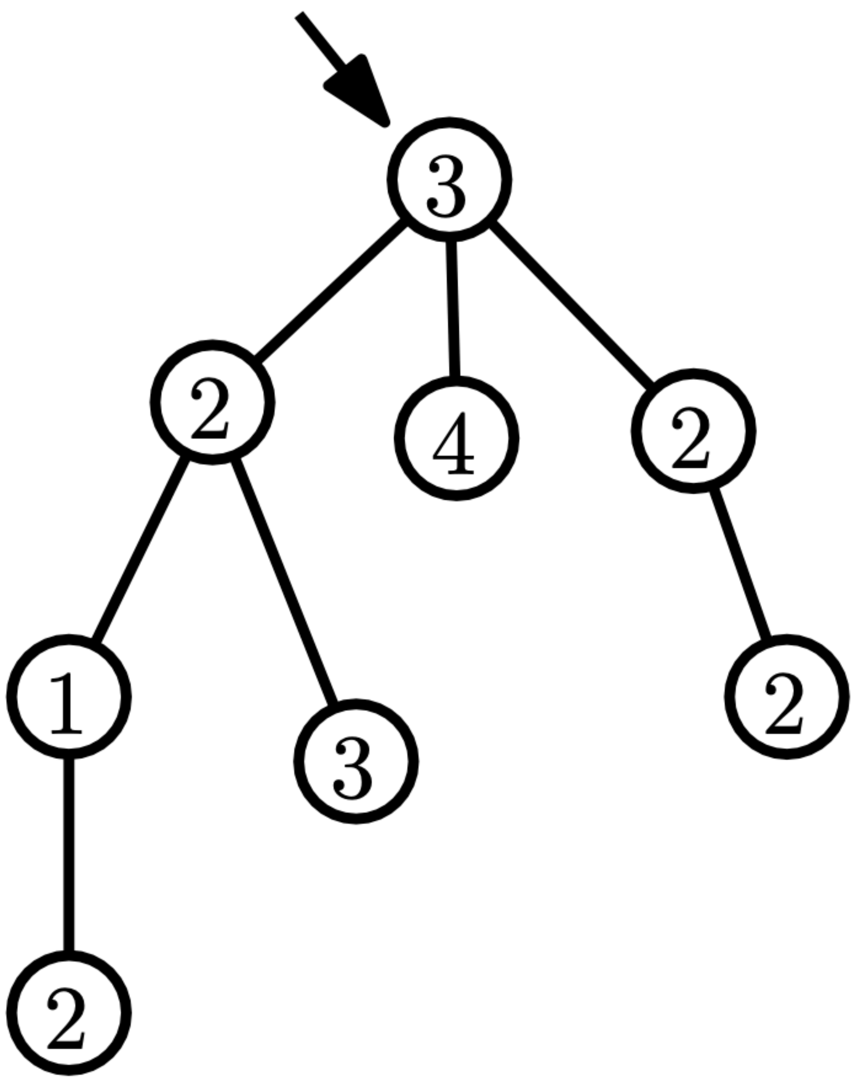

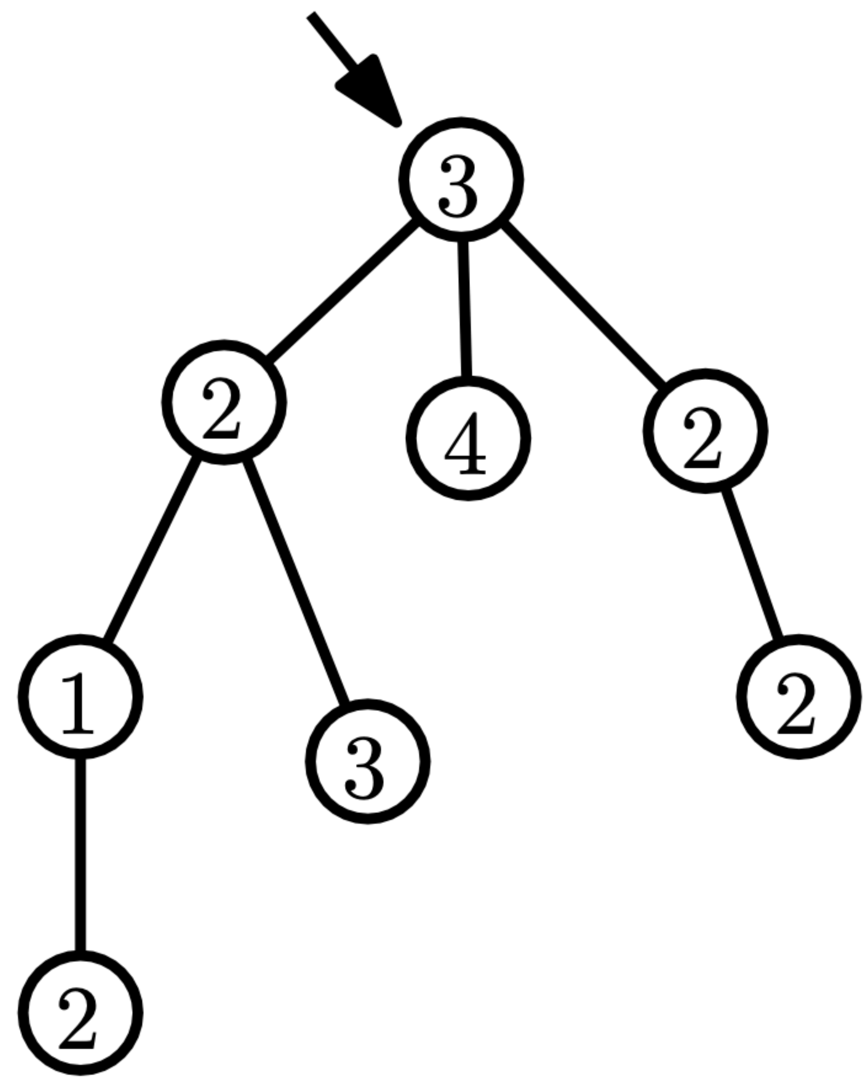

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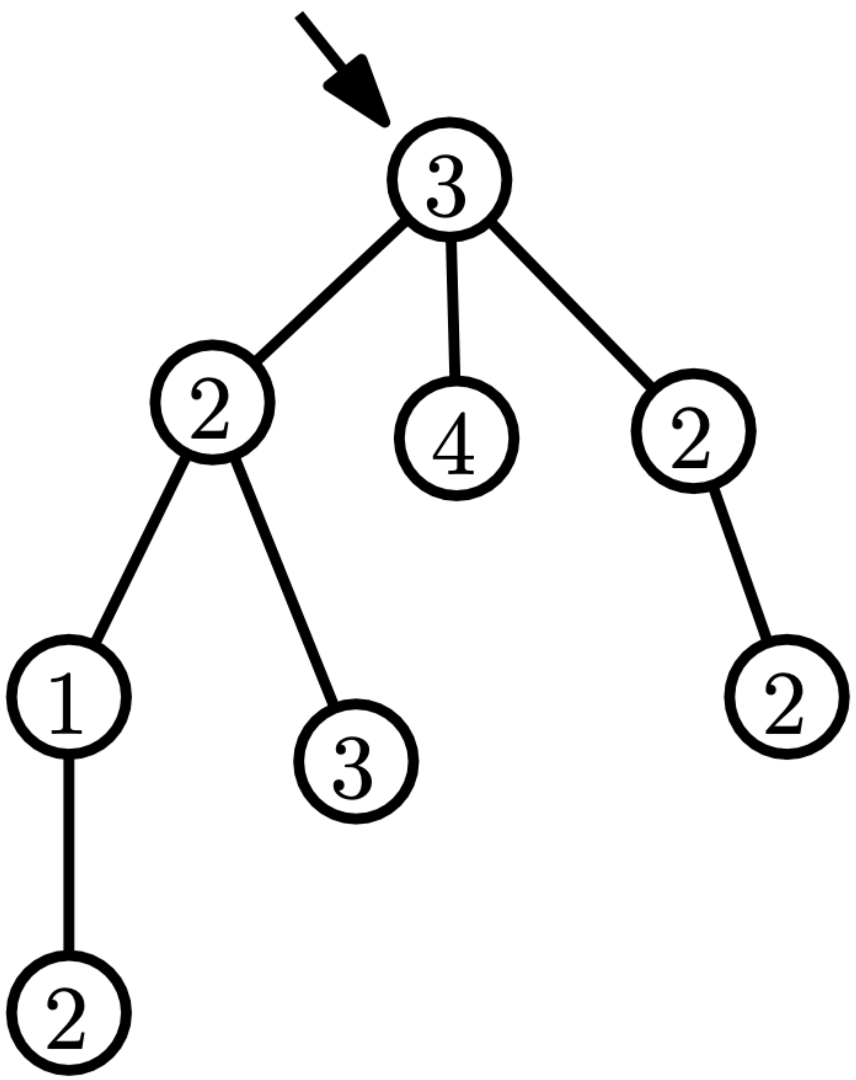

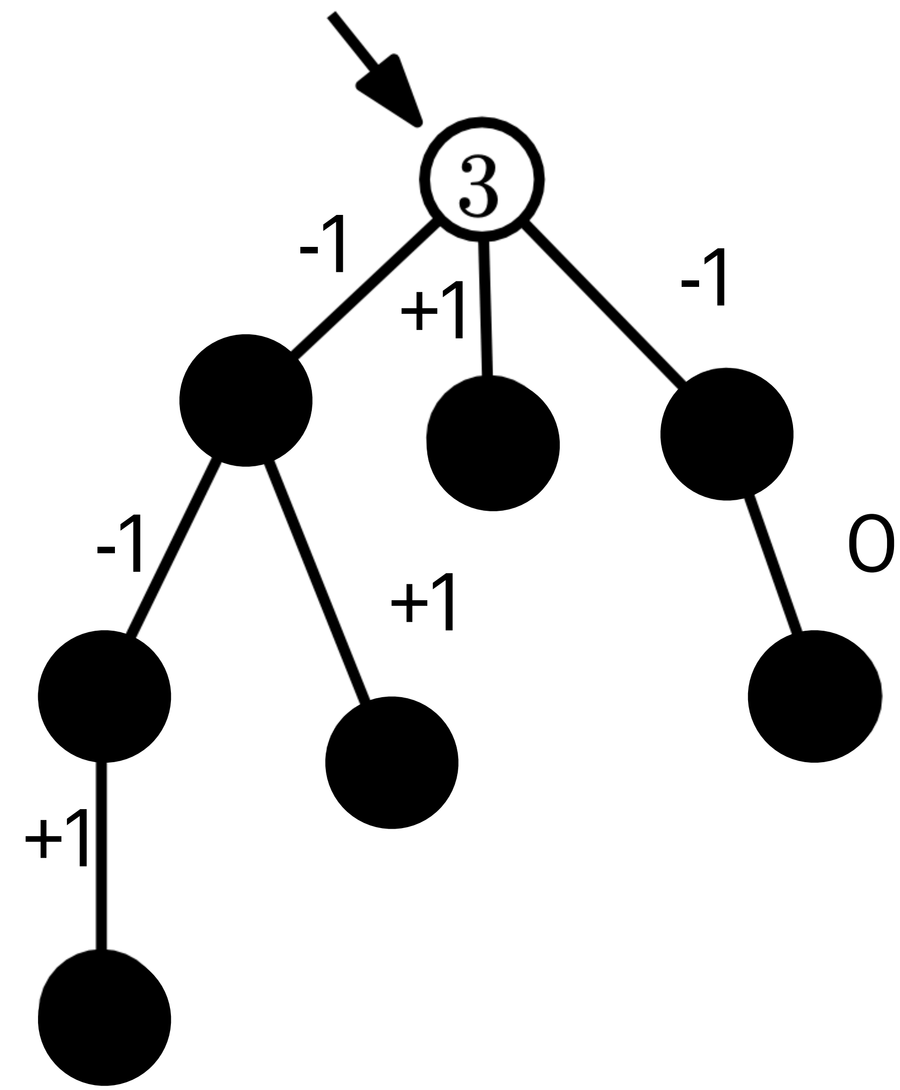

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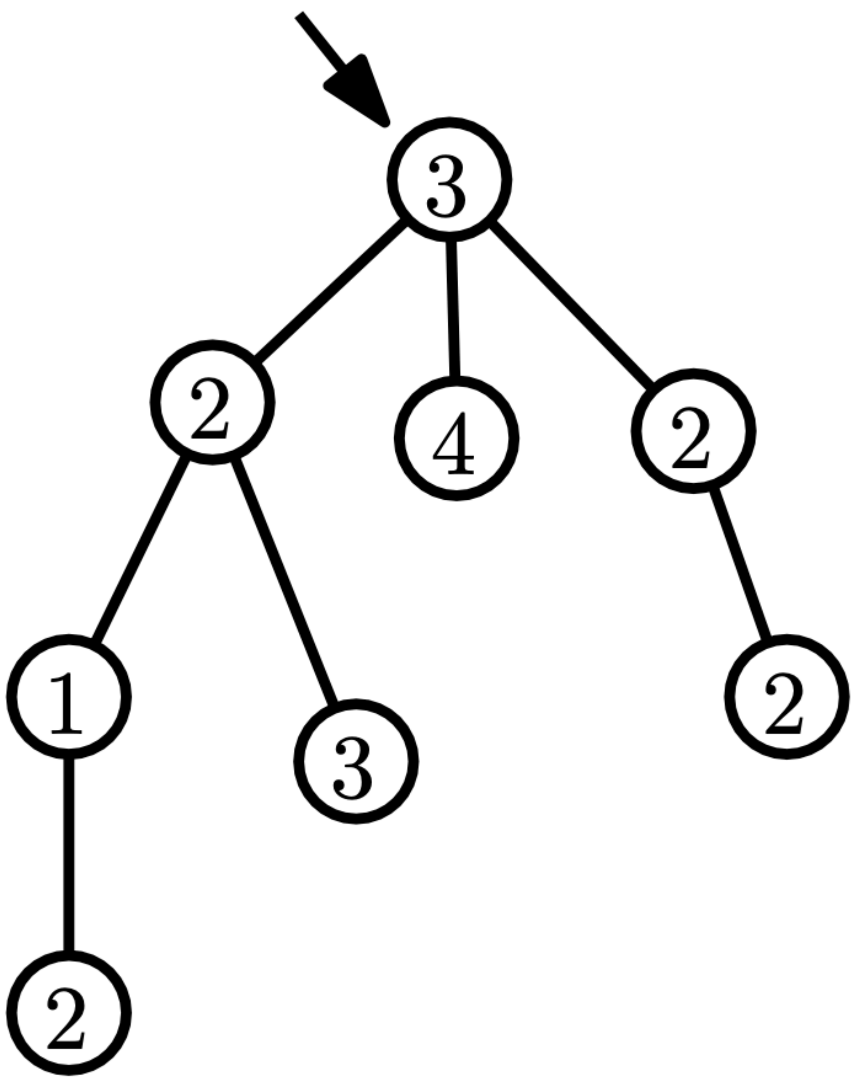

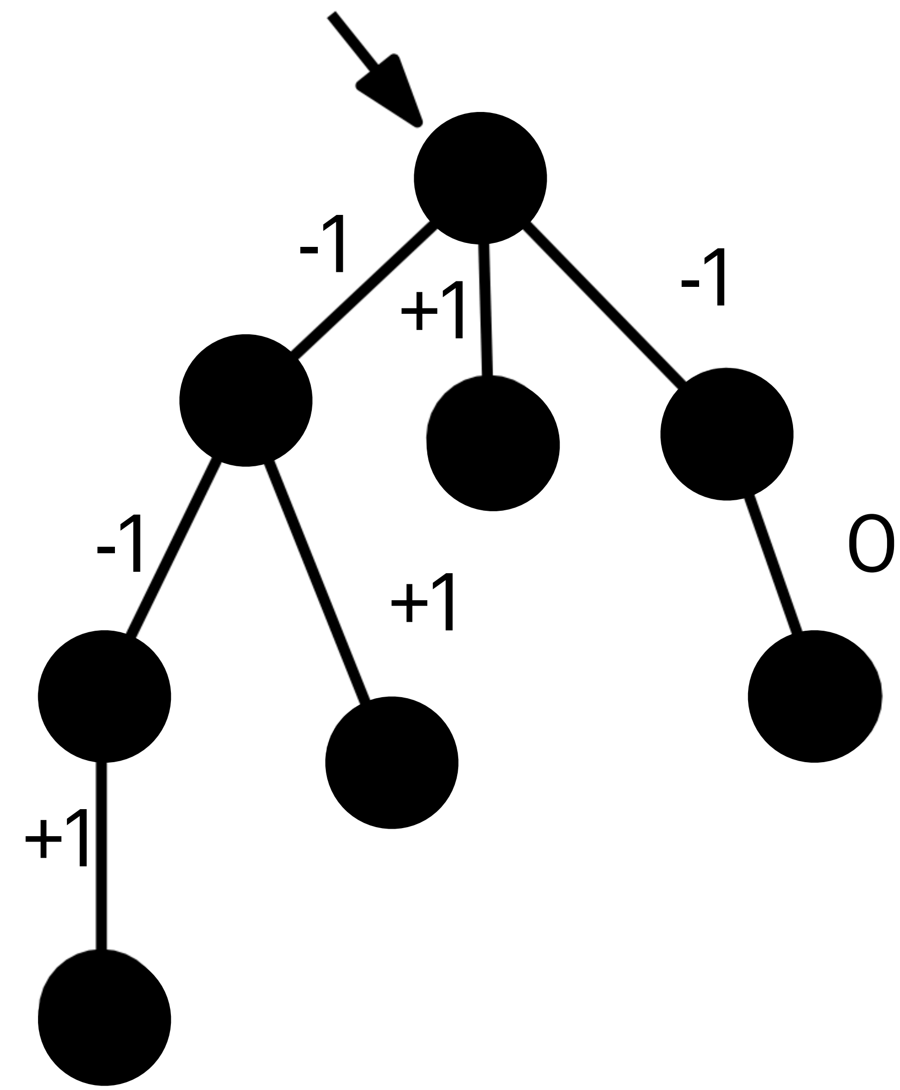

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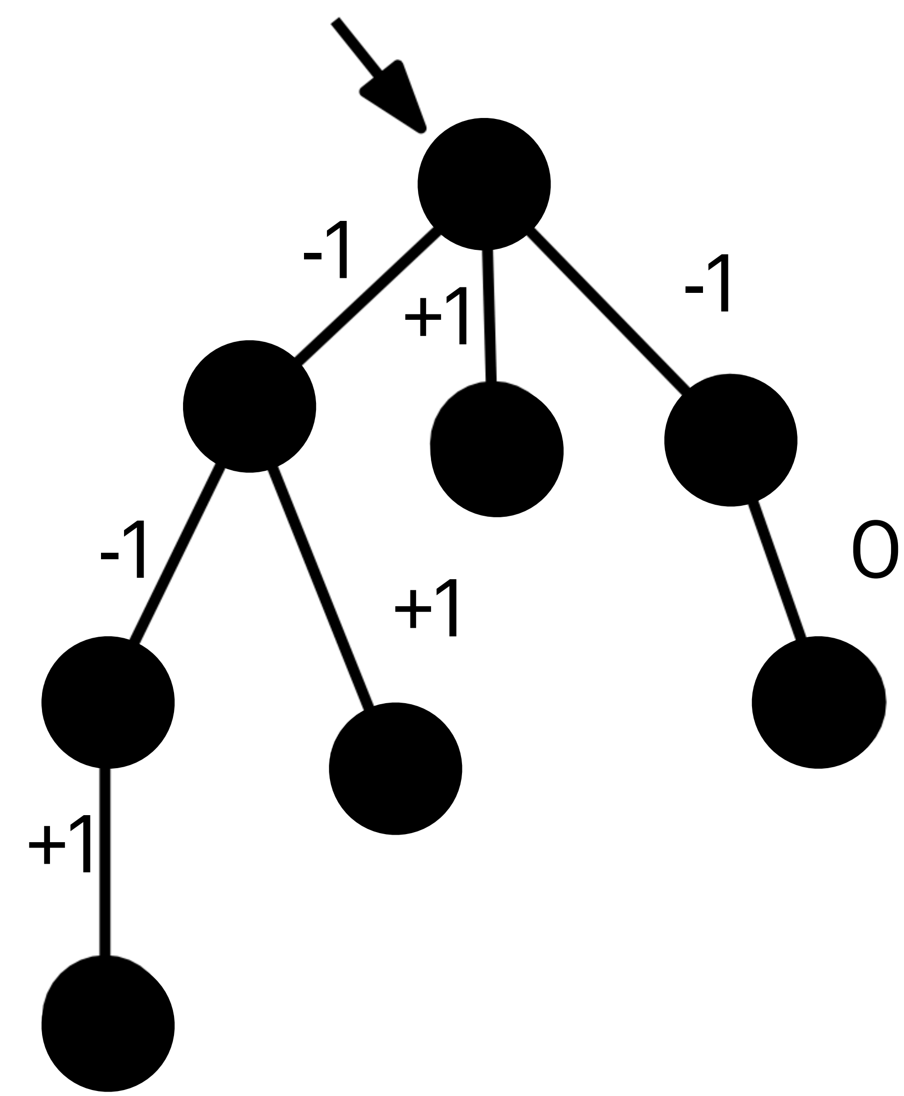
$$\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$$

rooted plane trees with n edges



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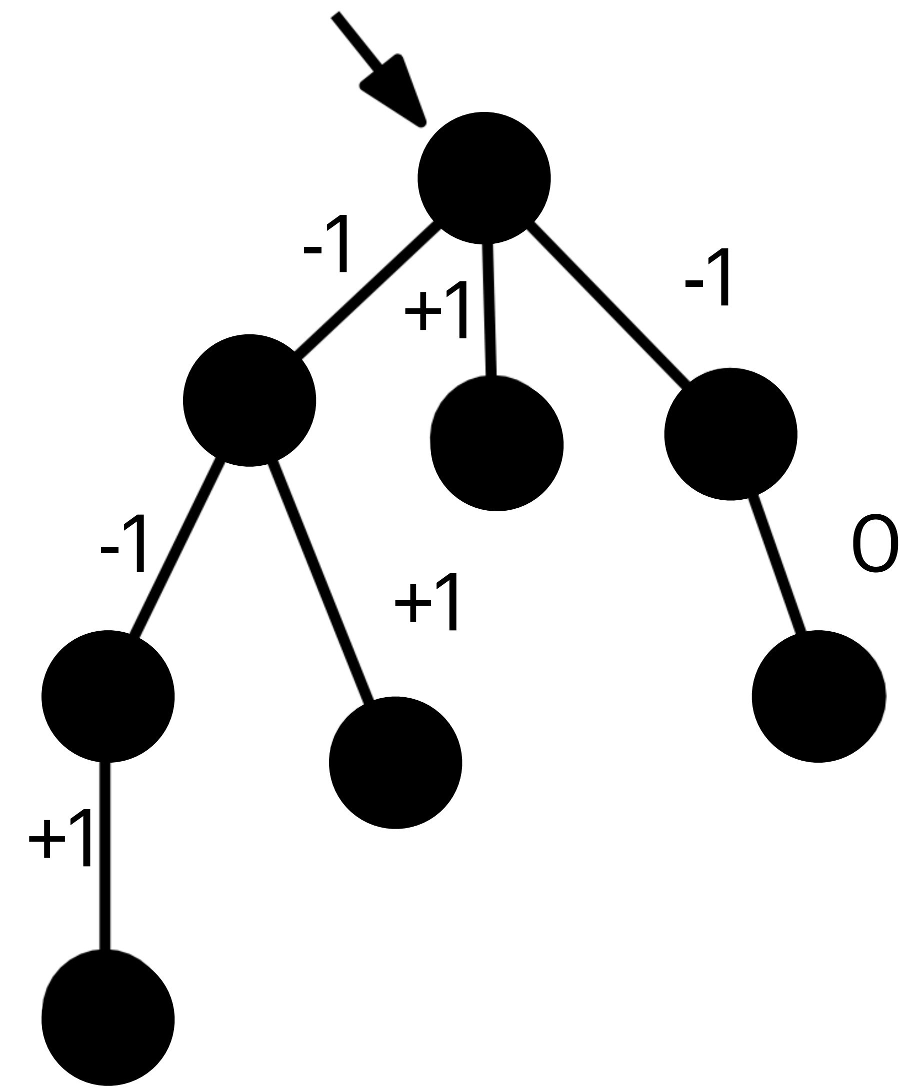

So

$$3^n \cdot \mathrm{Cat}_n = \frac{3^n}{n+1}\binom{2n}{n}$$

rooted well-labelled trees with n edges

(Bijective) enumeration of rooted maps

Theorem [Tutte 1963] Tutte's bijection sends rooted maps with n edges to rooted quadrangulations with n faces.

[Schaeffer 15, Theorem 10 and Corollary 7] The CVS bijection sends rooted quadrangulations with n faces and a marked vertex to rooted well-labelled trees with n edges and an additional global label in $\{+1, -1\}$.

(Bijective) enumeration of rooted maps

number of rooted maps with n edges

$$m_n = q_n$$

number of rooted quadrangulations with n faces

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Quadrangulation with n faces → $n+2$ vertices.

$\frac{3^n}{n+1}\binom{2n}{n}$ rooted well-labelled trees with n edges.

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$$(n+2)\, q_n = 2 \cdot \frac{3^n}{n+1}\binom{2n}{n}$$

(Bijective) enumeration of rooted maps

number of rooted maps with n edges

$$m_n = q_n$$

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$$m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} = \frac{2 \cdot 3^n \, (2n)!}{n! \, (n+2)!} .$$

$$(n+2) \, q_n = 2 \cdot \frac{3^n}{n+1} \binom{2n}{n}$$

(Bijective) enumeration of rooted maps

Tutte's + CVS bijections =>

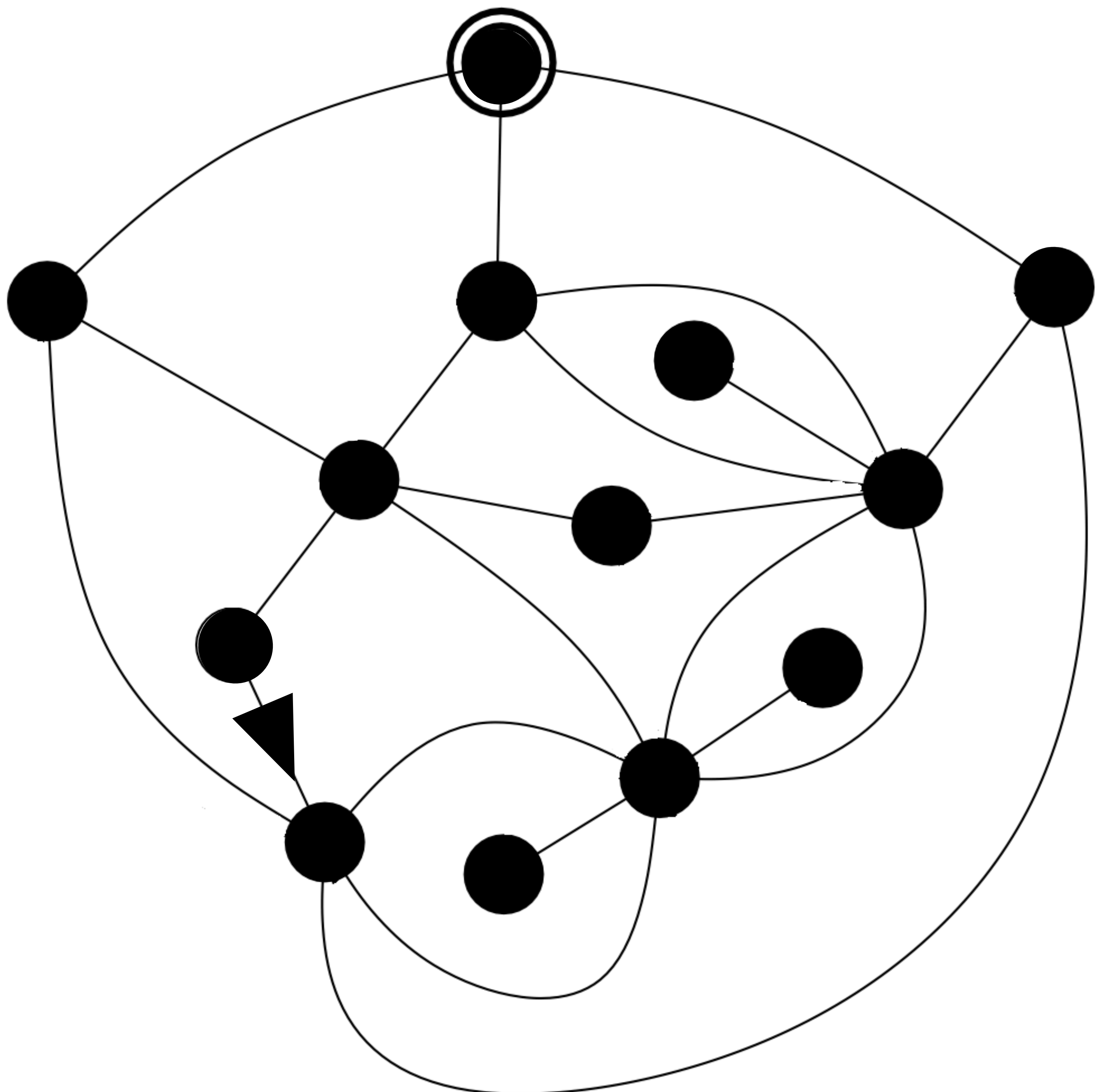
$$m_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} = \frac{2 \cdot 3^n \, (2n)!}{n! \, (n+2)!}.$$

CVS construction: step 1 (1/2)

[Schaeffer 15, Theorem 10 and Corollary 7] The CVS bijection sends rooted quadrangulations with n faces and a marked vertex to rooted well-labelled trees with n edges with an additional global label in $\{+1, -1\}$.

CVS construction: step 1 (1/2)

Take a rooted quadrangulations with n faces and a marked vertex.

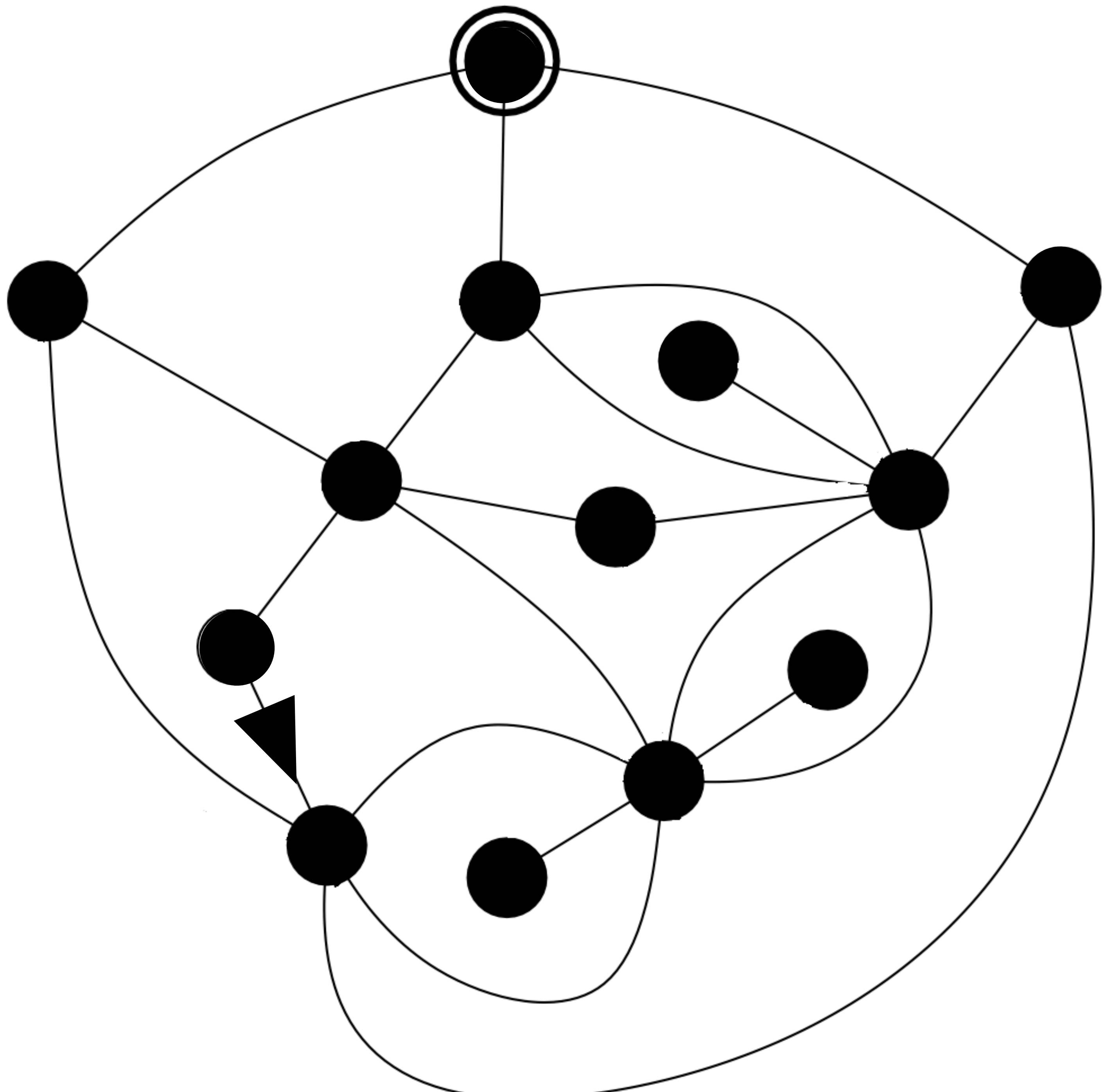


[Schaeffer 15, Figure 1.18]

CVS construction: step 1 (1/2)

Take a rooted quadrangulations with n faces and a marked vertex.

1. Label each vertex by its distance to the marked vertex.



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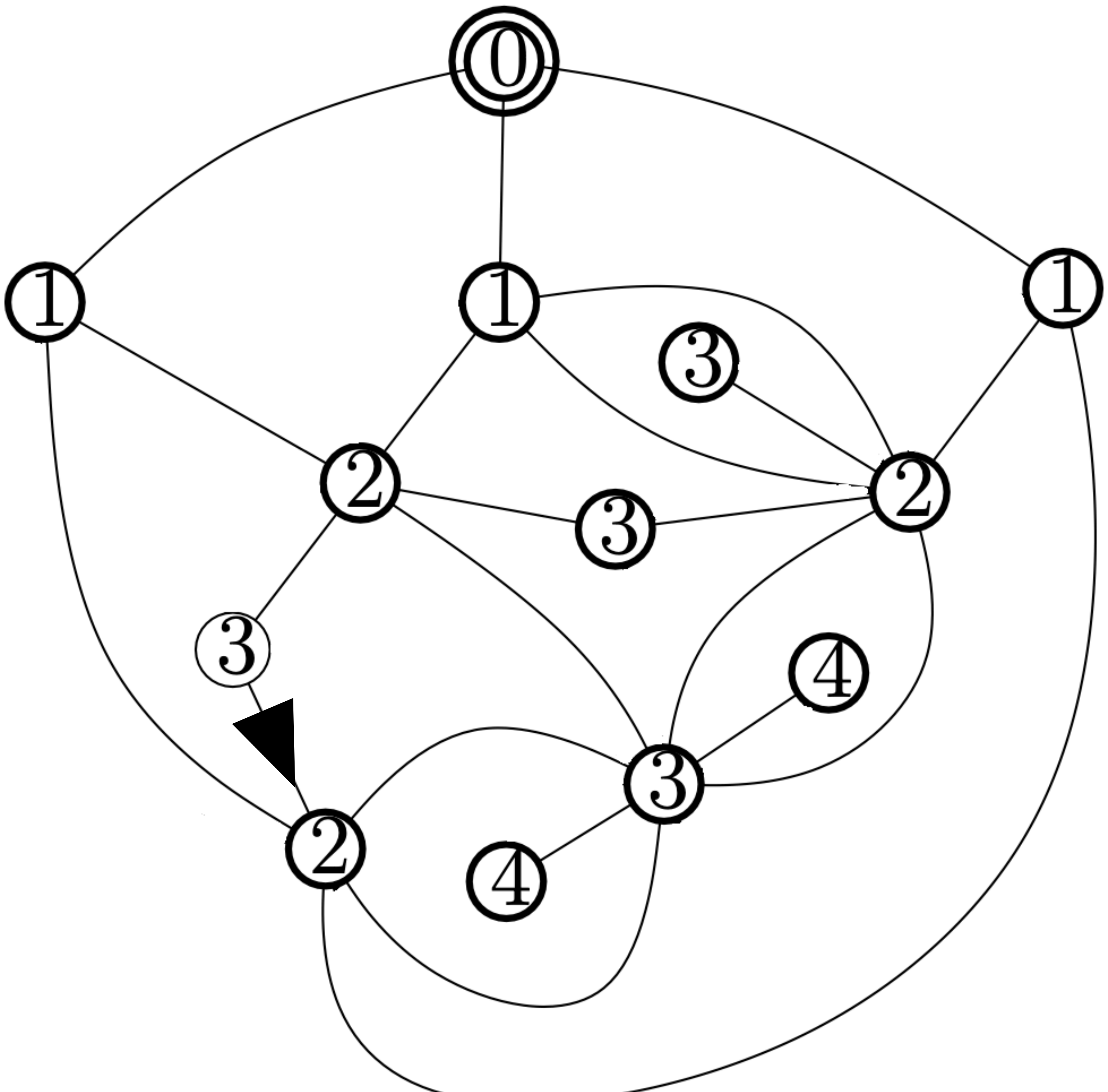


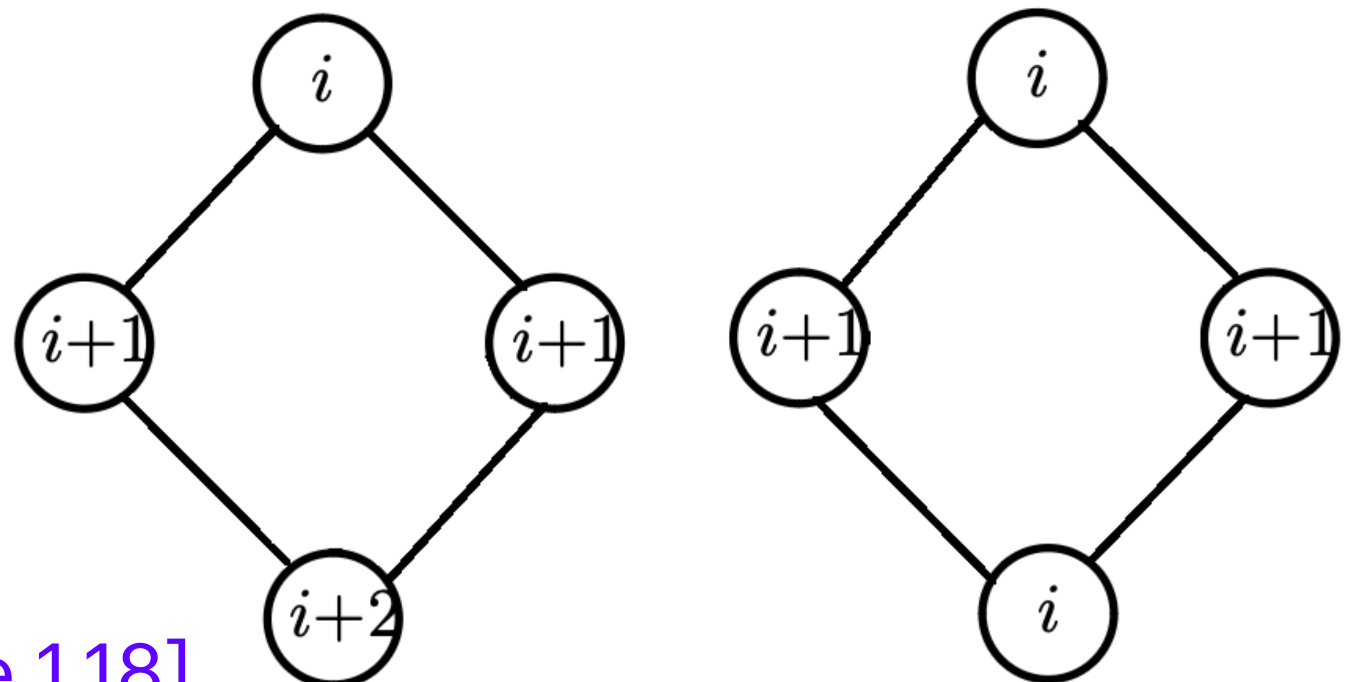
[Schaeffer 15, Figure 1.18]

CVS construction: step 1 (2/2)

After step 1., only two possibilities for the faces: along an edge, the labels vary

- At most by 1 (distance);
- By 1 mod 2 as quadrangulations are bipartite.

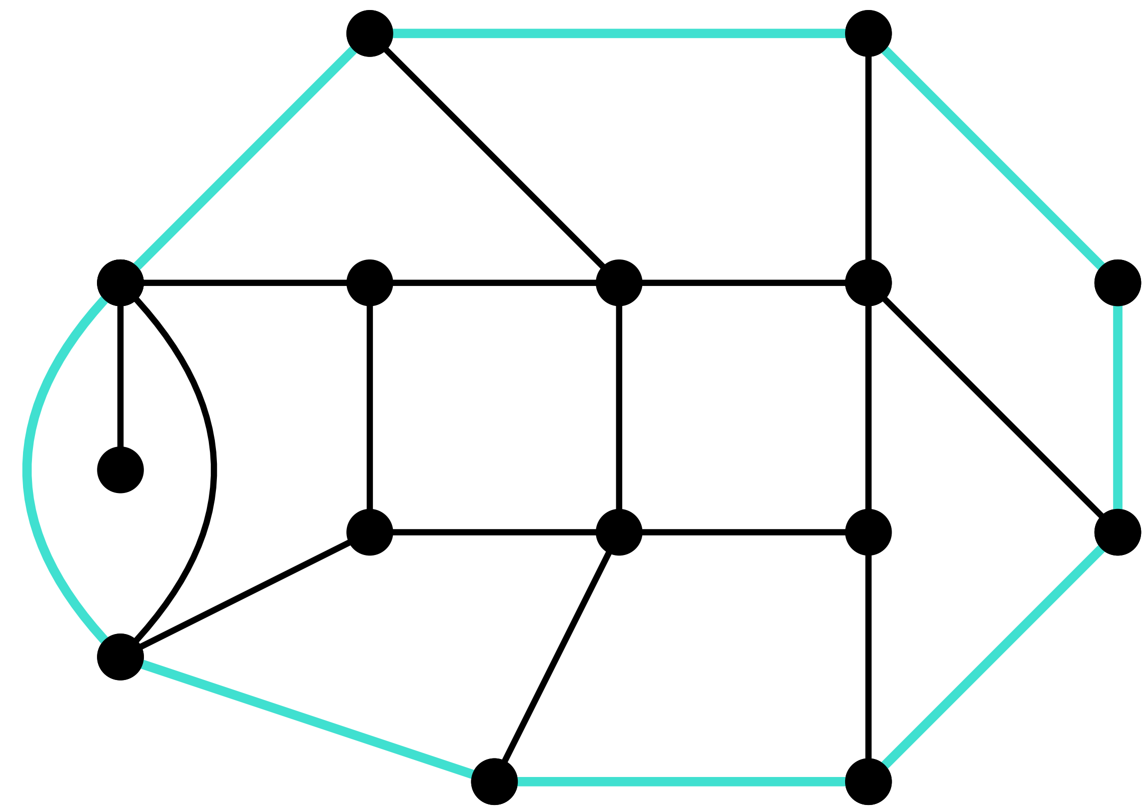
→ By exactly 1.



[Schaeffer 15, Figure 1.18]

<u>Proposition</u> All quadrangulations are bipartite = their vertices can be decomposed into black and white vertices such that there is no monochromatic edge.

CVS construction: step 1 (2/2)



Proof. All cycles have even length.

Cycle separates the sphere into 2 connected components.

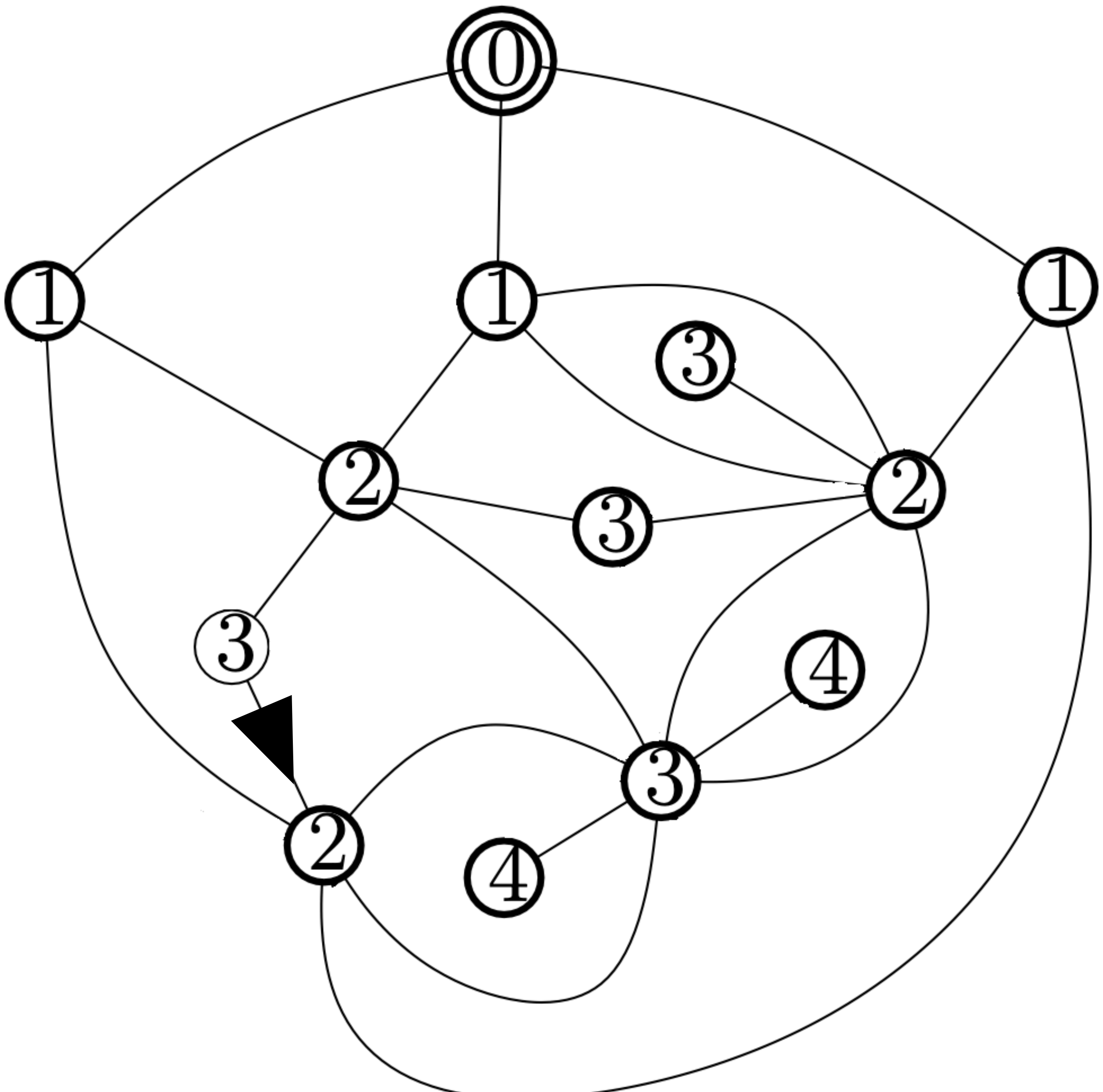
$$\sum_{f \in \text{side}} \deg(f) = 2\,|\text{black edges}| + |\text{blue edges}| = 0\ [2]\,.$$

<u>Proposition</u> All quadrangulations are bipartite = their vertices can be decomposed into black and white vertices such that there is no monochromatic edge.

CVS construction: complete edition

Take a rooted quadrangulations with n faces and a marked vertex.

1. Label each vertex by its distance to the marked vertex.

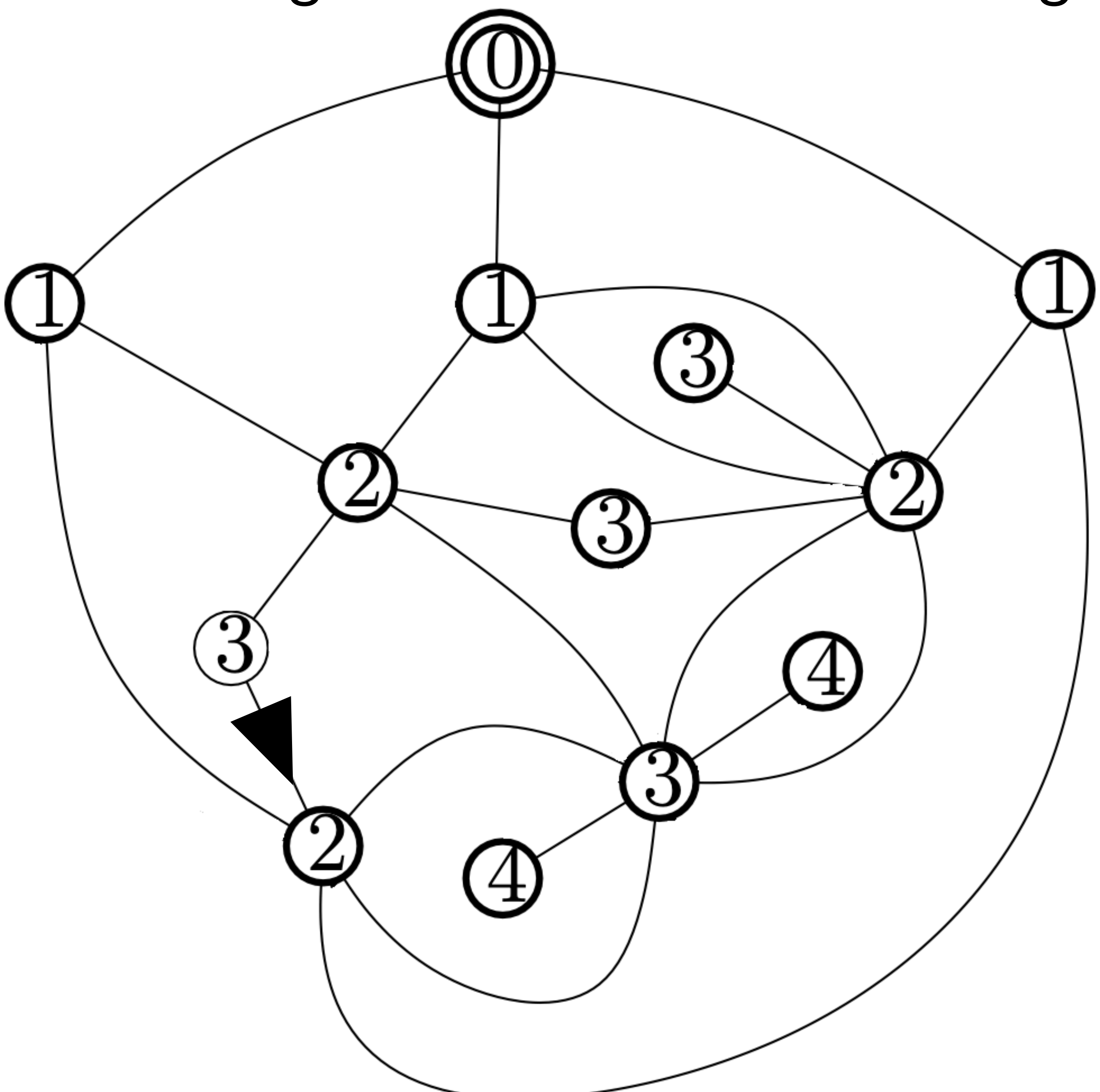


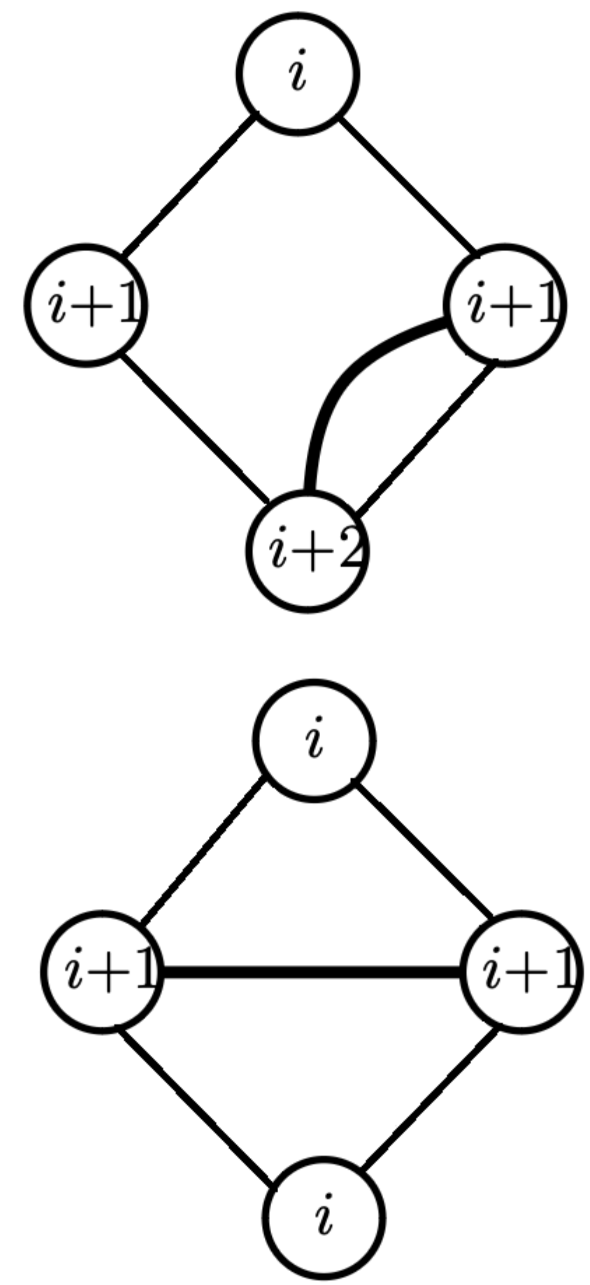
[Schaeffer 15, Figure 1.18]

CVS construction: complete edition

Take a rooted quadrangulations with n faces and a marked vertex.

1. Label each vertex by its distance to the marked vertex.
2. Add edges in each face according to the following rules.





[Schaeffer 15, Figure 1.18]

CVS construction: complete edition

Take a rooted quadrangulations with n faces and a marked vertex.

1. Label each vertex by its distance to the marked vertex.
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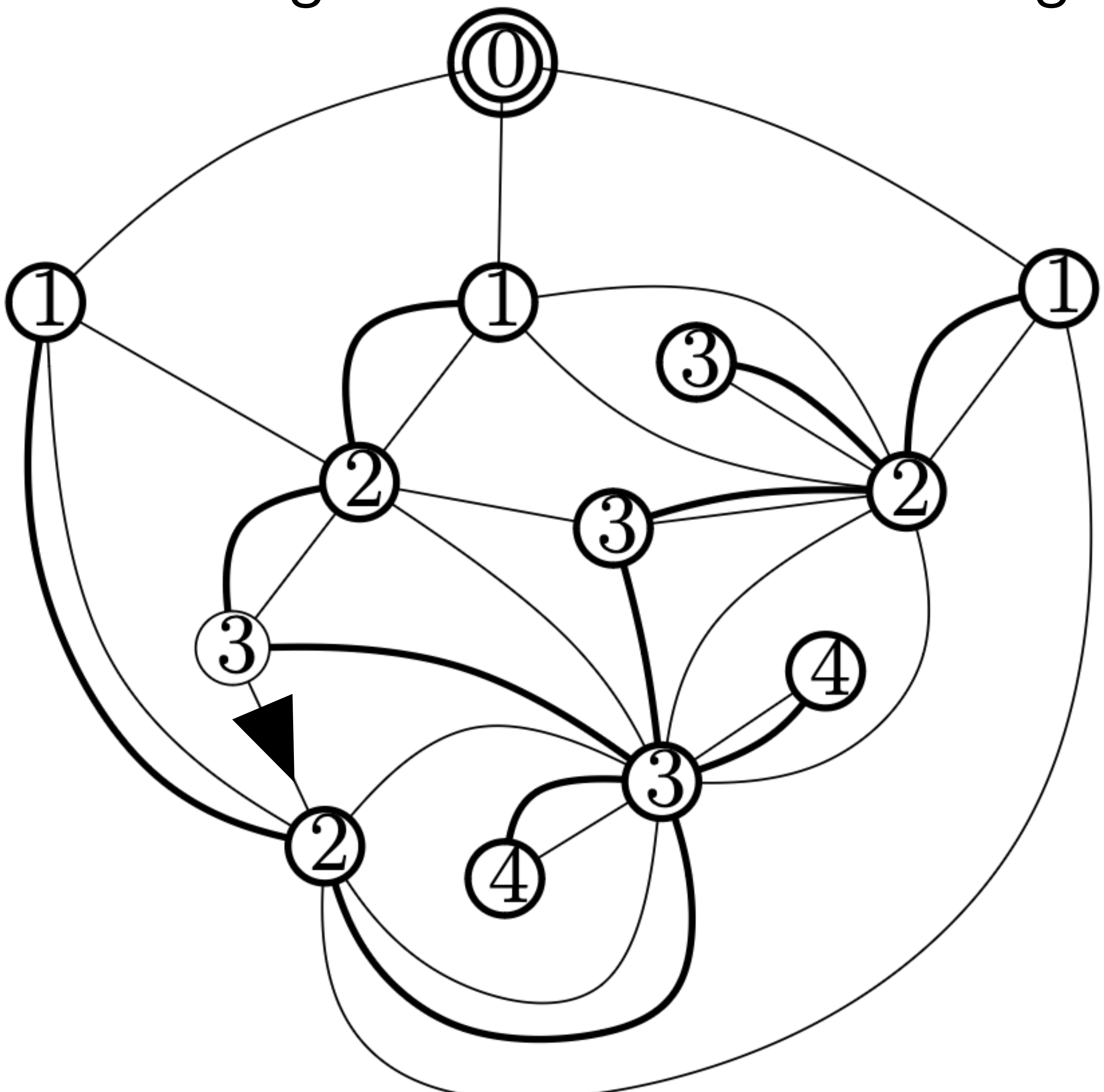

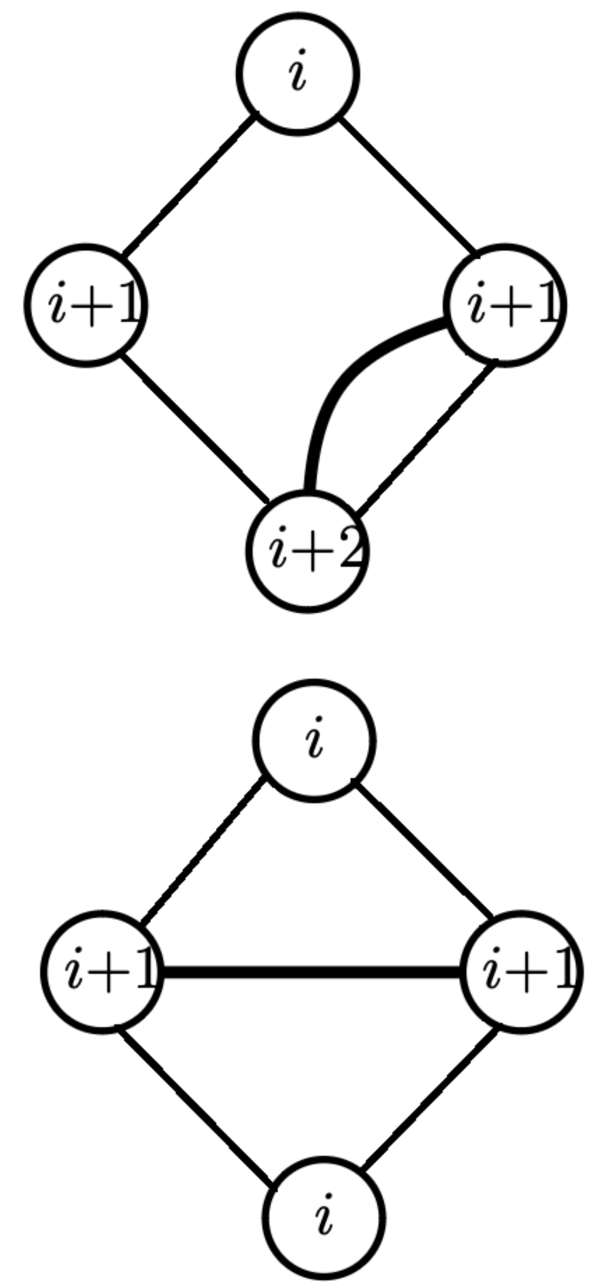

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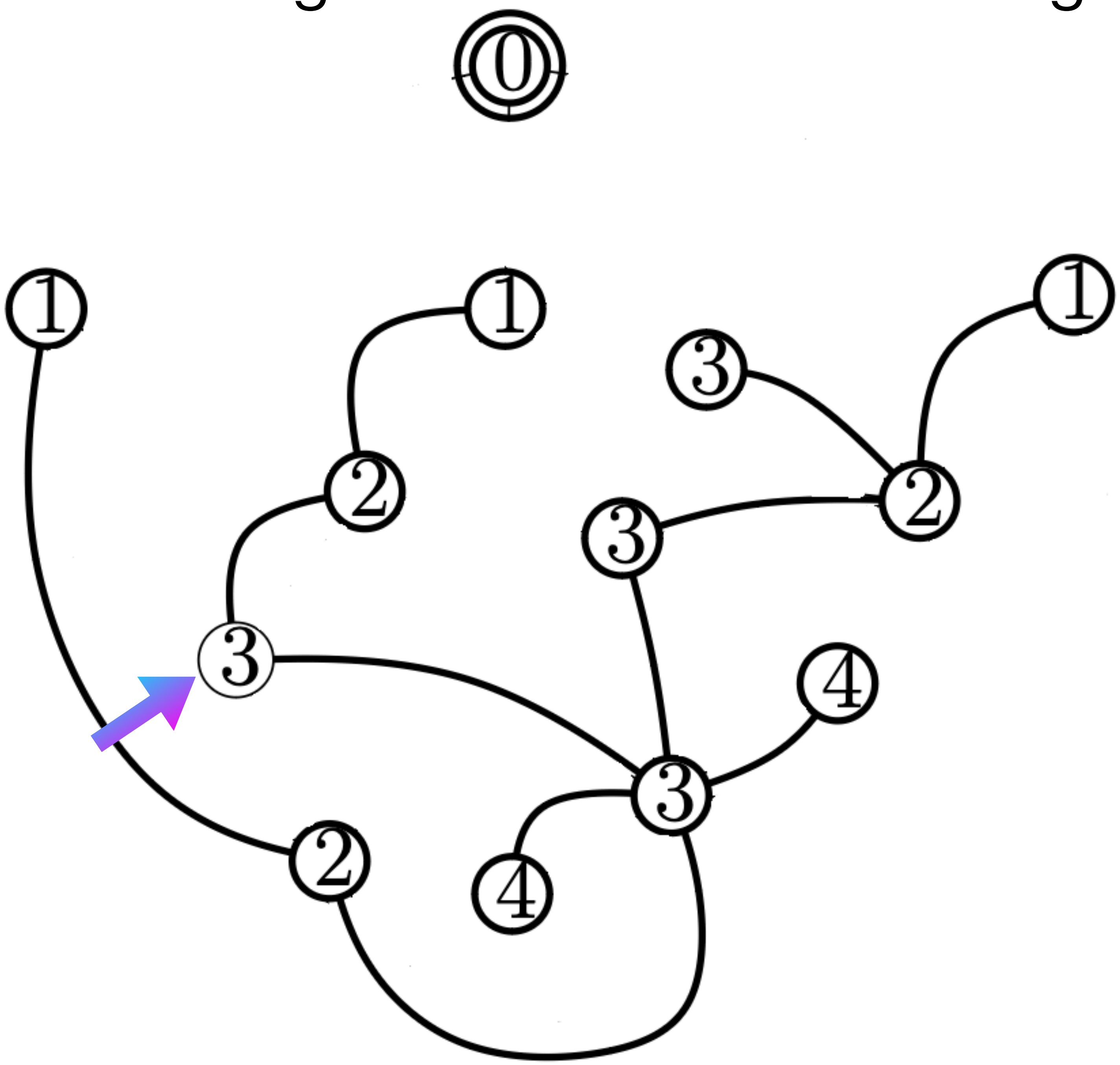

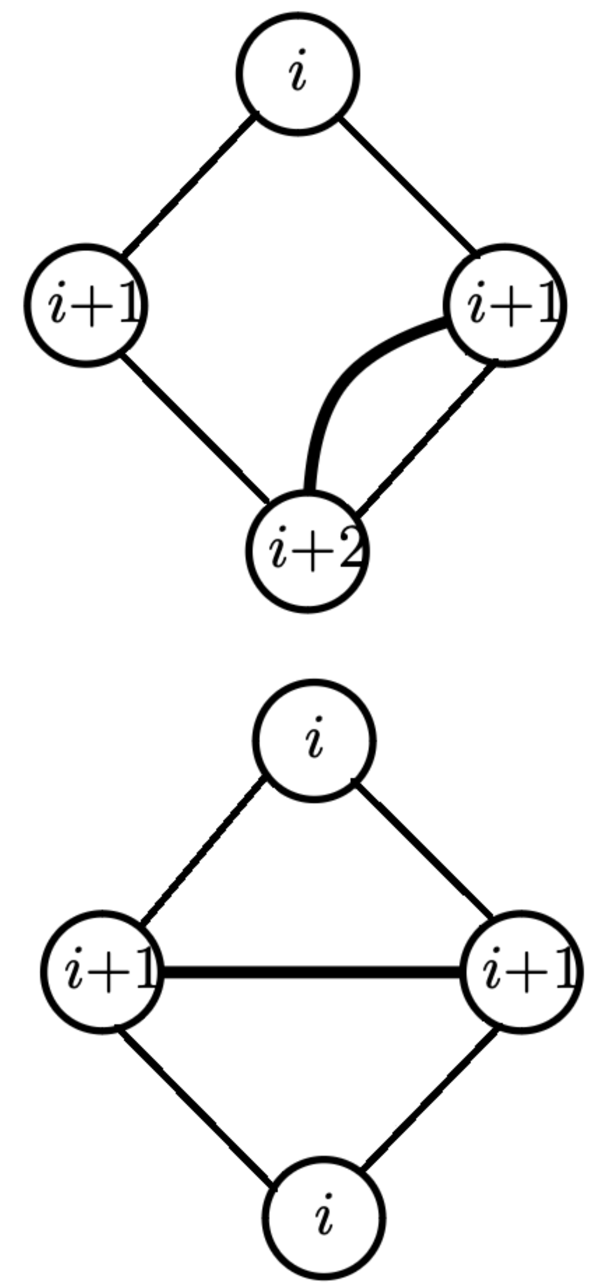

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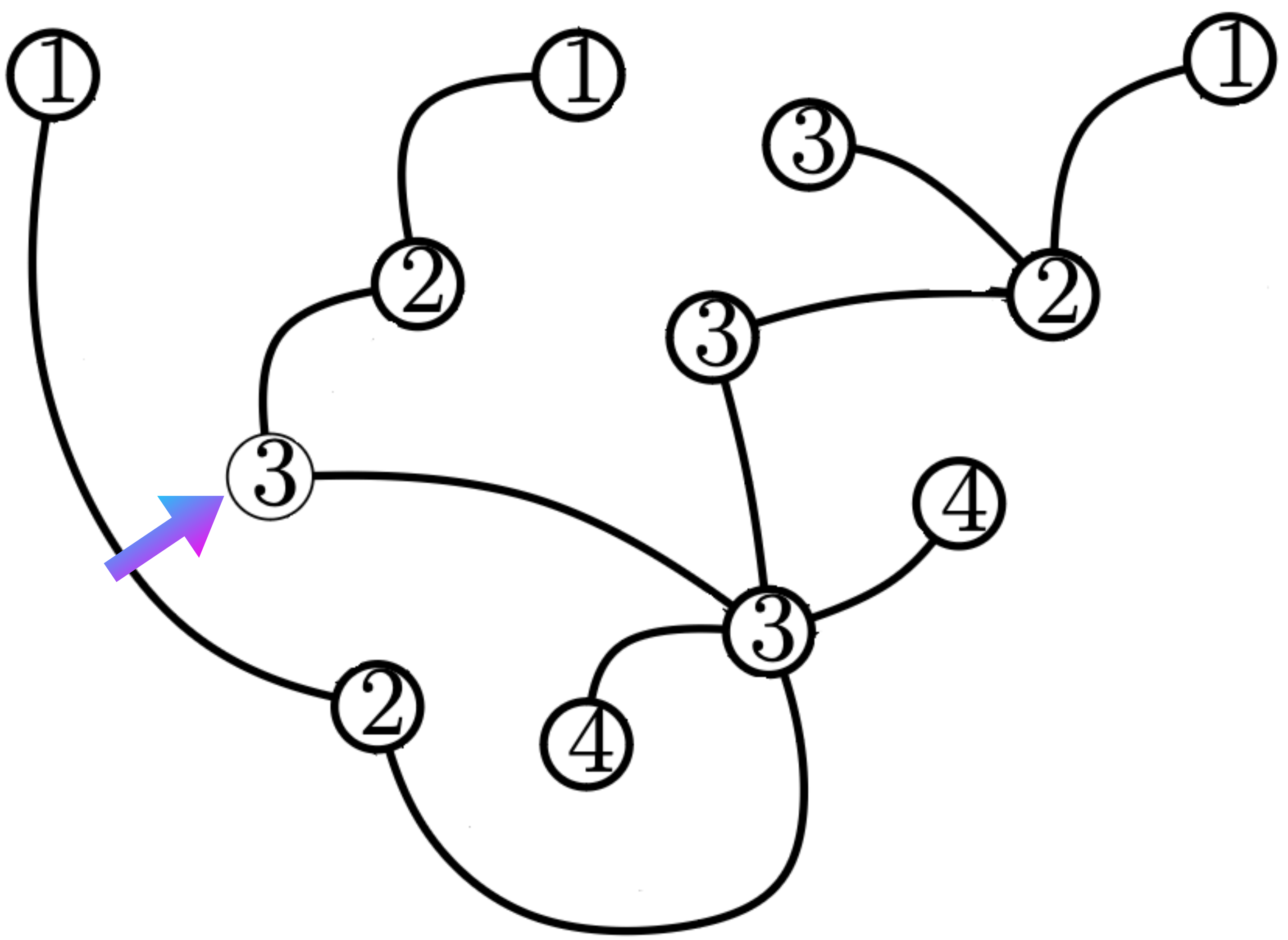
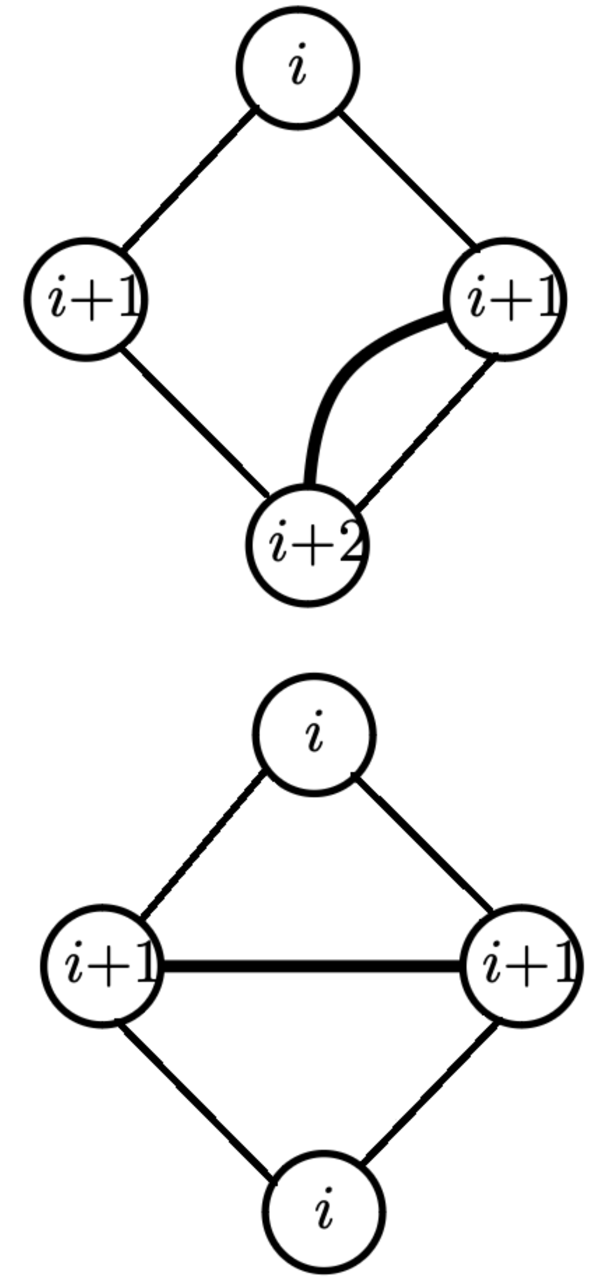




[Schaeffer 15, Figure 1.18]

Proof

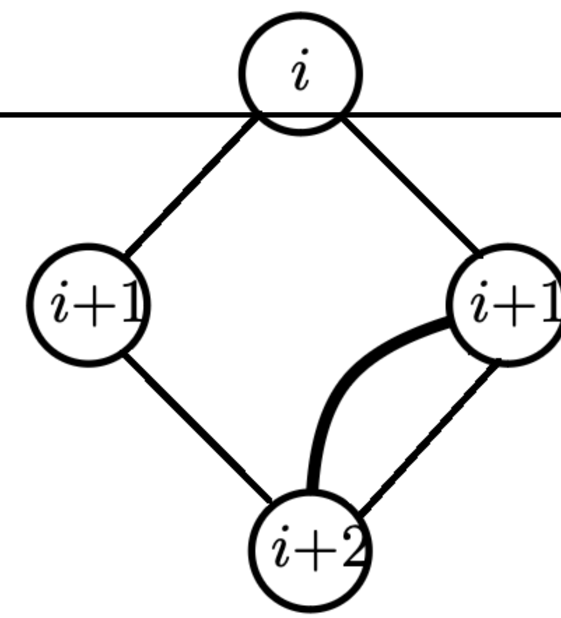
[Schaeffer 15, Theorem 10 and Corollary 7] The CVS bijection sends rooted quadrangulations with n faces and a marked vertex to rooted well-labelled trees with n edges with an additional global label in $\{+1, -1\}$.

Proof outline

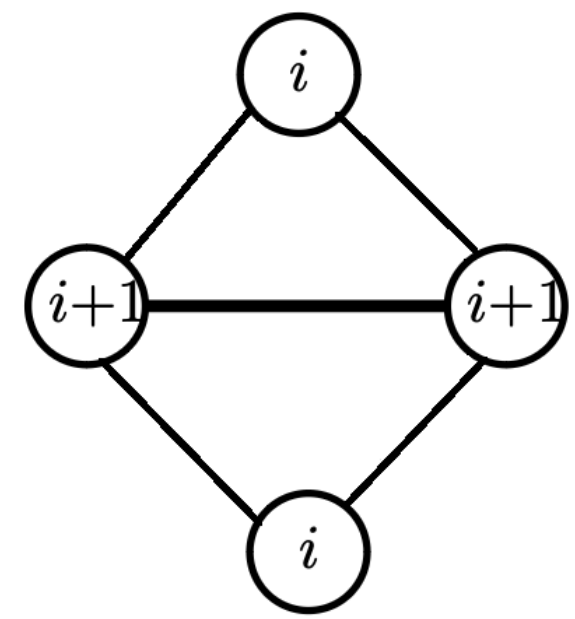
1. The construction produces a rooted well-labelled structure with n edges with an additional global label in $\{+1, -1\}$;
2. This structure is a tree.
3. The construction is invertible.

Proof: step 1

[Schaeffer 15, Theorem 10 and Corollary 7] The CVS bijection sends rooted quadrangulations with n faces and a marked vertex to rooted well-labelled trees with n edges with an additional global label in $\{+1, -1\}$.

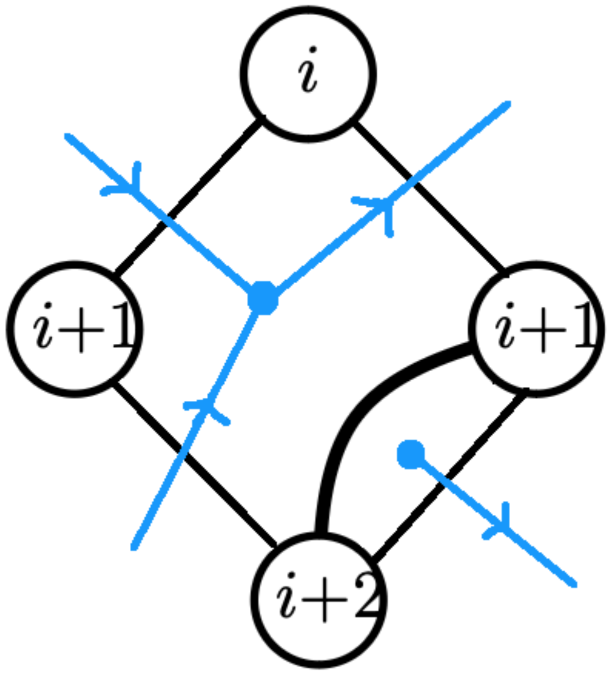


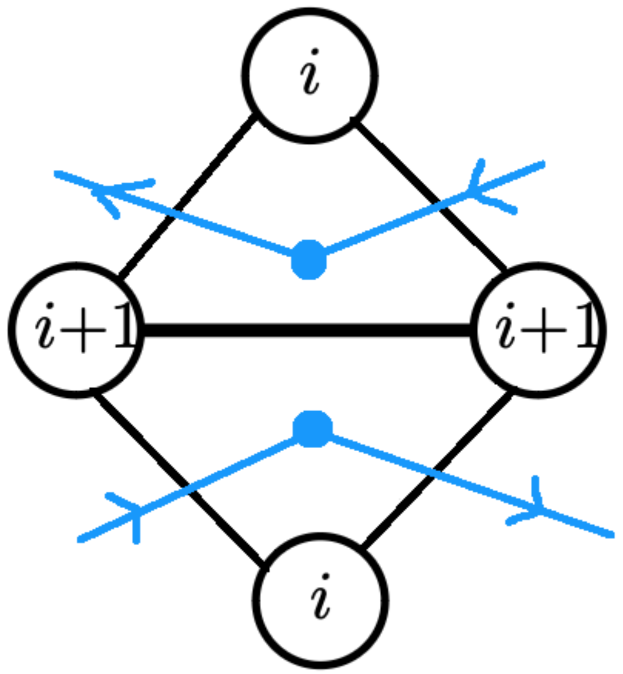
1. Clearly, the result of the construction :
- Has 1 edge per face of the initial quadrangulation;
- Is well-labelled:
  - Vertex-labelled by positive integers;
  - At least 1 node labelled 1.



Proof: step 2 (1/3)

2. The result is a tree:





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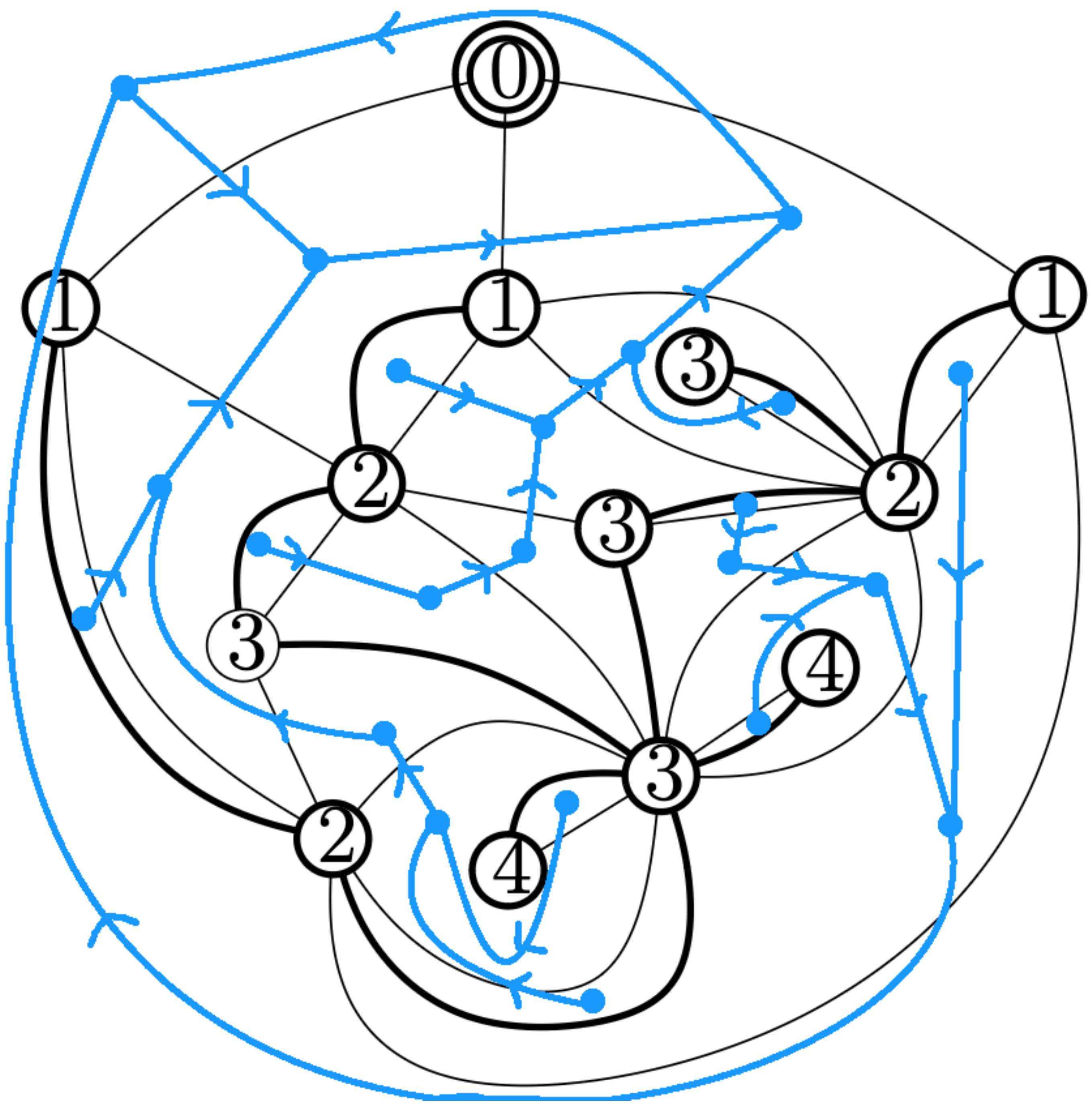

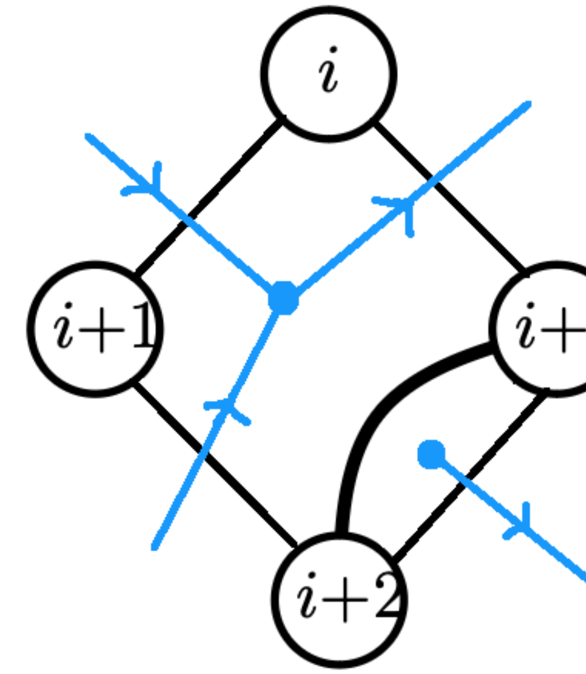

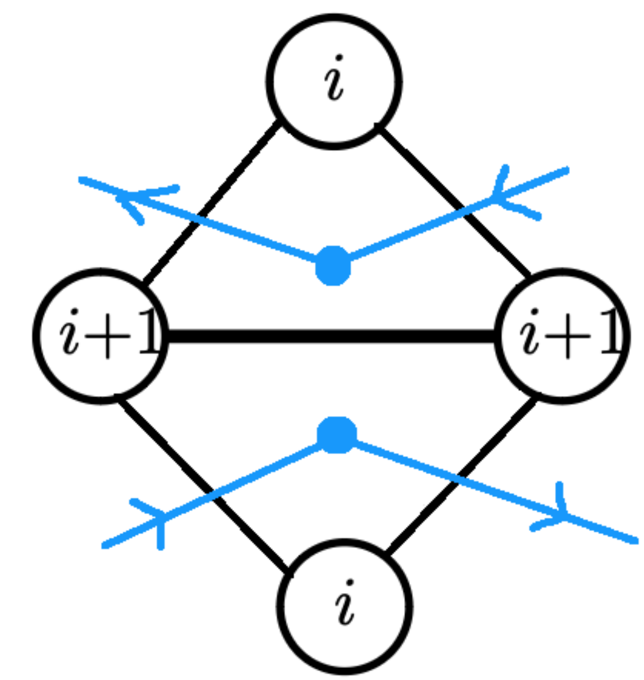

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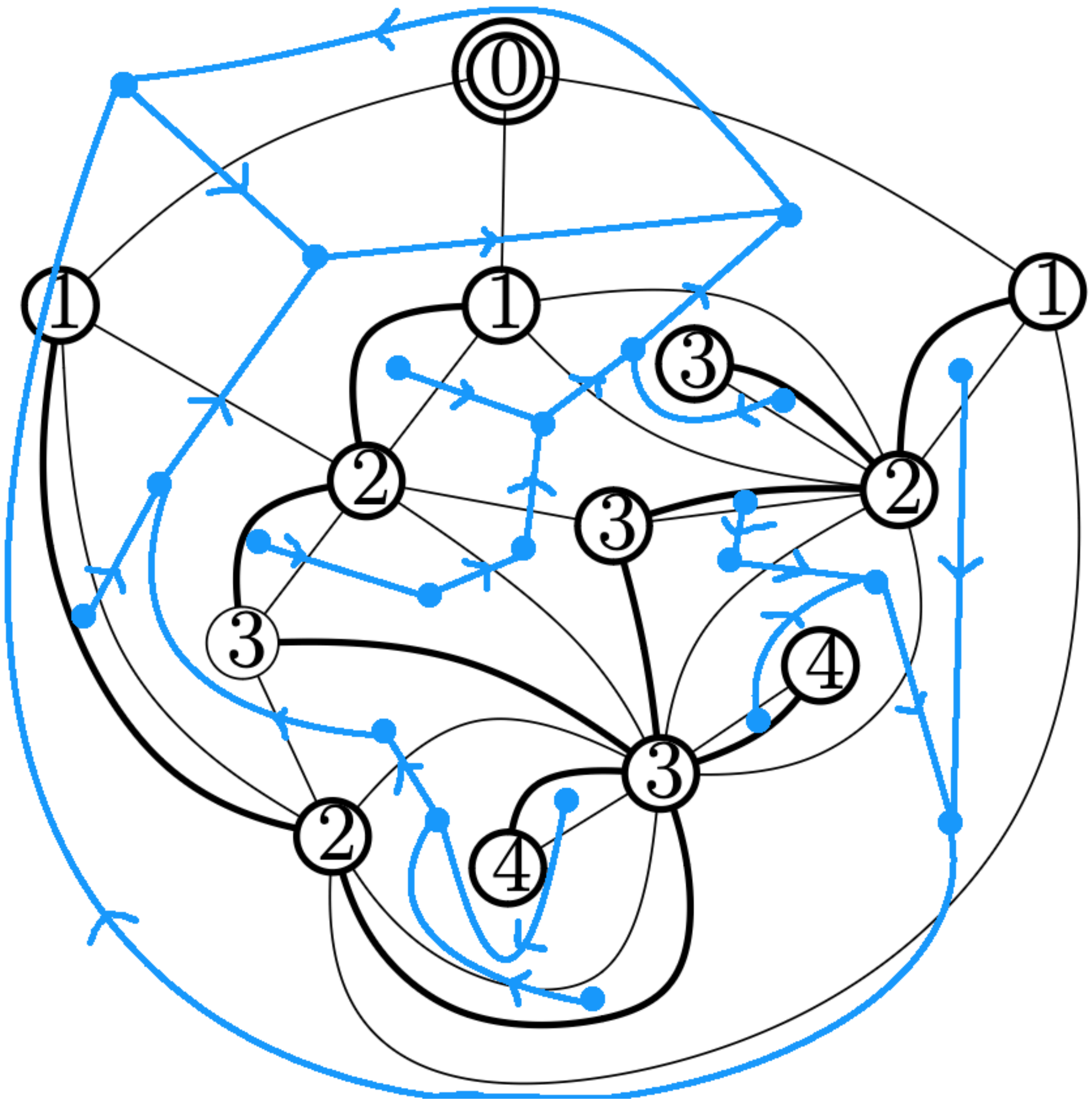
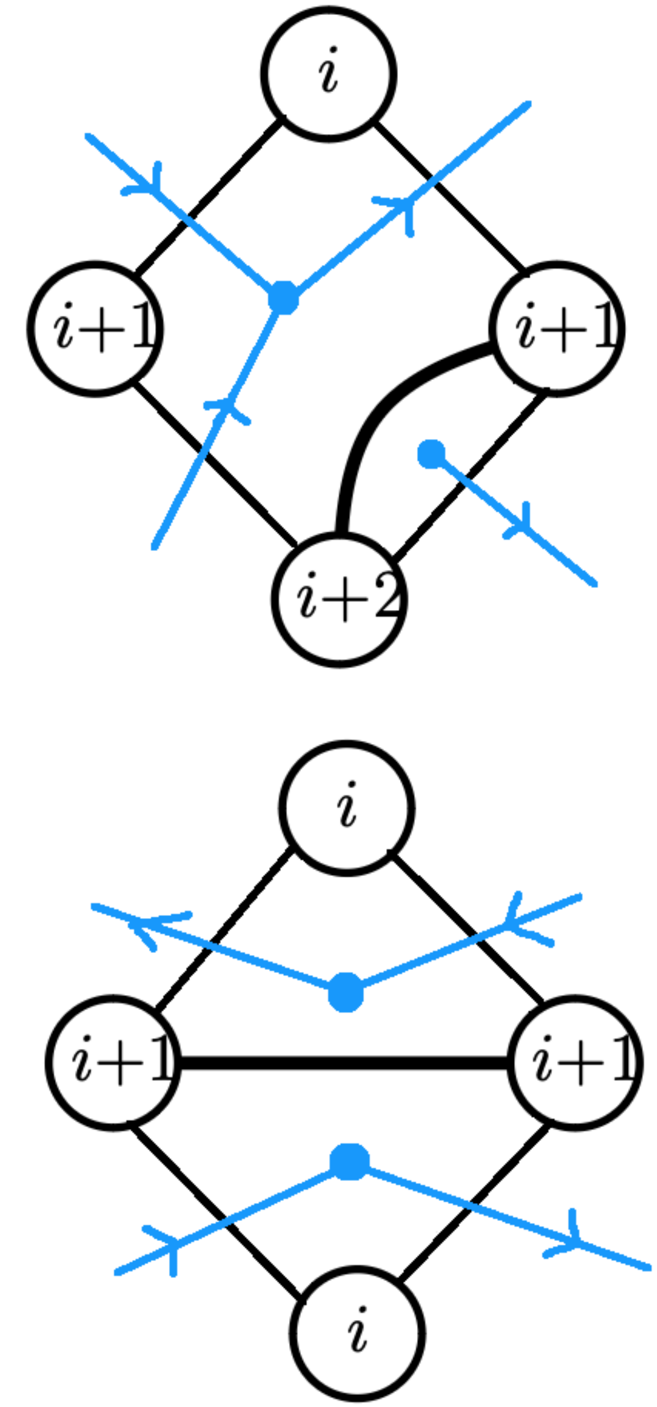
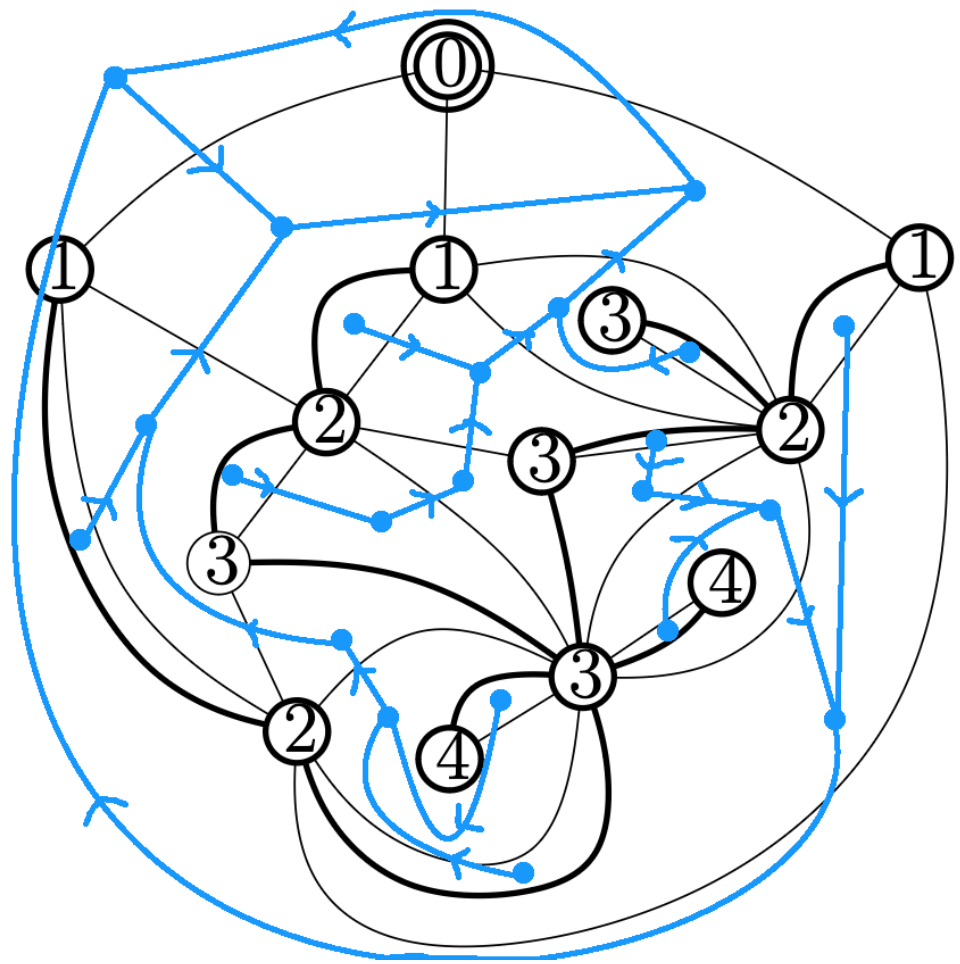
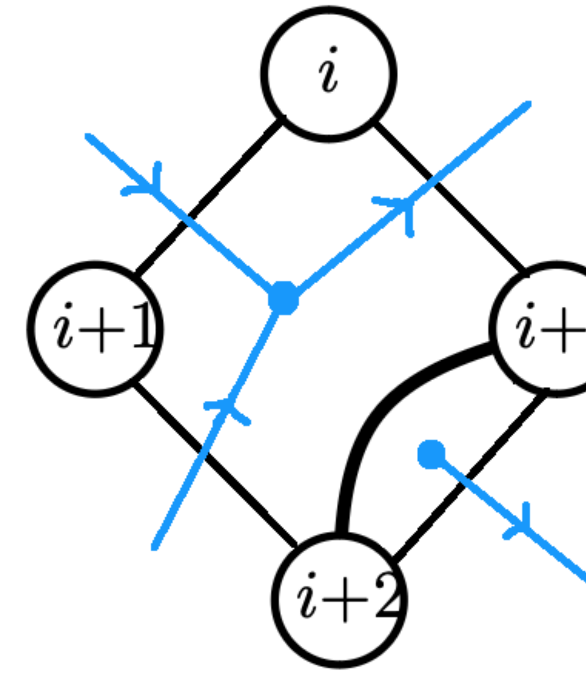




Labels on the left of blue edges are non-increasing.
There is one outgoing edge per blue vertex.

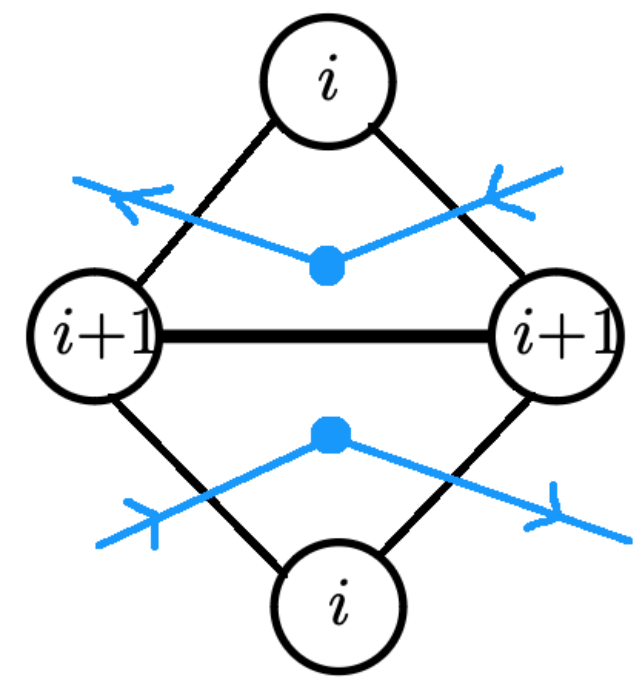
Proof: step 2 (2/3)





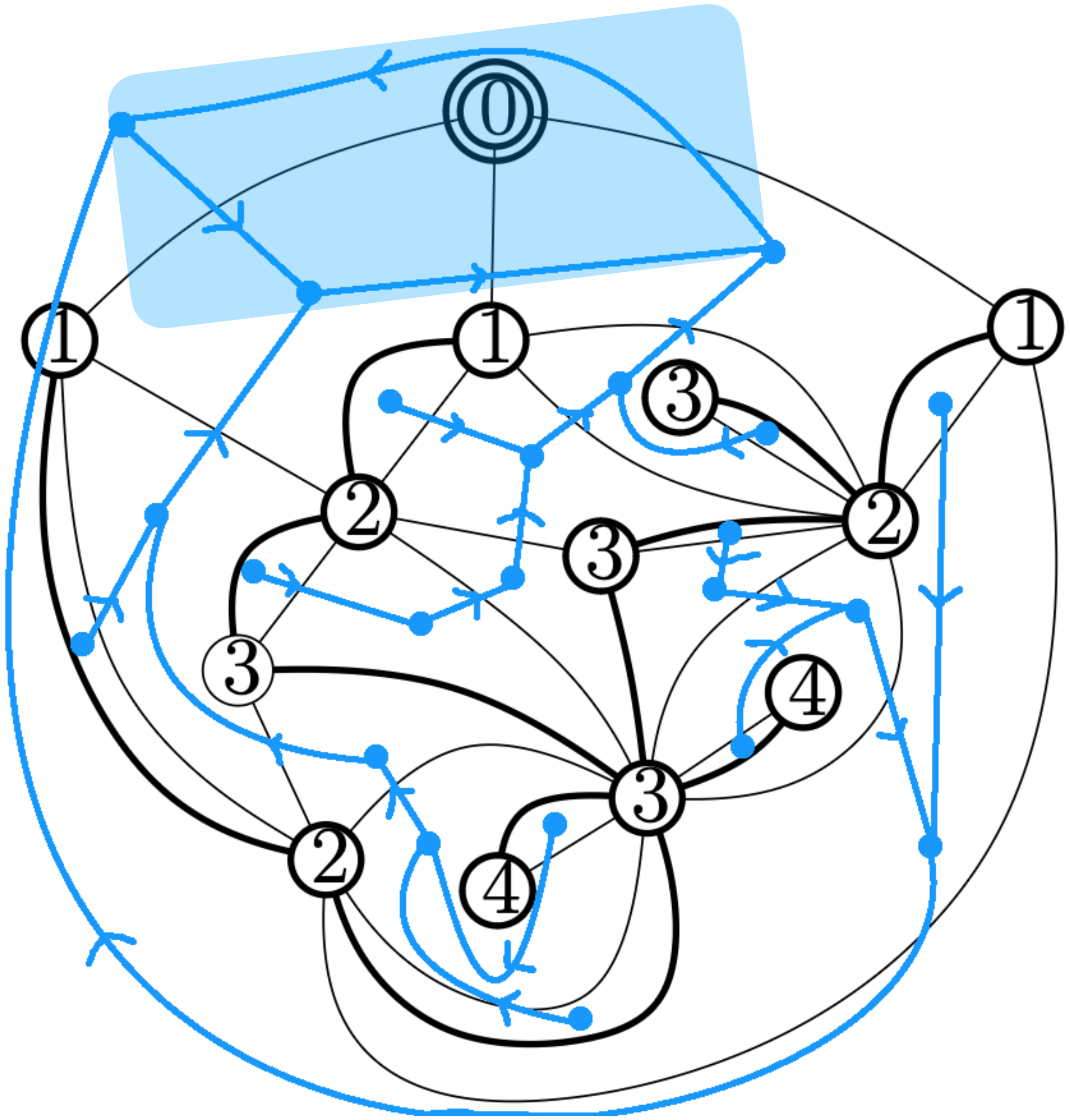
A blue cycle
- Is oriented (1 outgoing edge per vertex);
- Constant label on its left.

=> Is around 1 vertex.



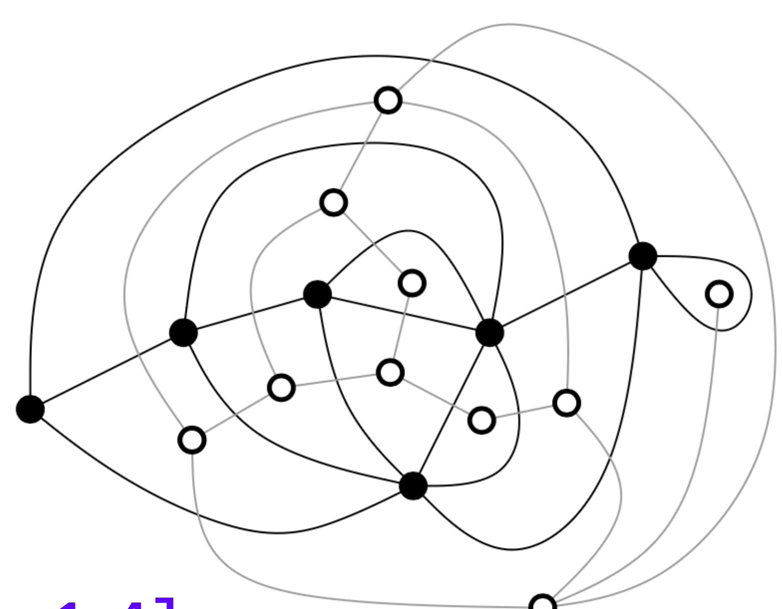
=> There is exactly 1 blue cycle, around the marked vertex.

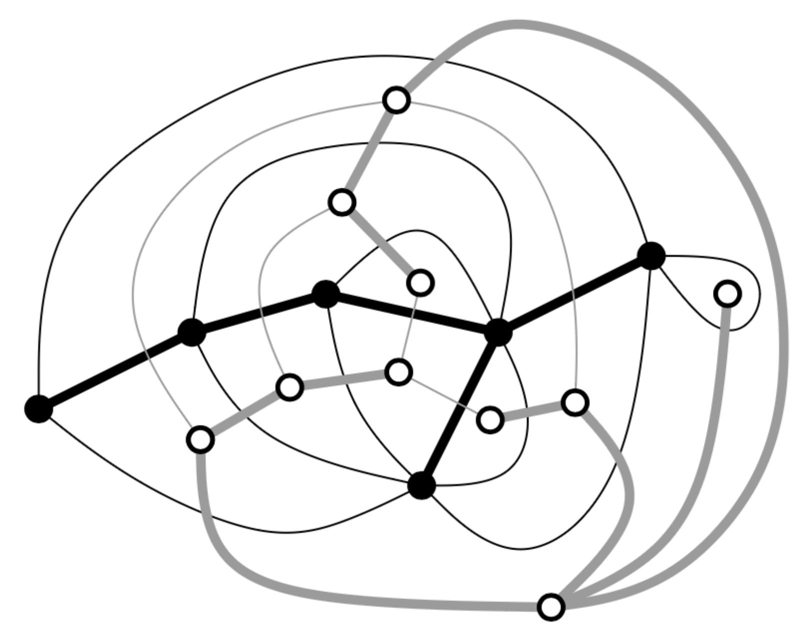
Proof: step 2 (3/3)



=> Contracting the cycle, we get a spanning tree for the dual (of black+bold black), whose "dual" is bold-black

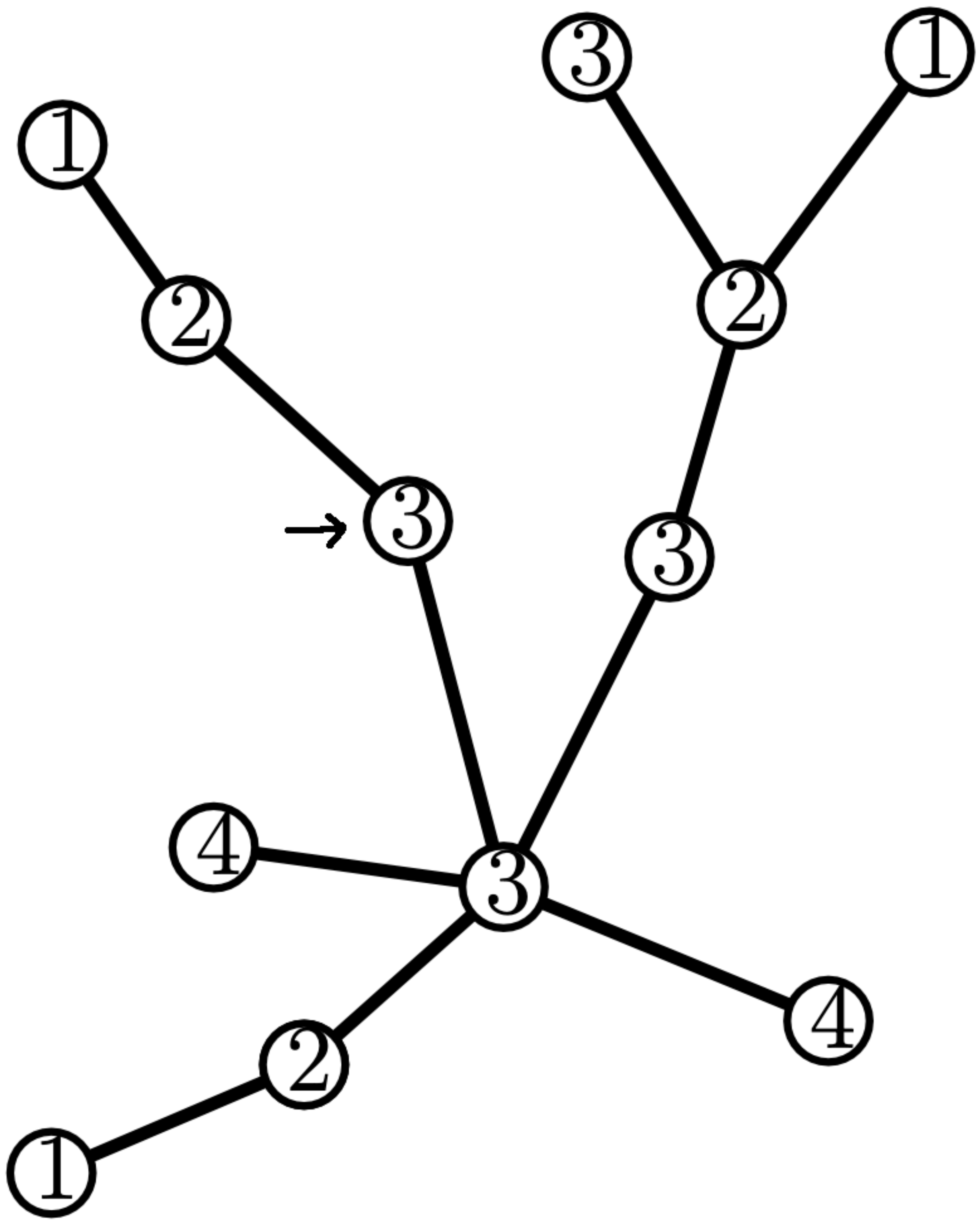
=> Spanning tree!





[Schaeffer 15, Figure 1.4]

Proof: step 3 (1/2)

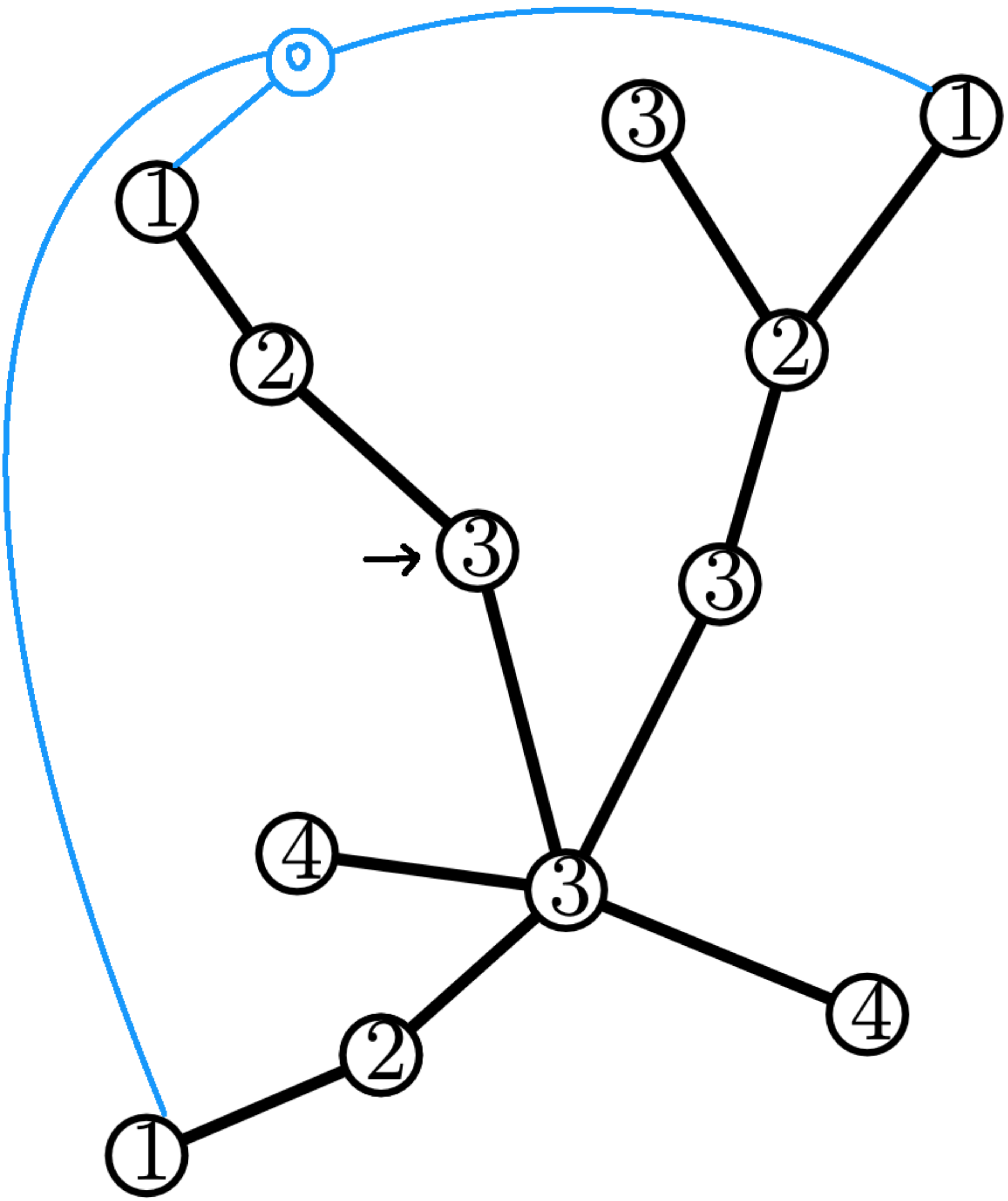


3. The construction is bijective.

Take a rooted well-labelled trees with n edges with an additional global label in $\{+1, -1\}$.

1. Add a 0 vertex and connect it to all the 1.
2. Go through the tree clockwise and connect every corner $i \geq 1$ to the next corner $i-1$.
3. Root the first added edge according to the global label.

Proof: step 3 (1/2)



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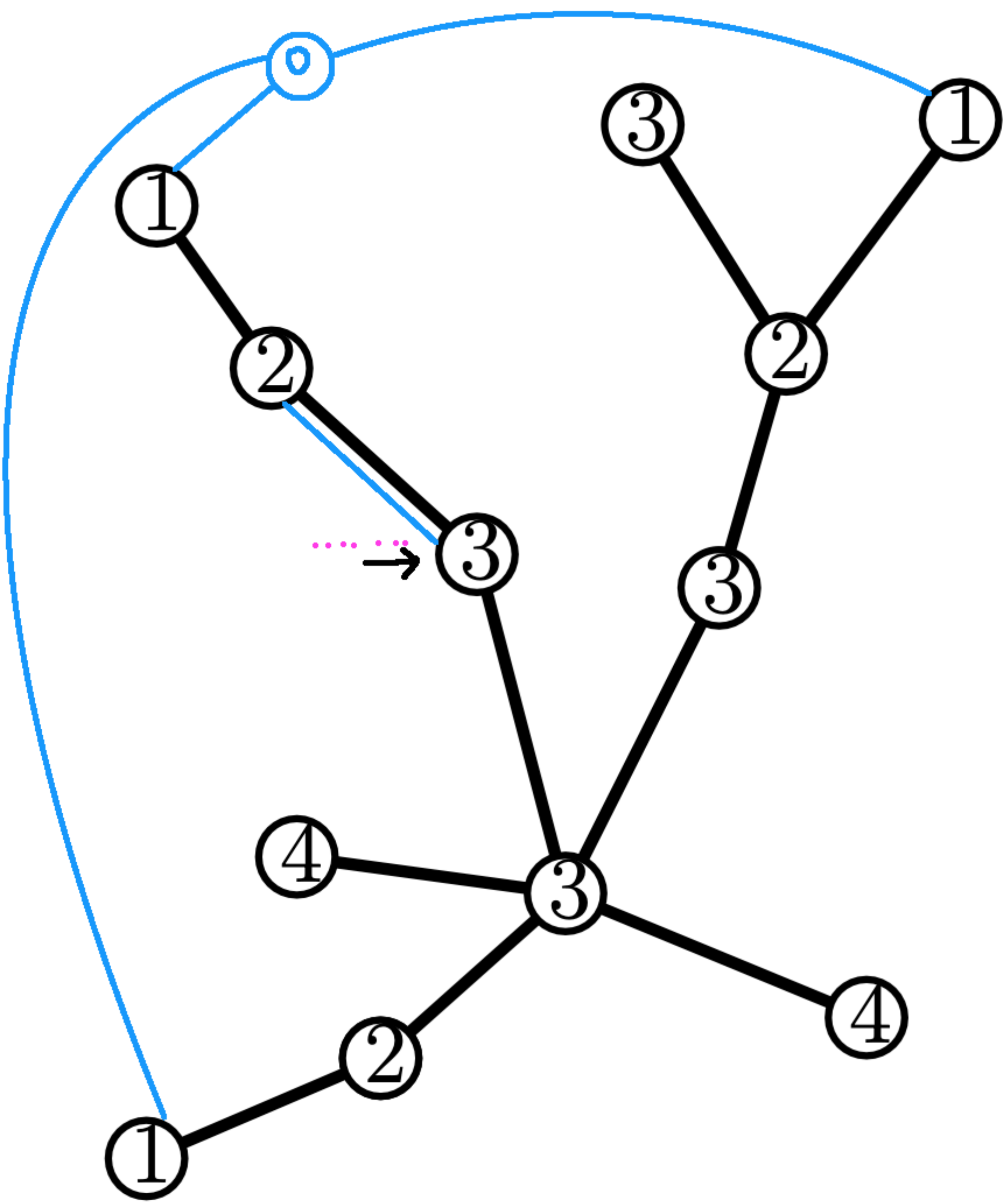

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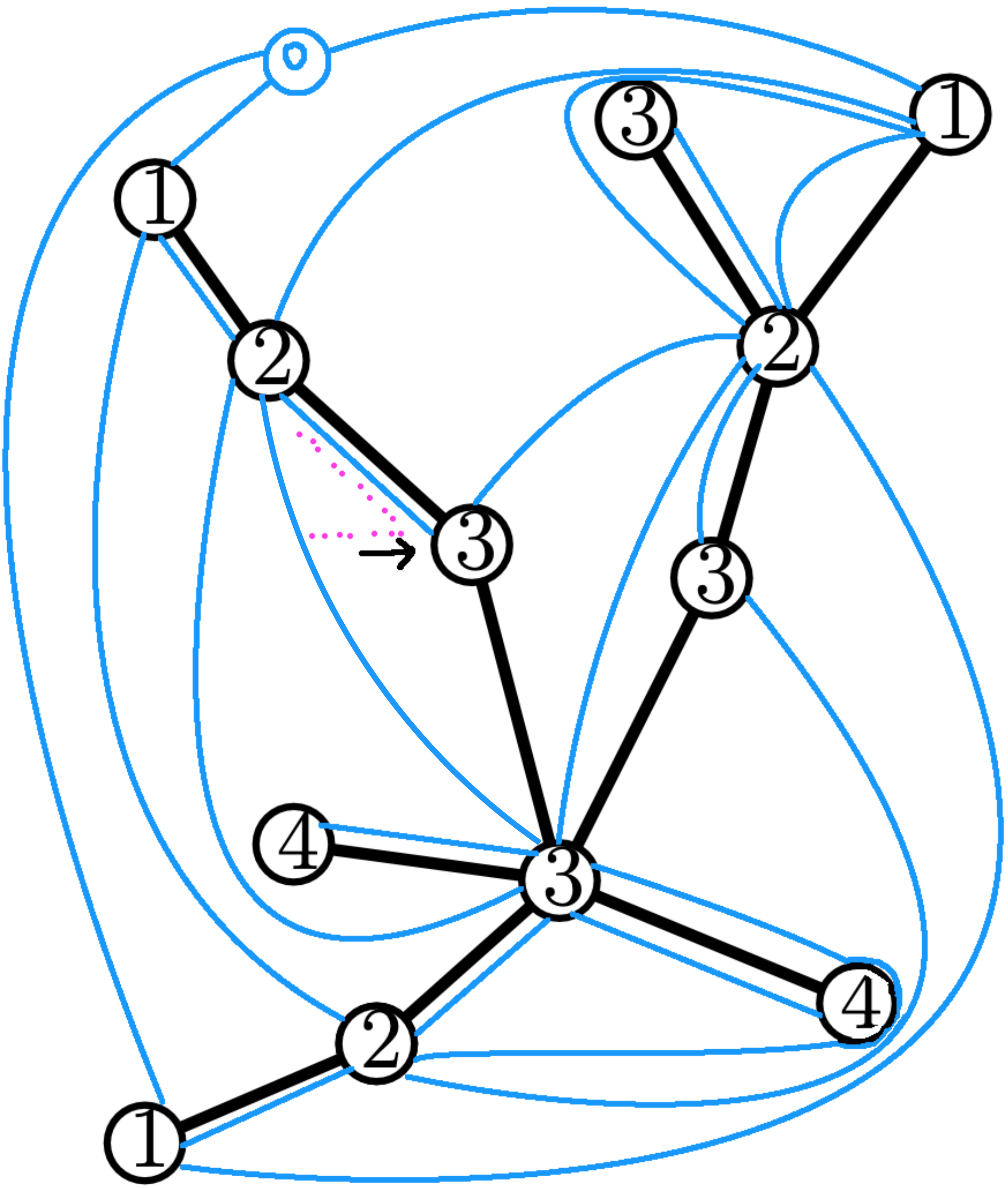

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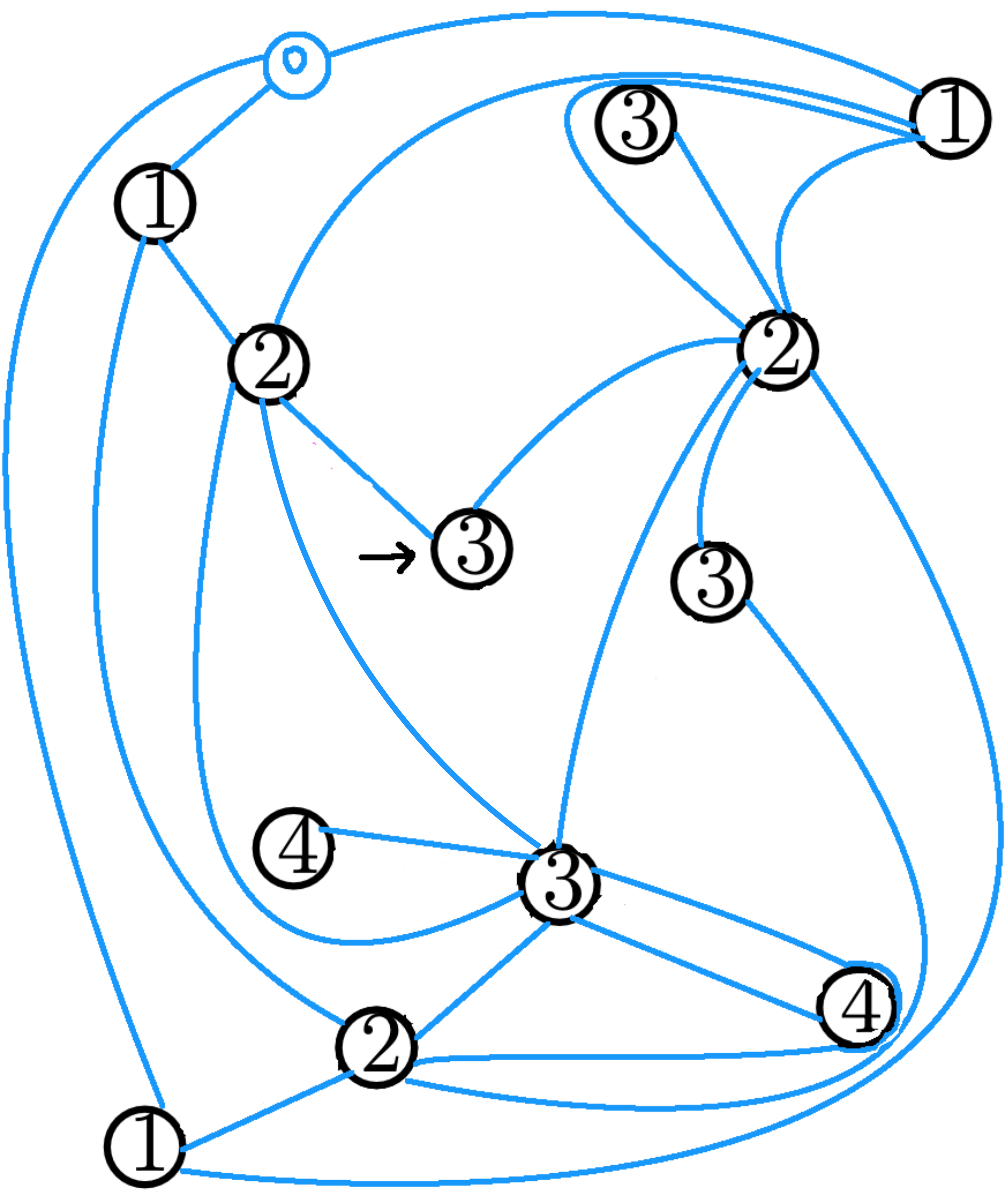

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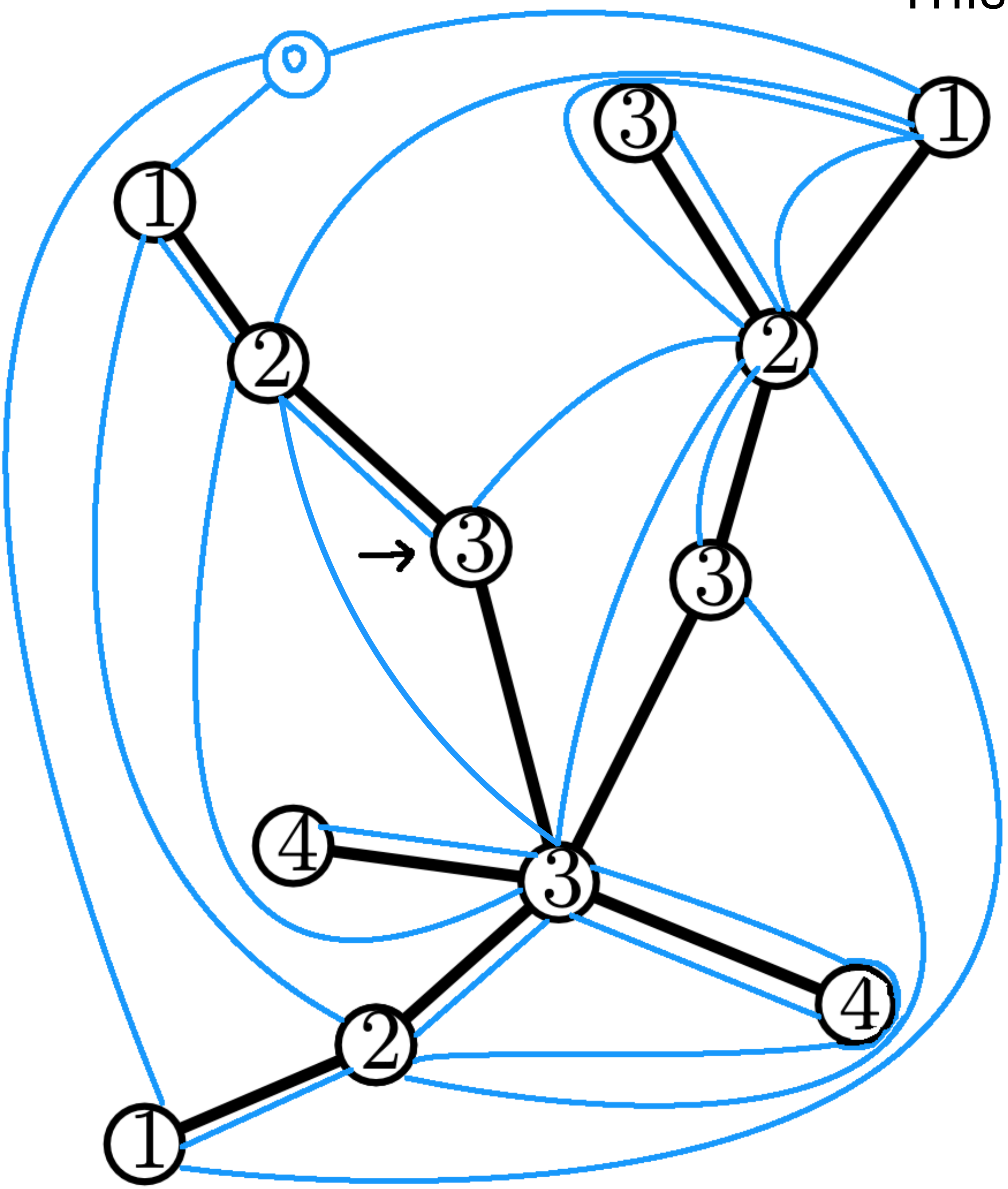
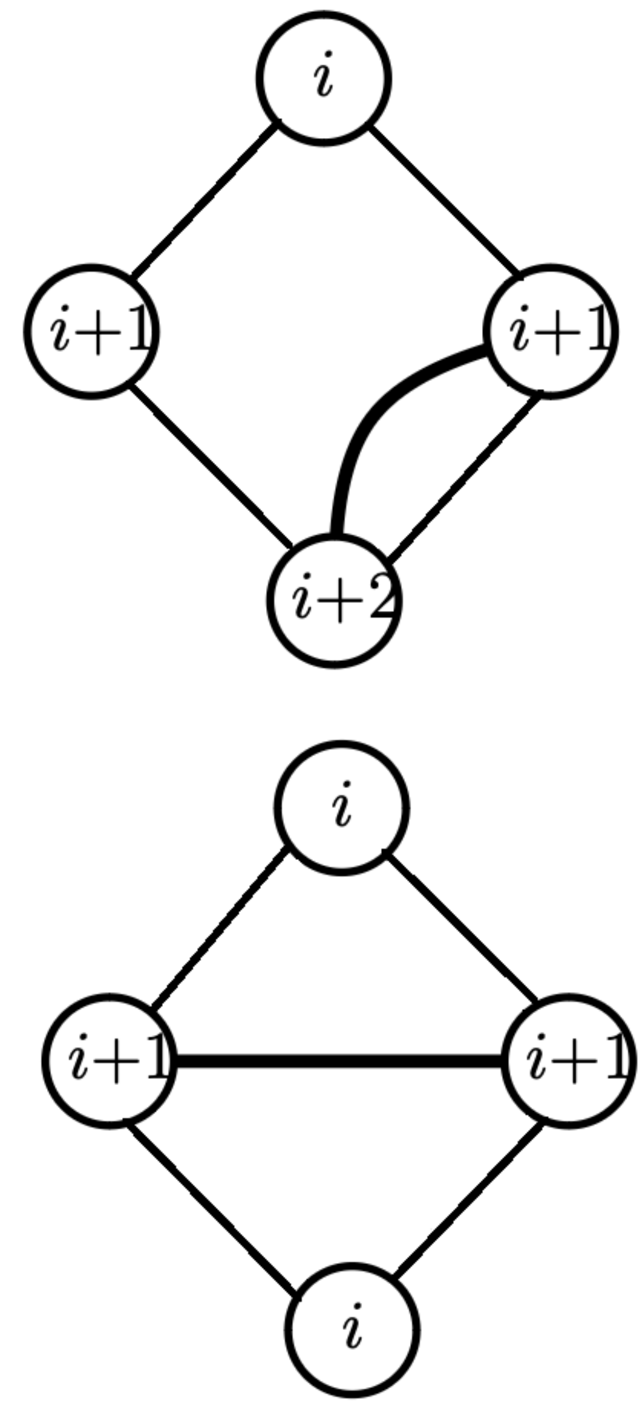
Proof: step 3 (2/2)

This is indeed the inverse construction.





Proof

This concludes the proof!

[Schaeffer 15, Theorem 10 and Corollary 7] The CVS bijection sends rooted quadrangulations with n faces and a marked vertex to rooted well-labelled trees with n edges with an additional global label in $\{+1, -1\}$.

CVS bijection for distances in planar maps

Labels in the well-labelled tree give distances to the marked vertex.

As $n \to \infty$,

- Height uniform (well-labelled) tree with n edges: $n^{1/2}$;
- Random label along a path of size $n^{1/2}$ follows CLT, variations in $(n^{1/2})^{1/2} = n^{1/4}$.

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<u>Theorem</u> [Chassaing—Schaeffer 04] In a uniform rooted quadrangulation with n faces, the distance between the root vertex and a uniform vertex behaves in $\Theta(n^{1/4})$ as $n \to \infty$.

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Extended to random uniform planar maps by an isometric bijection [Ambjørn—Budd 13].

Scaling limit for uniform (rooted) planar maps

<u>Theorem</u> [Le Gall 13, Miermont 13, Bettinelli—Jacob—Miermont 14] Let $\mathbf{M}_n$ be a uniform rooted quadrangulation with n faces or uniform rooted maps with n edges. Then, there exists $c > 0$ such that

$$\frac{c}{n^{1/4}} \mathbf{M}_n \xrightarrow[n \to \infty]{(d),\ GH} \mathcal{S}_e.$$

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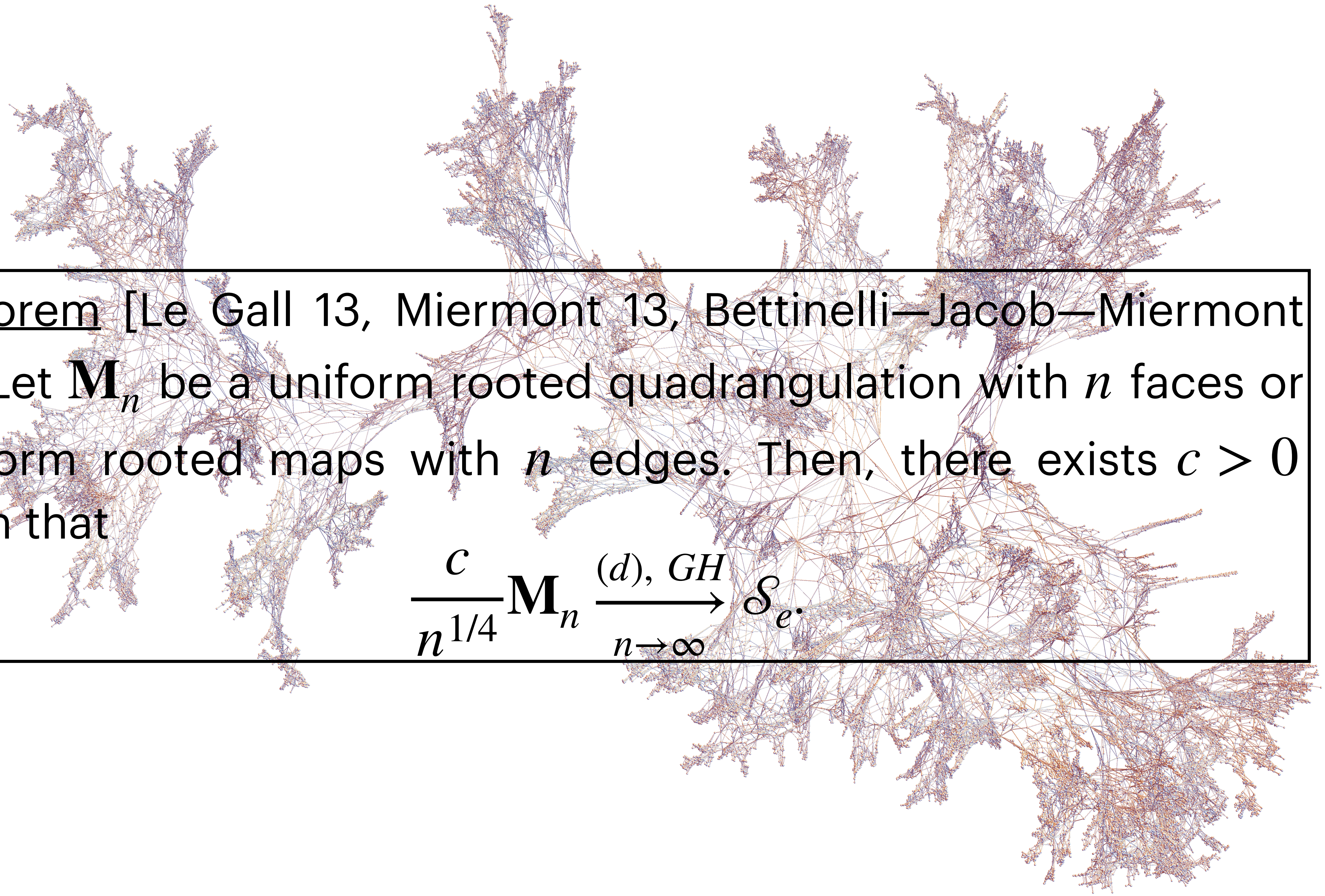

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Scaling limit for uniform (rooted) planar maps

Brownian Sphere $\mathcal{S}_e$

IV. Bouttier—Di Francesco—Guitter's bijection (BDG)

BDG bijection

[BDG 04]

Abstract

We extend Schaeffer's bijection between rooted quadrangulations and well-labeled trees to the general case of Eulerian planar maps with prescribed face valences to obtain a bijection with a new class of labeled trees, which we call mobiles.

BDG bijection

[BDG 04]

Abstract

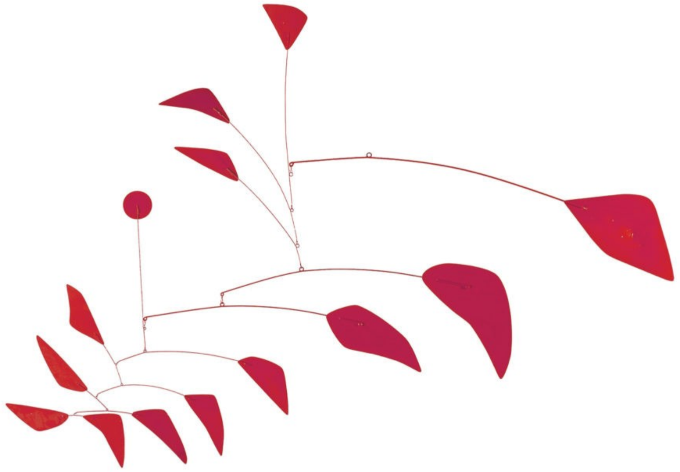
We extend Schaeffer's bijection between rooted quadrangulations and well-labeled trees to the general case of Eulerian planar maps with prescribed face valences to obtain a bijection with a new class of labeled trees, which we call mobiles.

Eulerian = face-bicolored maps.
Special case considered here = bipartite maps *(= vertices can be decomposed into black and white vertices such that there is no monochromatic edge)*.

Mobiles

IKEA, *KLAPPA*

Alexander Calder, *Big Red*



Christel Sadde, *Les confettis*

Mobiles

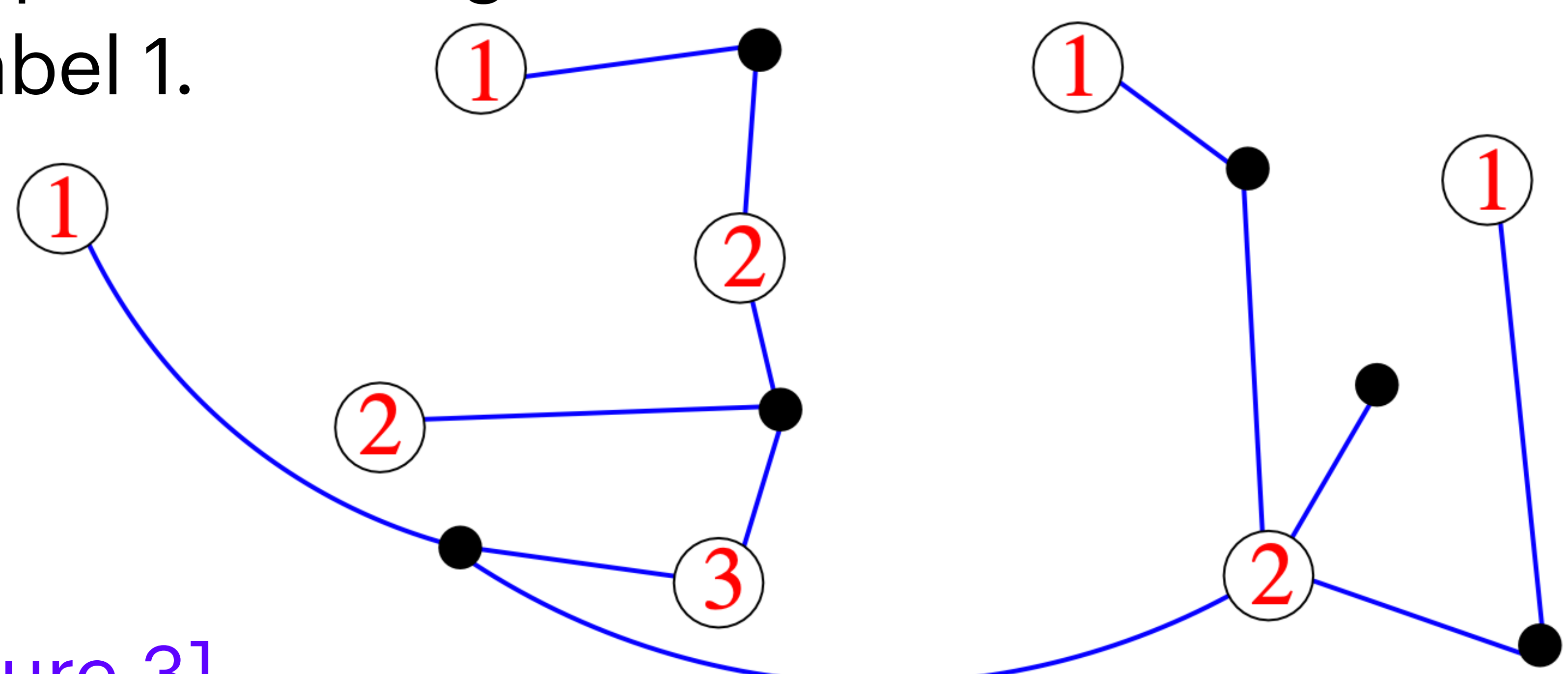
Mobile = plane tree such that

- Vertices are either unlabelled or labelled by an integer;
- Edges are between unlabelled and labelled vertices;
- Two labelled vertices v, v' adjacent to the same unlabelled vertex and consecutive in clockwise direction satisfy

$$\ell(v') \geq \ell(v) - 1.$$

Mobile is well-labelled if

- Each vertex carries a positive integer label;
- There is a vertex of label 1.



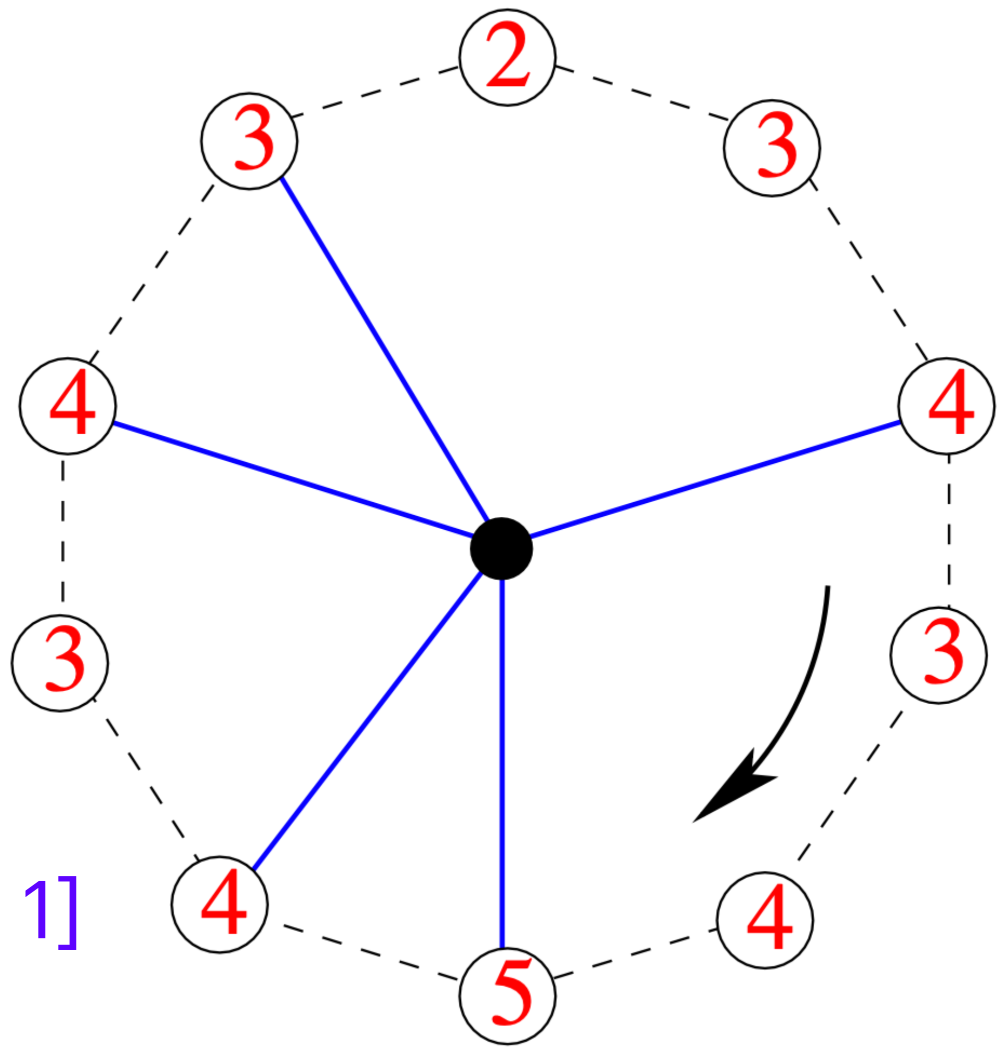
[BDG 04, Figure 3]

BDG construction (1/2)

Theorem [BDG 04] The BDG bijection sends bipartite planar maps with a marked vertex and n faces to well-labeled mobiles with n vertices.

Take a bipartite map with n faces and a marked vertex.

1. Label each vertex by its distance to the marked vertex.
2. Add a vertex in each face and connect it to the vertices immediately followed clockwise by a smaller label.



[BDG 04, Figure 1]

BDG construction (2/2)

Take a bipartite map with n faces and a marked vertex.

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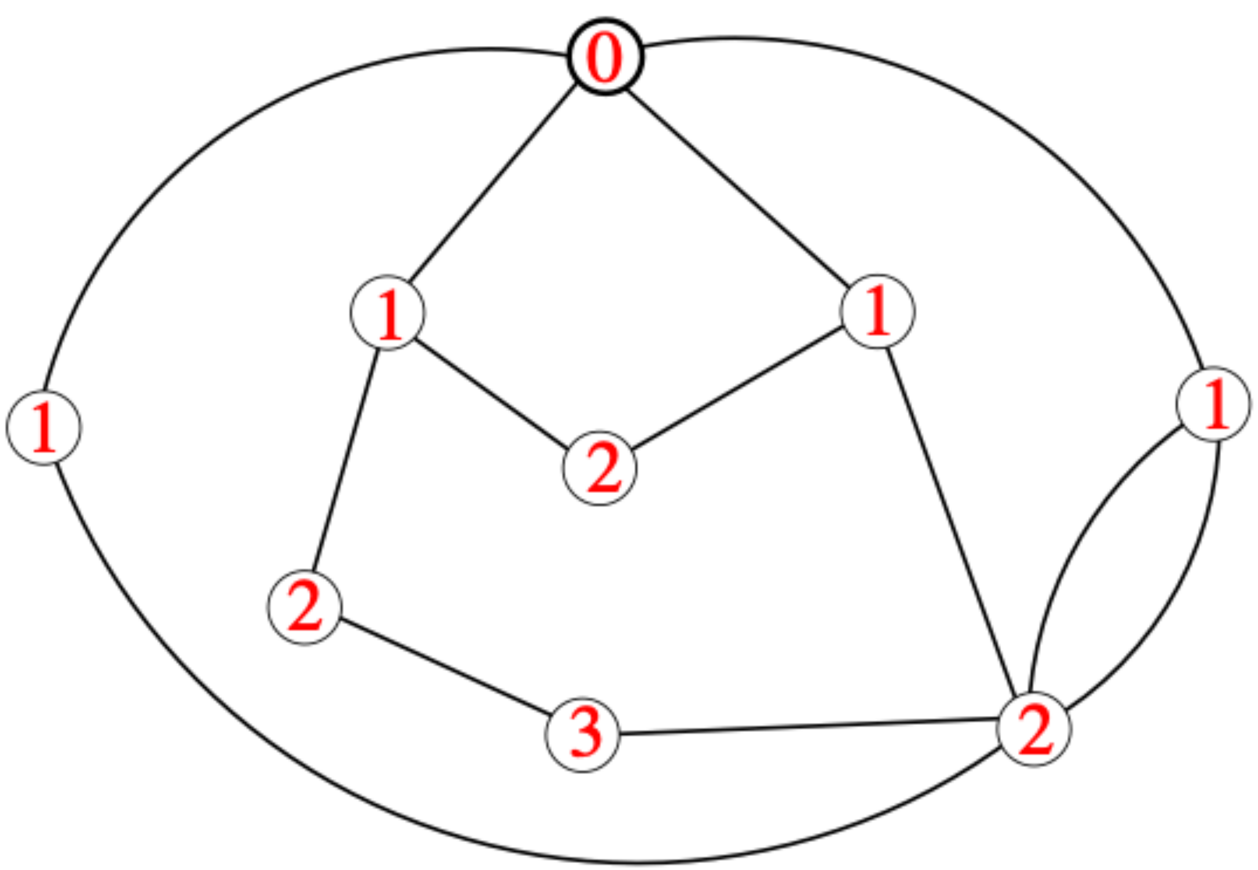
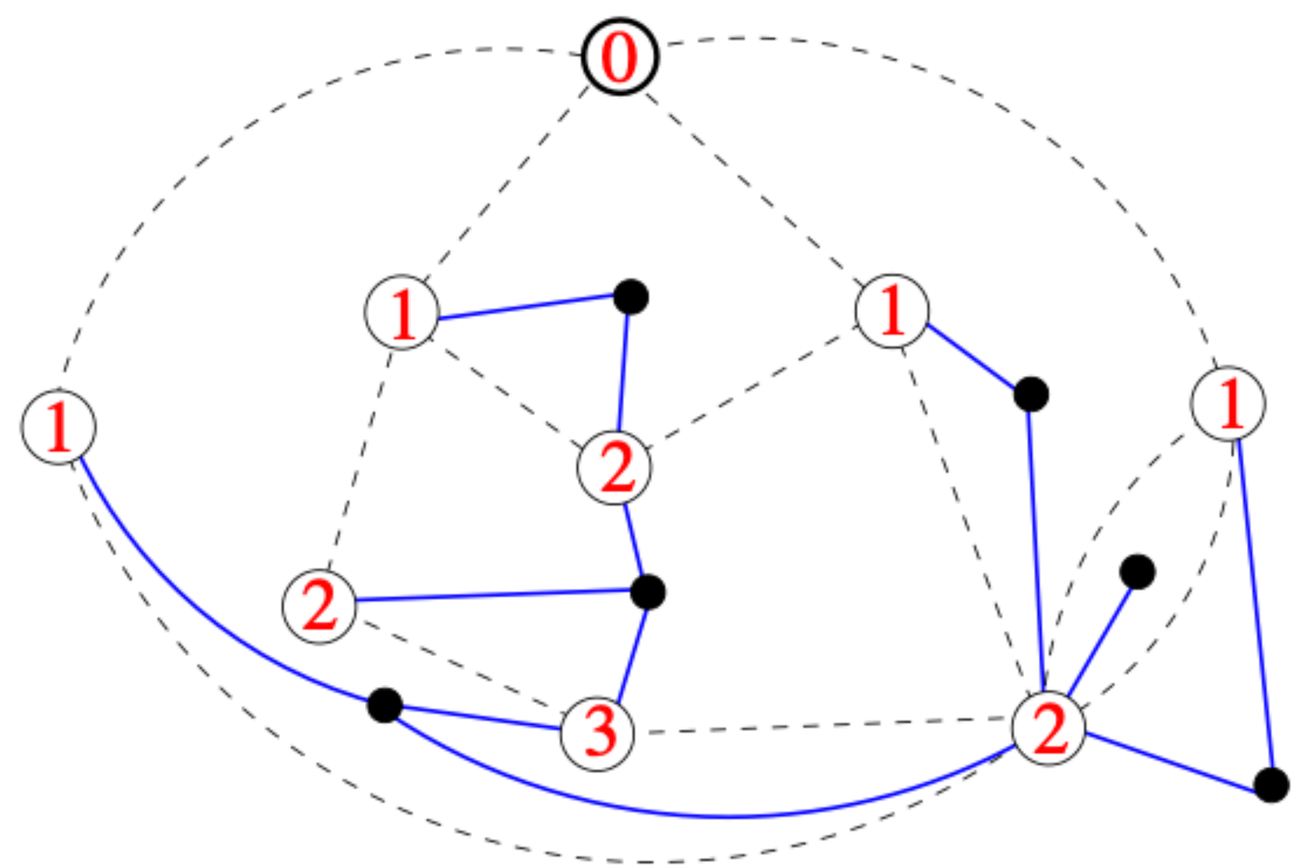
2. Add a vertex in each face and connect it to the vertices immediately followed clockwise by a smaller label.



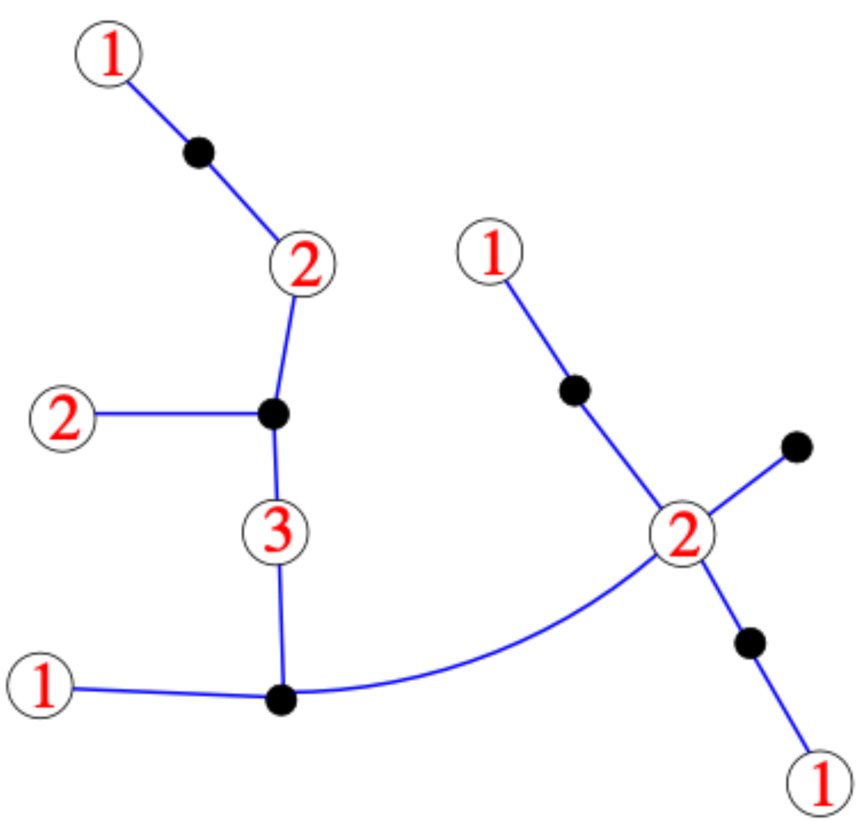


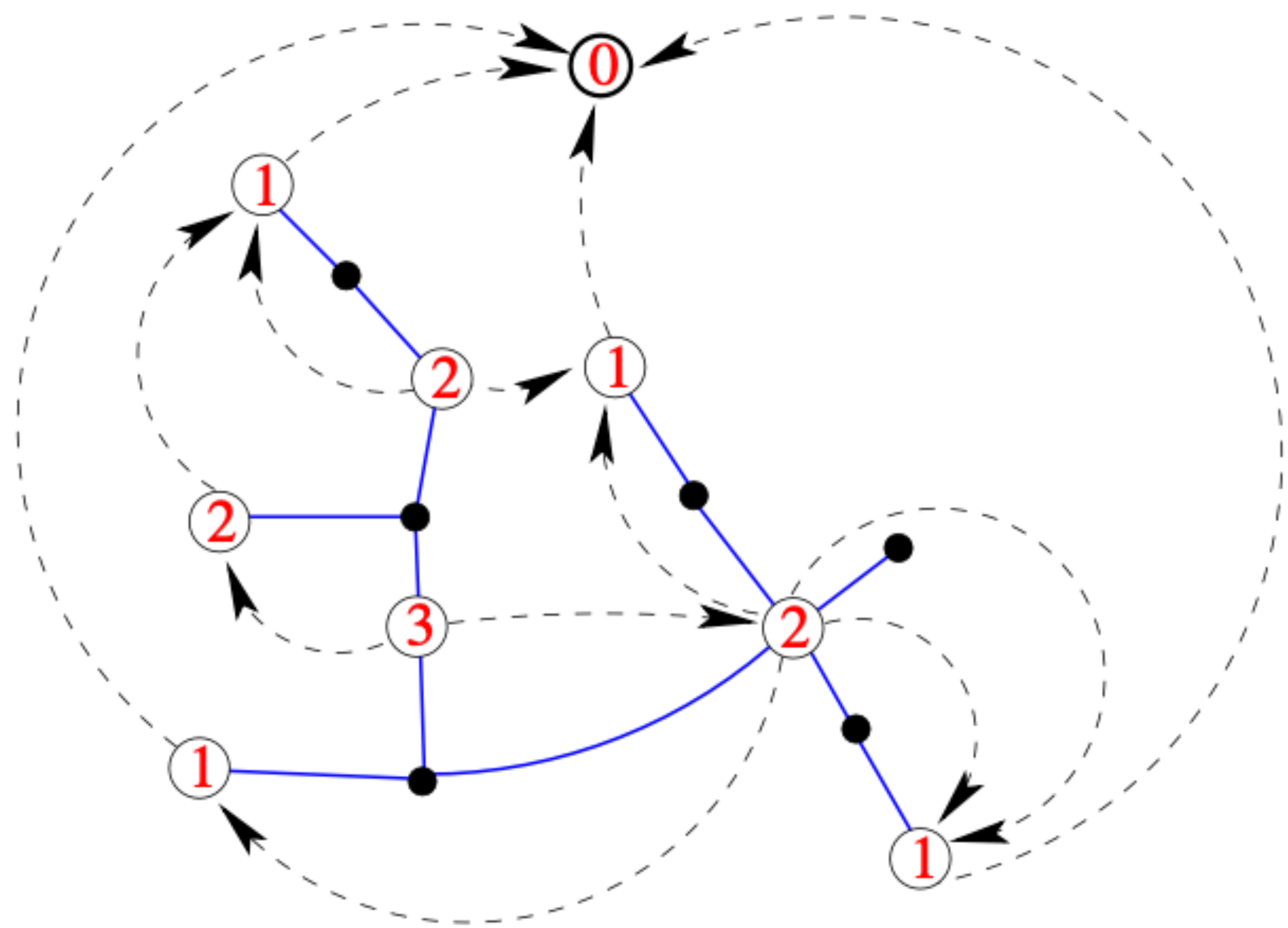
[BDG 04, Figure 3]

Inverse construction

Same as for CVS!

1. Add a 0 vertex and connect it to all the 1.
2. Go through the tree clockwise and connect every corner labelled $i \geq 1$ to the next corner $i - 1$.





[BDG 04, Figure 4]

V. Conclusion

Other bijections for planar maps

- Bijections for tree-rooted maps and other decorated maps;
- Seemingly unrelated objects are in bijection with (families of) maps: families of Tamari intervals, families of λ-terms, fighting fish, 2-stack-sortable permutations…
- Decomposition of maps into smaller blocks;
- Algebraic representations of maps: triplets of permutations, ramified coverings;
- …

Thank you!