

Phase transitions of block-weighted planar maps

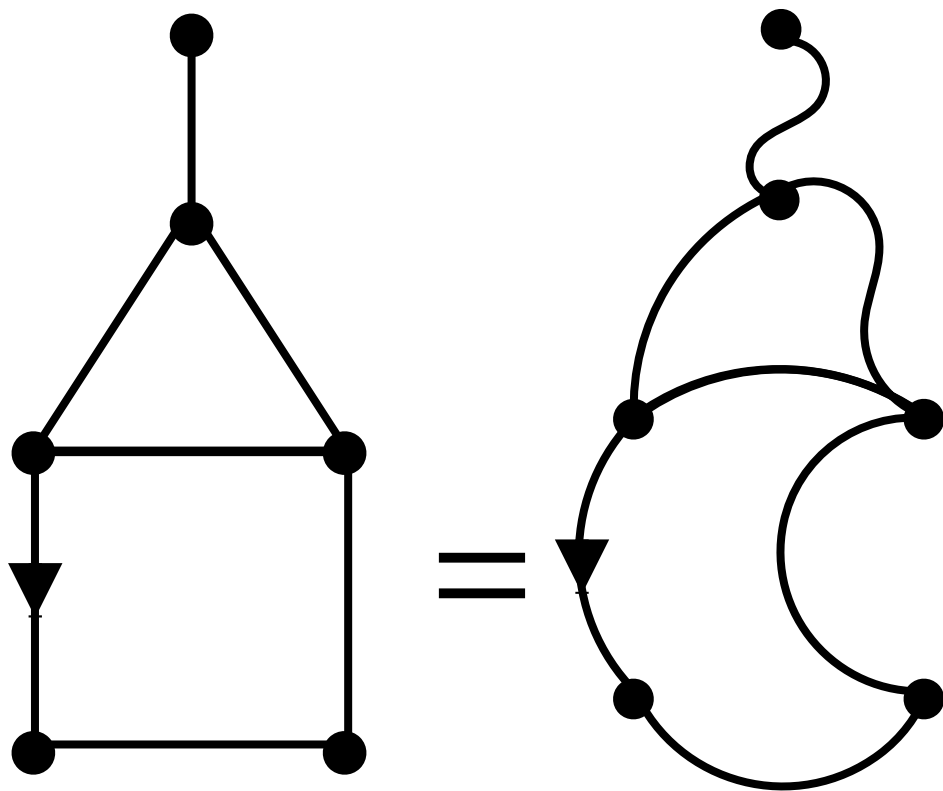
SFB Kick-off meeting
24 September 2024

Zéphyr Salvy (he/they)

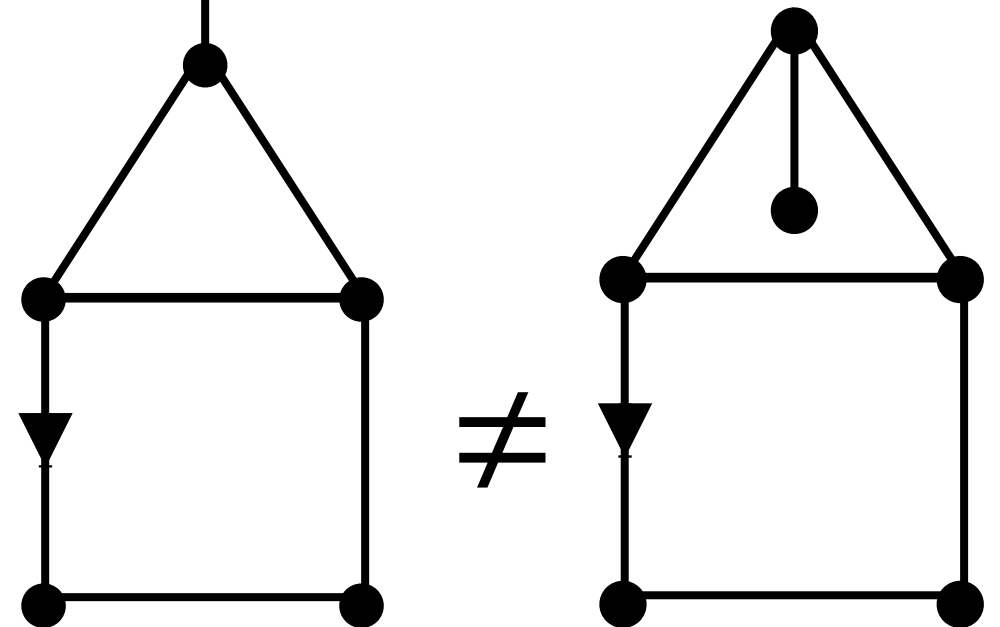
Hopefully soon: TU Wien

Planar maps

Planar map \mathfrak{m} = embedding on the sphere of a connected planar graph, considered up to homeomorphisms



Planar map = planar graph +
cyclic order on neighbours

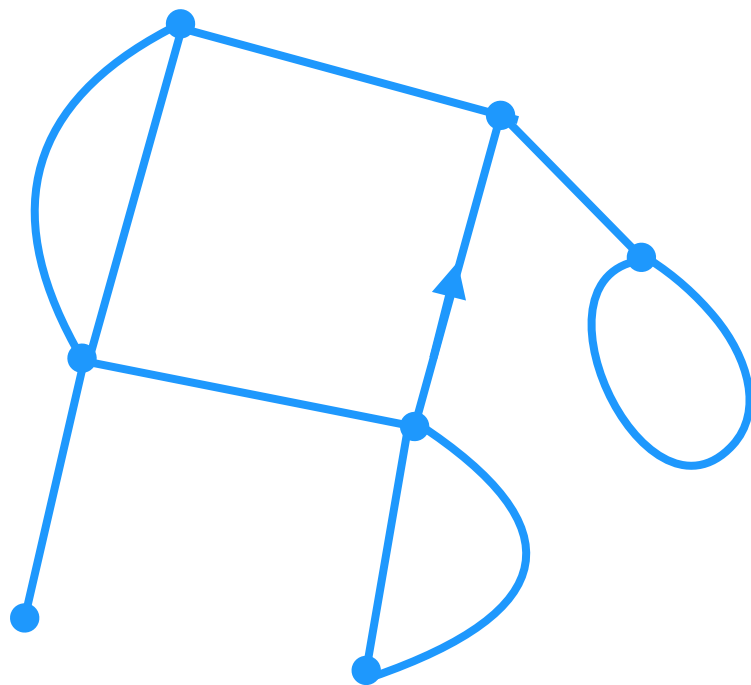


- **Rooted** planar map = map endowed with a marked oriented edge (represented by an arrow);
- **Size** $|\mathfrak{m}|$ = number of edges;
- **Corner** (does not exist for graphs !) = space between two consecutive edges around a vertex (trigonometric order).

Universality results for planar maps

- Enumeration: $\kappa \rho^{-n} n^{-5/2}$ [Tutte 1963];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for quadrangulations [Le Gall 2013, Miermont 2013] and general maps [Bettinelli, Jacob, Miermont 2014];

Brownian Sphere \mathcal{S}_e

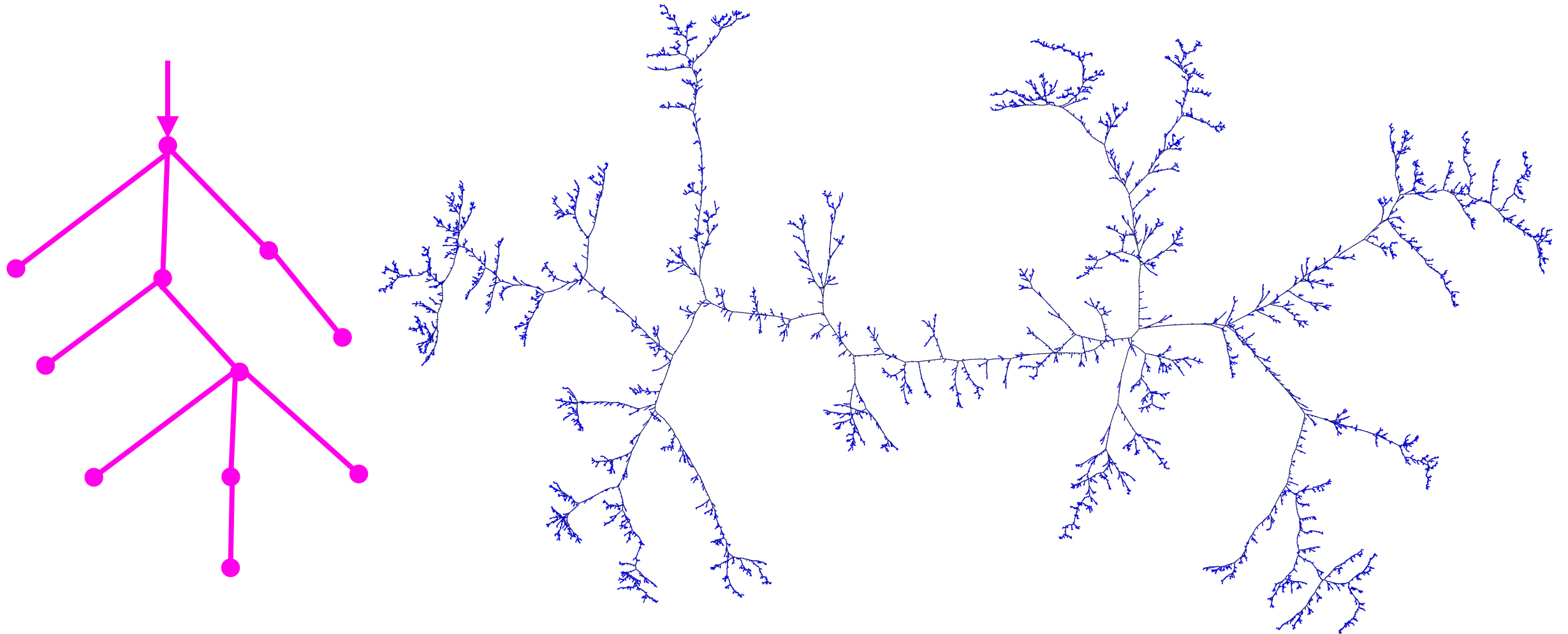


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- Scaling limit: Brownian sphere for quadrangulations [Le Gall 2013, Miermont 2013] and general maps [Bettinelli, Jacob, Miermont 2014];
- Universality:
 - Same enumeration [Drmota, Noy, Yu 2020];
 - Same scaling limit, e.g. for triangulations & $2q$ -angulations [Le Gall 2013], simple quadrangulations [Addario-Berry, Albenque 2017].

Universality results for plane trees

- Enumeration: $\kappa \rho^{-n} n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];



Universality results for plane trees

- Enumeration: $\kappa \rho^{-n} n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];
- Universality:
 - Same enumeration,
 - Same scaling limit, even for some classes of **maps**; e.g. outerplanar maps [Caraceni 2016], maps with a boundary of size $\gg n^{1/2}$ [Bettinelli 2015].

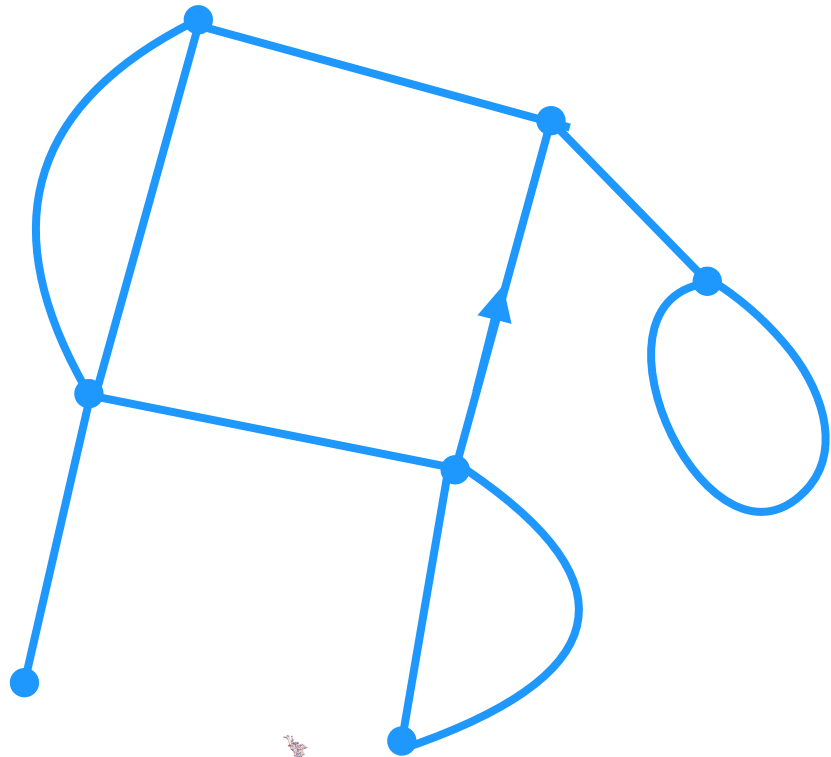
Models with (very) constrained boundaries

Motivation

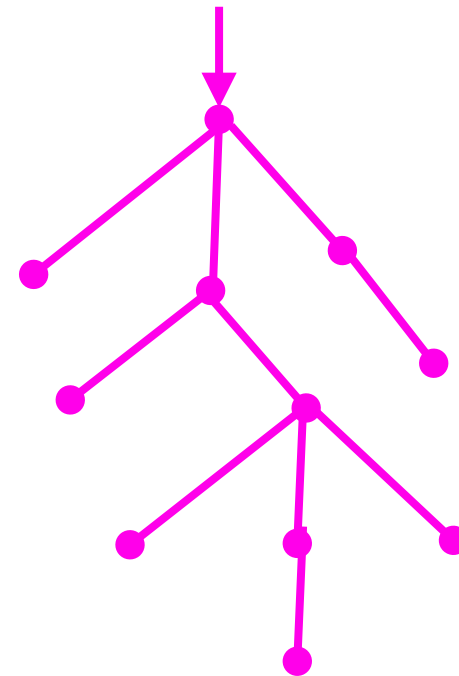
Inspired by [Bonzom
Delepoue Rivasseau 2015].

Two rich situations with universality results:

Planar maps



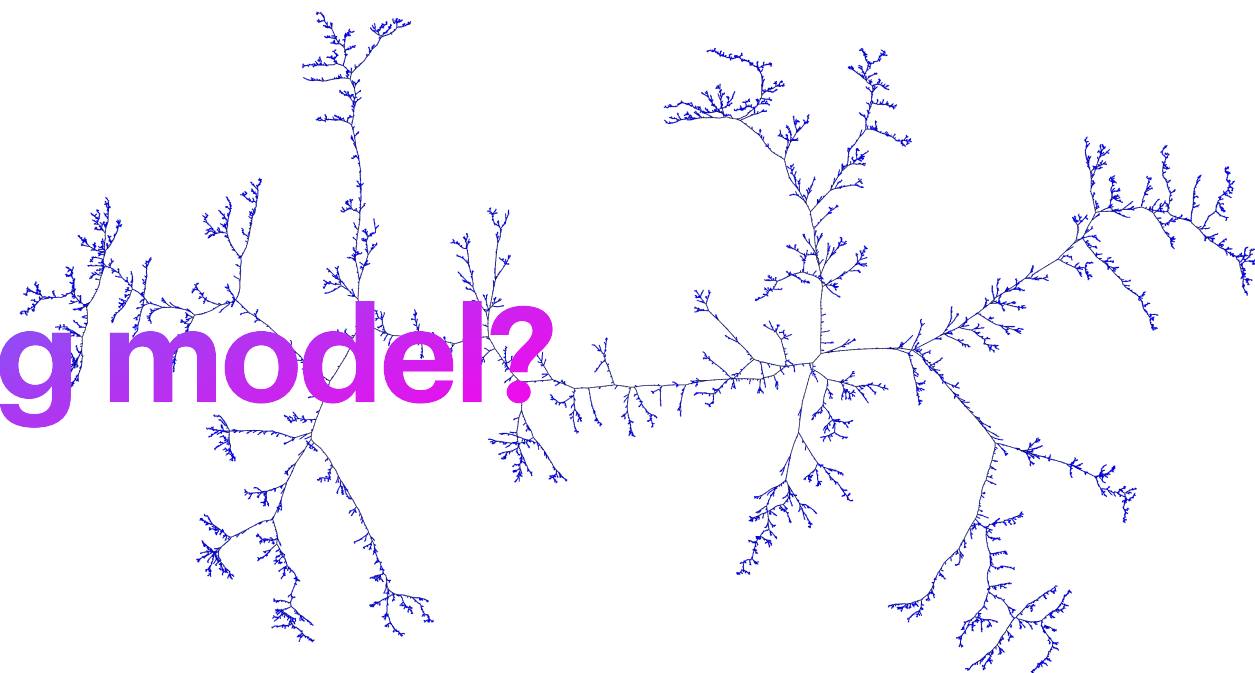
Plane trees



Interpolating model?



Brownian Sphere \mathcal{S}_e

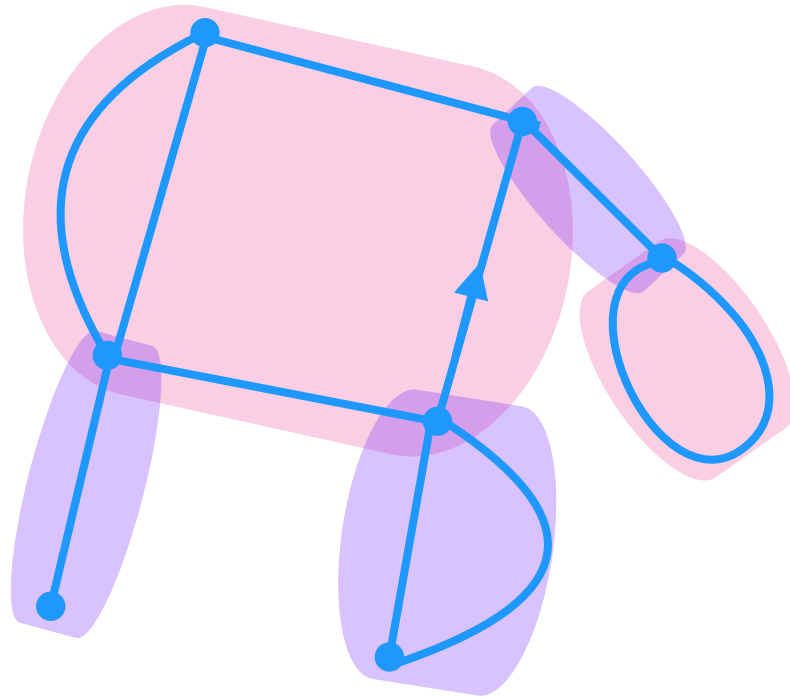


Brownian Tree \mathcal{T}_e

Model definition

2-connected = two vertices must be removed to disconnect.

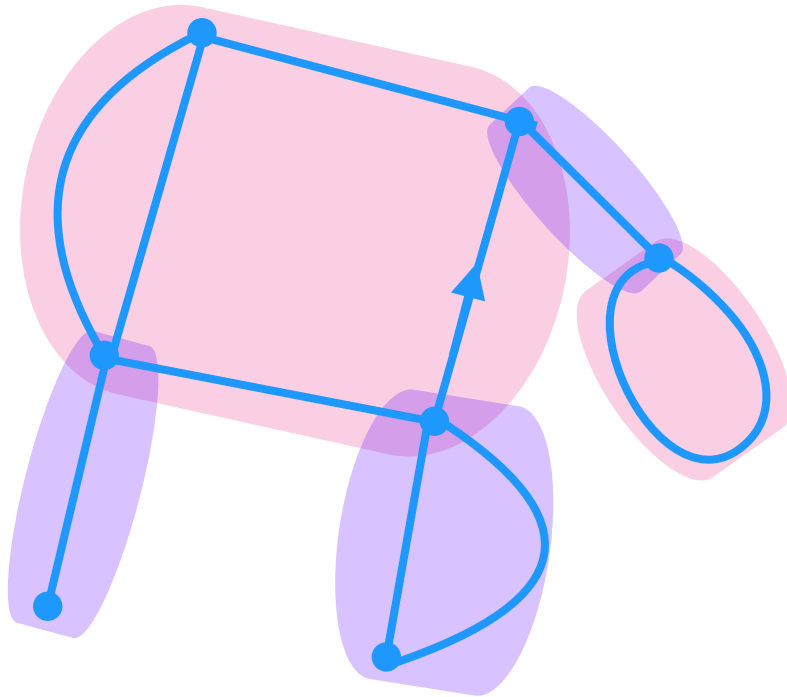
Block = maximal (for inclusion) 2-connected submap.



Model definition

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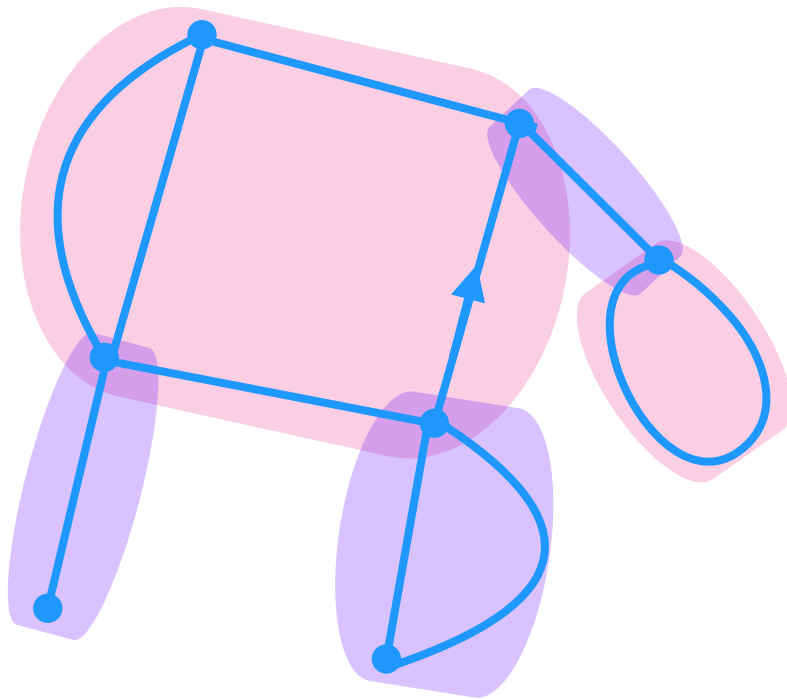


Condensation phenomenon: a large block concentrates a macroscopic part of the mass
[Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

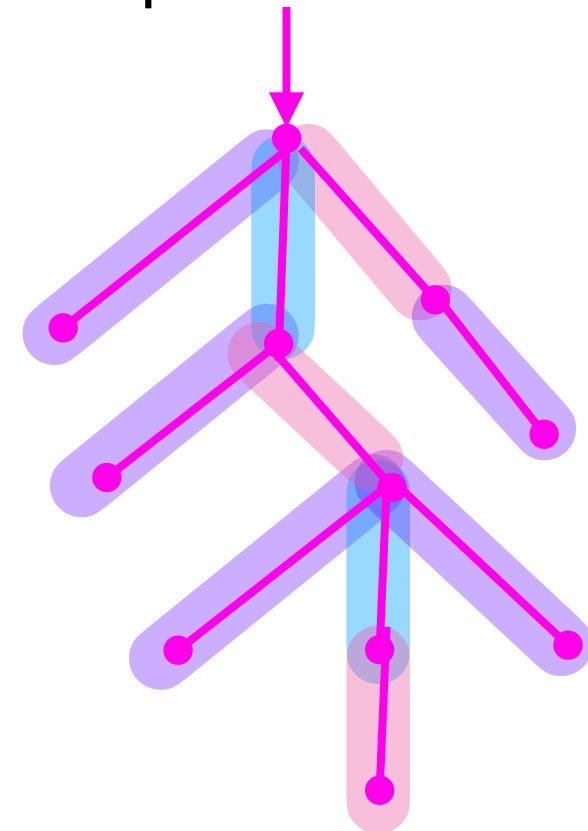
Model definition

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Condensation phenomenon: a large block concentrates a macroscopic part of the mass
[Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].



Only small blocks.

Interpolating model using blocks!

Outline of the talk

Phase transitions of block-weighted planar maps

I. Model

II. Block tree of a map and its applications

Interlude. Quadrangulations

III. Scaling limits

IV. Extension to other families of maps

V. Extension to tree-rooted maps

VI. Perspectives

with William
Fleurat



with Marie
Albenque & Éric
Fusy



I. Model

Model

Introduced by [Bonzom Delepoue Rivasseau 2015];

General setting in [Stufler 2020].

Goal: parameter that affects the typical number of blocks.

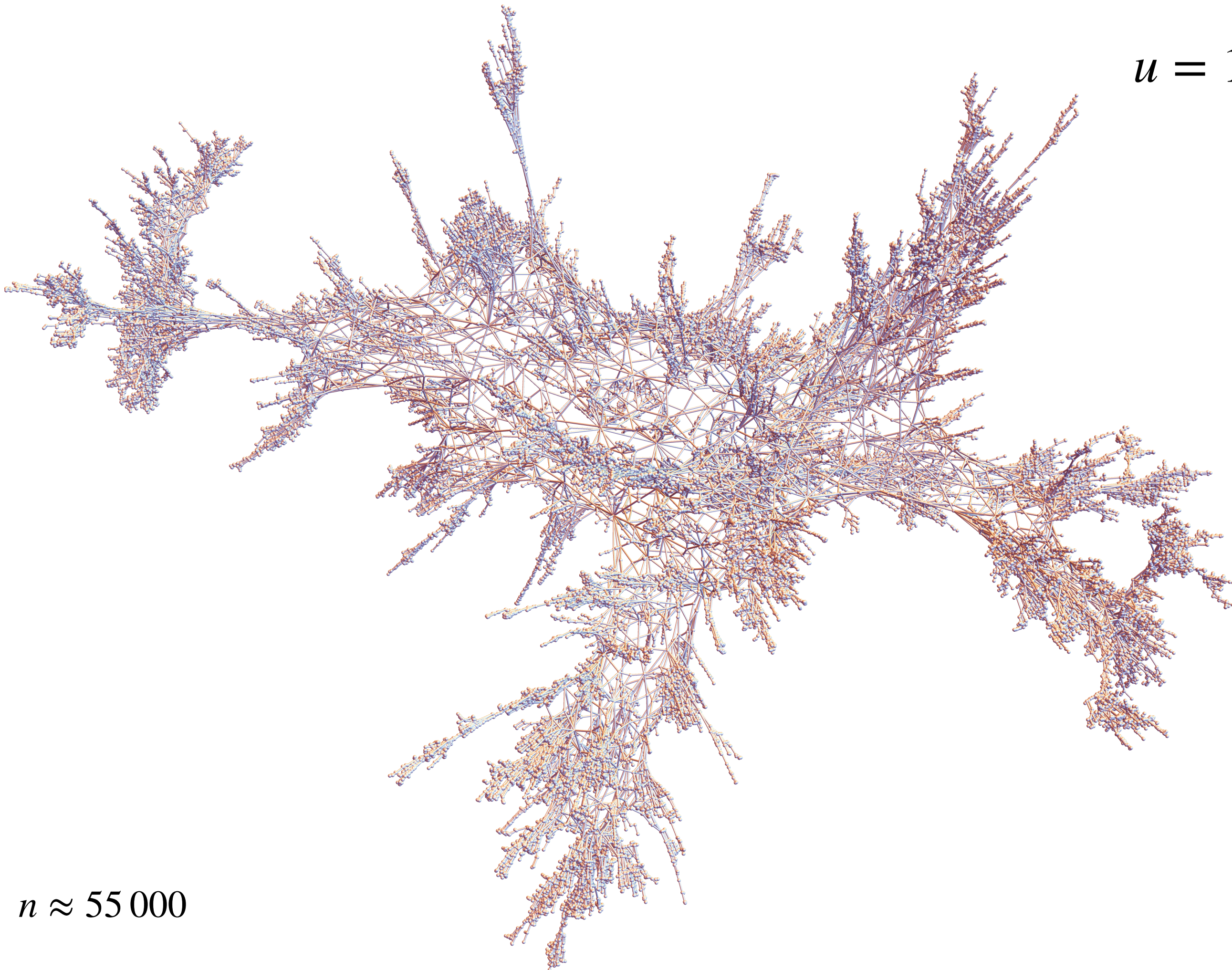
We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,
 $\mathcal{M}_n = \{\text{maps of size } n\}$,
 $\mathfrak{m} \in \mathcal{M}_n$,
 $Z_{n,u} = \text{normalisation.}$

- $u = 1$: uniform distribution on maps of size n ;
- $u \rightarrow 0$: minimising the number of blocks (=2-connected maps);
- $u \rightarrow \infty$: maximising the number of blocks (= trees!).

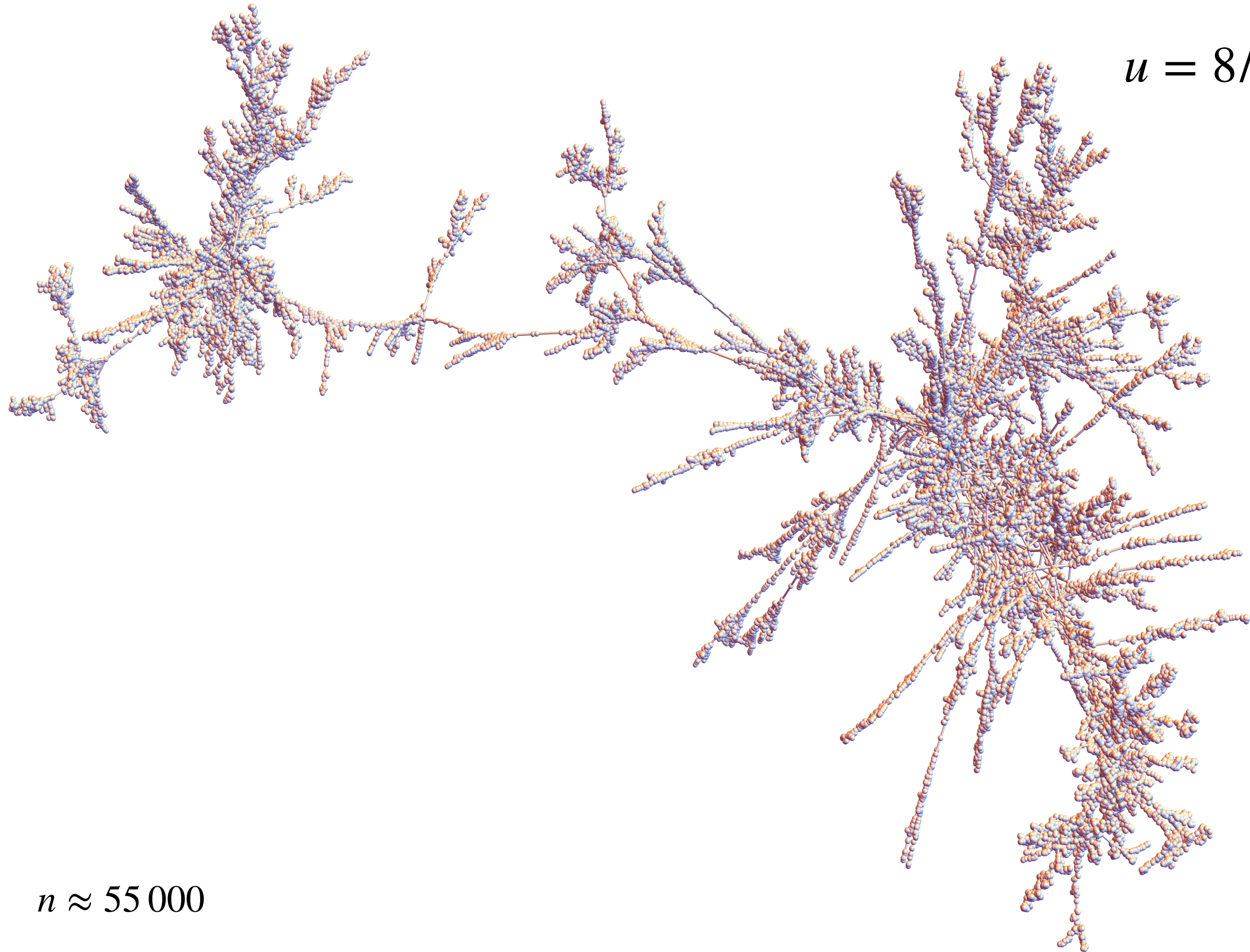
Given u , asymptotic behaviour when $n \rightarrow \infty$?

$$u = 1$$



$$n \approx 55\,000$$

$$u = 8/5$$



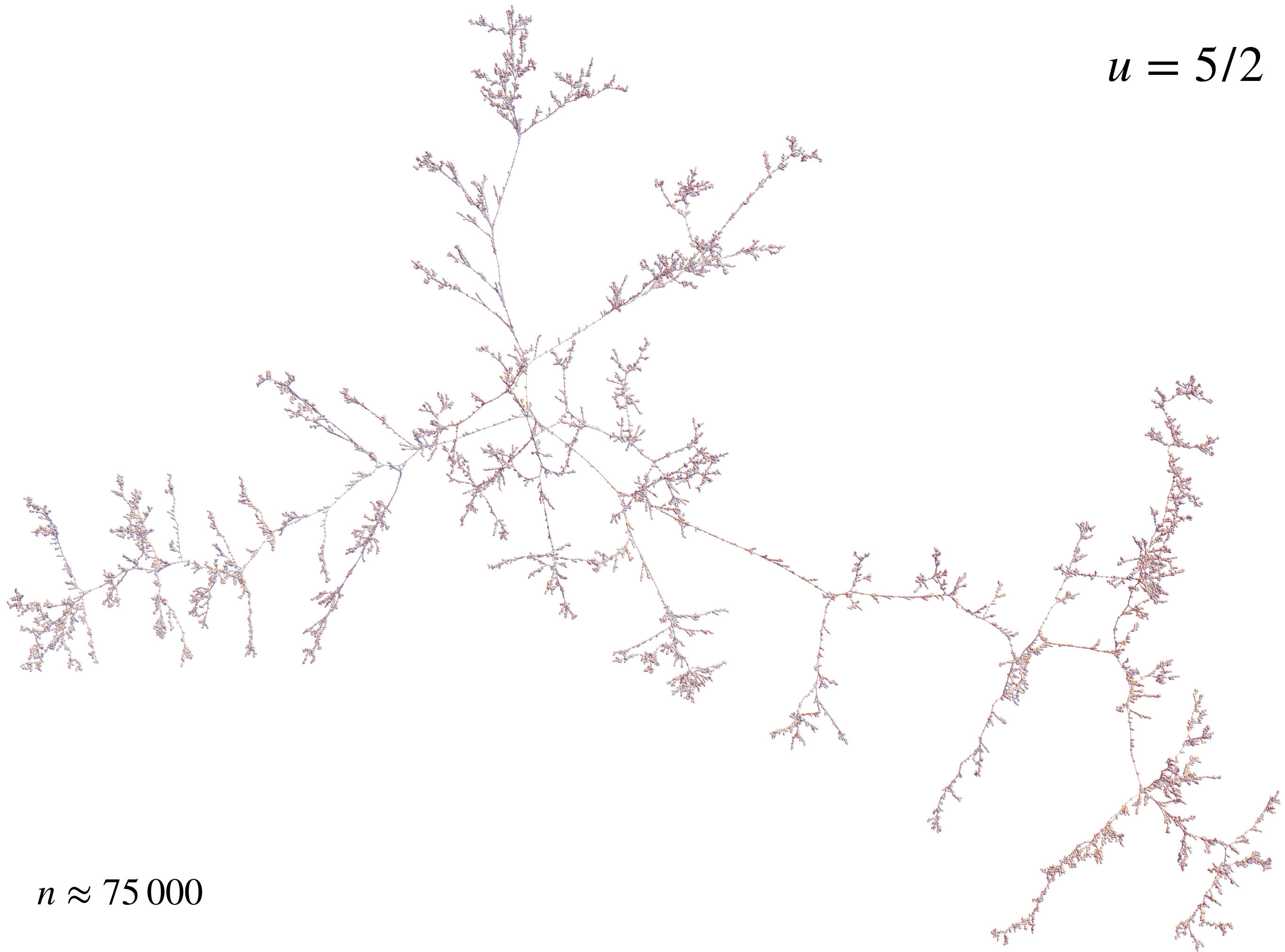
$$n \approx 55\,000$$

$$u = 9/5$$



$$n \approx 80\,000$$

$$u = 5/2$$



$$n \approx 75\,000$$

$$u = 5$$



$$n \approx 50\,000$$

Phase transition

Theorem [Fleurat, S. 24] Model exhibits a phase transition at $u = 9/5$. When $n \rightarrow \infty$:

- Subcritical phase $u < 9/5$: “general map phase” one huge block;
- Critical phase $u = 9/5$: a few large blocks;
- Supercritical phase $u > 9/5$: “tree phase” only small blocks.

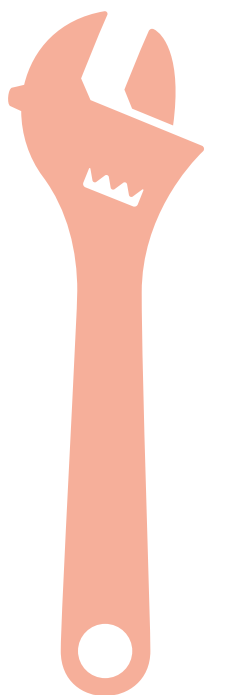
We obtain explicit results on enumeration, size of blocks and scaling limits in each case.

→ *A phase transition in block-weighted random maps*
W. Fleurat & Z. S., Electronic Journal of Probability, 2024

Results

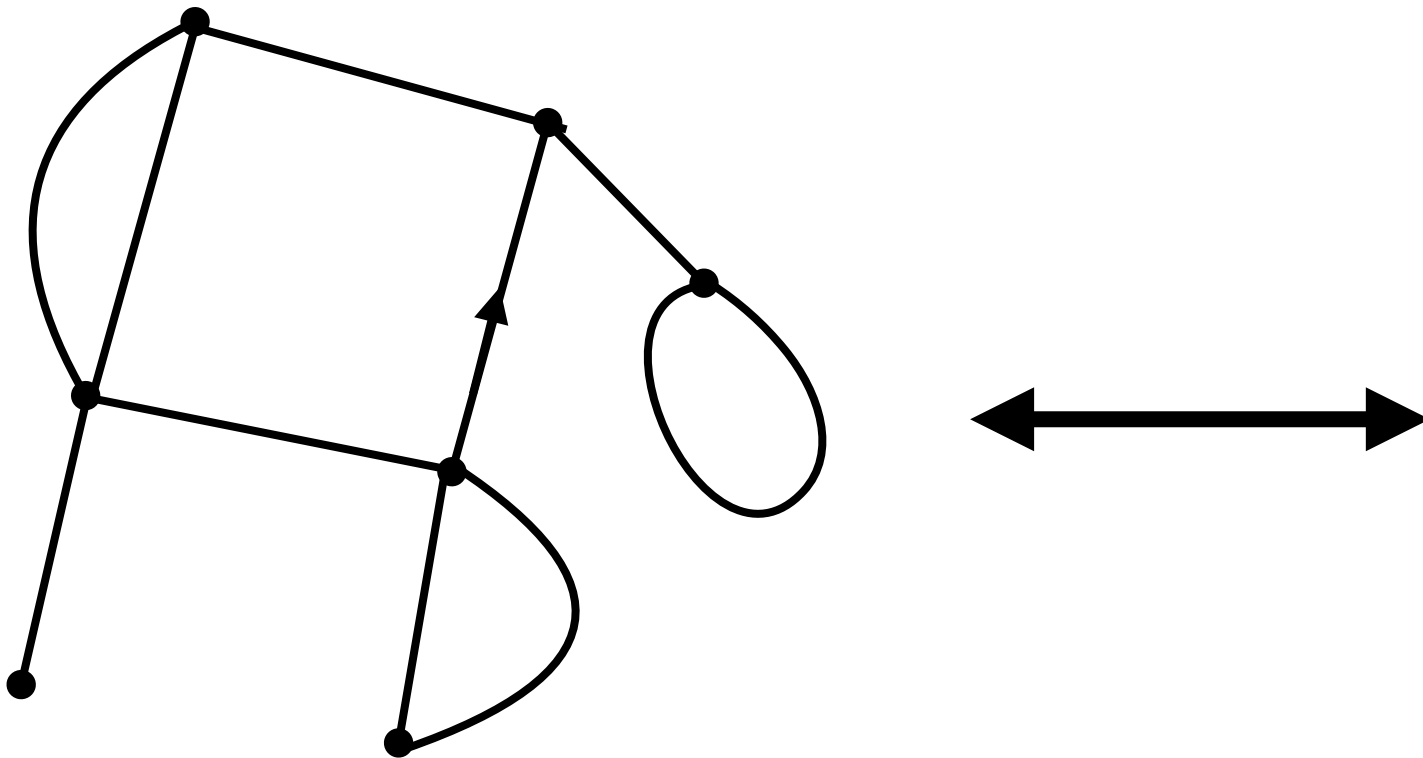
For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration			
Size of <ul style="list-style-type: none"> - the largest block - the second one 			
Scaling limit of M_n			

II. Block tree of a map and its applications



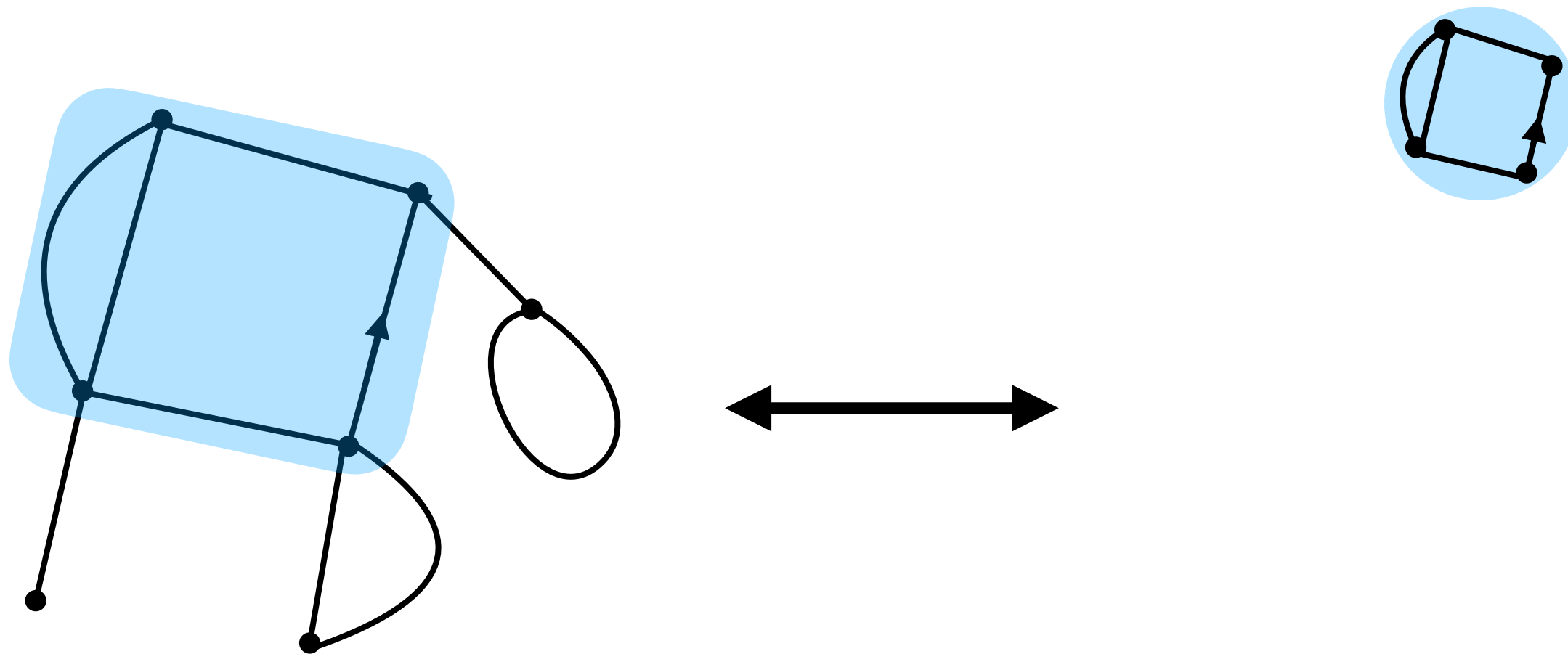
Decomposition of a map into blocks

Inspiration from [Tutte 1963]



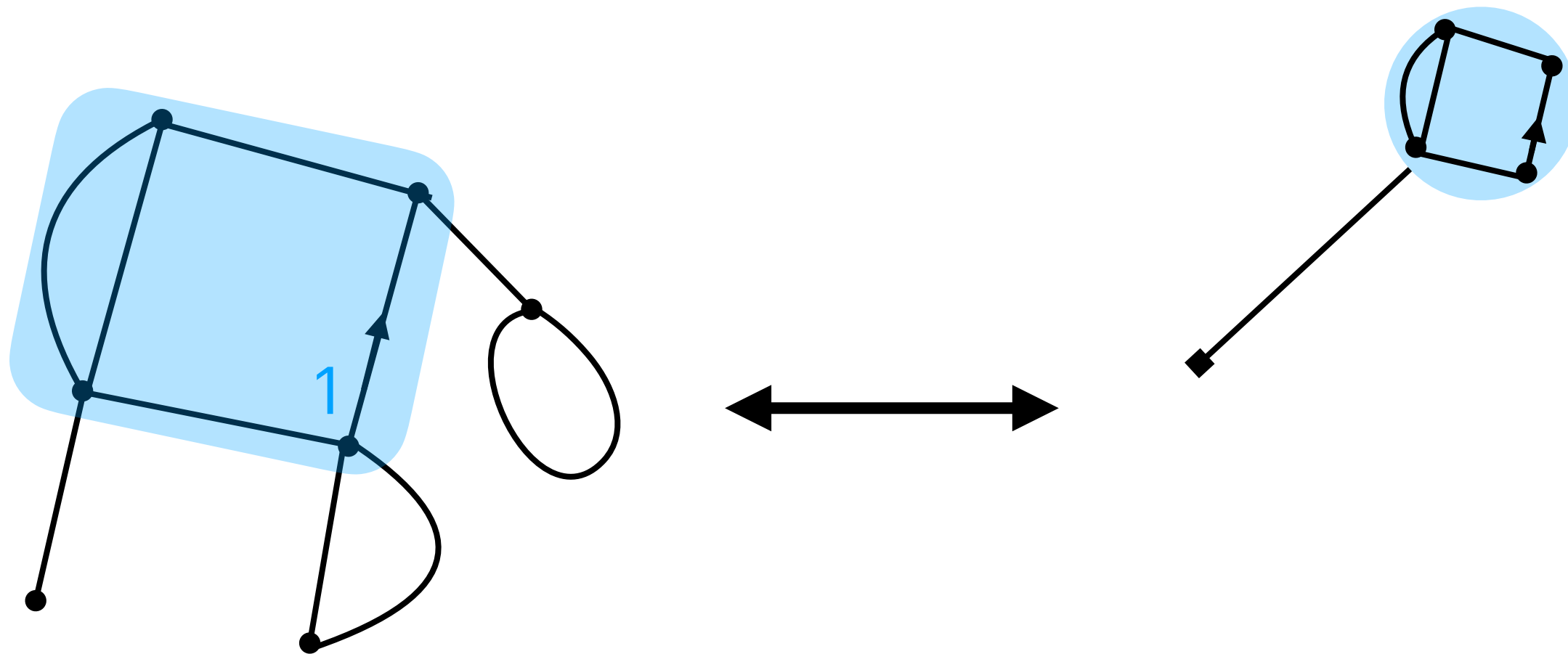
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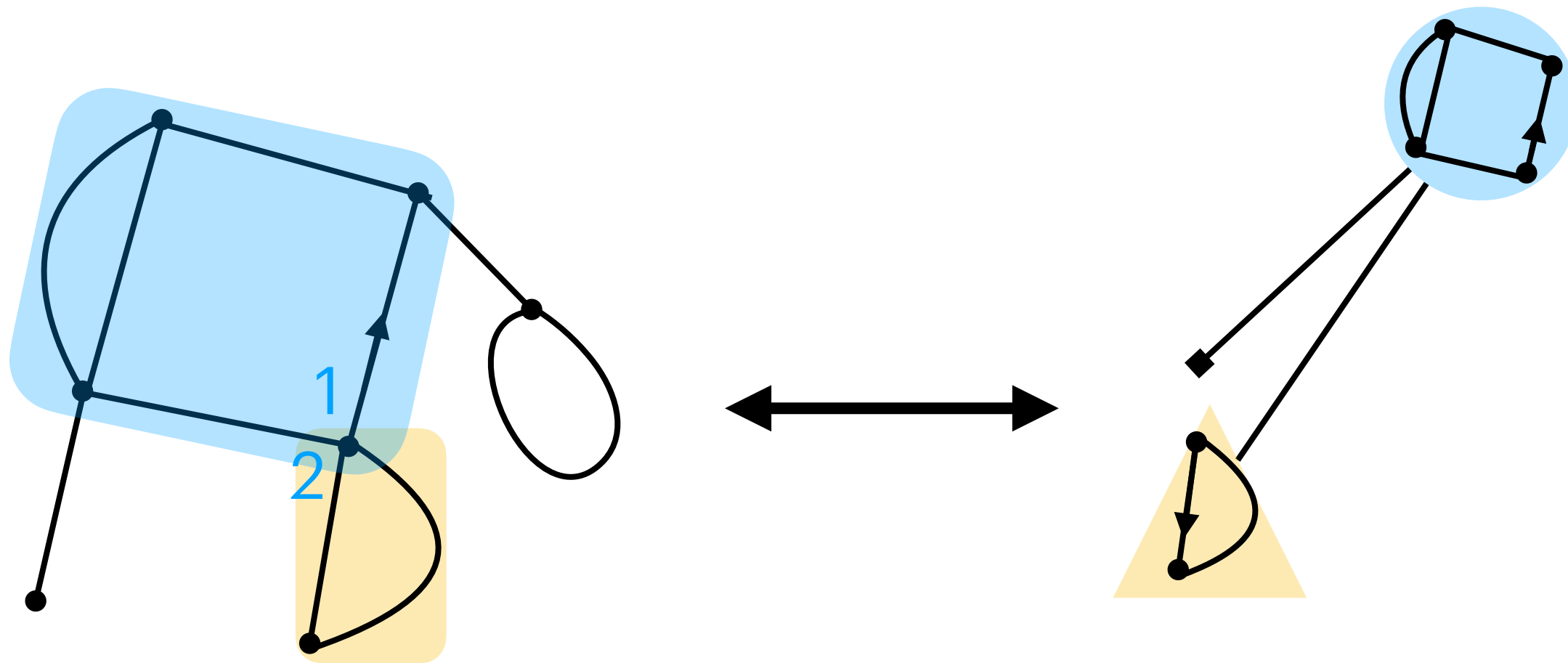
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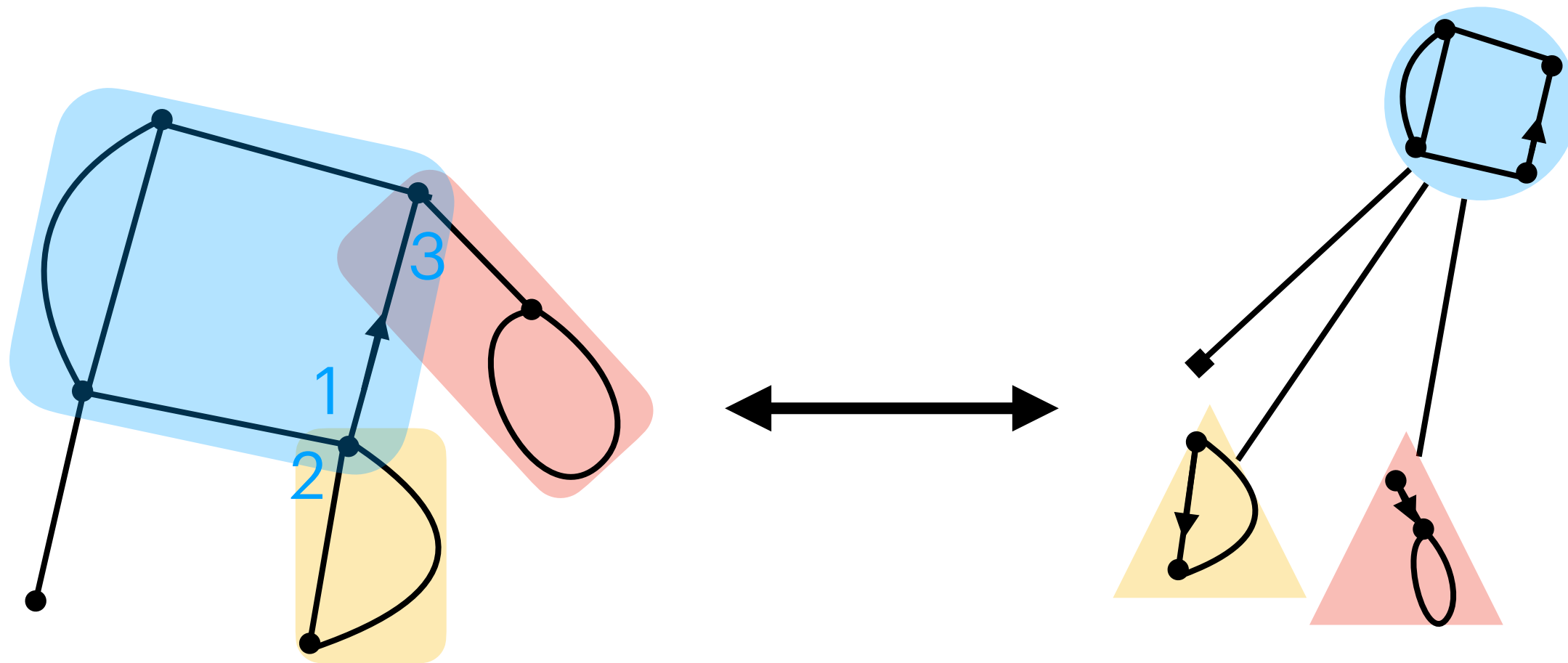
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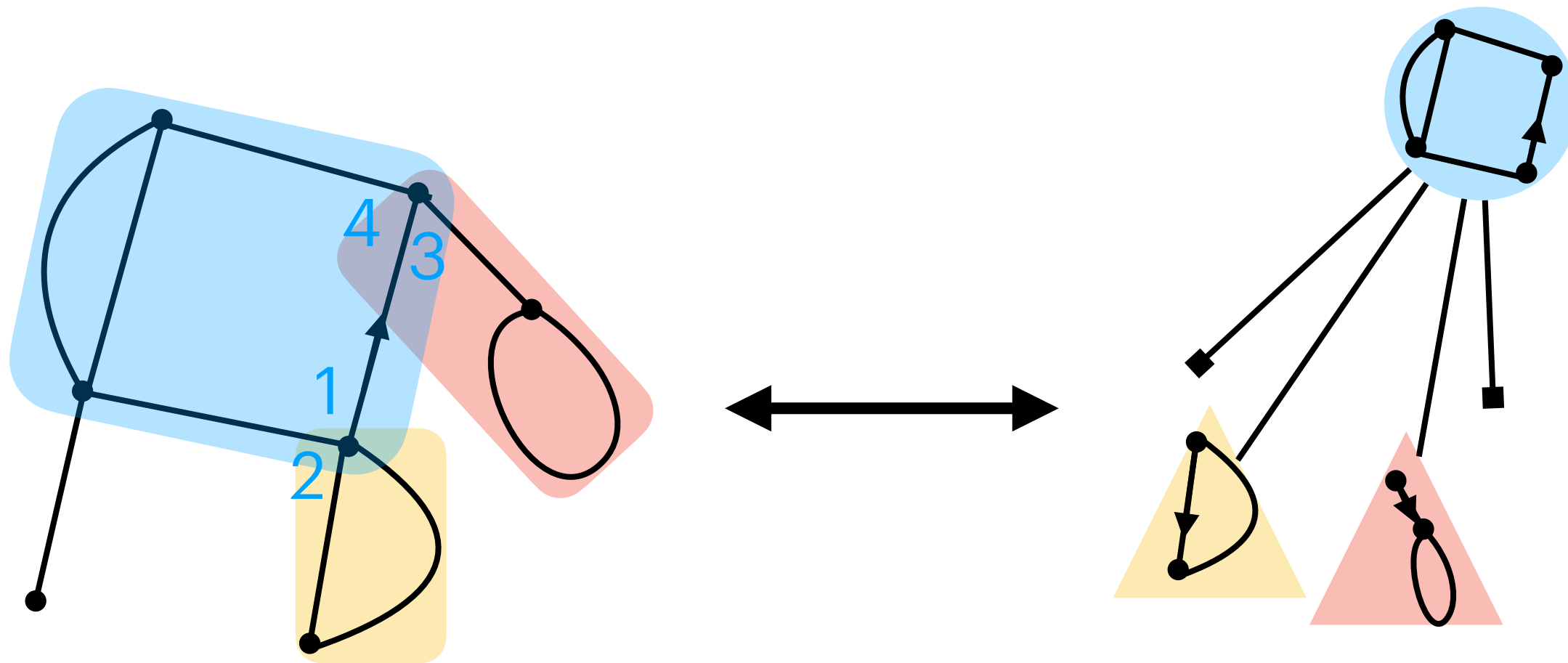
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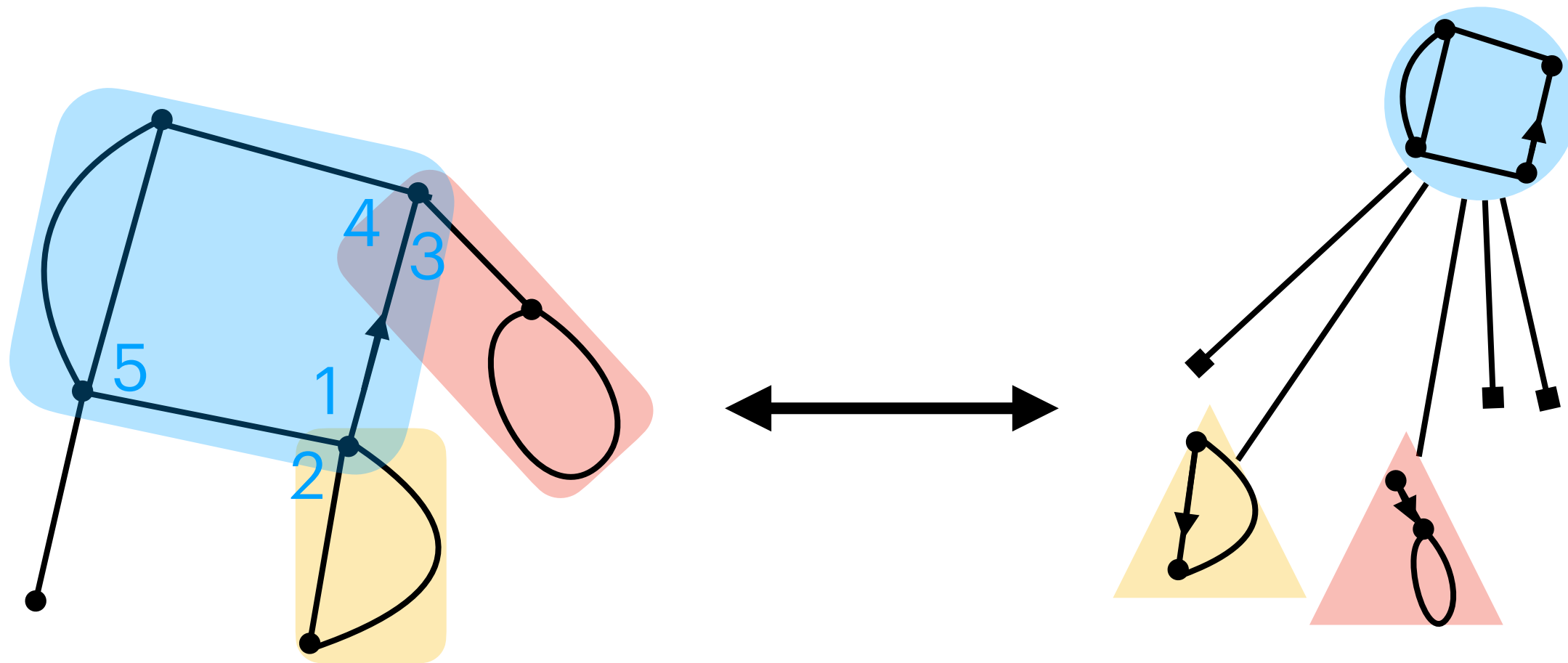
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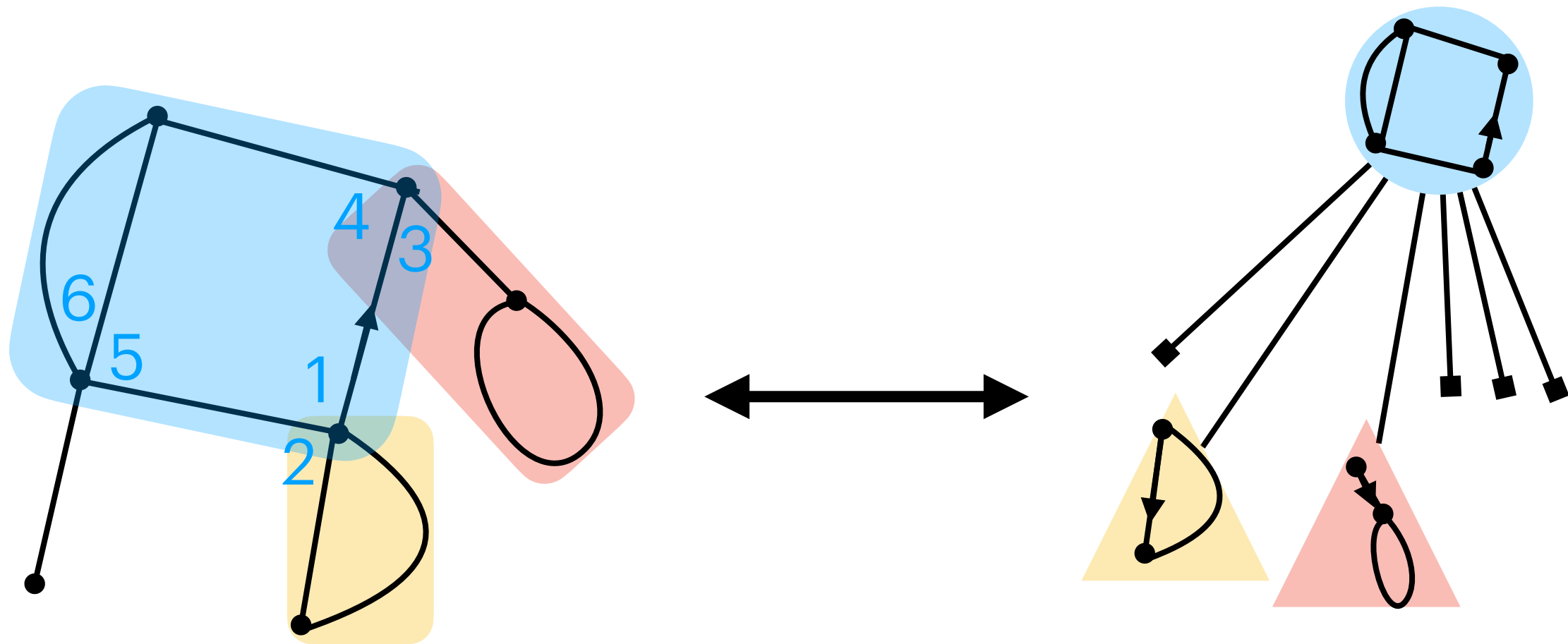
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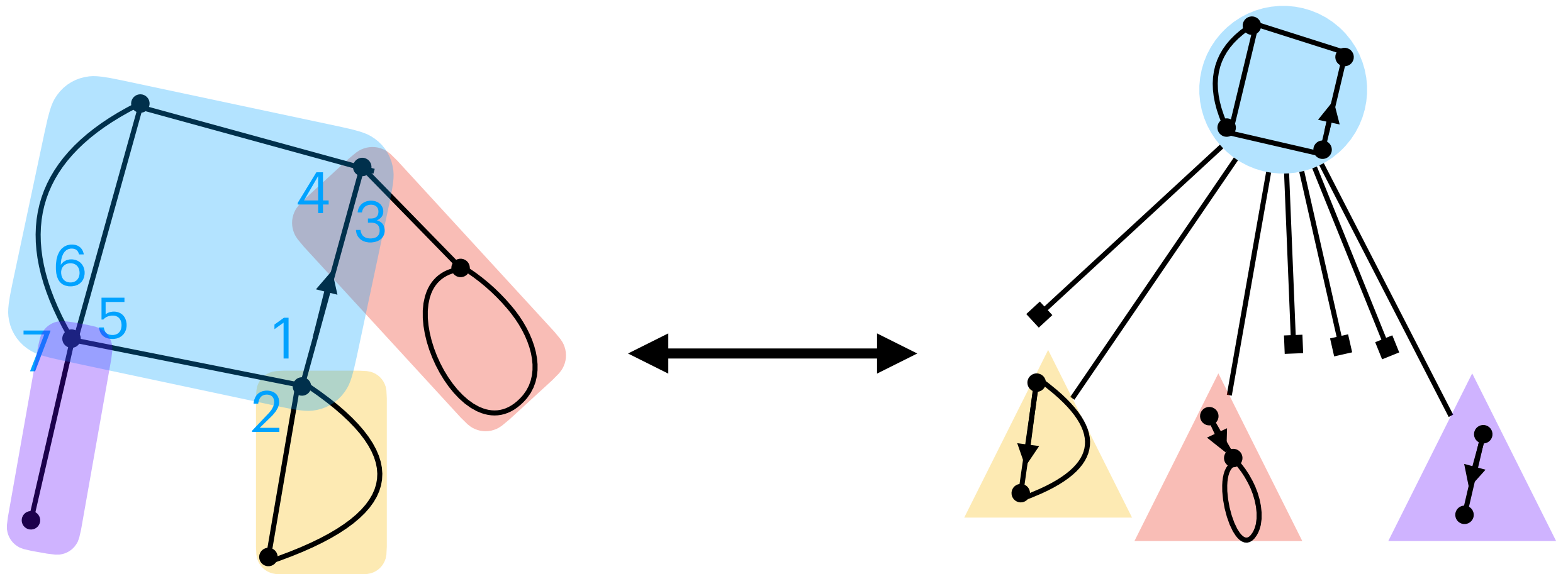
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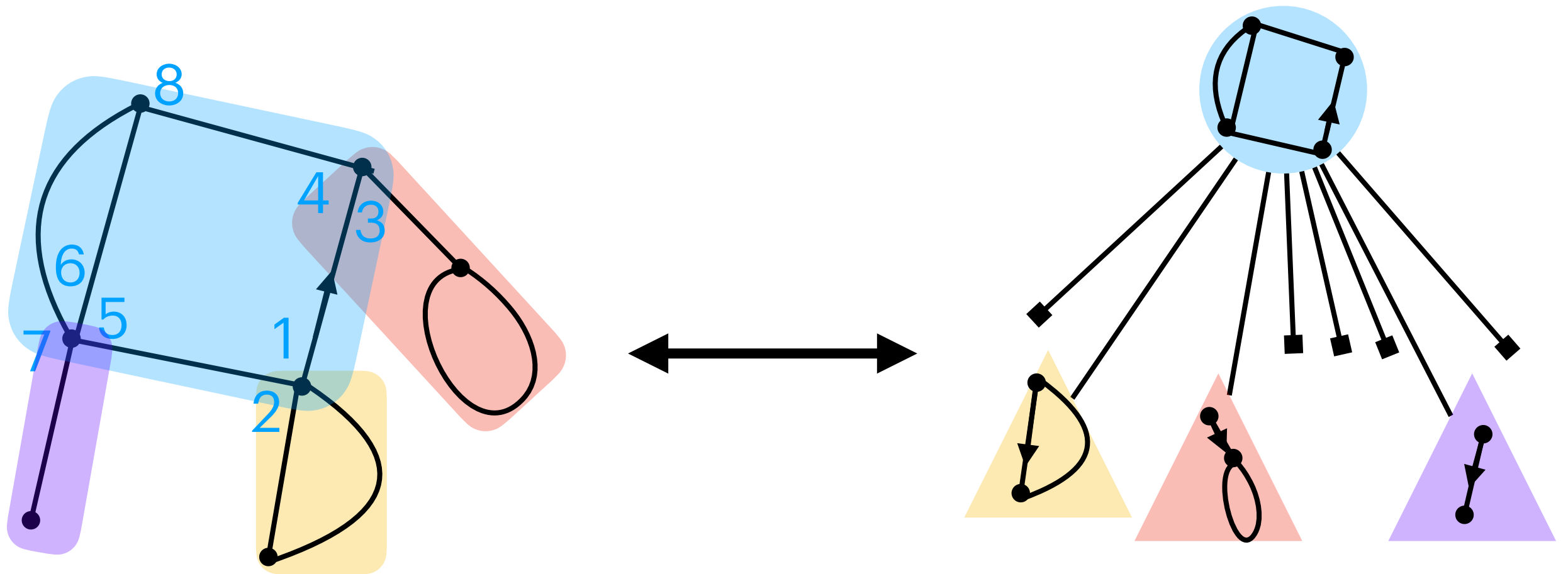
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Inspiration from [Tutte 1963]



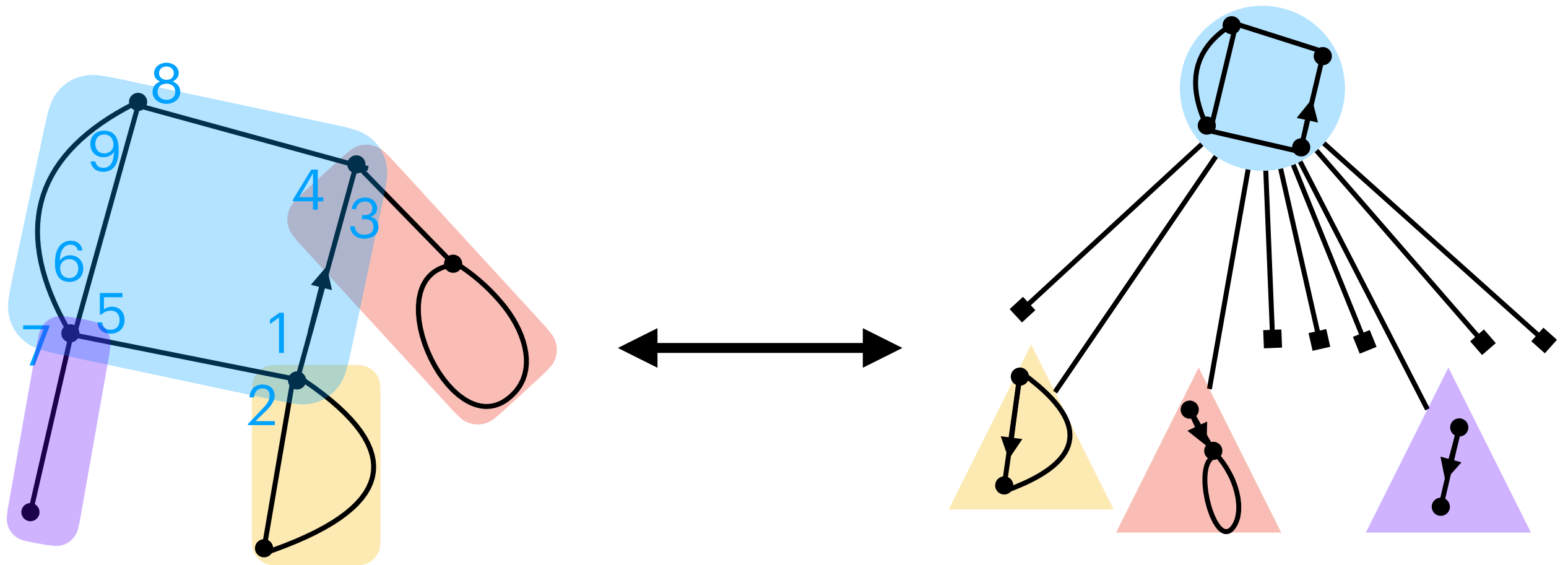
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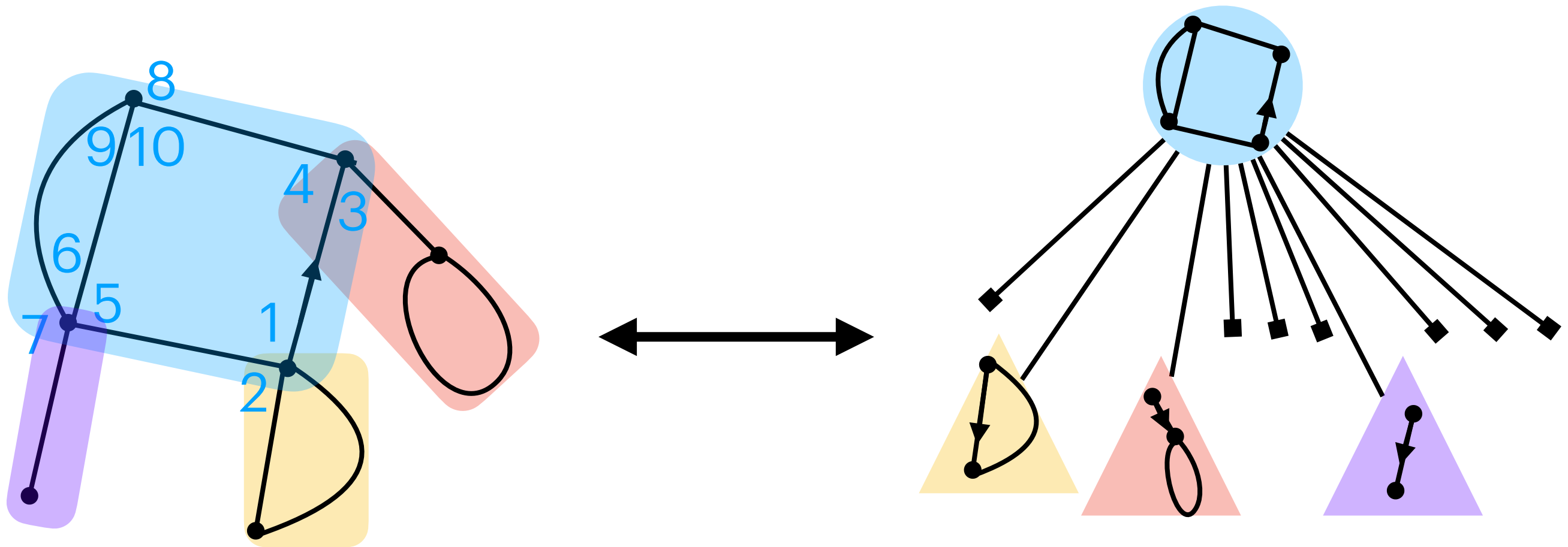
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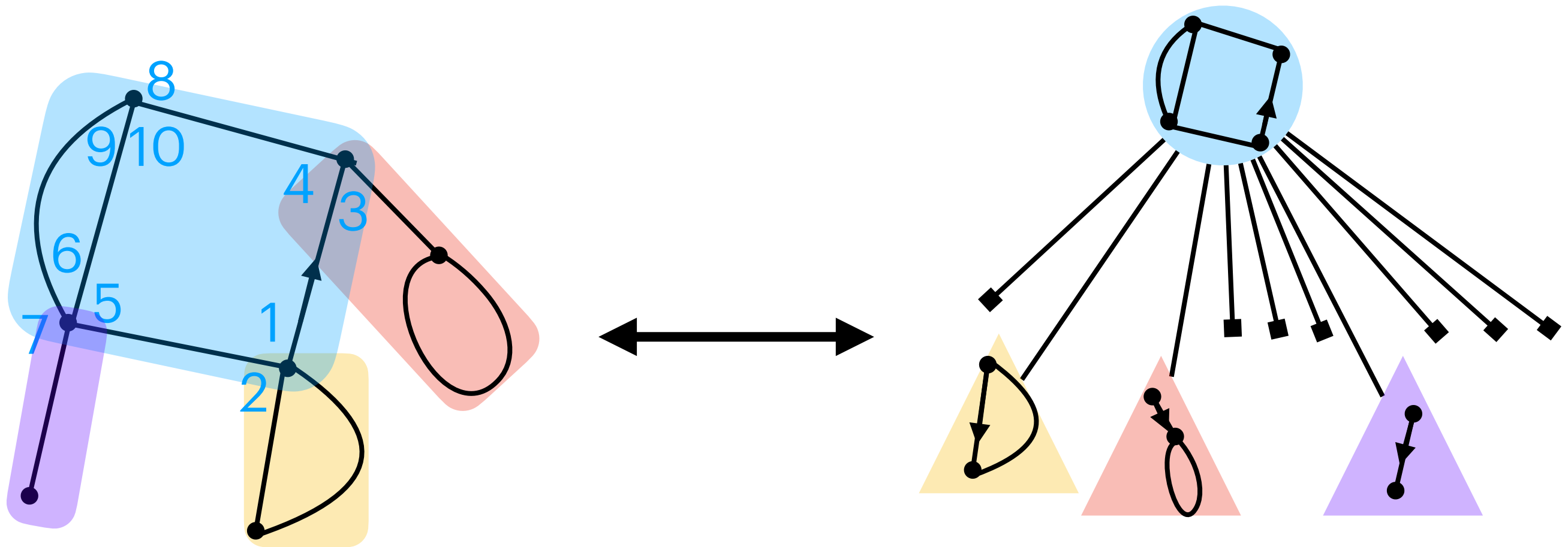
Decomposition of a map into blocks

Inspiration from [Tutte 1963]



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$$M(z) = B(zM^2(z))$$

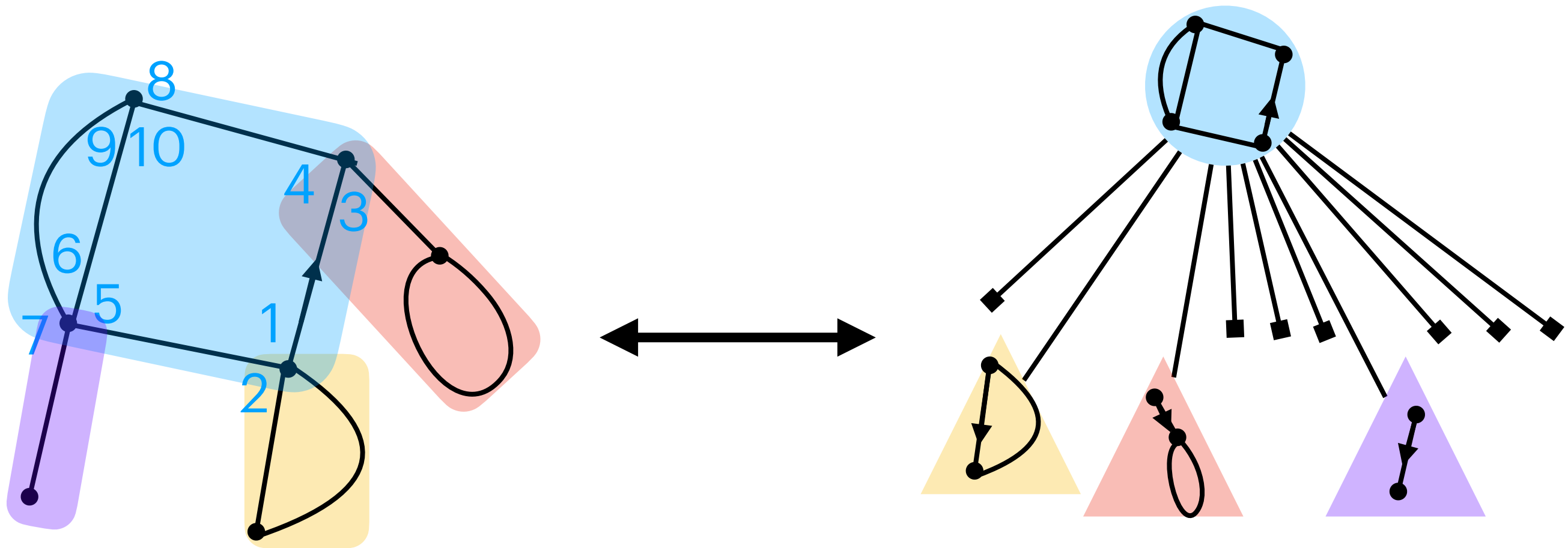
GS of maps

GS of 2-connected maps

Decomposition of a map into blocks

$$M(z, u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$

Inspiration from [Tutte 1963]



$$M(z) = B(zM^2(z))$$

GS of maps

GS of 2-connected maps

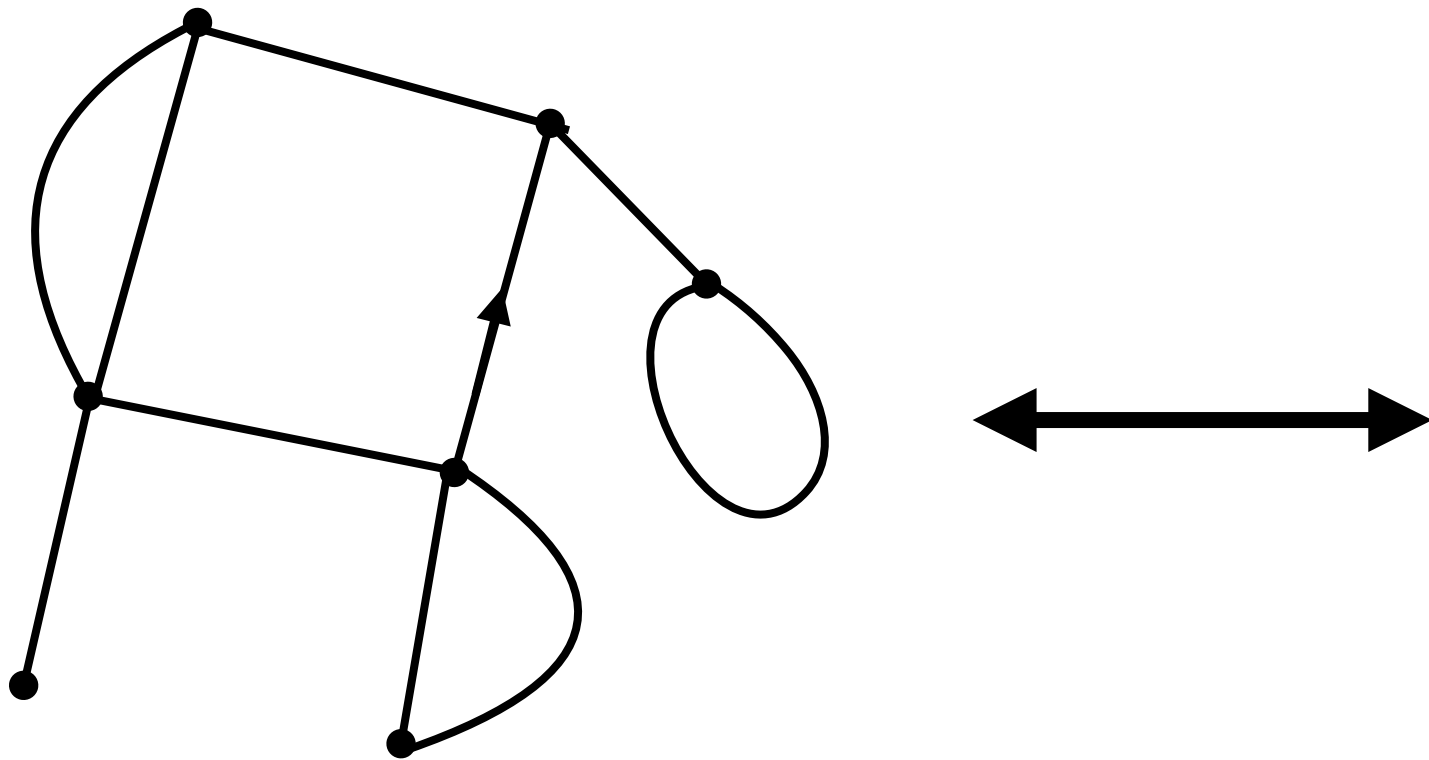
With a weight u on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration <small>[Bonzom 2016]</small>	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one			
Scaling limit of M_n			

Decomposition of a map into blocks

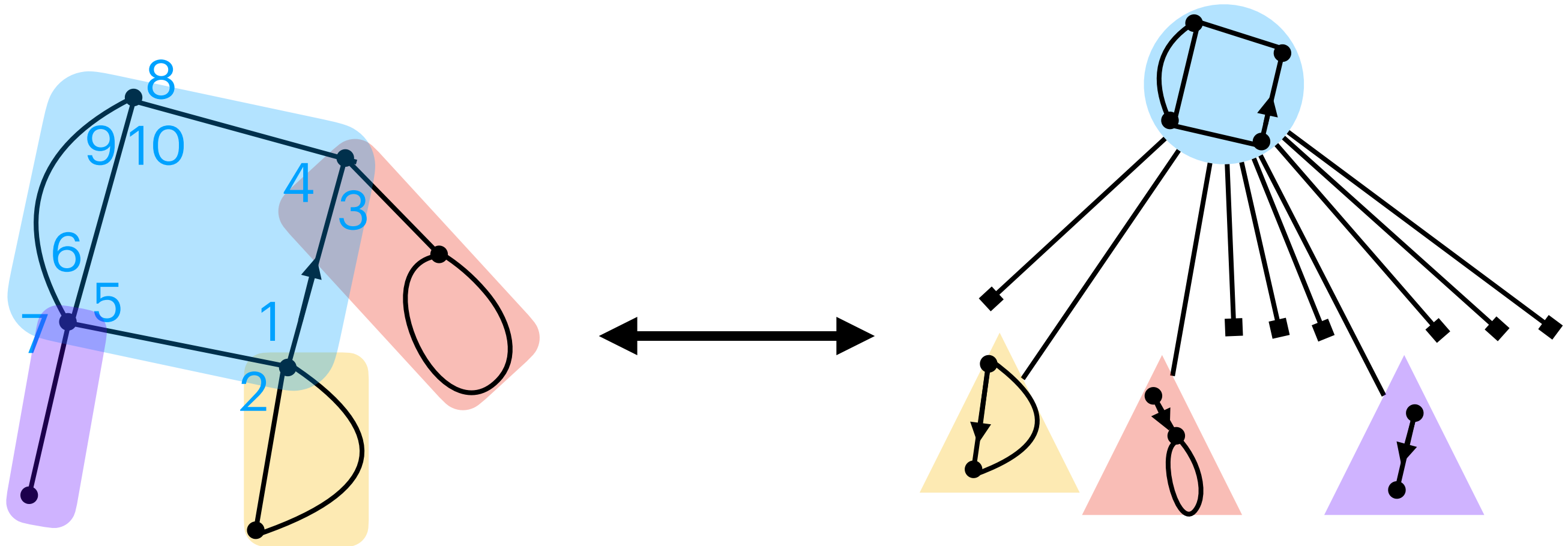
Inspiration from [Tutte 1963]



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

Decomposition of a map into blocks

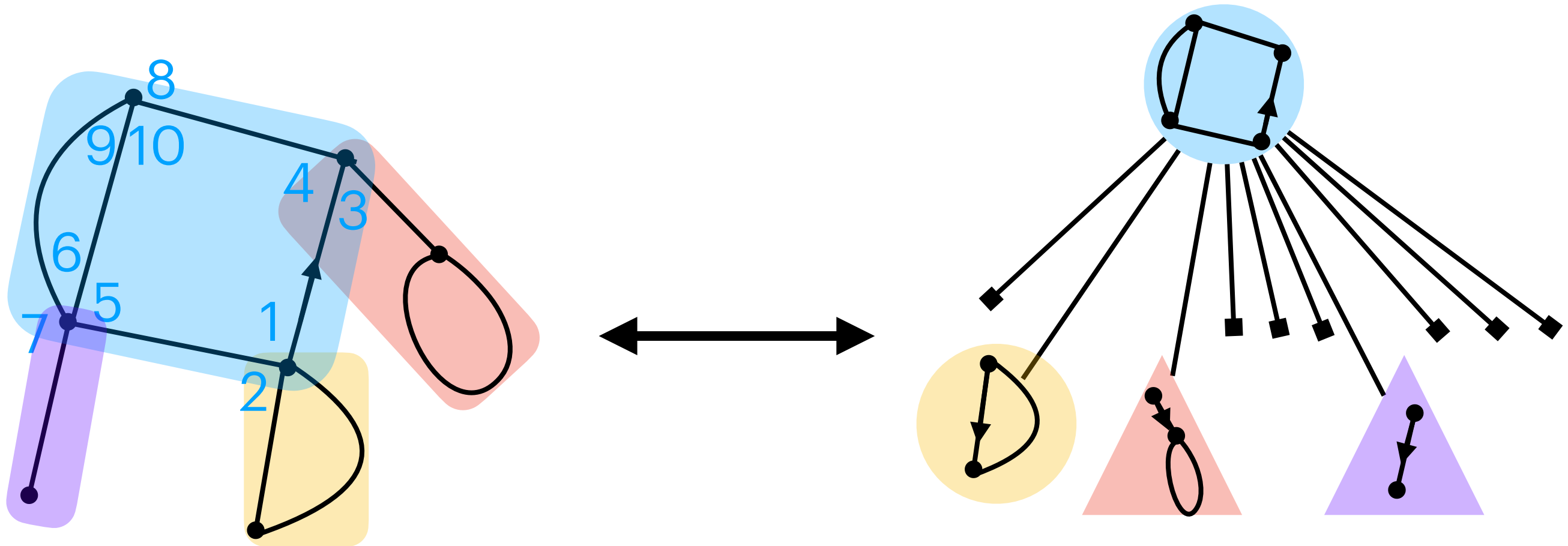
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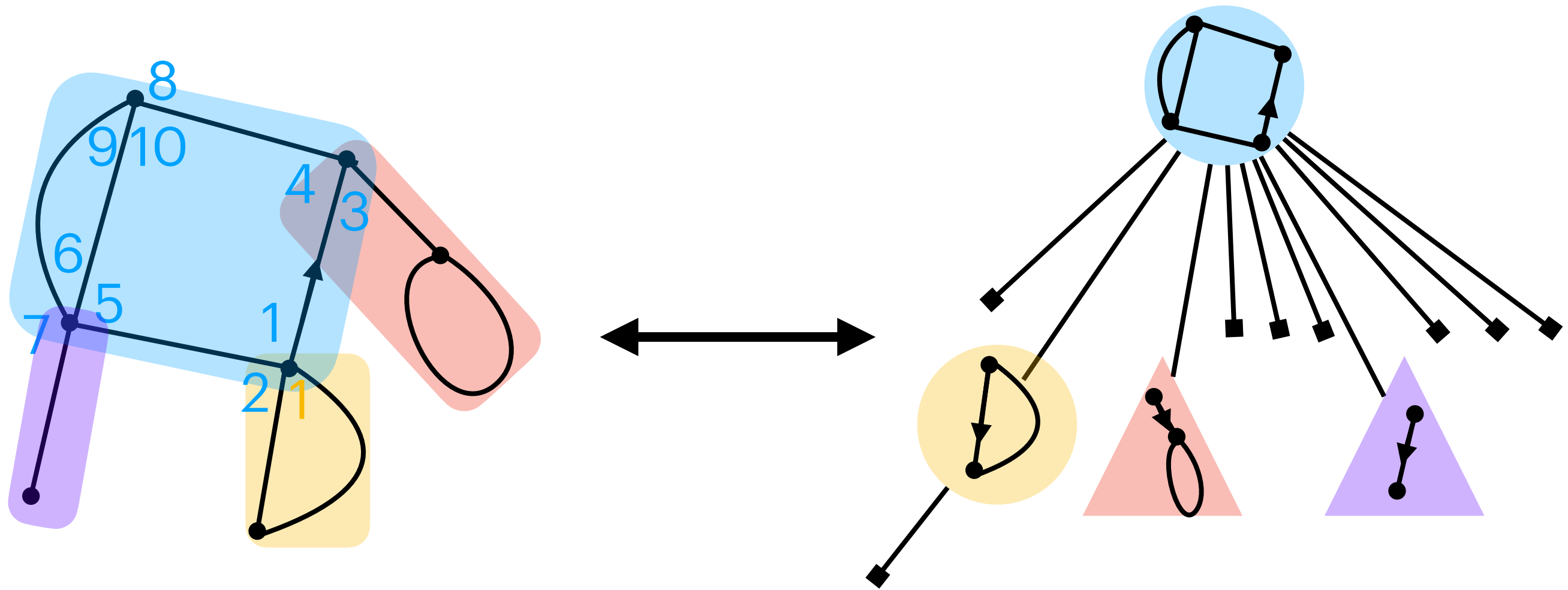
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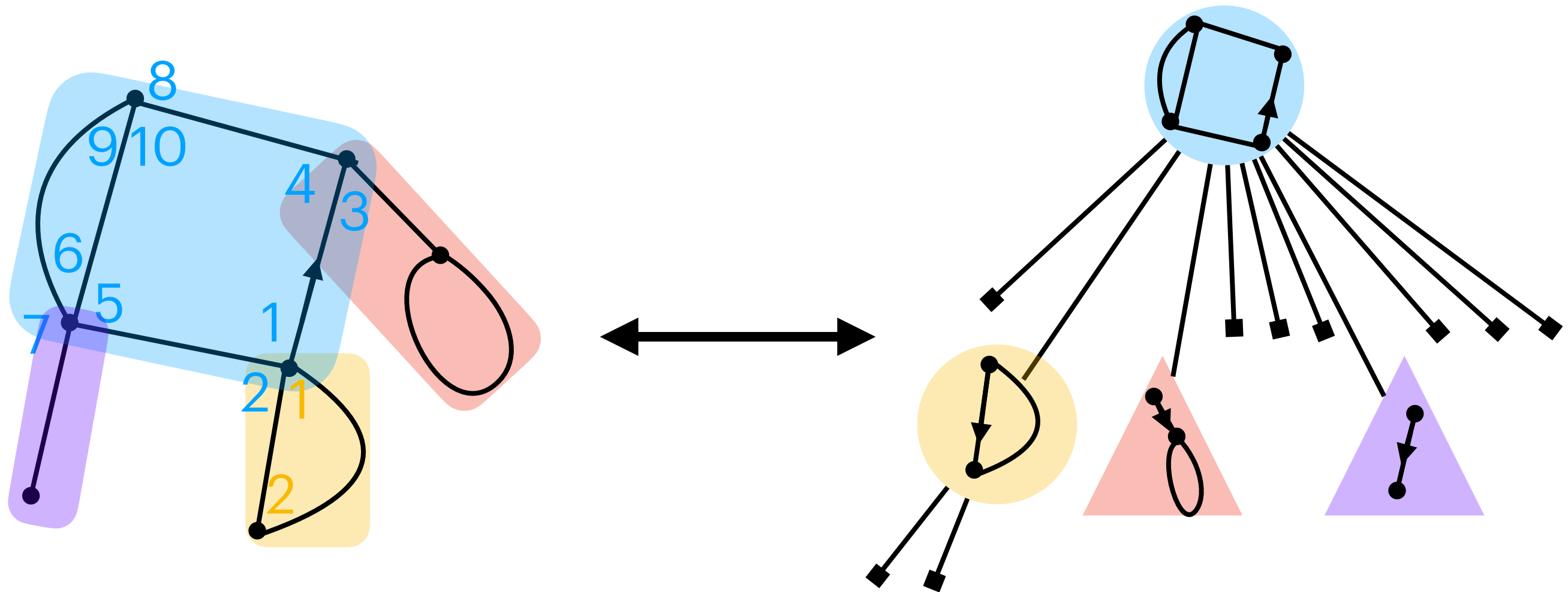
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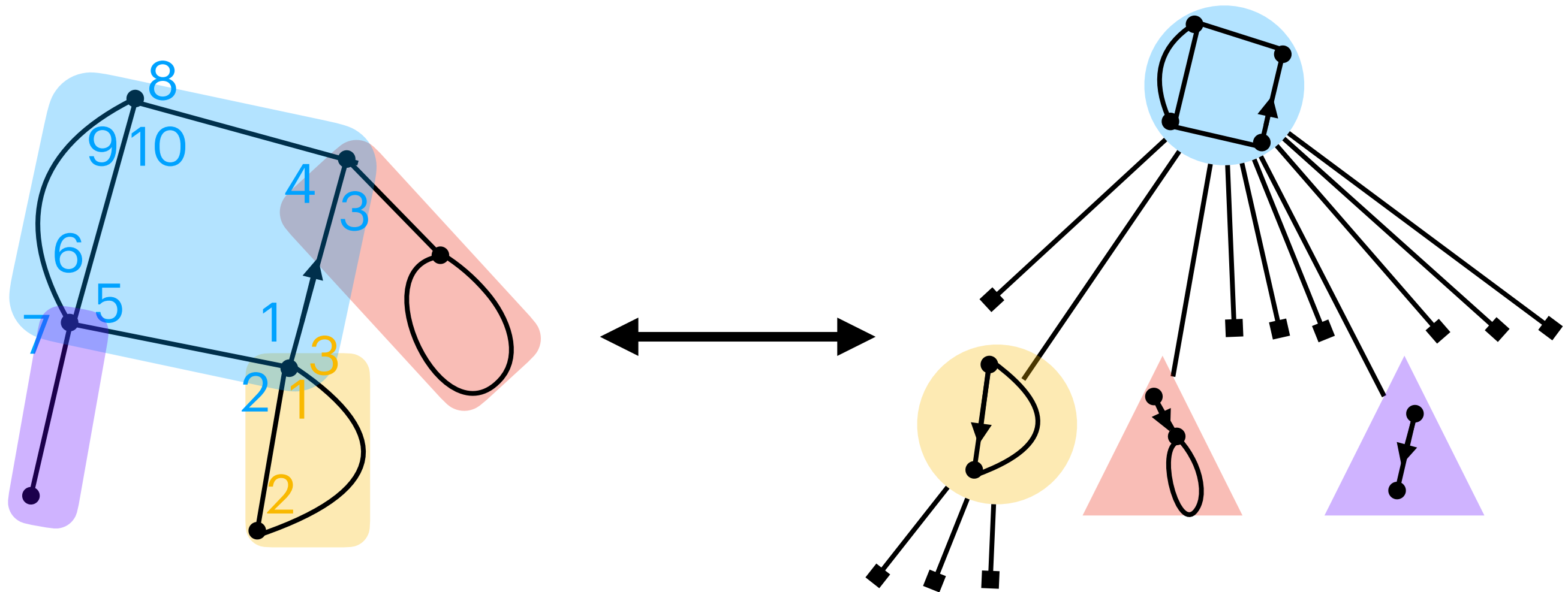
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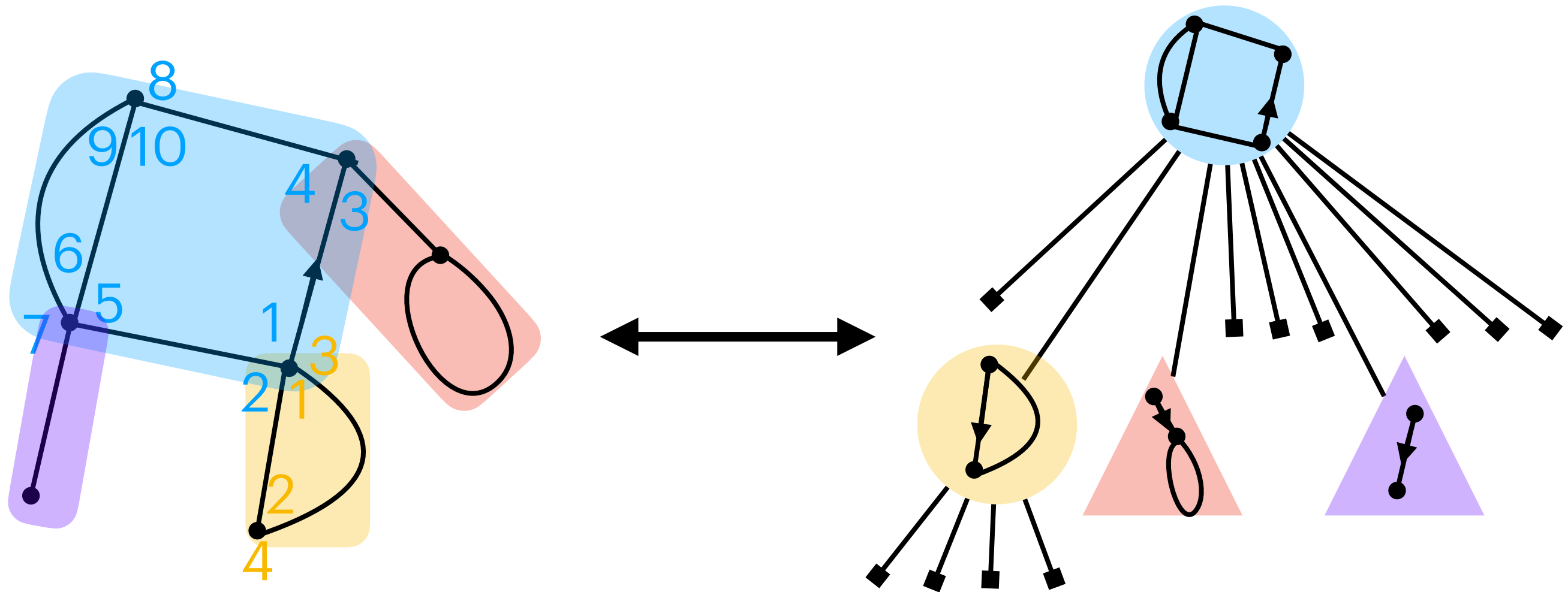
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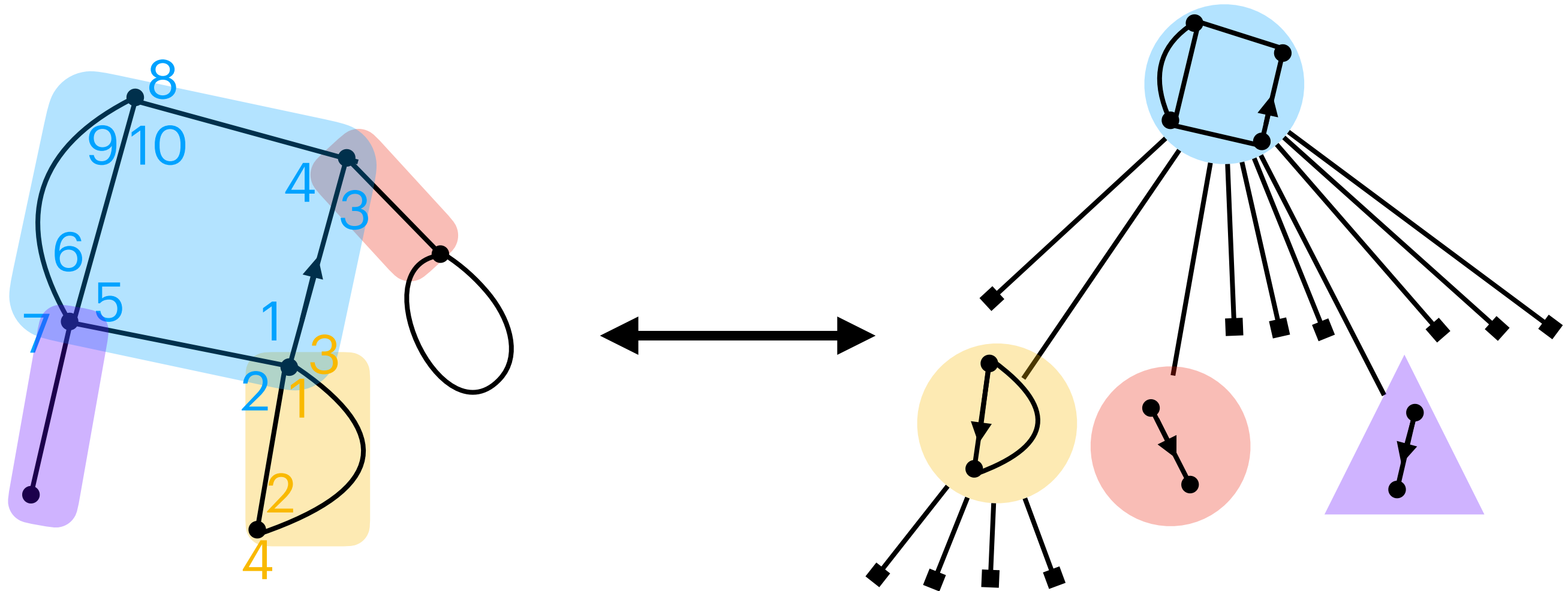
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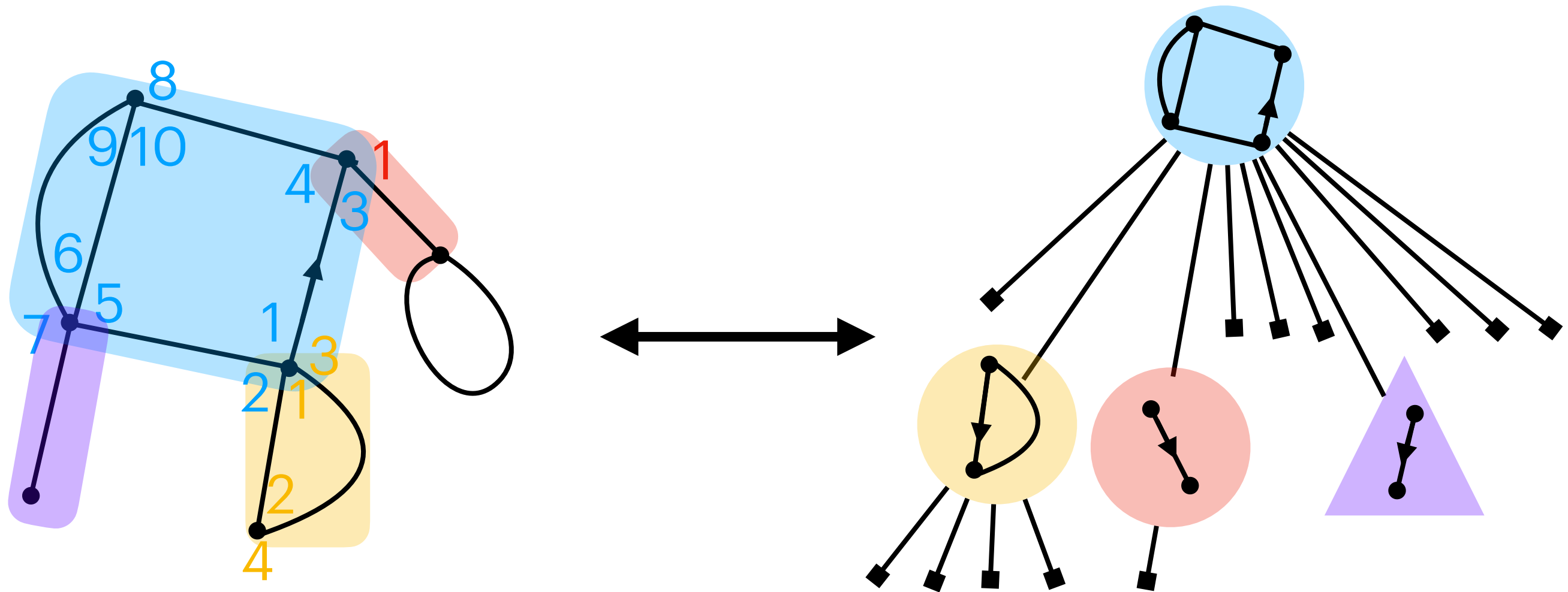
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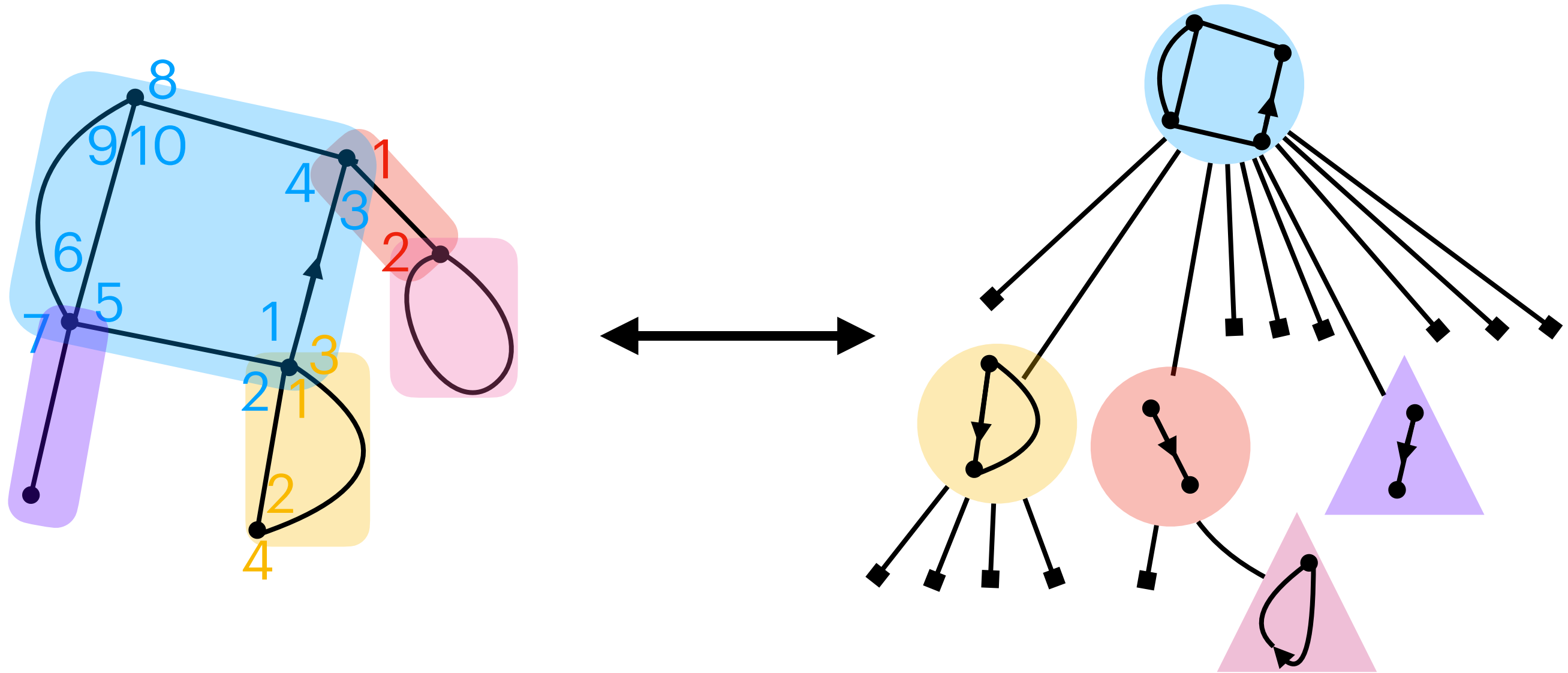
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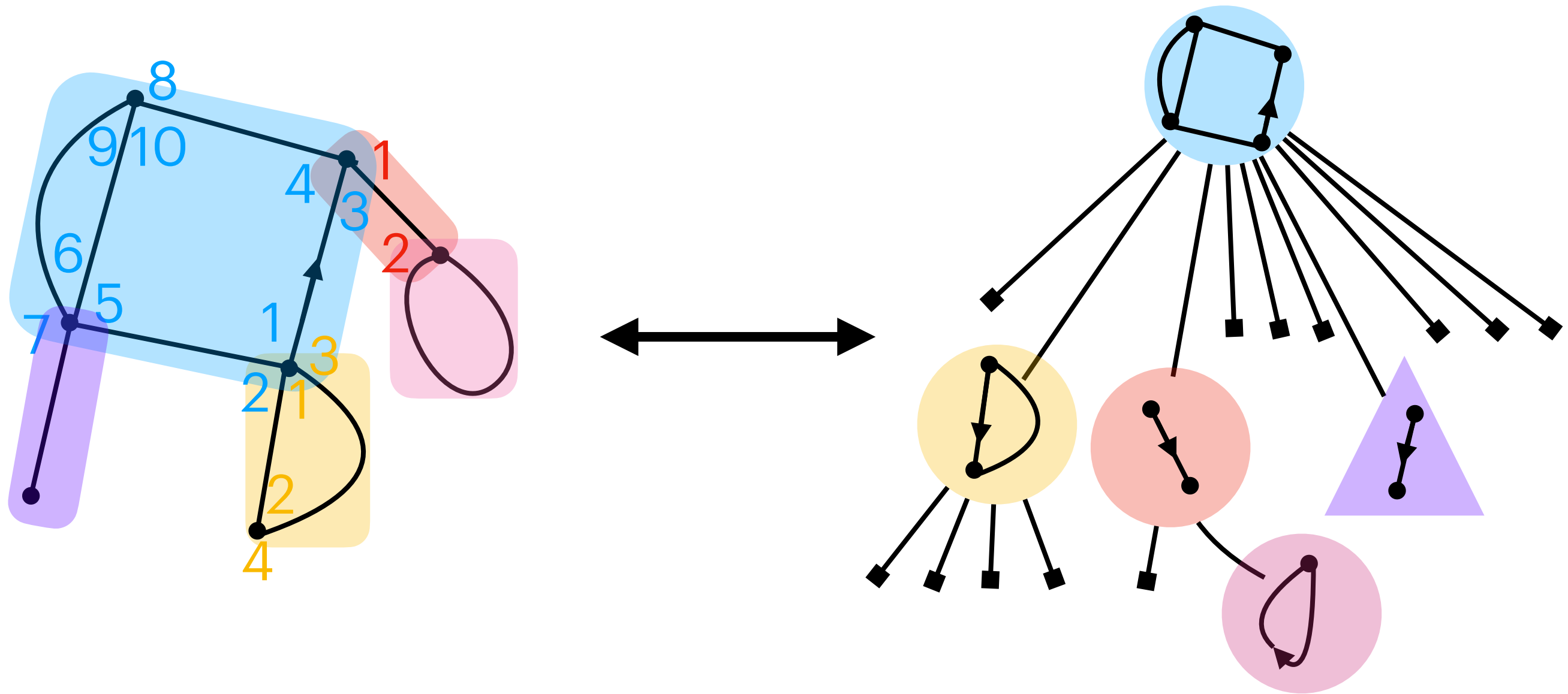
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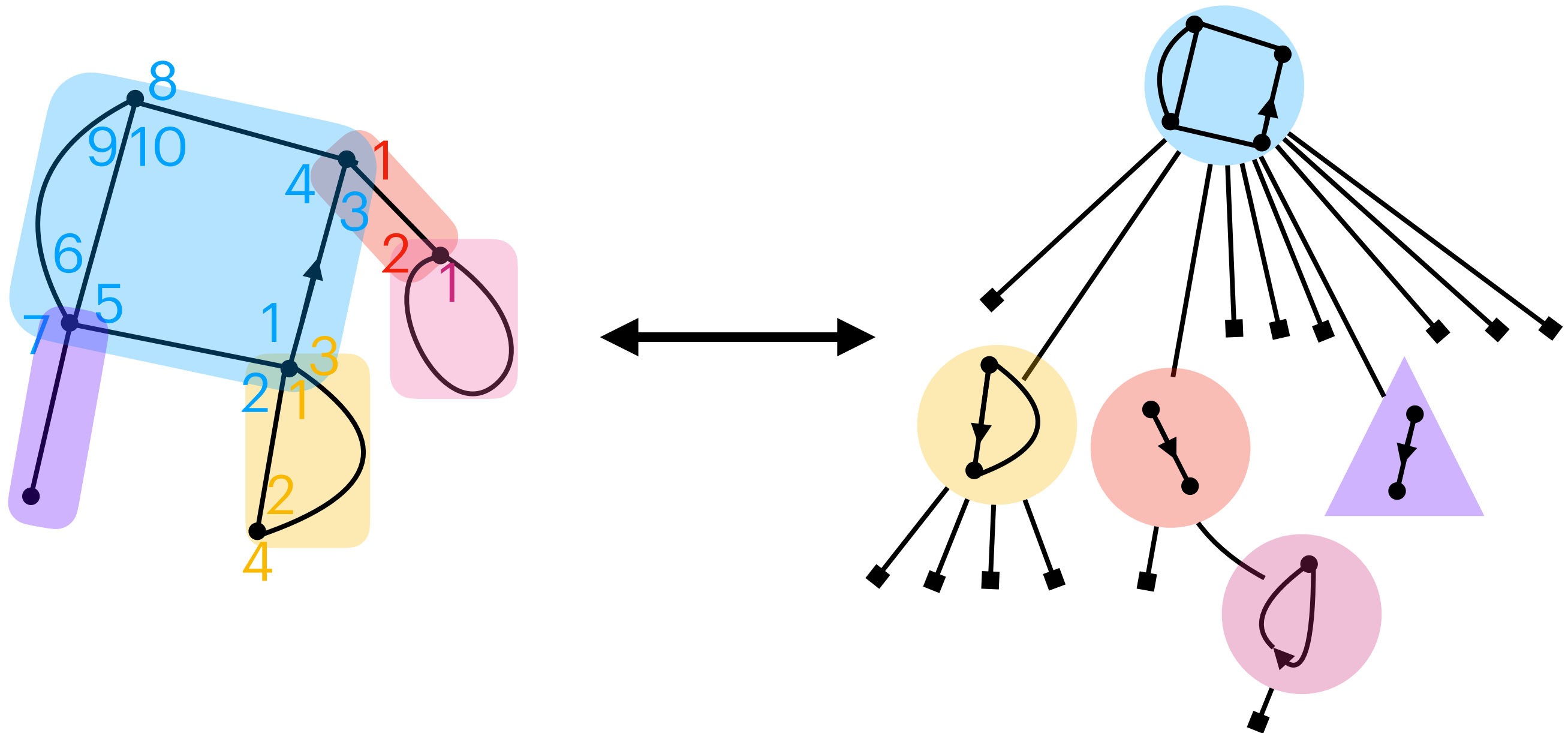
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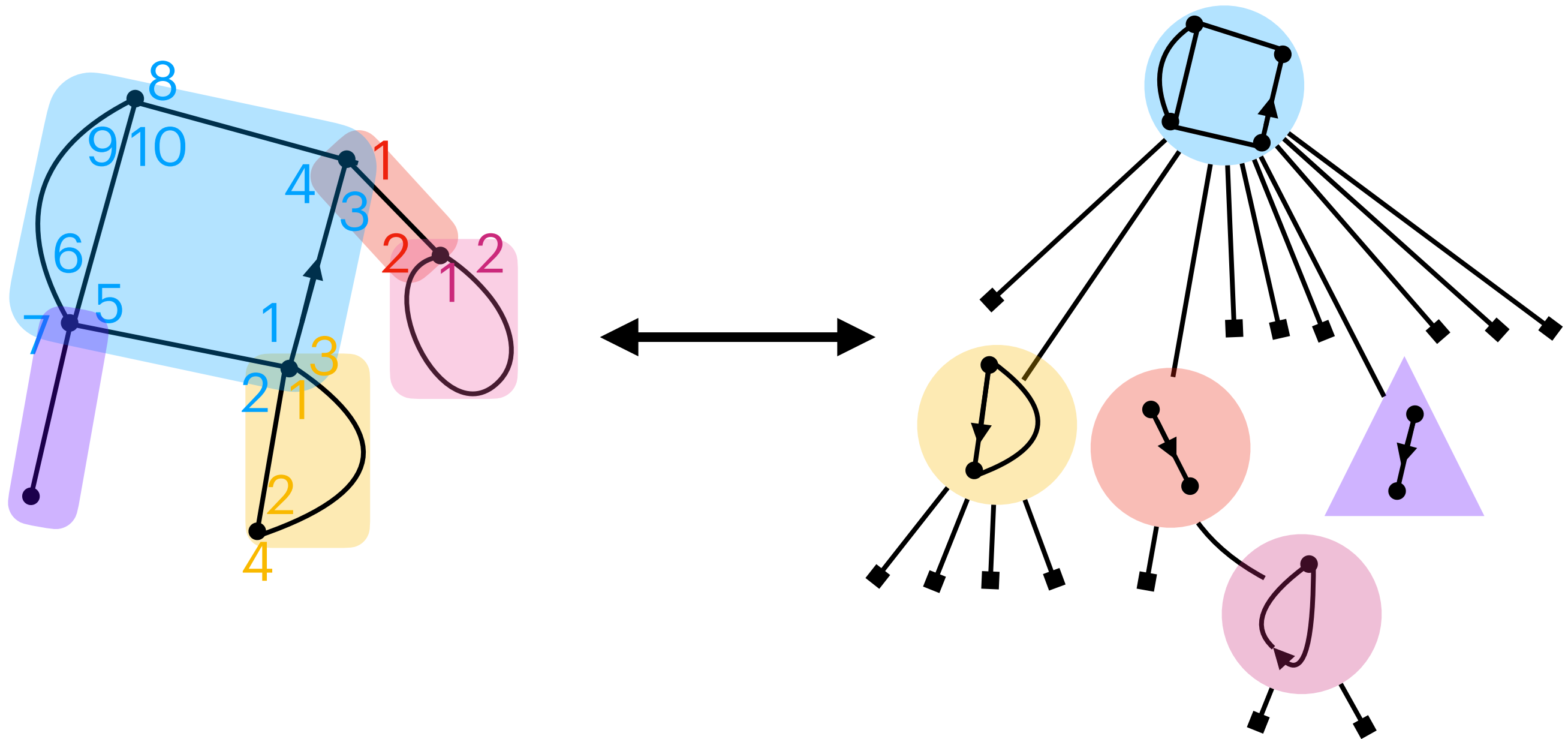
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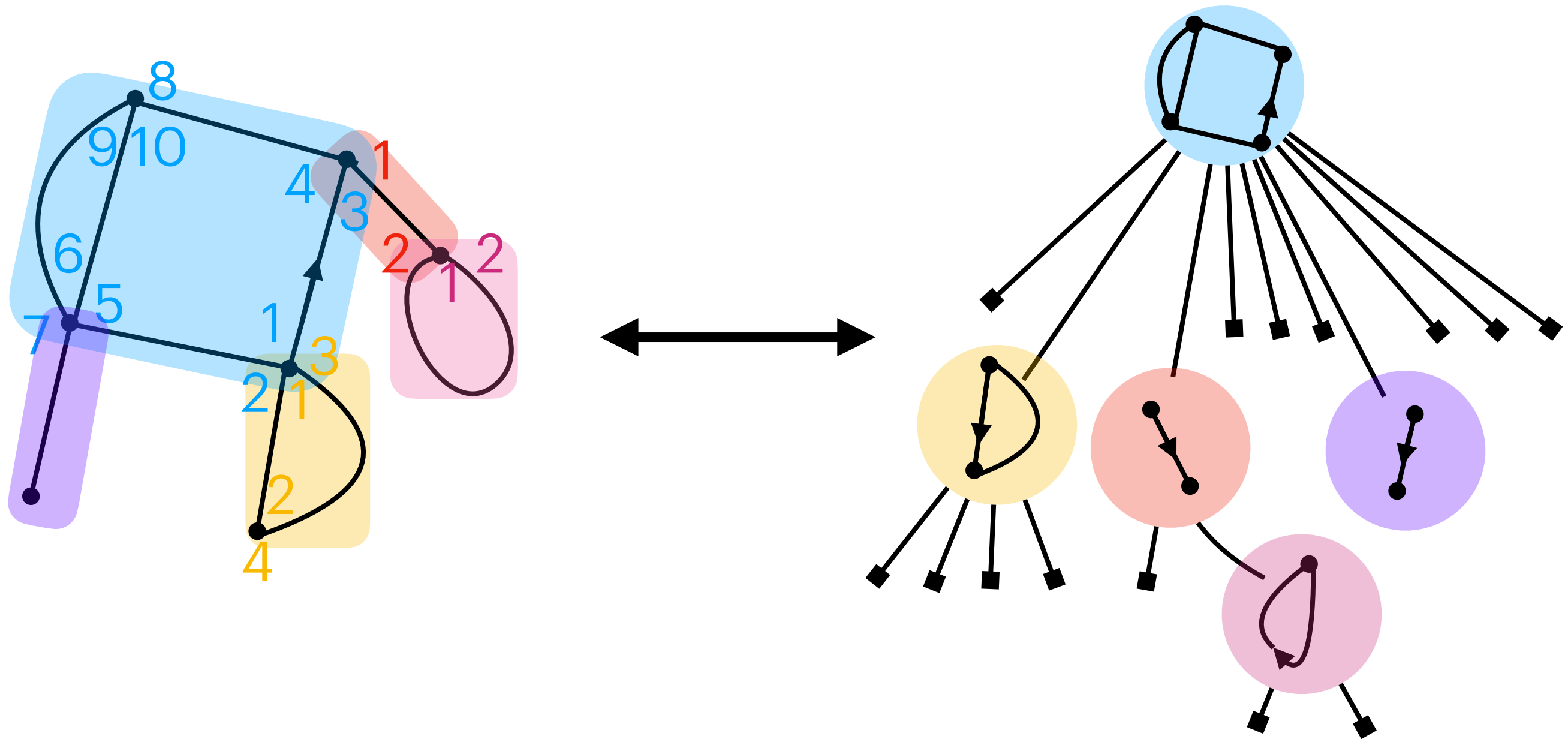
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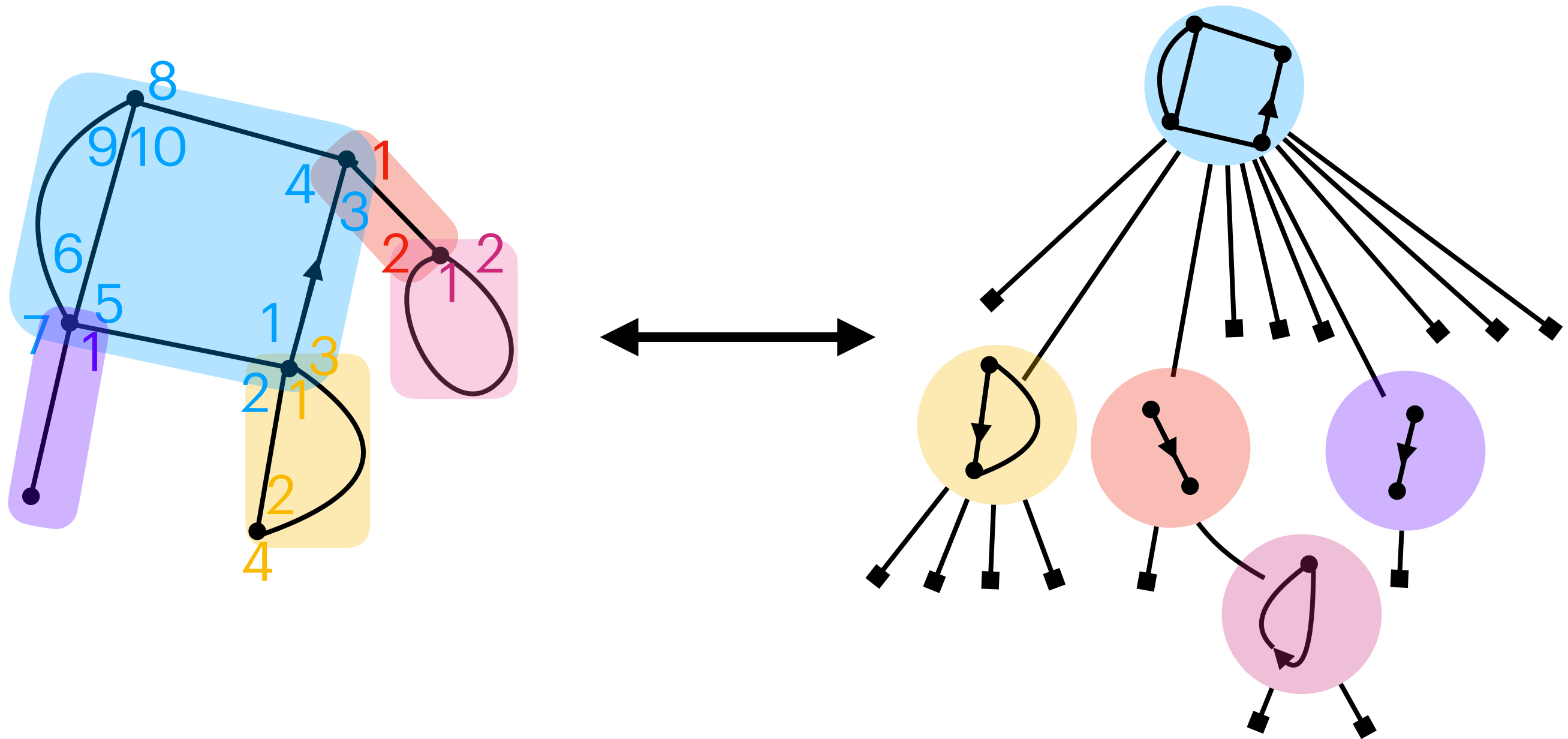
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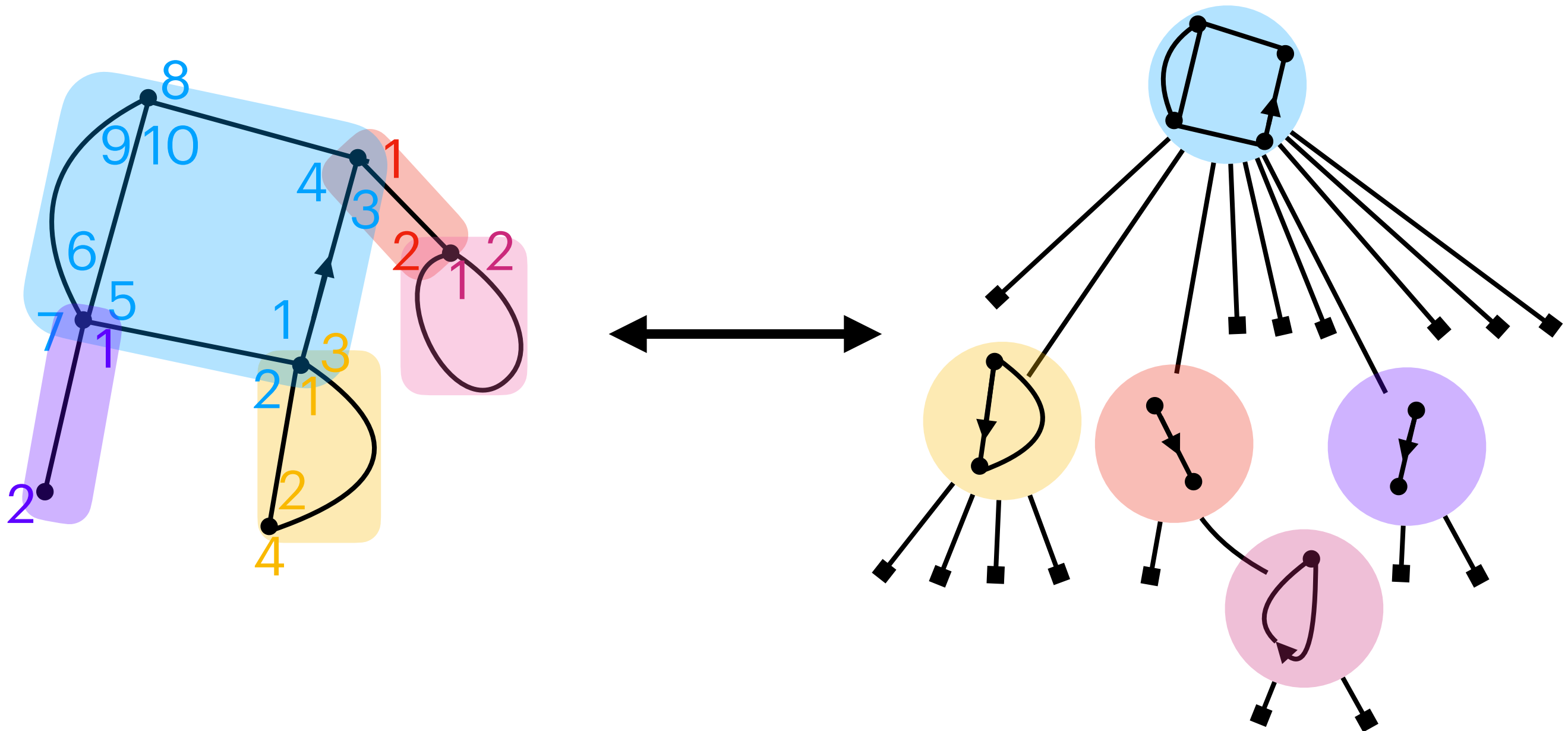
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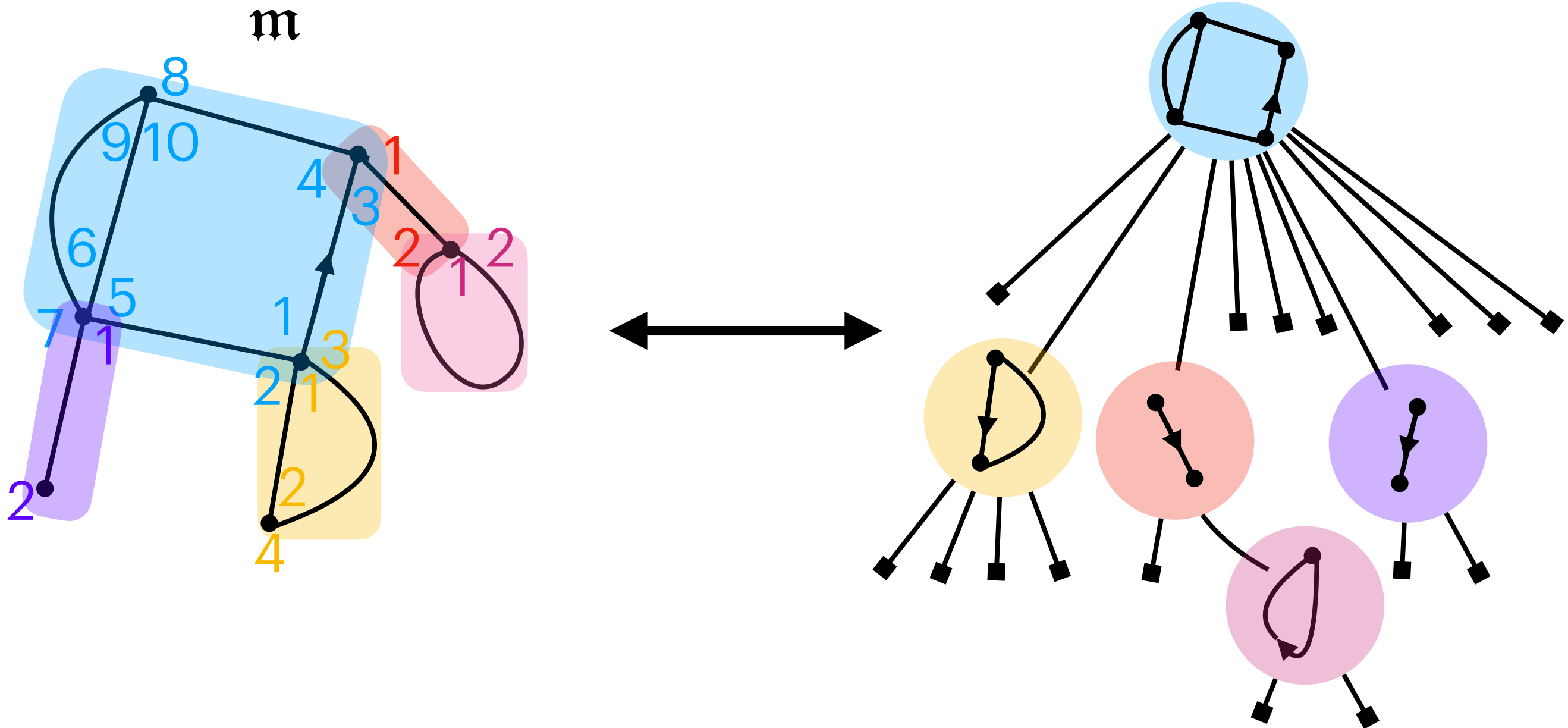
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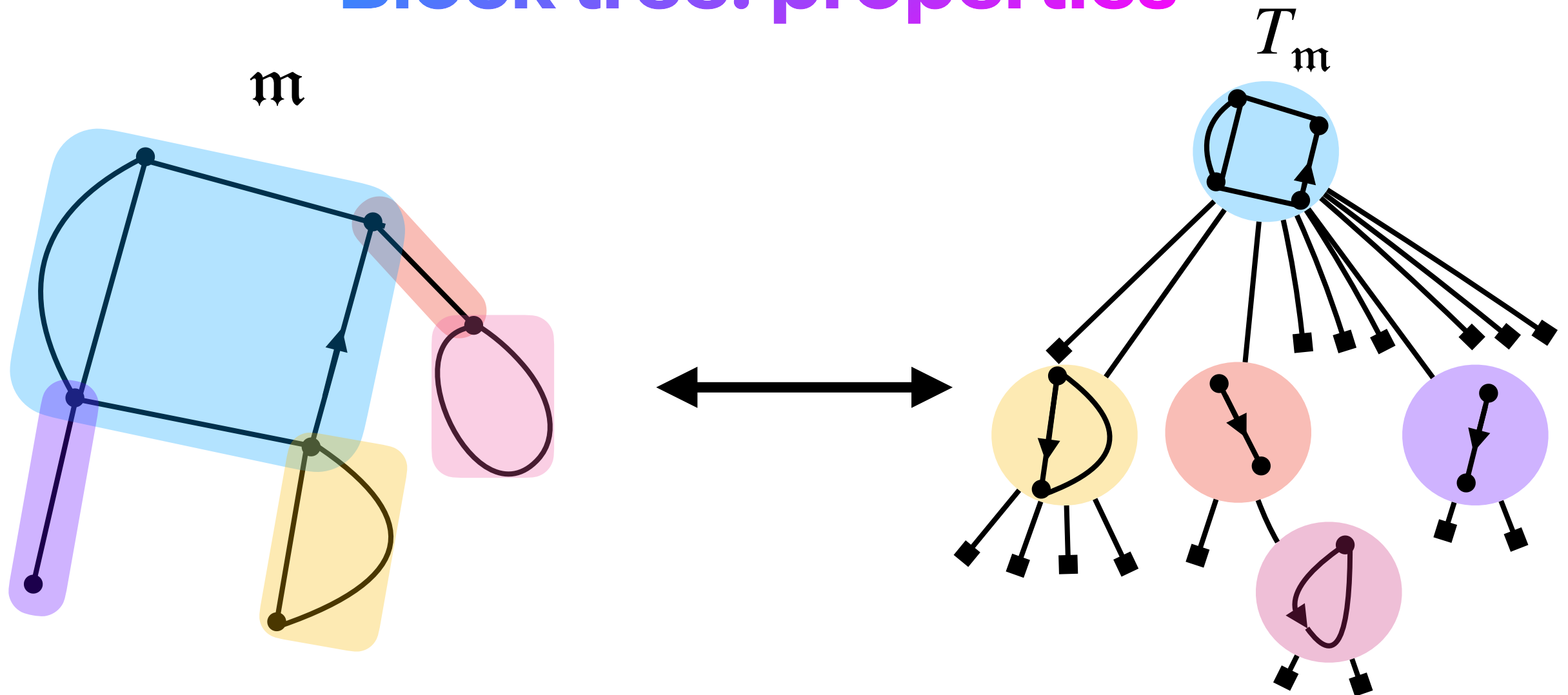
Decomposition of a map into blocks

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Block tree: properties



- \mathfrak{m} is entirely determined by $T_{\mathfrak{m}}$ and $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ where \mathfrak{b}_v is the block of \mathfrak{m} represented by v in $T_{\mathfrak{m}}$;
- Internal node (with k children) of $T_{\mathfrak{m}} \leftrightarrow$ block of \mathfrak{m} of size $k/2$.

T_{M_n} gives the block sizes of a random map M_n .

Block trees are BGW-trees

μ -Bienaymé-Galton-Watson (BGW) tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on \mathbb{N} .

Theorem [Fleurat, S. 24]

$u > 0$

If $M_n \hookrightarrow \mathbb{P}_{n,u'}$ then there exists an (explicit) $y = y(u)$ s.t.

T_{M_n} has the law of a BGW tree of reproduction law $\mu^{y,u}$

conditioned to be of size $2n$, with

$$\mu^{y,u}(\{2k\}) = \frac{B_k y^k u^{1_{k \neq 0}}}{uB(y) + 1 - u}.$$

Largest blocks?

- Degrees of T_{M_n} give the block sizes of the map M_n ;
- Largest degrees of a BGW tree are well-known [Janson 2012].

Rough intuition

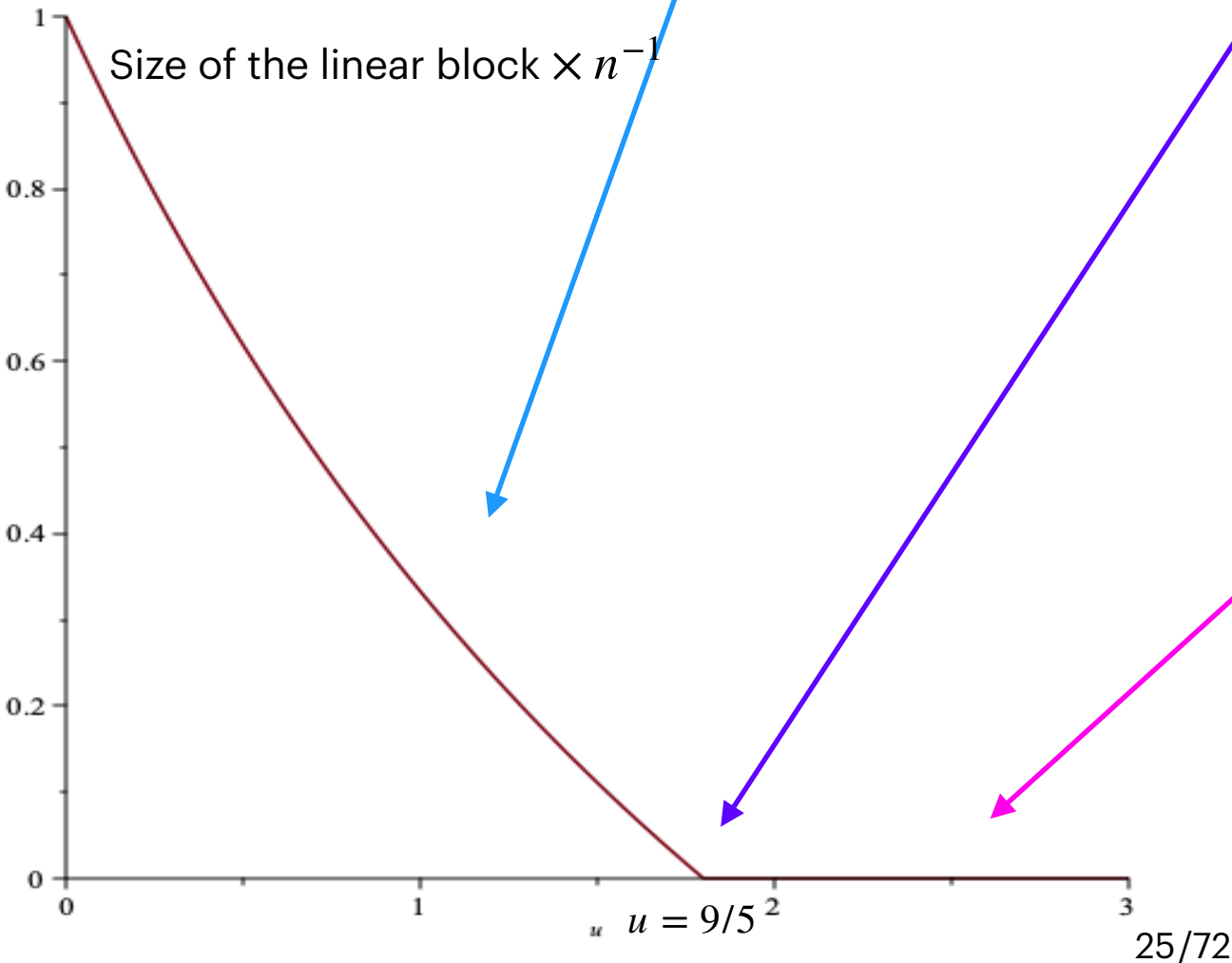
	$u < 9/5$	$u = 9/5$	$u > 9/5$
$\mu^{y(u),u}(\{2k\})$	$\sim c_u k^{-5/2}$		$\sim c_u \pi_u^k k^{-5/2}$
BGW tree	subcritical	critical	

Dichotomy between situations:

- Subcritical: condensation, cf [Jonsson Stefánsson 2011];
- Supercritical: behaves as maximum of independent variables.

Size $L_{n,k}$ of the k -th largest block

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
$L_{n,1}$	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2 \ln\left(\frac{4}{27y}\right)} - \frac{5 \ln(\ln(n))}{4 \ln\left(\frac{4}{27y}\right)} + O(1)$
$L_{n,2}$	$\Theta(n^{2/3})$ [Stufler 2020]		



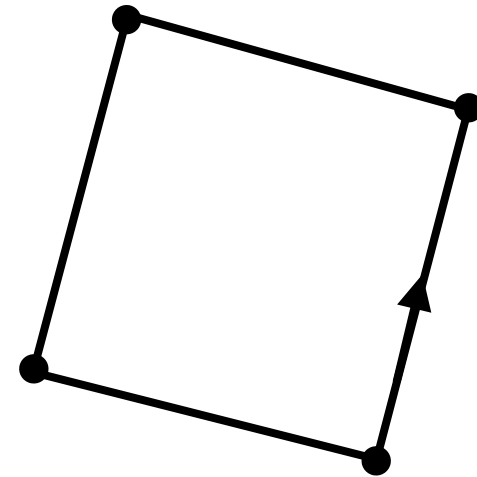
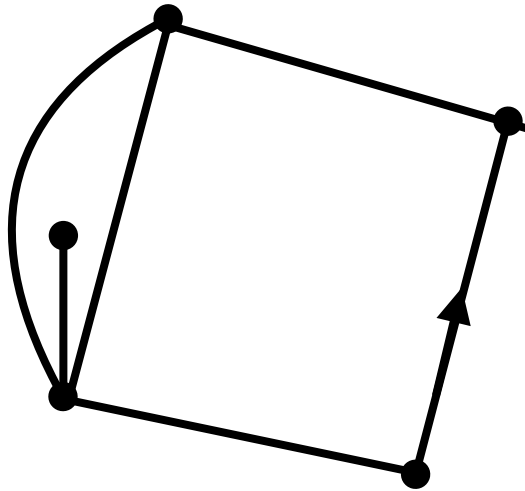
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2 \ln\left(\frac{4}{27y}\right)} - \frac{5 \ln(\ln(n))}{4 \ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

Interlude: quadrangulations

Quadrangulations

Def: map with all faces of degree 4.

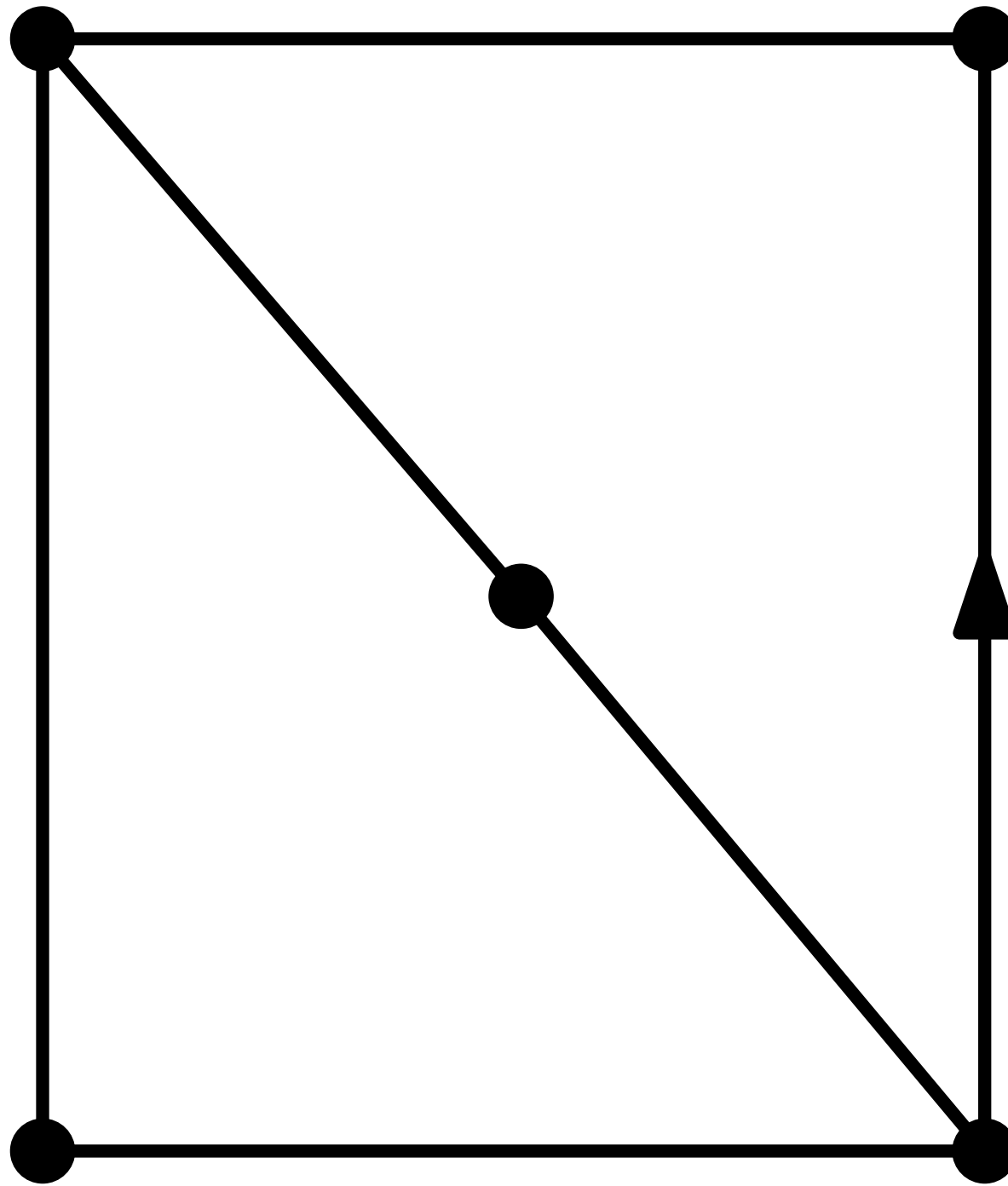


Simple quadrangulation = no multiple edges.

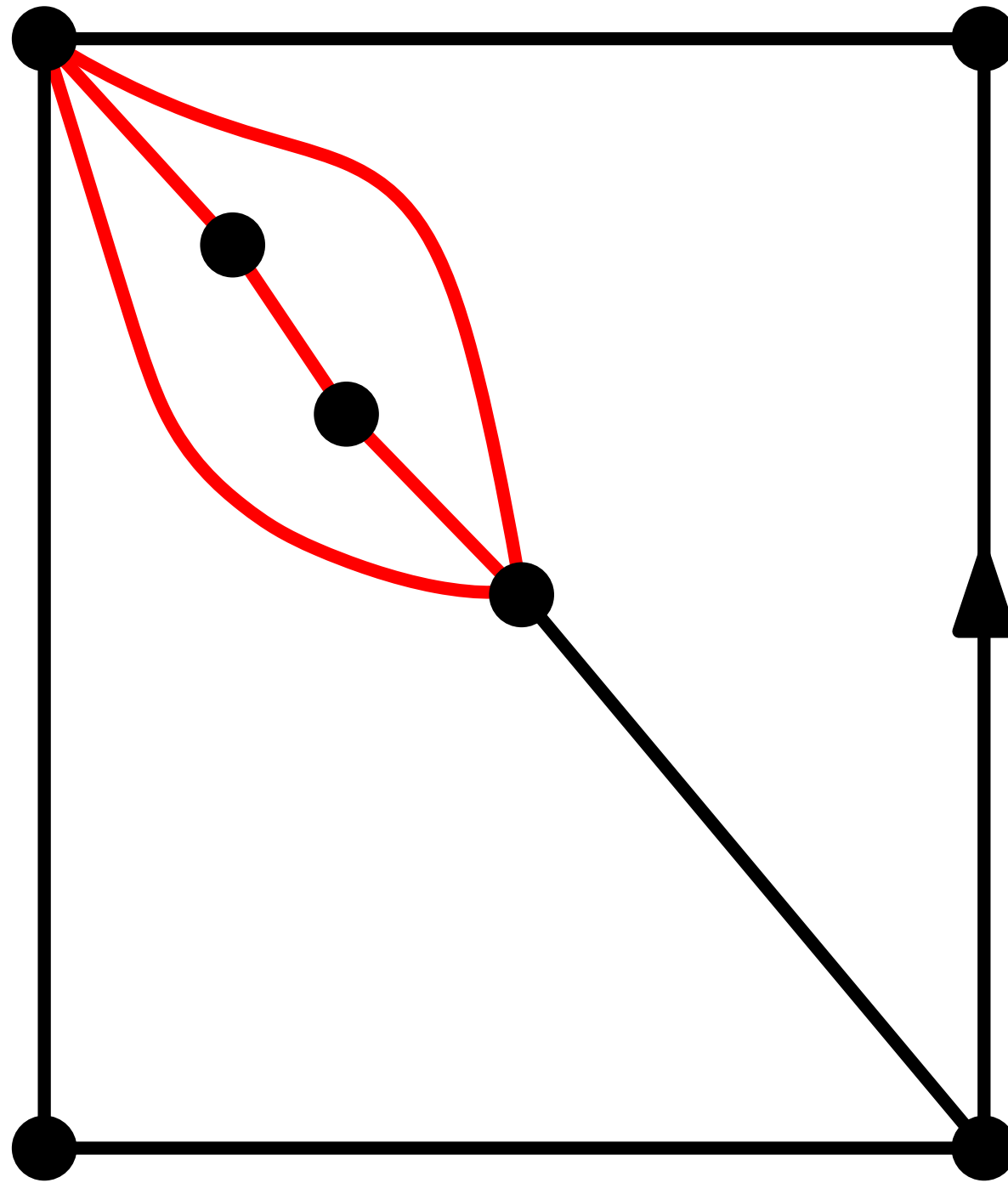
Size $|\mathbf{q}|$ = number of *faces*.

$$|V(\mathbf{q})| = |\mathbf{q}| + 2, |E(\mathbf{q})| = 2|\mathbf{q}|.$$

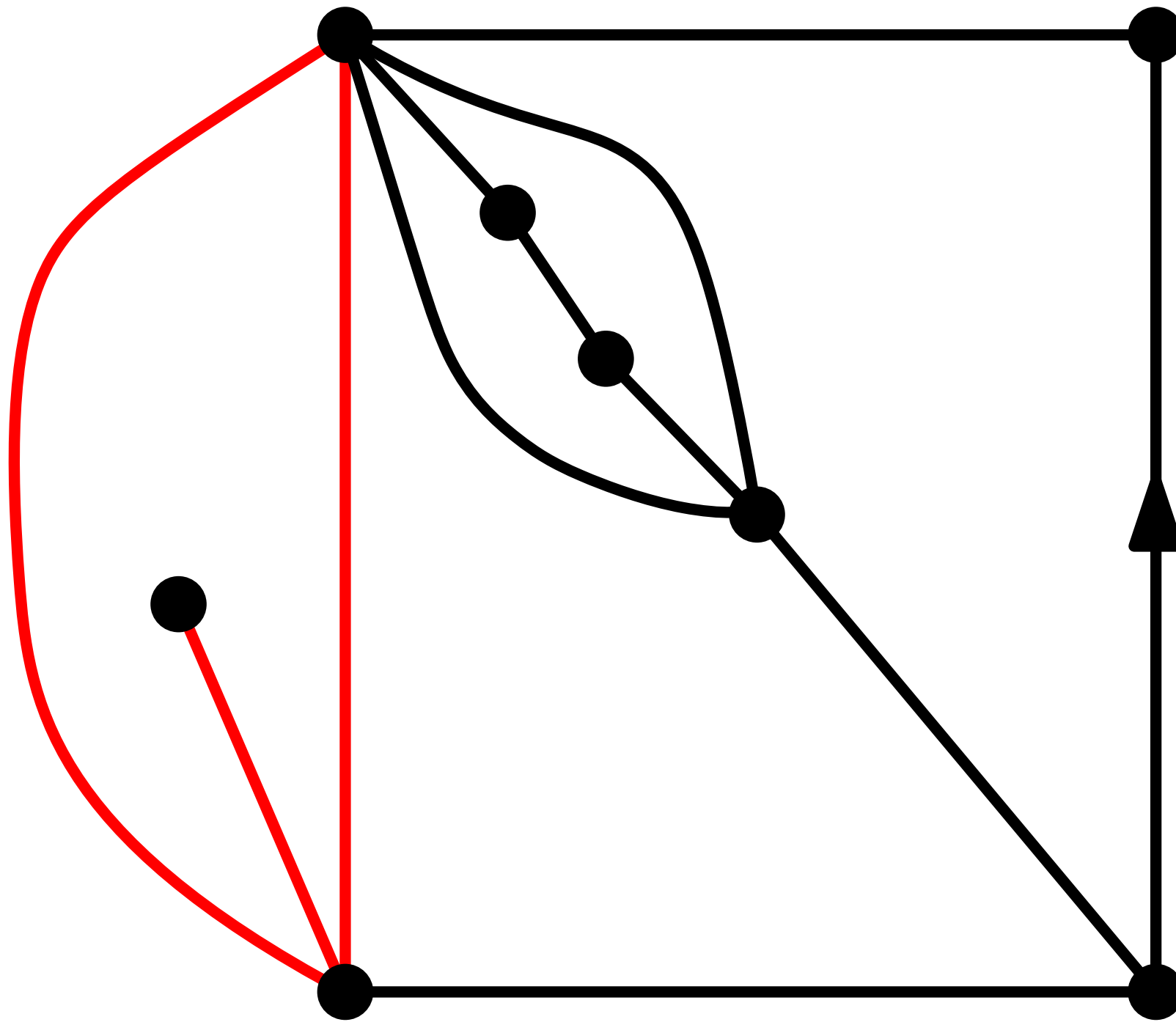
Construction of a quadrangulation from a simple core



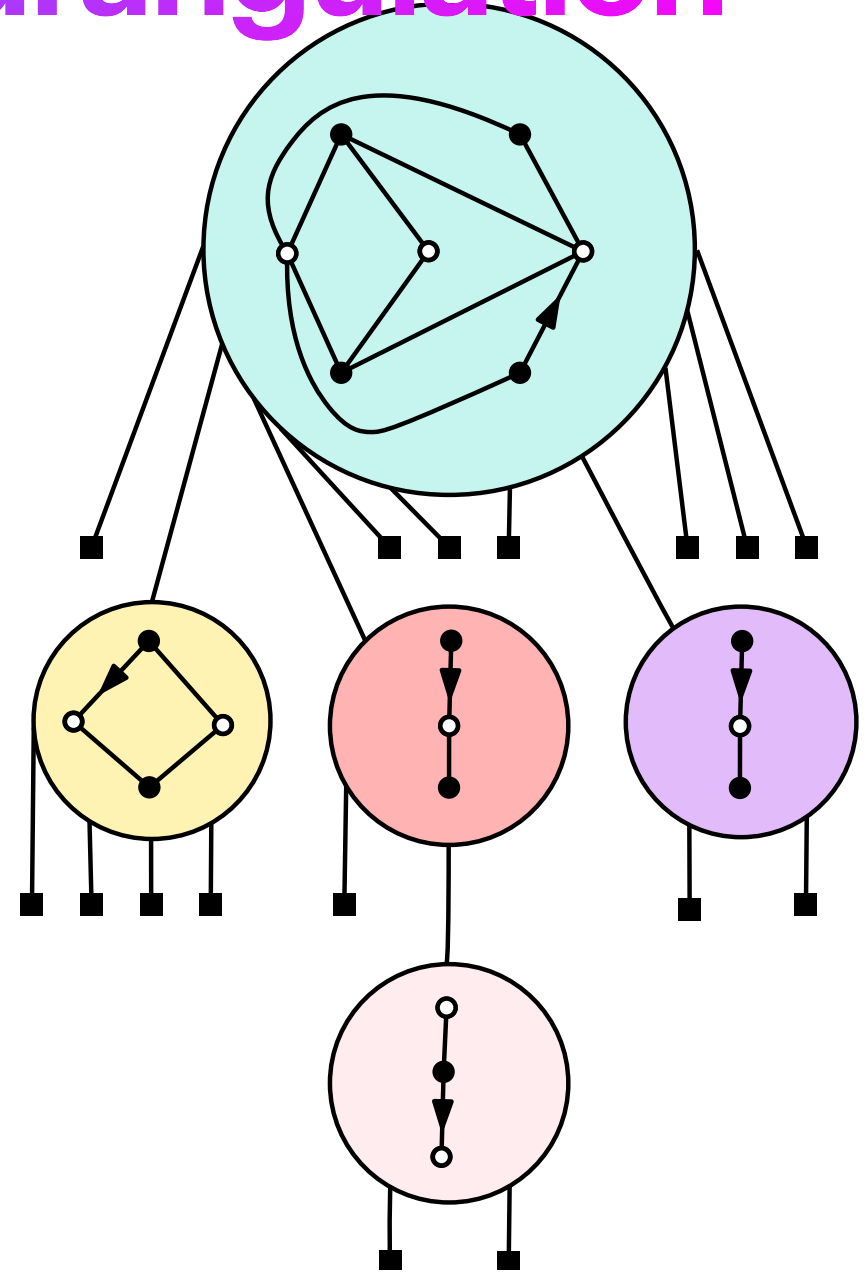
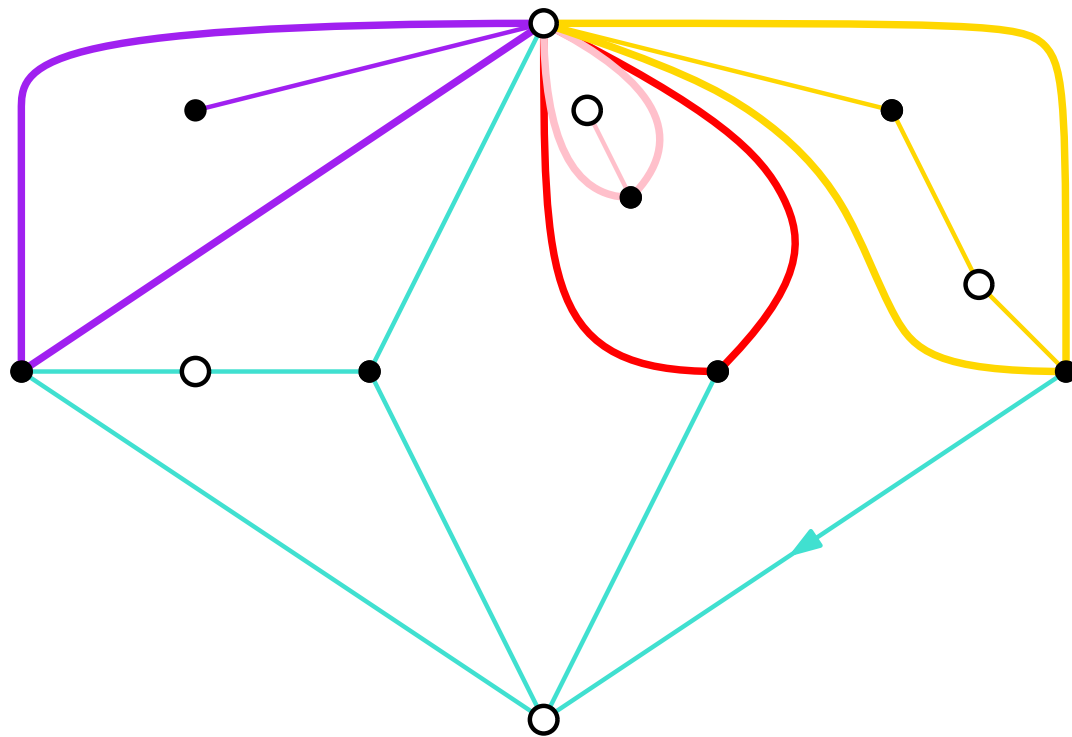
Construction of a quadrangulation from a simple core



Construction of a quadrangulation from a simple core



Block tree for a quadrangulation

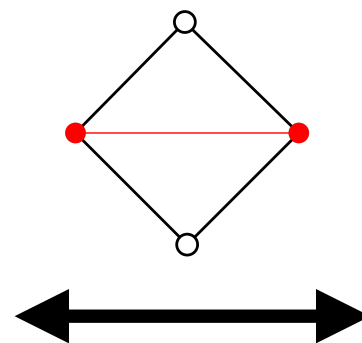
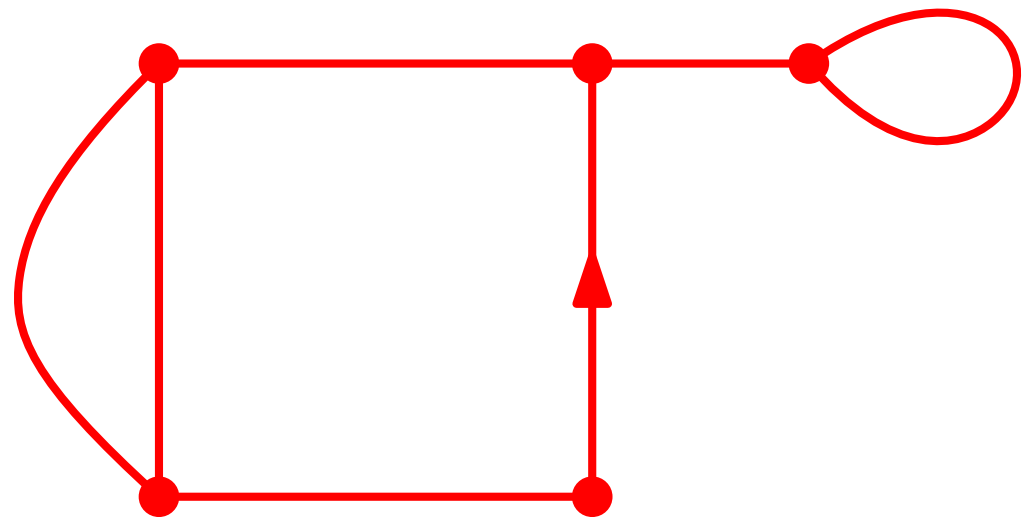


With a weight u on blocks: $Q(z, u) = uS(zQ^2(z, u)) + 1 - u$

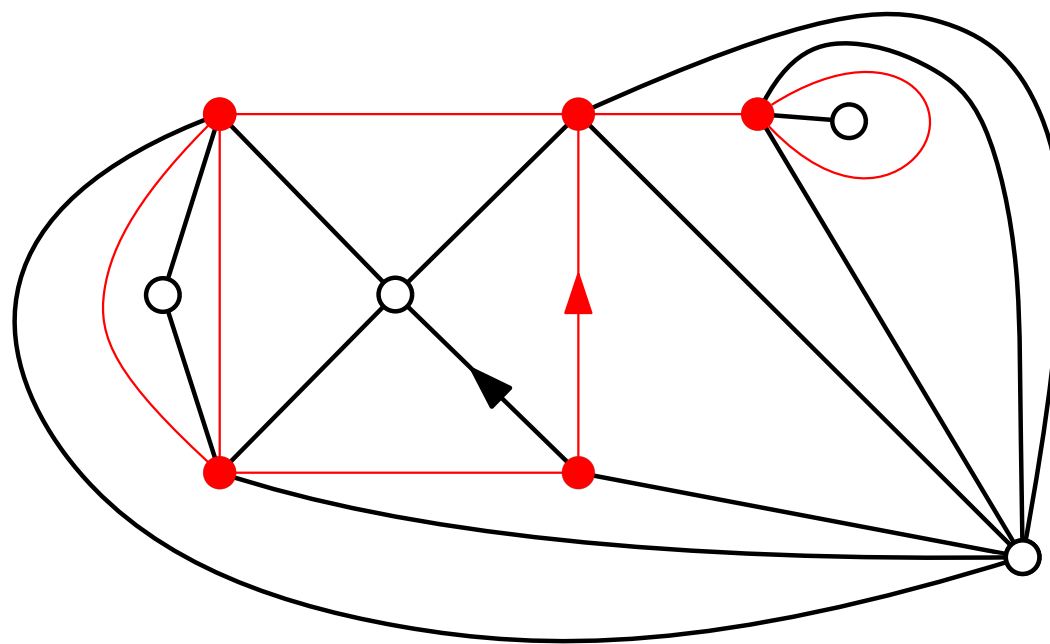
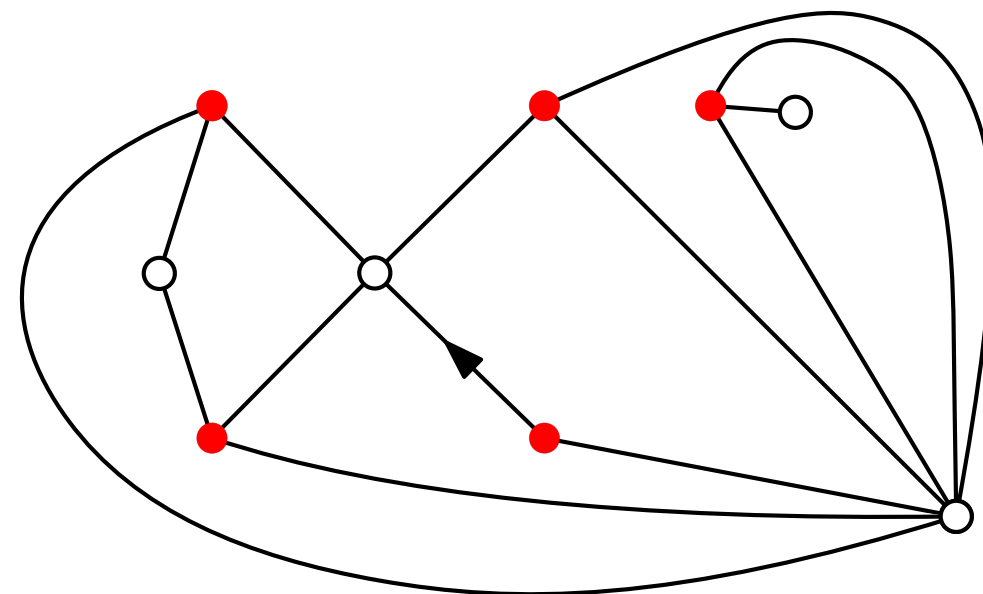
Remember: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

Tutte's bijection

Map

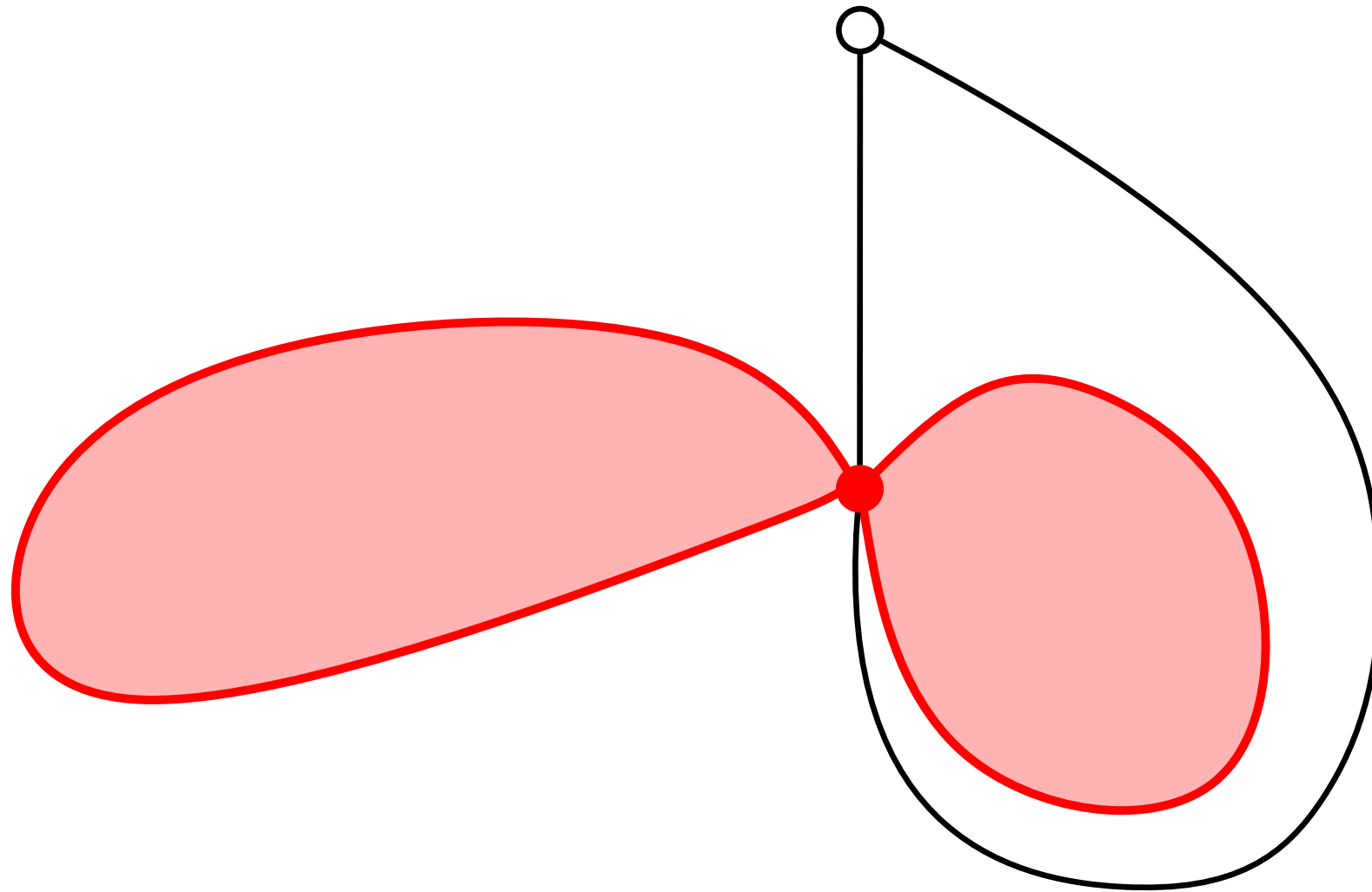


Quadrangulation



[Tutte 1963]

Tutte's bijection for 2-connected maps

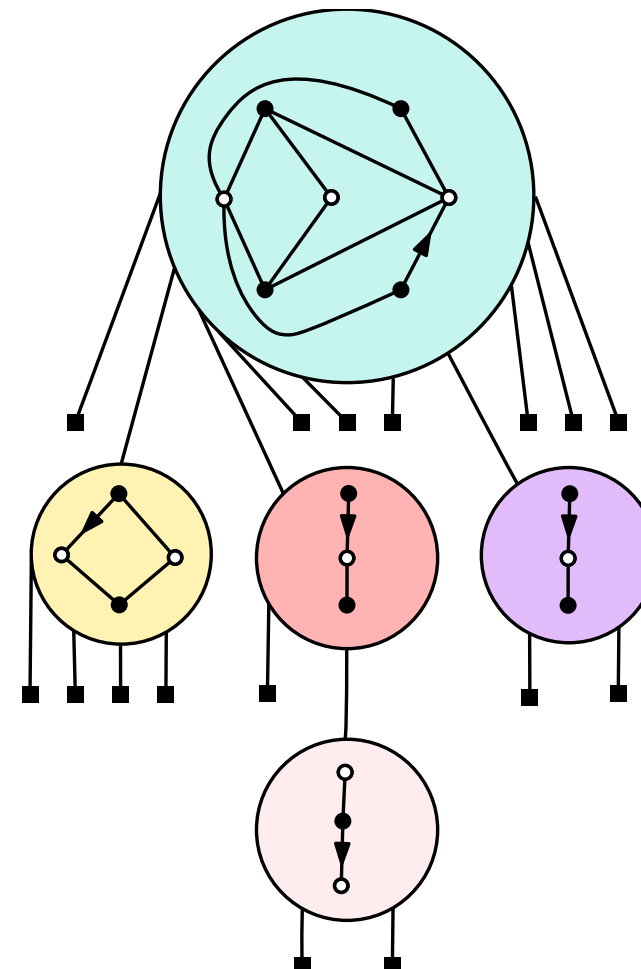
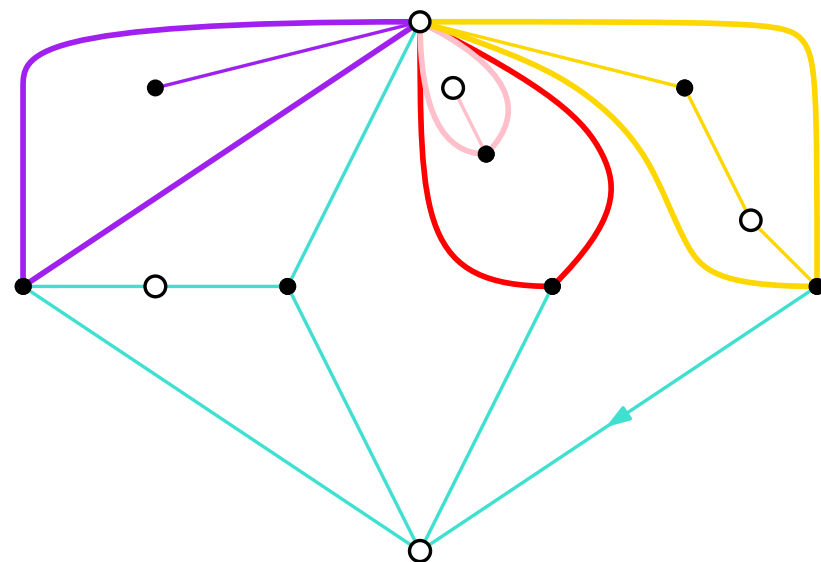
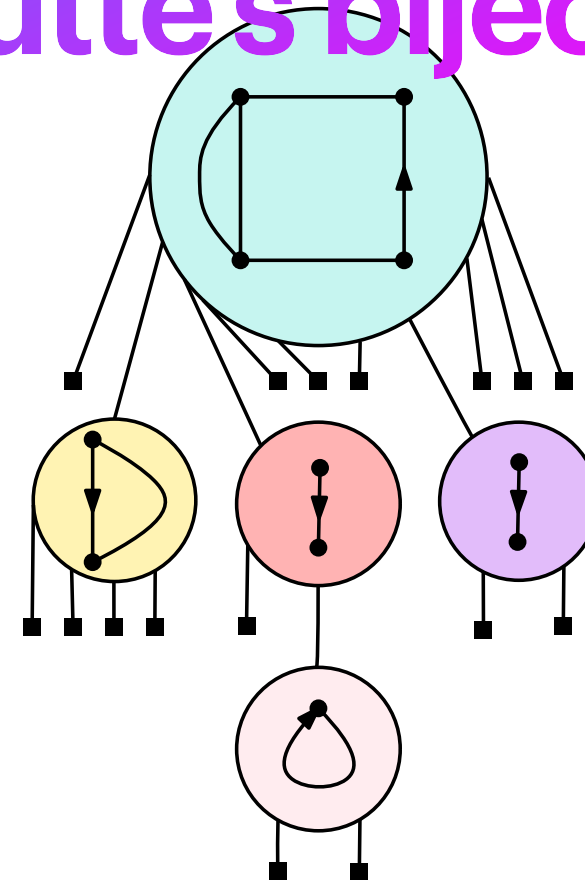
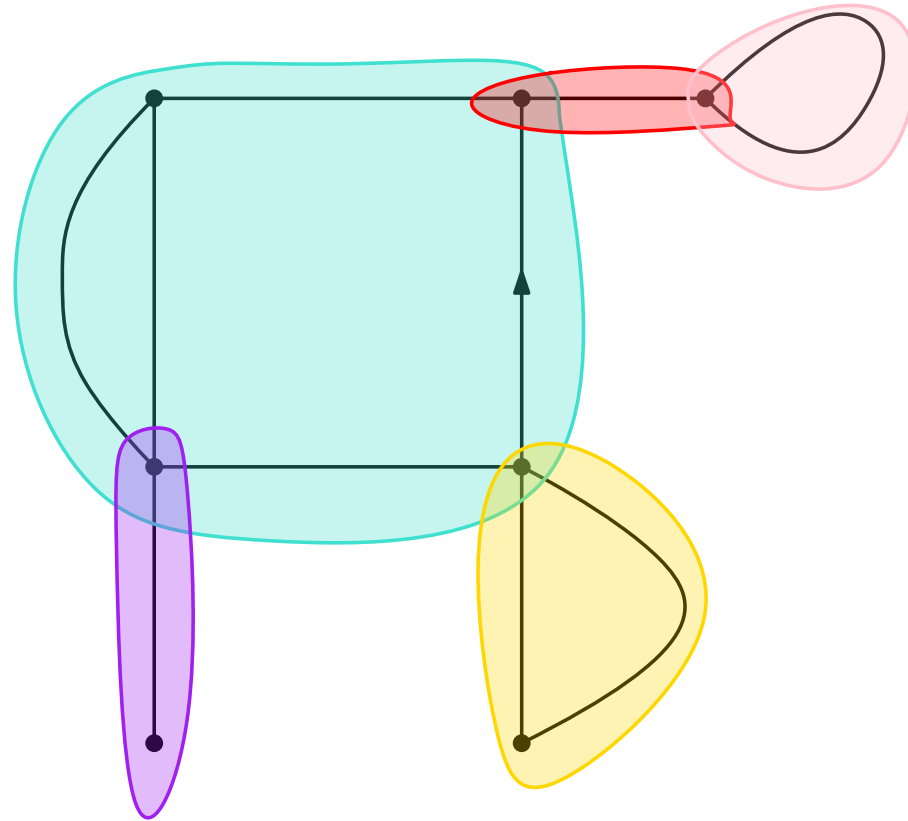


Cut vertex \Rightarrow multiple edge

2-connected maps \Leftrightarrow simple quadrangulations

[Brown 1965]

Block trees under Tutte's bijection



Implications on results

We choose: $\mathbb{P}_{n,u}(\mathbf{q}) = \frac{u^{\#blocks(\mathbf{q})}}{Z_{n,u}}$ where

$u > 0,$
 $\mathcal{Q}_n = \{\text{quadrangulations of size } n\},$
 $\mathbf{q} \in \mathcal{Q}_n,$
 $Z_{n,u} = \text{normalisation}.$

Results on the size of (2-connected) blocks can be transferred immediately for quadrangulations and their simple blocks.

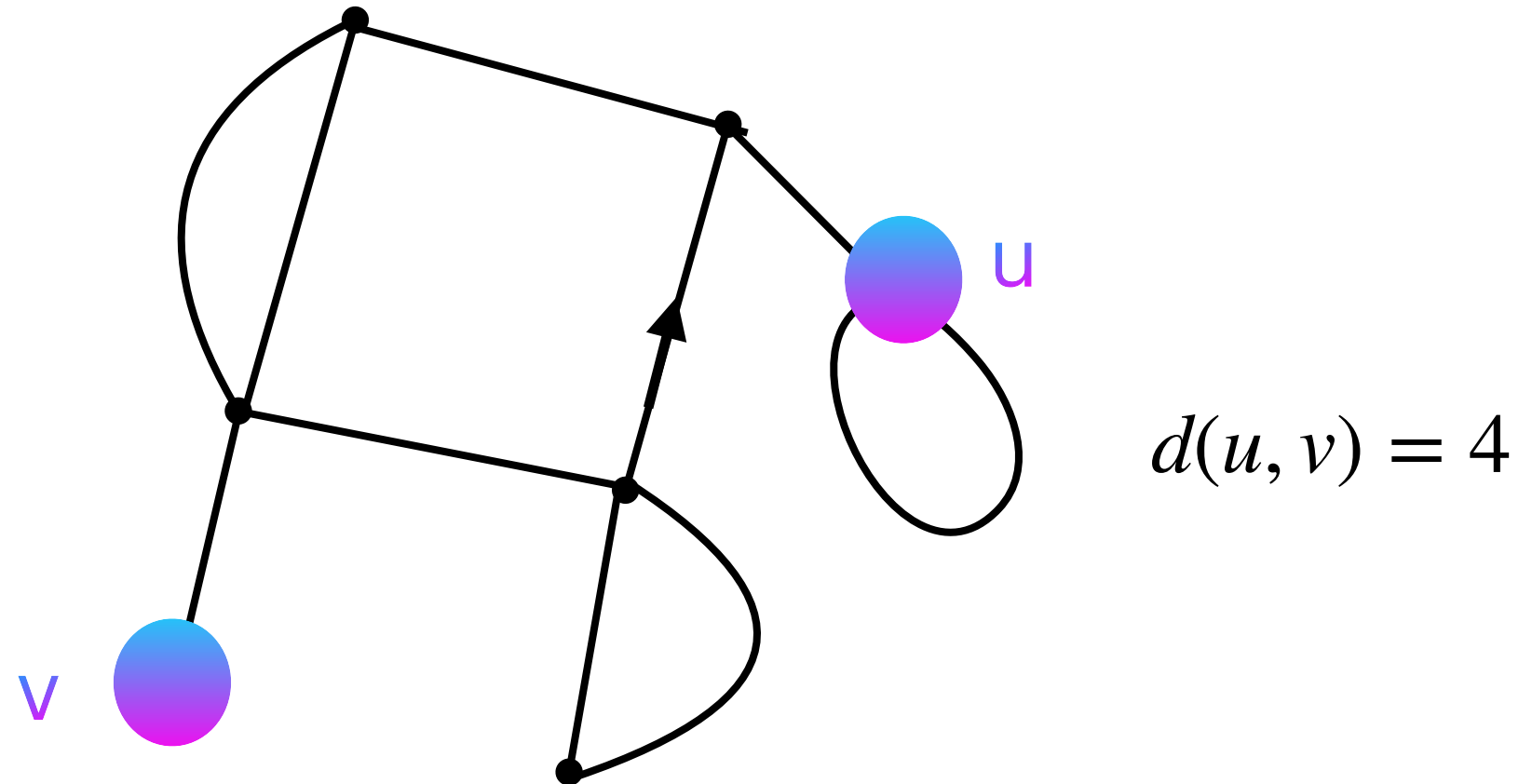
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016] for 2-c case	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2 \ln\left(\frac{4}{27y}\right)} - \frac{5 \ln(\ln(n))}{4 \ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

III. Scaling limits

Scaling limits

Convergence of the whole object considered as a metric space (with the graph distance), after renormalisation.



$M_n \hookrightarrow \mathbb{P}_{n,u}$ (map or quadrangulation)

What is the limit of the sequence of metric spaces $((M_n, d/n^?))_{n \in \mathbb{N}}$?

(Convergence for Gromov-Hausdorff(-Prokhorov) topology)

Scaling limit of supercritical and critical maps

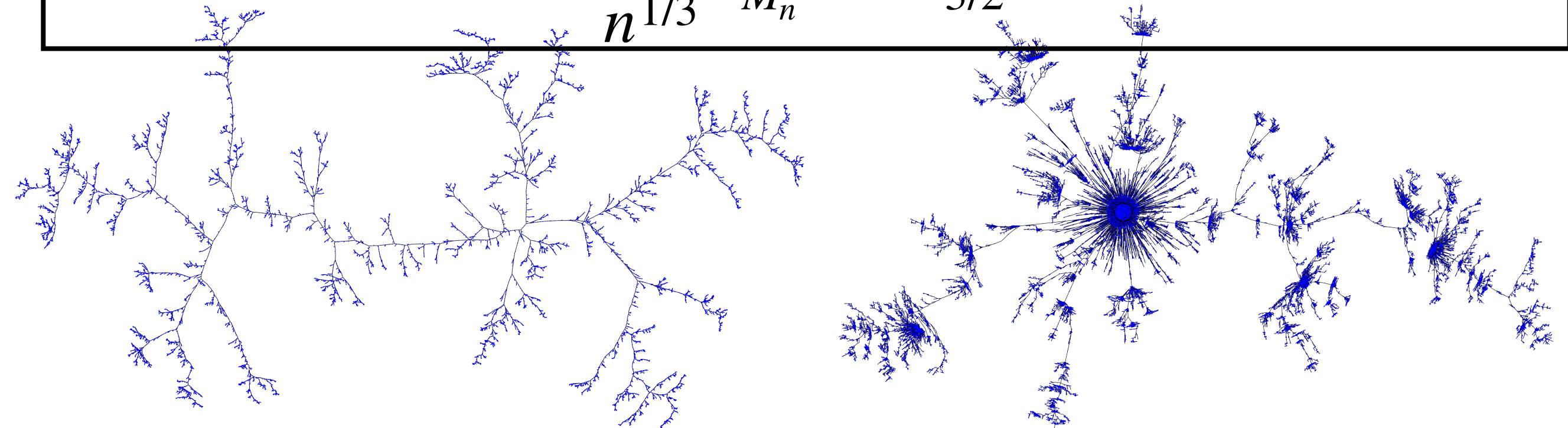
Lemma For $M_n \hookrightarrow \mathbb{P}_{n,u'}$

- If $u > 9/5$,

$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \rightarrow \mathcal{T}_e.$$

- If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \rightarrow \mathcal{T}_{3/2}.$$



Brownian Tree \mathcal{T}_e

Stable Tree $\mathcal{T}_{3/2}$

Scaling limit of supercritical and critical maps

Lemma For $M_n \hookrightarrow \mathbb{P}_{n,u'}$

- If $u > 9/5$,

$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \rightarrow \mathcal{T}_e.$$

- If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \rightarrow \mathcal{T}_{3/2}.$$

Proof Known scaling limits of critical BGW trees

- with finite variance [Aldous 1993, Le Gall 2006];
- infinite variance and polynomial tails [Duquesne 2003].

Scaling limit of supercritical and critical maps

Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u'}$

- [Stufler 2020] If $u > 9/5$,

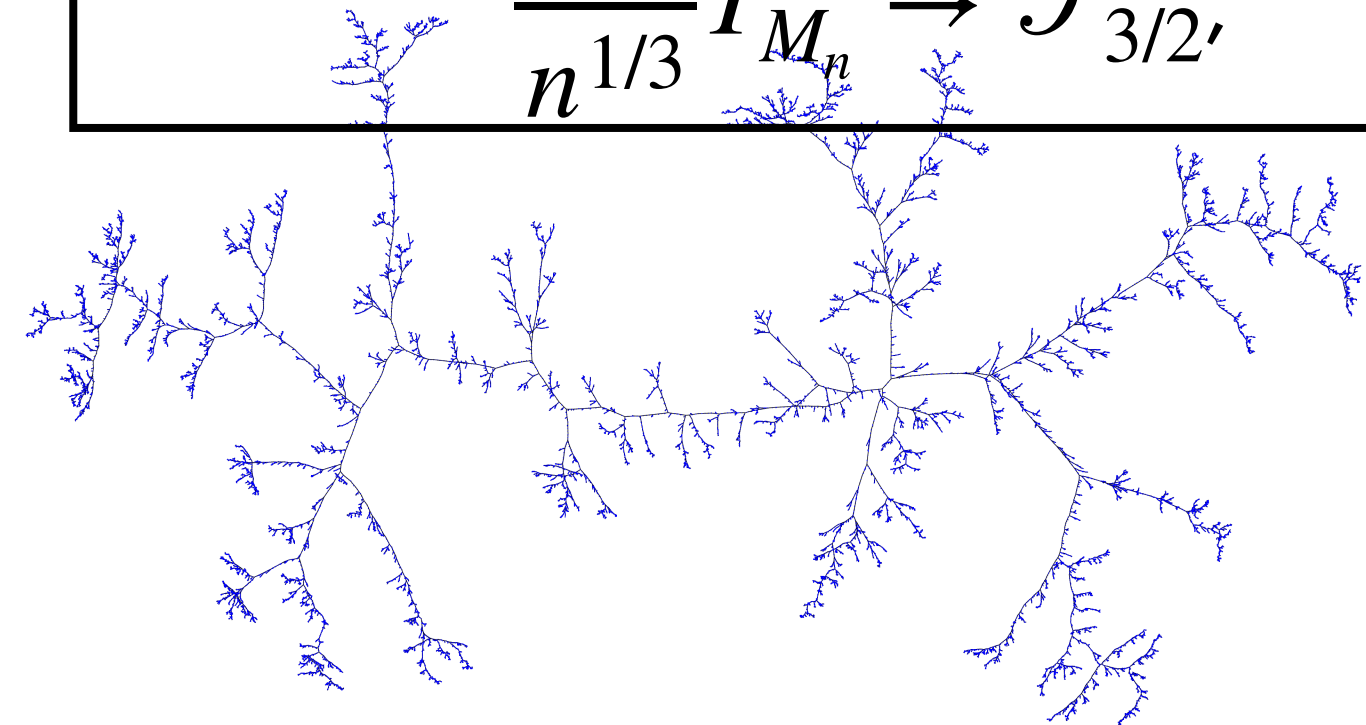
$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \rightarrow \mathcal{T}_e$$

$$\frac{C_3(u)}{n^{1/2}} M_n \rightarrow \mathcal{T}_e.$$

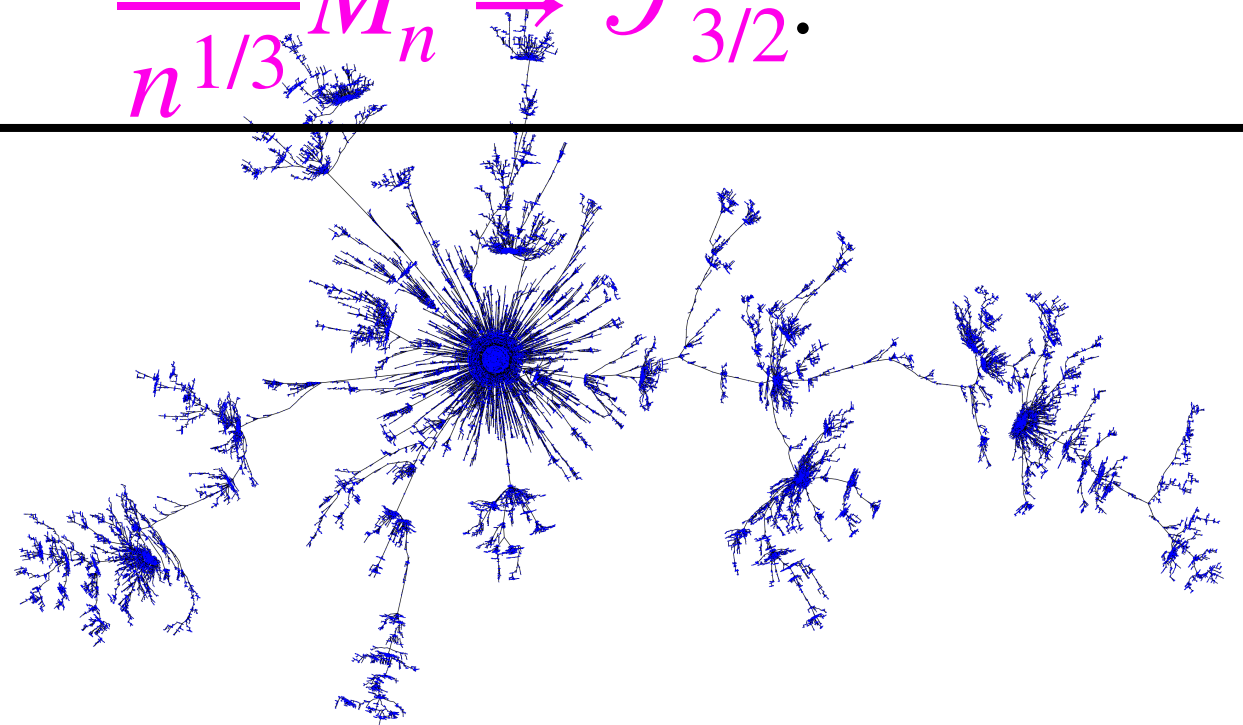
- [Fleurat, S. 24] If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \rightarrow \mathcal{T}_{3/2}$$

$$\frac{C_2}{n^{1/3}} M_n \rightarrow \mathcal{T}_{3/2}.$$



Brownian Tree \mathcal{T}_e



Stable Tree $\mathcal{T}_{3/2}$

Scaling limit of supercritical and critical maps

Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u}$

- [Stufler 2020] If $u > 9/5$,

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- [Fleurat, S. 24] If $u = 9/5$,

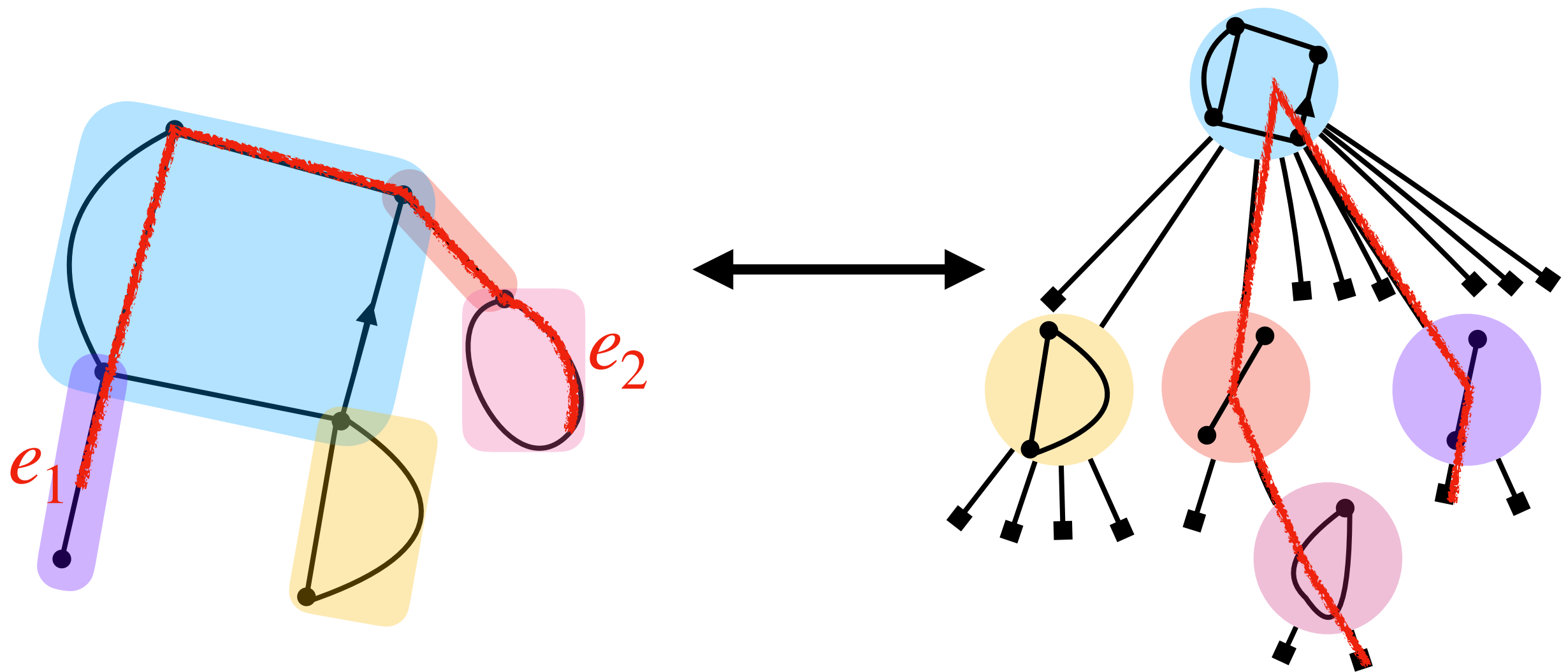
$$\frac{c_2}{n^{1/3}} T_{M_n} \rightarrow \mathcal{T}_{3/2},$$

$$\frac{C_2}{n^{1/3}} M_n \rightarrow \mathcal{T}_{3/2}.$$

Proof Distances in M_n behave like distances in T_{M_n} !

Supercritical and critical cases

Goal = show that distances in M_n behave like distances in T_{M_n} .



Let $\kappa = \mathbb{E}$ ("diameter" bipointed block). By a "law of large numbers"-type argument

$$d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2).$$

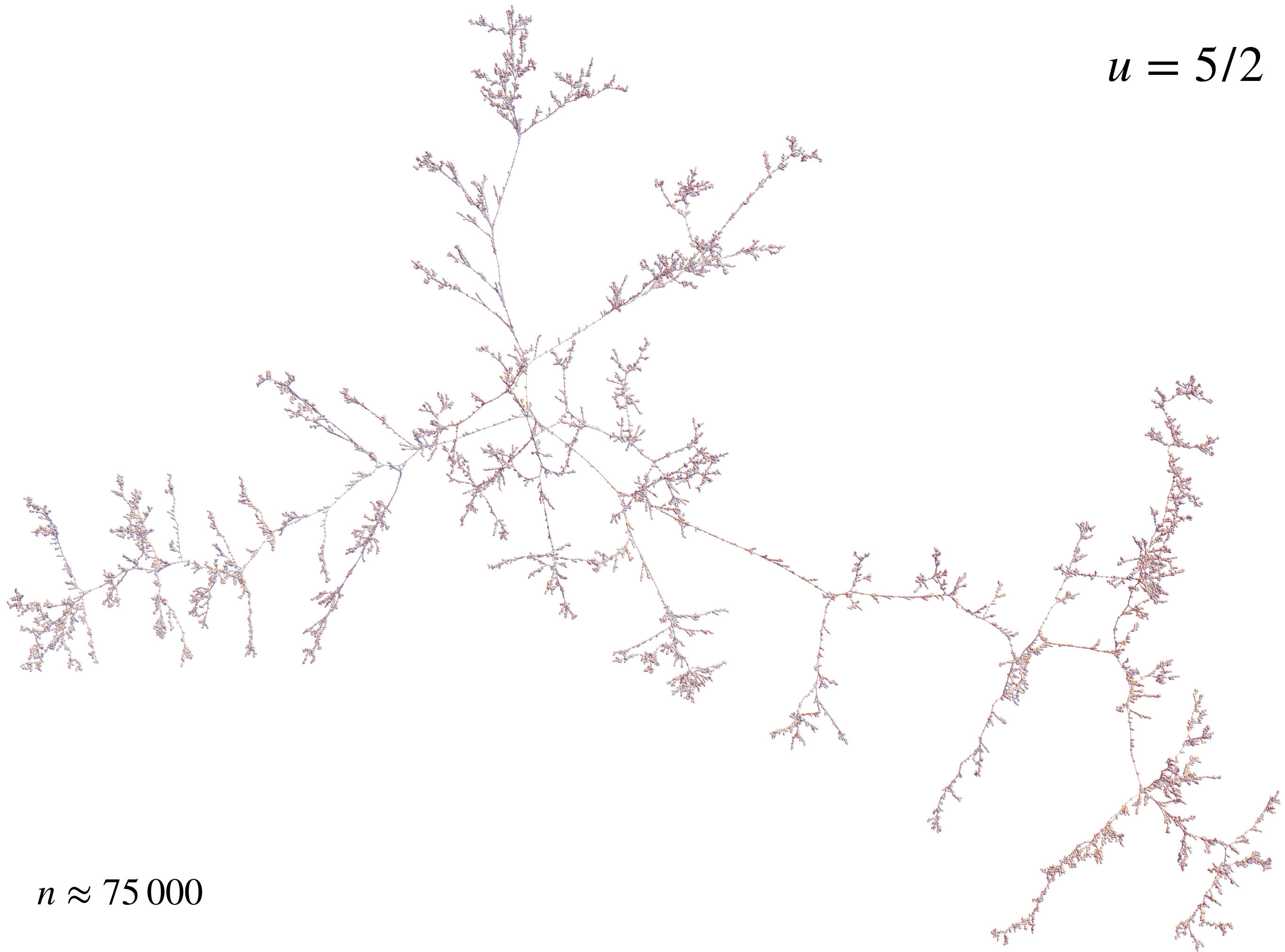
Difficult for the critical case => large deviation estimates

$$u = 9/5$$



$$n \approx 80\,000$$

$$u = 5/2$$



$$n \approx 75\,000$$

$$u = 5$$



$$n \approx 50\,000$$

Scaling limits of subcritical maps

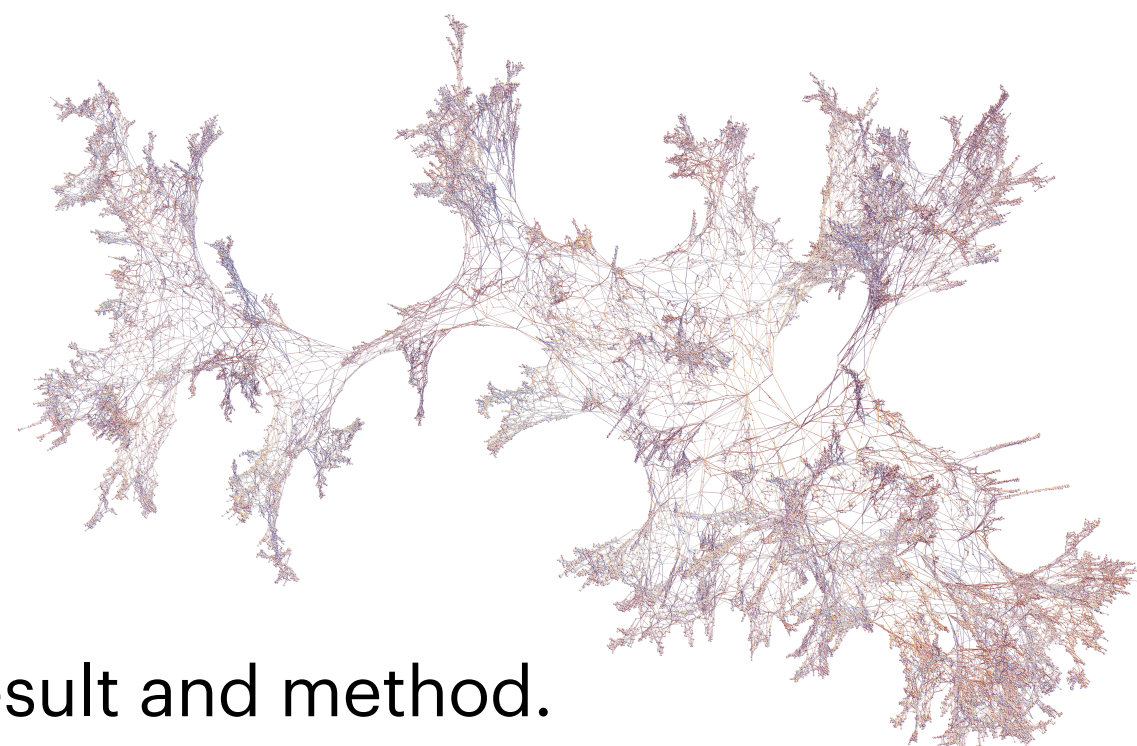
Theorem [Fleurat, S. 24] If $u < 9/5$, for $M_n \hookrightarrow \mathbb{P}_{n,u}$ and denoting $B(M_n)$ its largest block:

$$d_{GHP} \left(\frac{C_1(u)}{n^{1/4}} M_n, \frac{1}{n^{1/4}} B(M_n) \right) \rightarrow 0.$$

Brownian Sphere \mathcal{S}_e

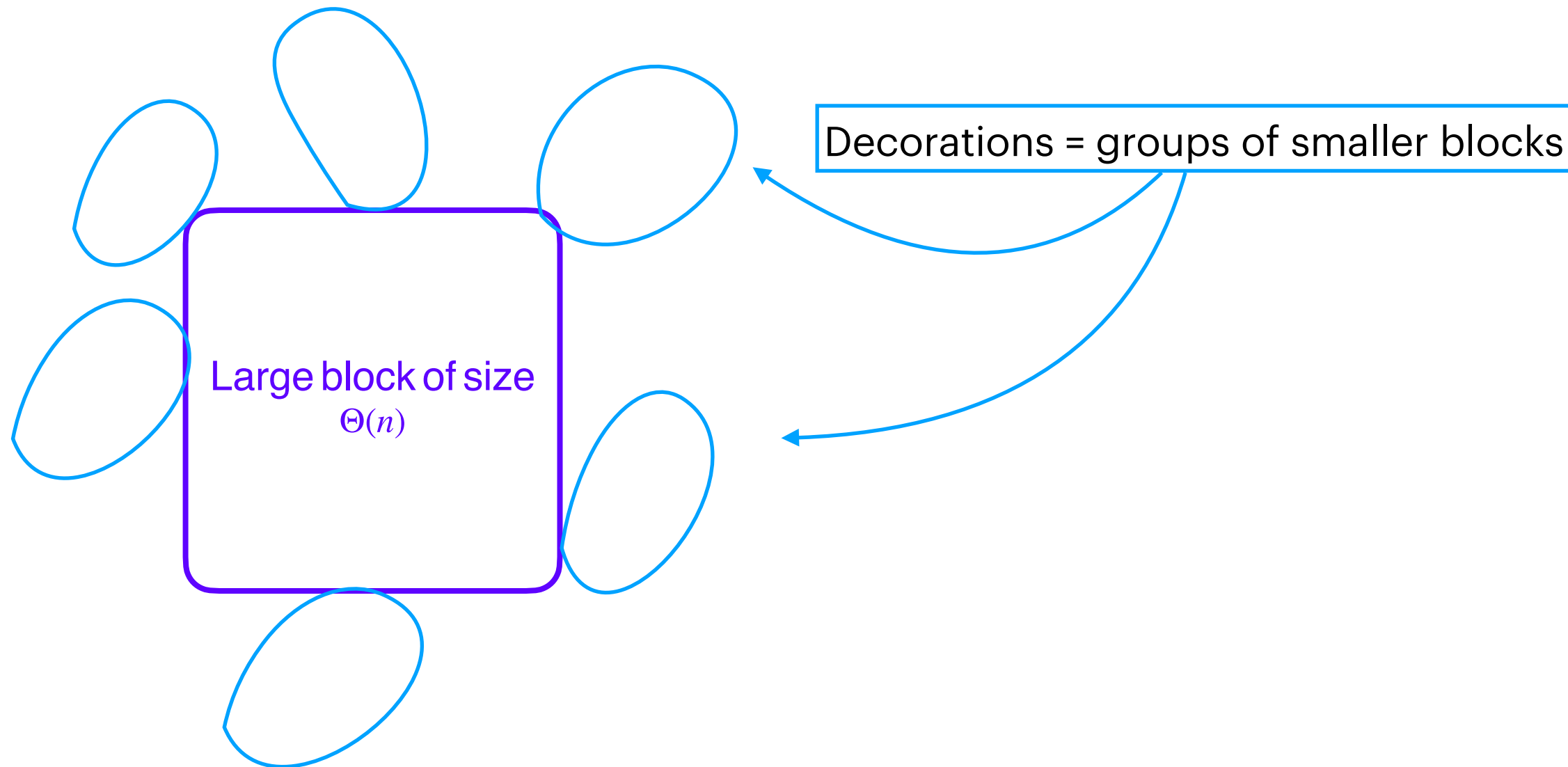
So, if $cn^{-1/4} B_n \rightarrow \mathcal{S}_e$, then

$$\frac{C_1(u)}{cn^{1/4}} M_n \rightarrow \mathcal{S}_e.$$



See [Addario-Berry, Wen 2019] for a similar result and method.

Subcritical case



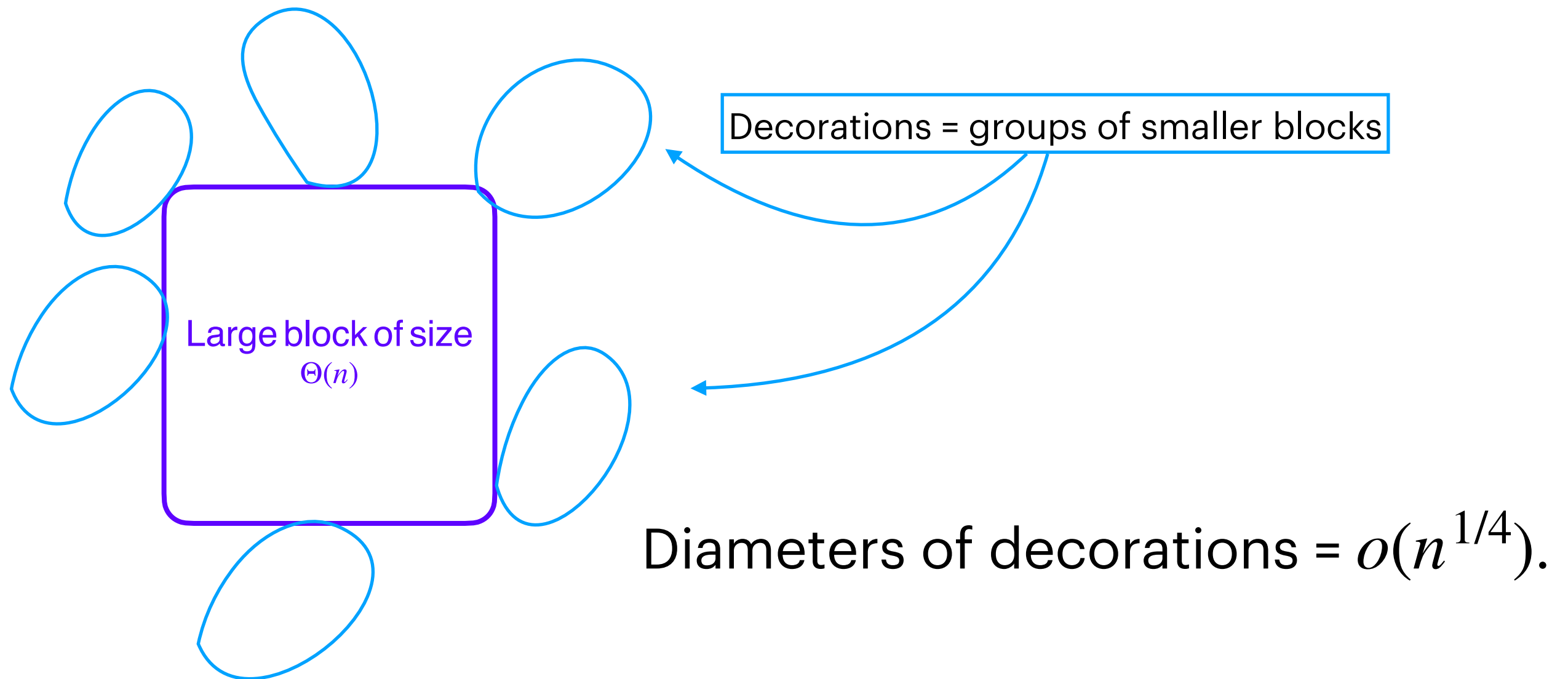
Diameter of a decoration \leq blocks to cross \times max diameter of blocks

$$\leq \text{diam}(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = \text{diam}(T_{M_n}) \times O(n^{1/6+\delta})$$

T_{M_n} is a subcritical BGW tree $= O(n^{1/6+2\delta}) = o(n^{1/4})$.

[Chapuy Fusy Giménez Noy 2015]

Subcritical case



The scaling limit of M_n (rescaled by $n^{1/4}$) is the scaling limit of uniform blocks!

Scaling limits of subcritical maps

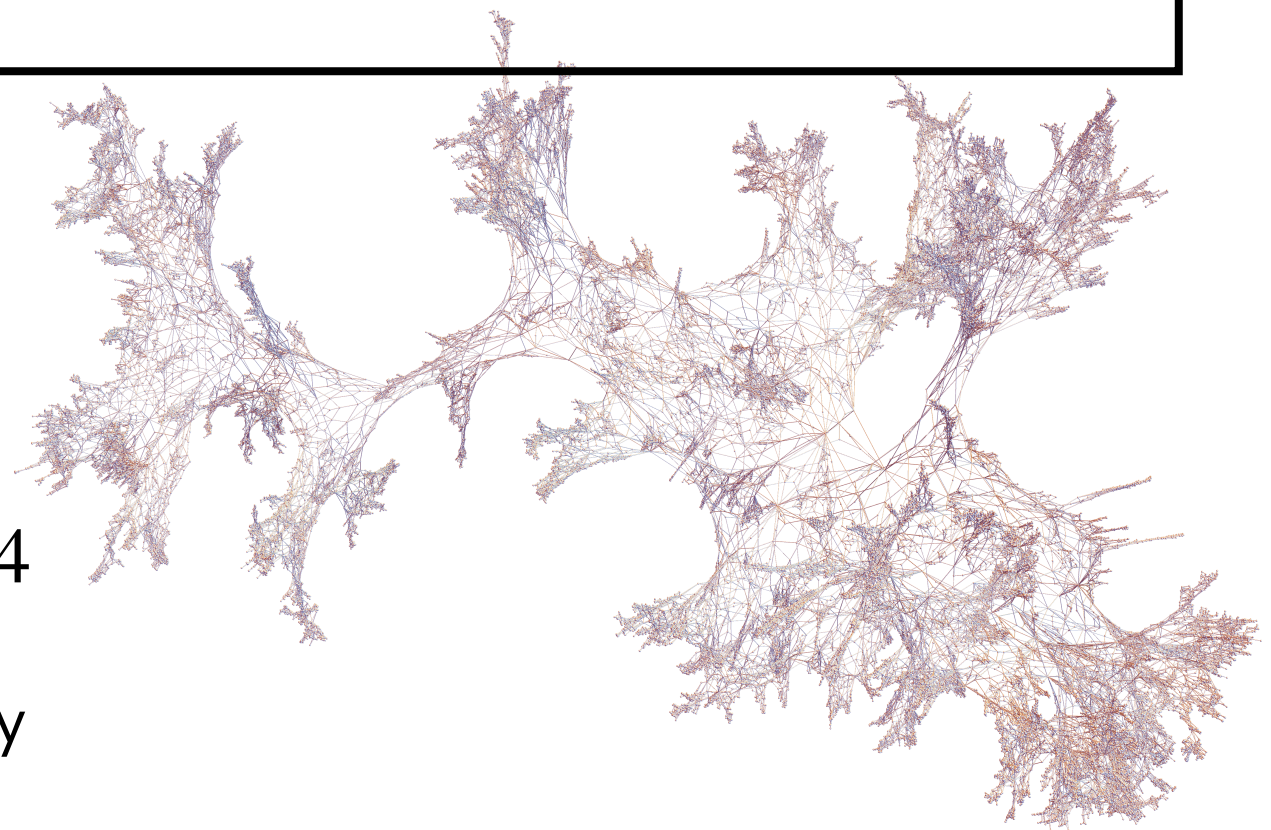
Theorem [Fleurat, S. 24] If $u < 9/5$, for $Q_n \hookrightarrow \mathbb{P}_{n,u}$ a quadrangulation:

$$\frac{C_1(u)}{n^{1/4}} Q_n \rightarrow \mathcal{S}_e.$$

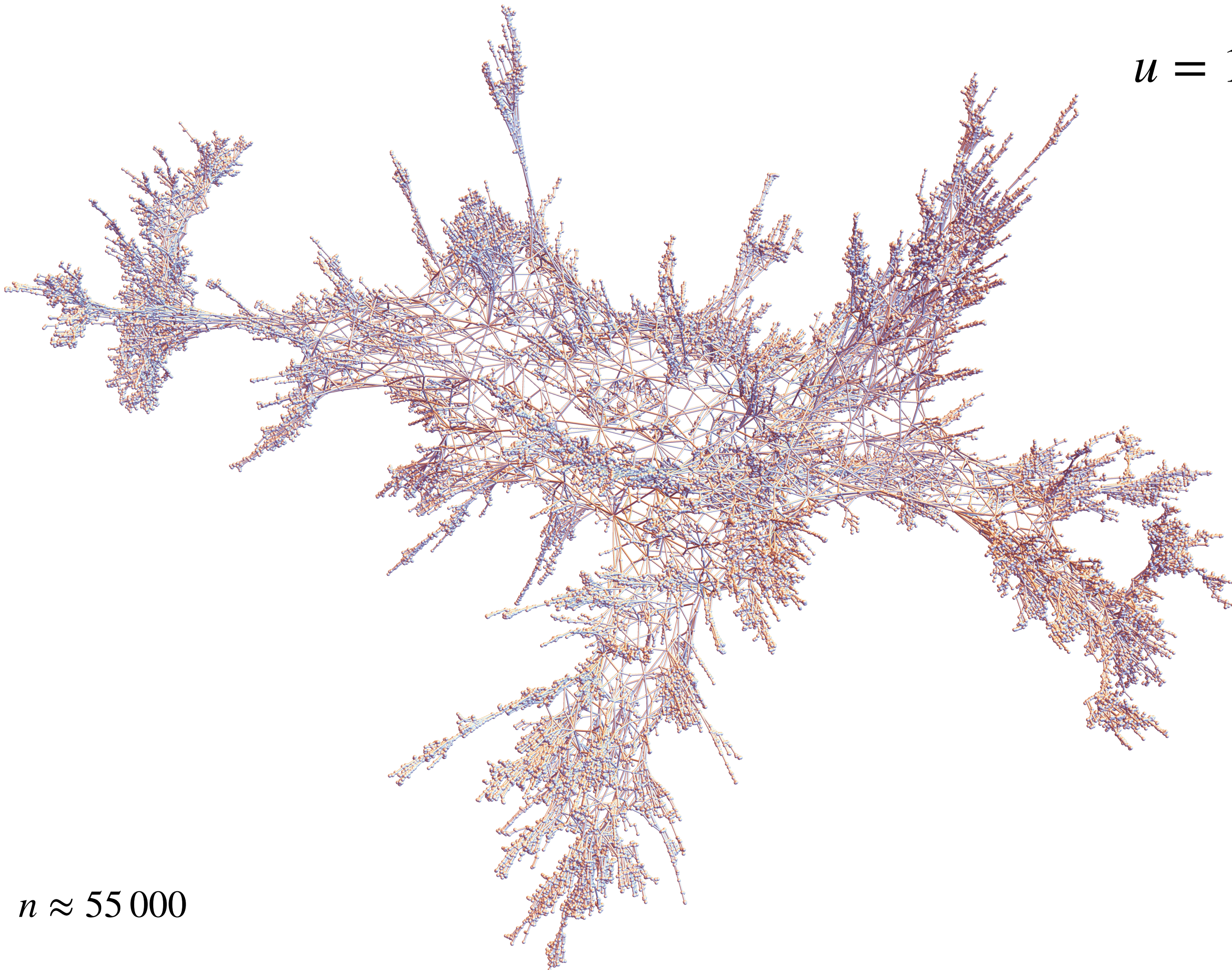
Moreover, Q_n and its simple core converge jointly to the same Brownian sphere.

Proof

- Previous theorem;
- Scaling limit of uniform simple quad. rescaled by $n^{1/4}$ = Brownian sphere [Addario-Berry Albenque 2017].

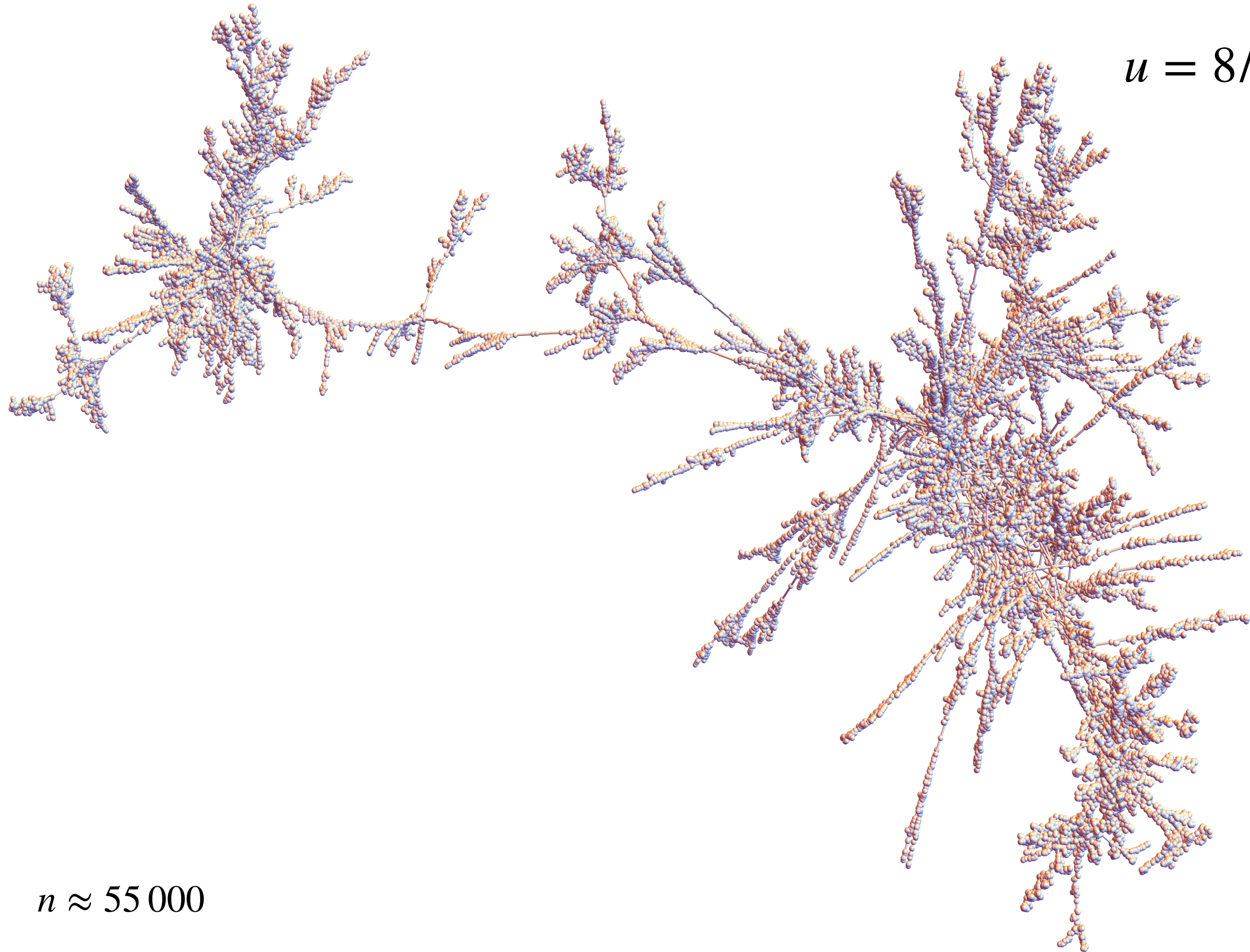


$$u = 1$$



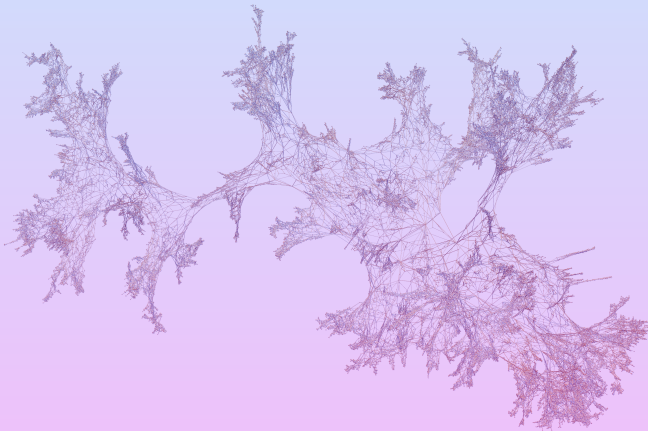
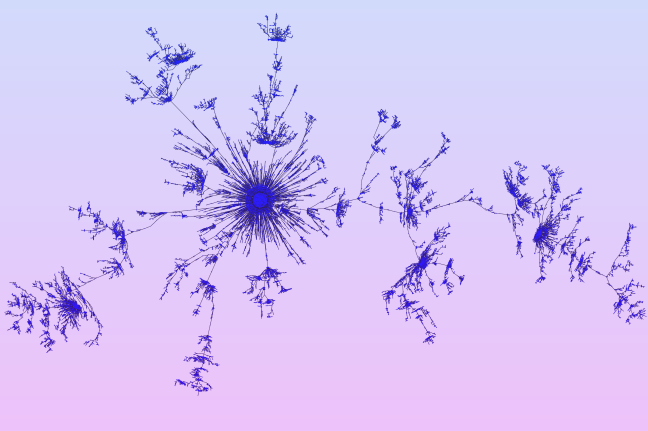
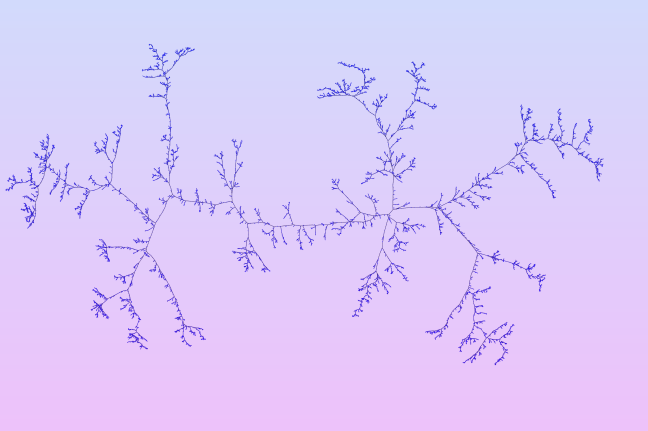
$$n \approx 55\,000$$

$$u = 8/5$$



$$n \approx 55\,000$$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2 \ln\left(\frac{4}{27y}\right)} - \frac{5 \ln(\ln(n))}{4 \ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n	$\frac{C_1(u)}{n^{1/4}} M_n \rightarrow \mathcal{S}_e$  <p>Assuming the convergence of 2-connected maps towards the Brownian sphere</p>	$\frac{C_2}{n^{1/3}} M_n \rightarrow \mathcal{T}_{3/2}$ 	$\frac{C_3(u)}{n^{1/2}} M_n \rightarrow \mathcal{T}_e$ <p>[Stufler 2020]</p> 

IV. Extension to other families of maps

Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$
all, $M_1(z)$	bridgeless, $M_2(z)$ or loopless	$z/(1 - z(1 + M))^2$	$z(1 + M)^2$
loopless $M_2(z)$	simple $M_3(z)$	$z(1 + M)$	—
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1 + M)^2$	—
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1 + M)$	—
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1 + M)$
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1 + M)$	—
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1 - z(1 + M))^2$	$z(1 + M)^2$
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1 + M)^2$	—
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1 + M)$	—
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1 + M)^3$	—
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1 + M)^2$	—

Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$	u_C
all, $M_1(z)$	bridgeless, $M_2(z)$ or loopless	$z/(1 - z(1 + M))^2$	$z(1 + M)^2$	
loopless $M_2(z)$	simple $M_3(z)$	$z(1 + M)$	—	81/17
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1 + M)^2$	—	9/5
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1 + M)$	—	135/7
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1 + M)$	
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1 + M)$	—	36/11
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1 - z(1 + M))^2$	$z(1 + M)^2$	
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1 + M)^2$	—	52/27
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1 + M)$	—	68/3
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1 + M)^3$	—	16/7
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1 + M)^2$	—	64/37

→ *Unified study of the phase transition for block-weighted random planar maps* Z. Salvy (Eurocomb'23)

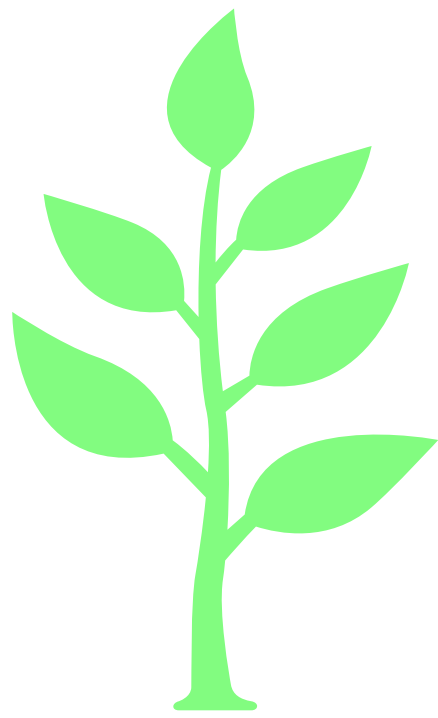
Statement of the results

Theorem [S. 23] Model of the preceding table without coreless maps exhibits a phase transition at some explicit u_C .

When $n \rightarrow \infty$:

- Subcritical phase $u < u_C$: “general map phase” one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: “tree phase” only small blocks.

We obtain explicit results on enumeration and size of blocks in each case.

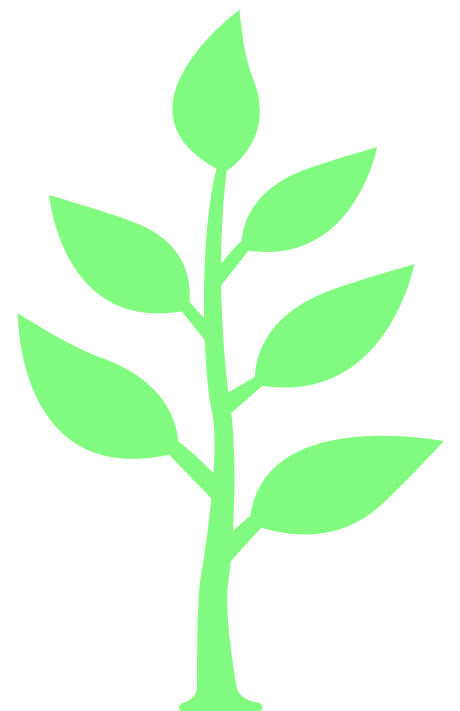


V. Extension to tree-rooted maps

Decorated maps are interesting

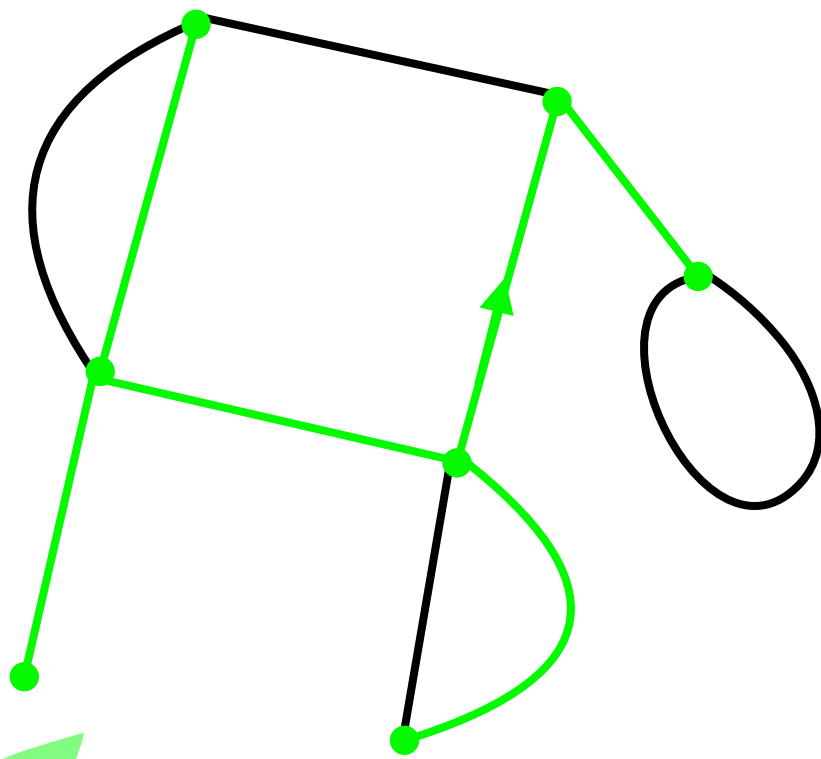
Theoretical physics point of view:

- Undecorated maps: “pure gravity” case (nothing happens on the surface);
- Decorated maps: things happen! new asymptotic behaviours! new universality classes! excitement!

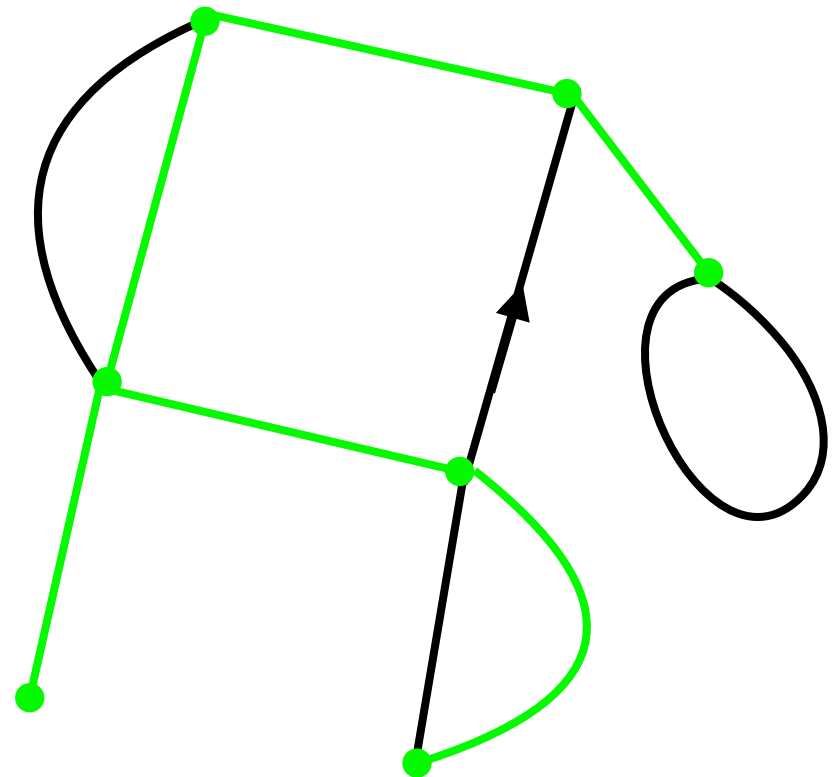


Tree-rooted maps

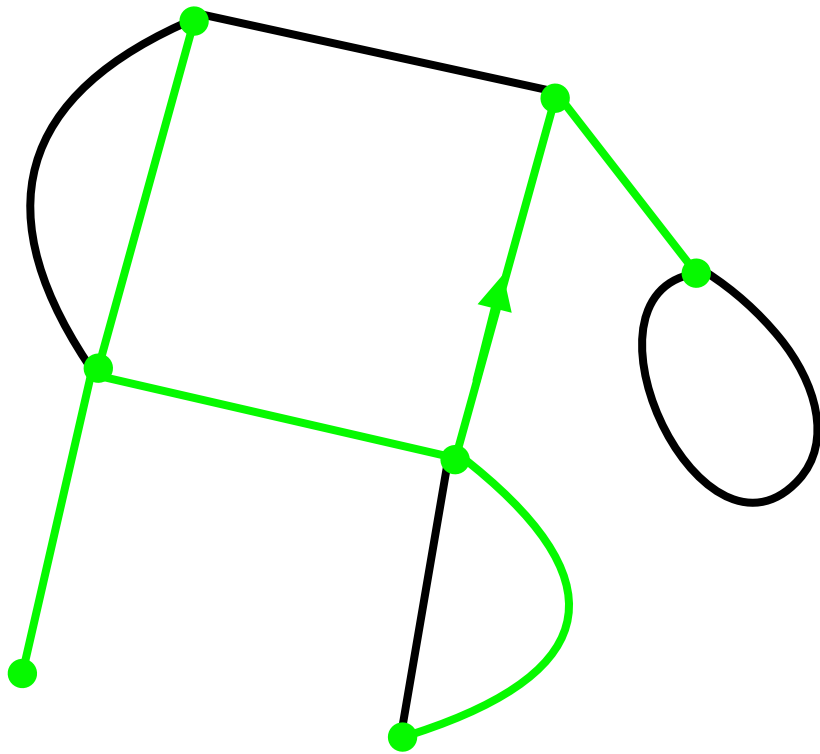
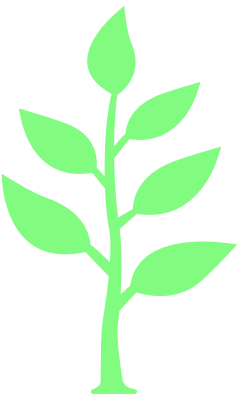
= (rooted planar) maps endowed with a spanning tree.



\neq



Tree-rooted maps



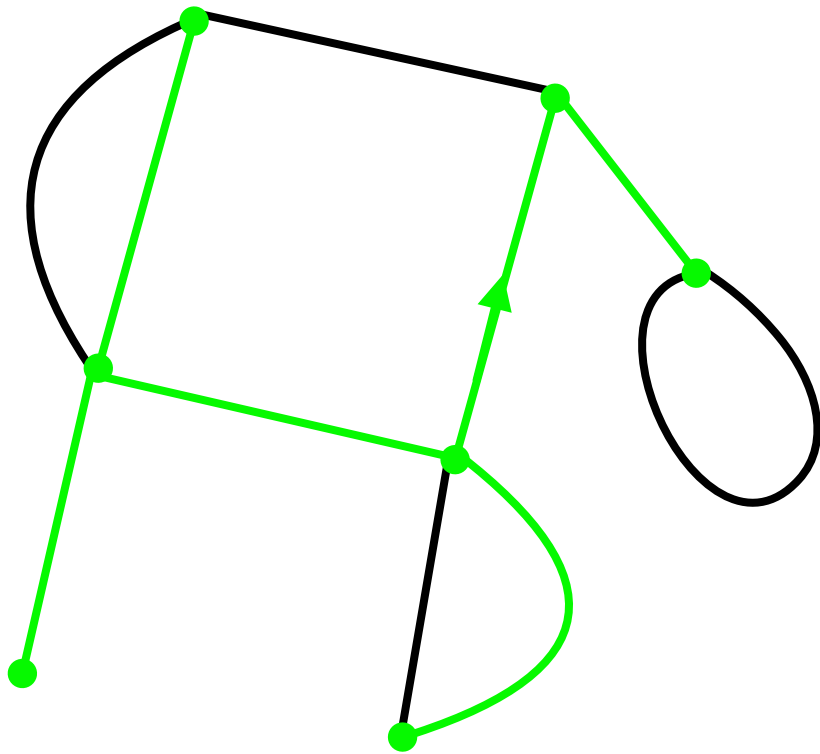
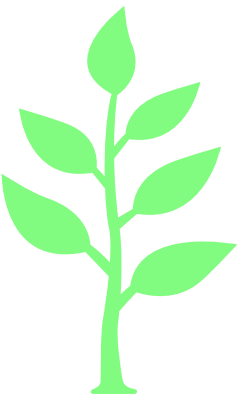
- Combinatorics well understood :
Mullin's bijection;

$$[z^n]M(z) = \text{Cat}_n \text{Cat}_{n+1}$$

[Mullin 67]

- Geometry not so much.

Tree-rooted maps



- Combinatorics well understood :
Mullin's bijection;

$$[z^n]M(z) = \text{Cat}_n \text{Cat}_{n+1}$$

[Mullin 67]

- Geometry not so much.

We want a phase transition in tree-rooted maps.

=> Block-weighted tree-rooted maps.

Model

Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

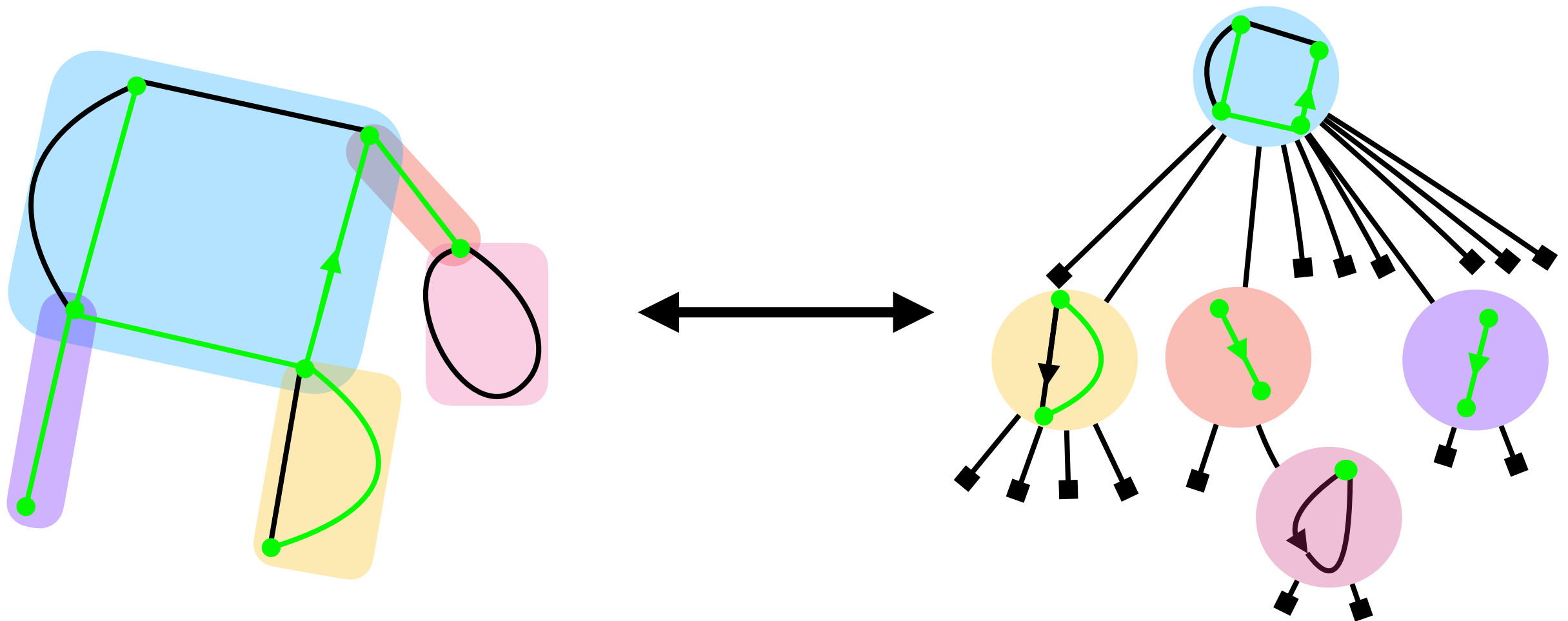
$u > 0,$
 $\mathcal{M}_n = \{\text{tree-rooted maps of size } n\},$
 $\mathfrak{m} \in \mathcal{M}_n,$
 $Z_{n,u} = \text{normalisation.}$

- $u = 1$: uniform distribution on tree-rooted maps of size n ;
- $u \rightarrow 0$: minimising the number of blocks (=2-connected tree-rooted maps);
- $u \rightarrow \infty$: maximising the number of blocks (= tree-rooted trees!).

Given u , asymptotic behaviour when $n \rightarrow \infty$?

Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



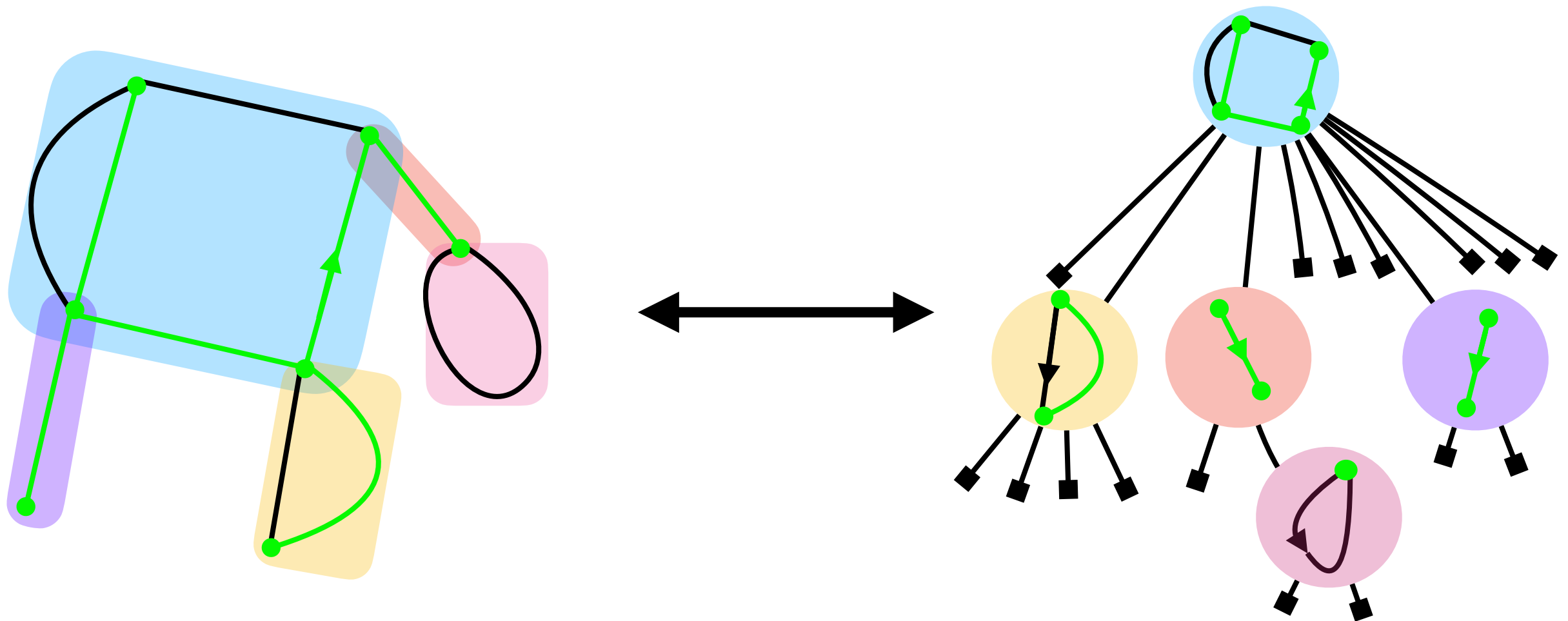
$$M(z) = B(zM^2(z))$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



$$M(z, u) = uB(zM^2(z, u)) + 1 - u$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

So everything should be easy, right?

Tree-rooted maps are not so nice

$$M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \text{ so}$$

Tree-rooted maps are not so nice

$$M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \text{ so}$$

- $[z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3};$

Tree-rooted maps are not so nice

$$M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \text{ so}$$


$$\bullet [z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3}; \quad \bullet \rho_M = \frac{1}{16};$$

Tree-rooted maps are not so nice

$$M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \text{ so}$$

$$\bullet [z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3}; \quad \bullet \rho_M = \frac{1}{16};$$

$$\bullet M(\rho_M) = 8 - \frac{64}{3\pi} \simeq 1.2 \text{ so } M \text{ is not algebraic...}$$

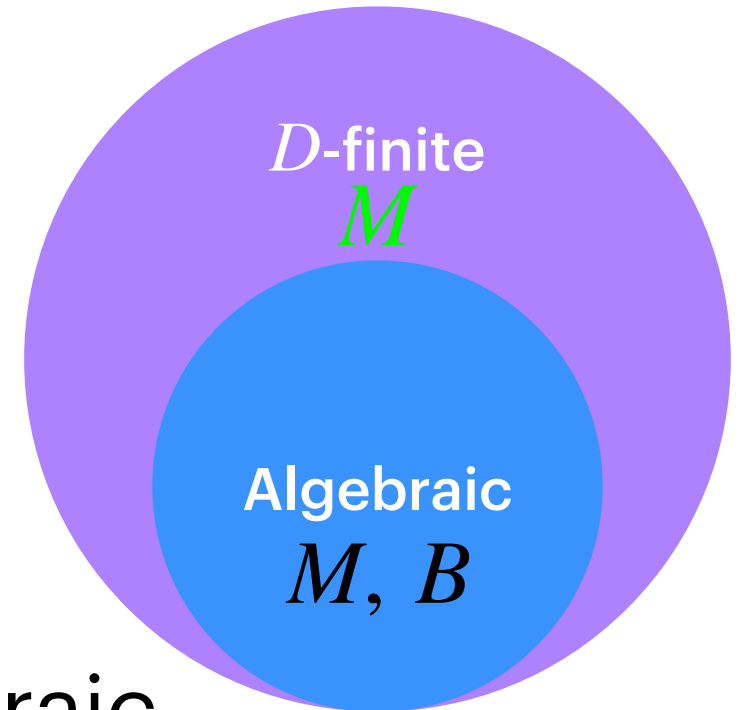

$$P(z, M(z)) = 0$$

Tree-rooted maps are not so nice

$$M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \text{ so}$$

$$\bullet [z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3}; \quad \bullet \rho_M = \frac{1}{16};$$

$$\bullet M(\rho_M) = 8 - \frac{64}{3\pi} \simeq 1.2 \text{ so } M \text{ is not algebraic...}$$



$$P(z, M(z)) = 0$$

↙

Tree-rooted maps are not so nice

$$M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \text{ so}$$

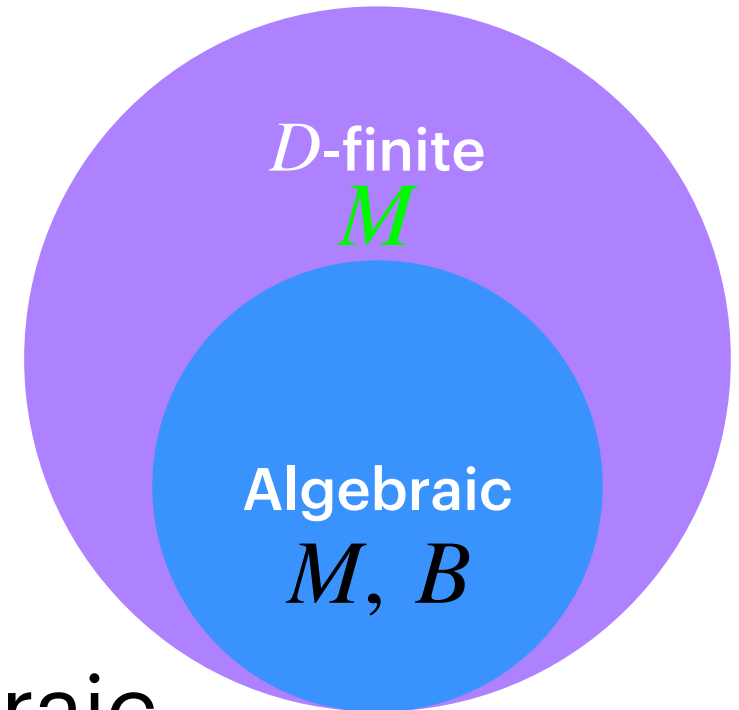
$$\bullet [z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3}; \quad \bullet \rho_M = \frac{1}{16};$$

$$\bullet M(\rho_M) = 8 - \frac{64}{3\pi} \simeq 1.2 \text{ so } M \text{ is not algebraic...}$$

• Fortunately, it is still *D-finite*

$$P(z, M(z)) = 0$$

$$P_0(z) \frac{\partial^2 M}{\partial z^2}(z) + P_1(z) \frac{\partial M}{\partial z}(z) + P_2(z) M(z) + P_3(z) = 0.$$



2-connected tree-rooted maps are naughty

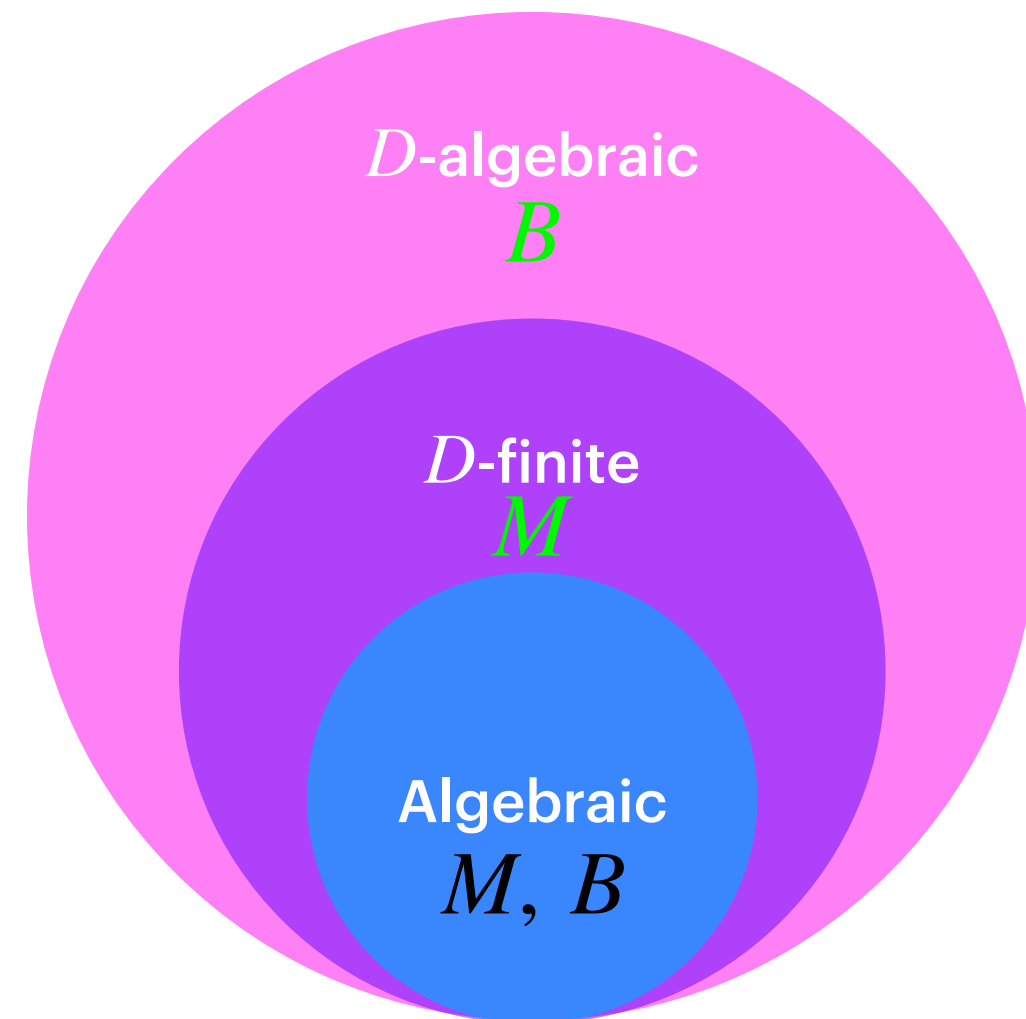
Using $M(z) = B(zM^2(z))$ and the properties of M , we show

2-connected tree-rooted maps are naughty

Using $M(z) = B(zM^2(z))$ and the properties of M , we show

- $\rho_B = \rho_M M^2(\rho_M) = \frac{4(3\pi - 8)^2}{9\pi^2} \approx 0.091$

is not algebraic so B is not D -finite



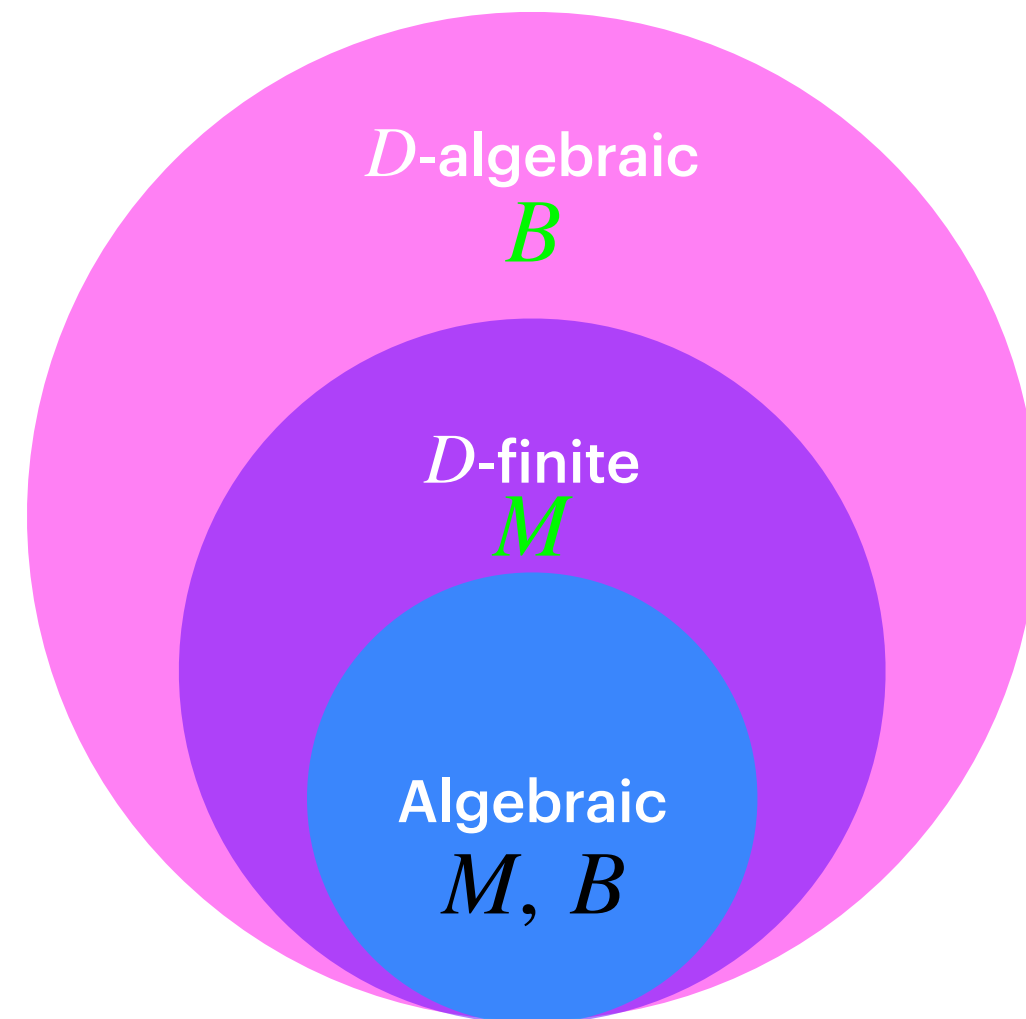
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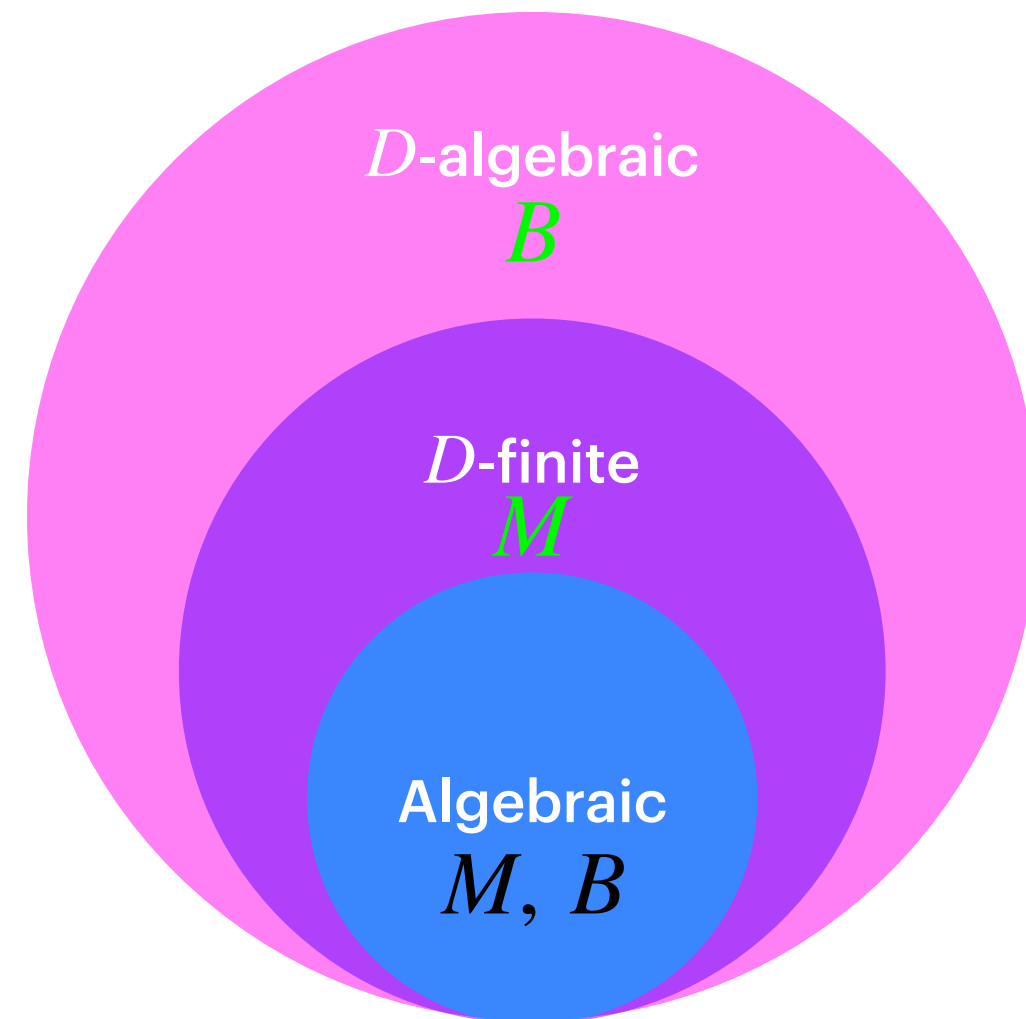
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- B is D -algebraic

$$P\left(\frac{\partial^2 B}{\partial y^2}(y), \frac{\partial B}{\partial y}(y), B(y), y\right) = 0.$$



Enumeration of 2-connected tree-rooted maps

Using $M(z) = B(zM^2(z))$ and the properties of M , we show

Theorem [Albenque, Fusy, S. 24]

$$[y^n]B(y) \sim \frac{4(3\pi - 8)^3}{27\pi(4 - \pi)^3} \times \rho_B^{-n} \times n^{-3}.$$

Phase transition

Theorem [Albenque, Fusy, S. 24] Model exhibits a phase

transition at $u_C = \frac{9\pi(4 - \pi)}{420\pi - 81\pi^2 - 512} \simeq 3.02$.

When $n \rightarrow \infty$:

- Subcritical phase $u < u_C$: “general tree-rooted map phase” one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: “tree phase” only small blocks.

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration			
Size of <ul style="list-style-type: none"> - the largest block - the second one 			
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of <ul style="list-style-type: none"> - the largest block - the second one 			
Scaling limit of M_n			

Results

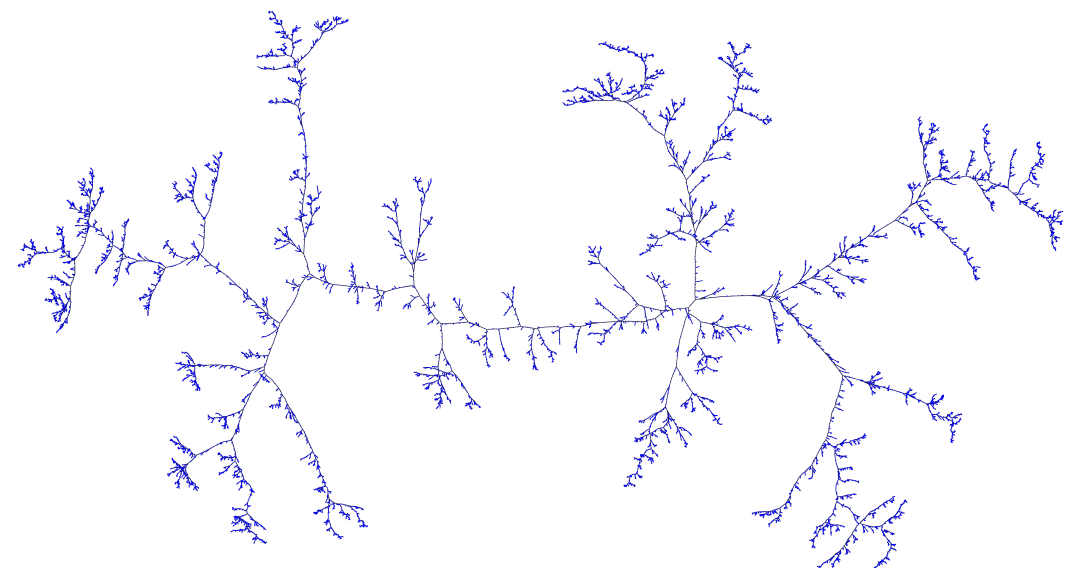
For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3 \ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3 \ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
Scaling limit of M_n	Ordered atoms of a Poisson Point Process		

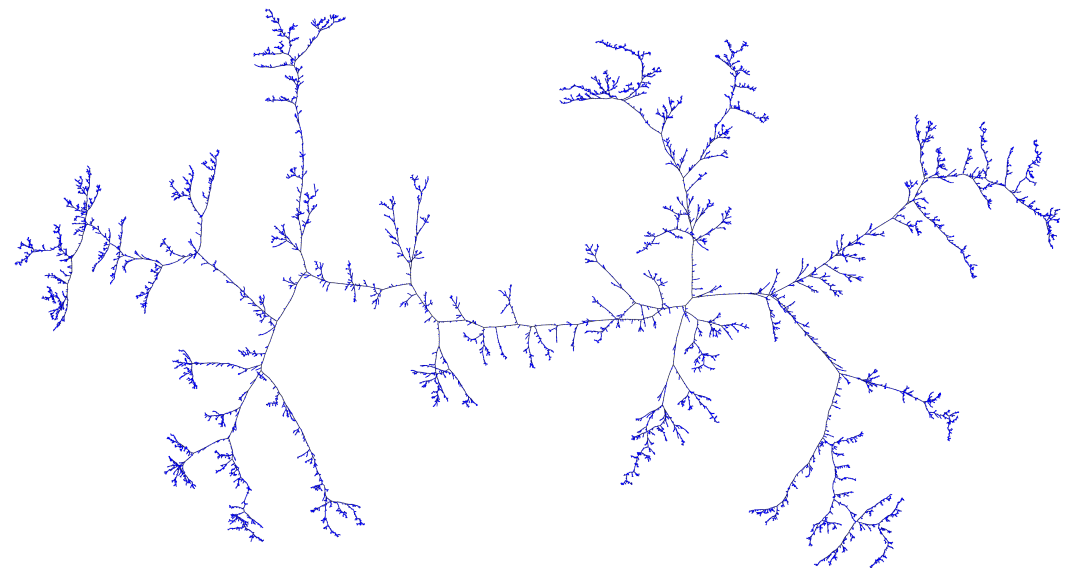
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3 \ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
Scaling limit of M_n	?	$\frac{C_2 \ln(n)^{1/2}}{n^{1/2}} M_n \rightarrow \mathcal{T}_e$	$\frac{C_3(u)}{n^{1/2}} M_n \rightarrow \mathcal{T}_e$ [Stufler 2020]



Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3 \ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
Scaling limit of M_n	?	$\frac{C_2 \ln(n)^{1/2}}{n^{1/2}} M_n \rightarrow \mathcal{T}_e$	$\frac{C_3(u)}{n^{1/2}} M_n \rightarrow \mathcal{T}_e$ [Stufler 2020]



Interlude: tree-rooted quadrangulations

CANCELLED

**Interlude: tree-rooted
quadrangulations**

$M(z) = Q(z)$ does not hold!

VI. Perspectives

Extensions to more involved decompositions

Block-weighted

- Maps into loopless blocks;
- 2-connected maps into 3-connected blocks...

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1 - z(1 + M))^2$	$z(1 + M)^2$
loopless $M_2(z)$	simple $M_3(z)$	$z(1 + M)$	—
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1 + M)^2$	—
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1 + M)$	—
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1 + M)$
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1 + M)$	—
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1 - z(1 + M))^2$	$z(1 + M)^2$
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1 + M)^2$	—
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1 + M)$	—
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1 + M)^3$	—
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1 + M)^2$	—

Extensions to more involved decompositions

Block-weighted

- Tree-rooted quadrangulations;
- Forested maps;
- Maps endowed with a Potts model / Ising model;
- 2-oriented quadrangulations (resp. 3-oriented triangulations) decomposed into irreducible blocks...

Critical window?

Phase transition very sharp \Rightarrow what if $u = 9/5 \pm \varepsilon(n)$?

- Block size results still hold if $u_n = 9/5 - \varepsilon(n)$, $\varepsilon^3 n \rightarrow \infty$;
- For $u_n = 9/5 + \varepsilon(n)$, this is the case as well: when $\varepsilon^3 n \rightarrow \infty$

$$L_{n,1} \sim 2.7648 \varepsilon^{-2} \ln(\varepsilon^3 n)$$

(analogous to [Bollobás 1984]’s result for Erdős-Rényi graphs!);

- Results exist for scaling limits in ER graphs [Addario-Berry, Broutin, Goldschmidt 2010], open question in our case.

Is there a critical window? If so, what is its width?

Thank you!