

Phase transition for tree-rooted maps

AofA 2024 @ Bath, UK
June 17, 2024

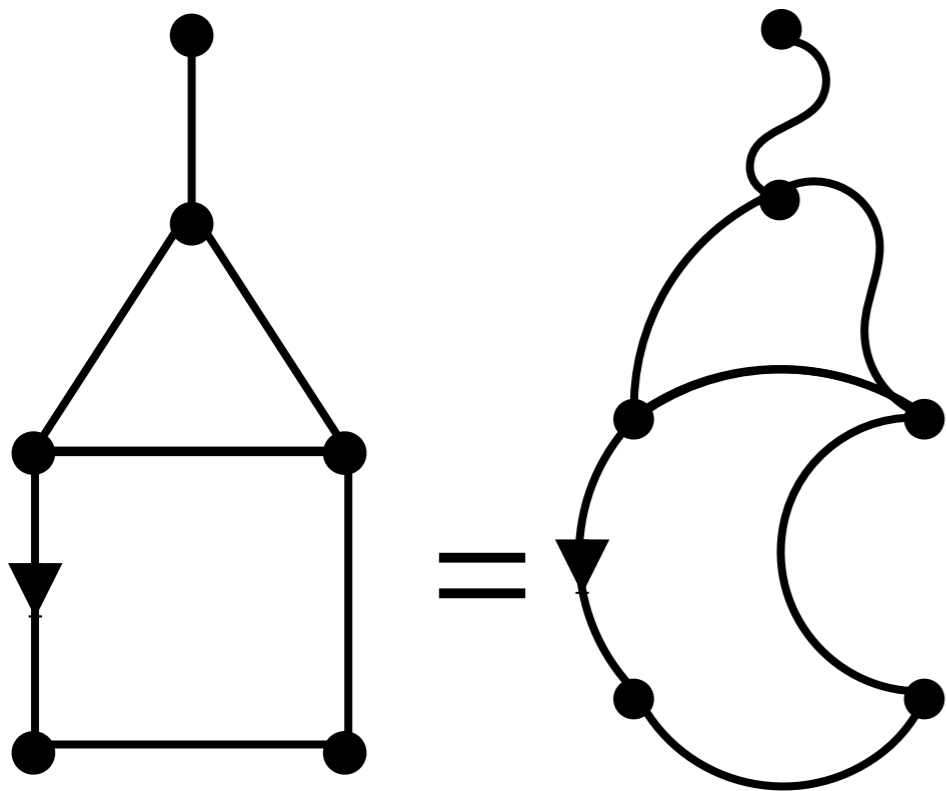
Zéphyr Salvy (he/they)

Joint work with Marie Albenque & Éric Fusy

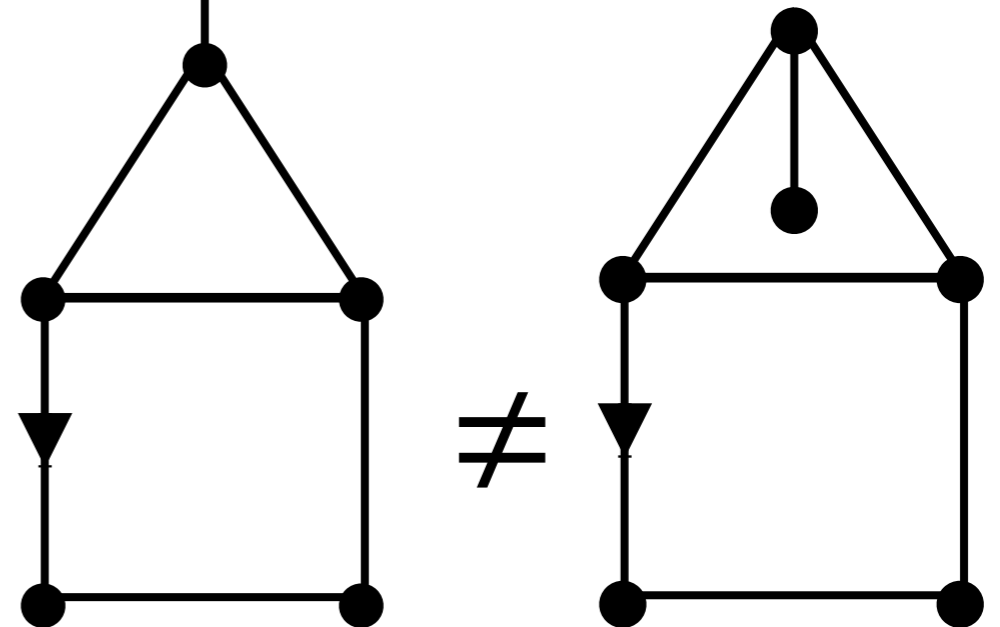
LIGM, Université Gustave Eiffel, France

Planar maps

Planar map \mathfrak{m} = embedding on the sphere of a connected planar graph, considered up to homeomorphisms



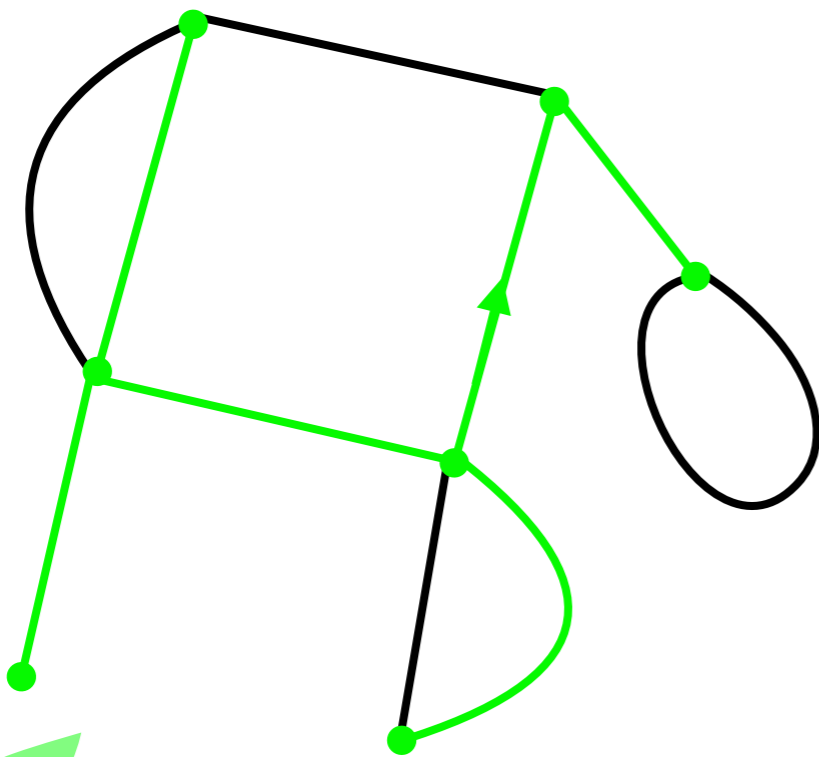
Planar map = planar graph +
cyclic order on neighbours



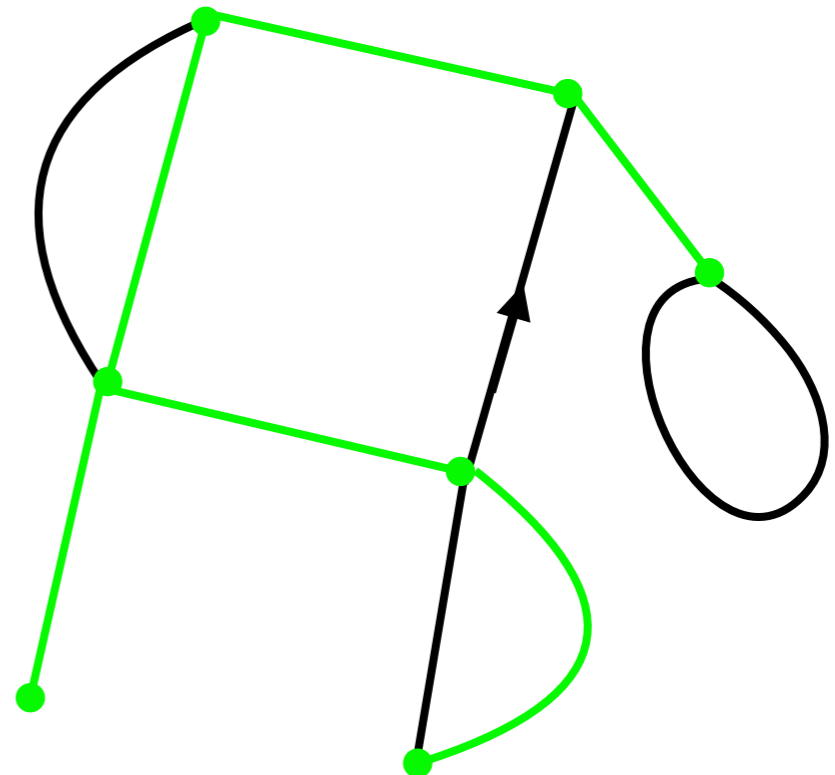
- **Rooted** planar map = map endowed with a marked oriented edge (represented by an arrow);
- **Size** $|\mathfrak{m}|$ = number of edges;
- **Corner** (does not exist for graphs !) = space between two consecutive edges around a vertex (trigonometric order).

Tree-rooted maps

= (rooted planar) maps endowed with a spanning tree.



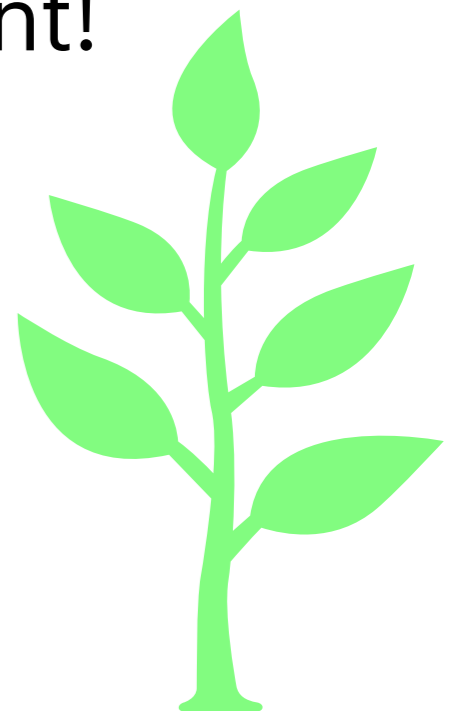
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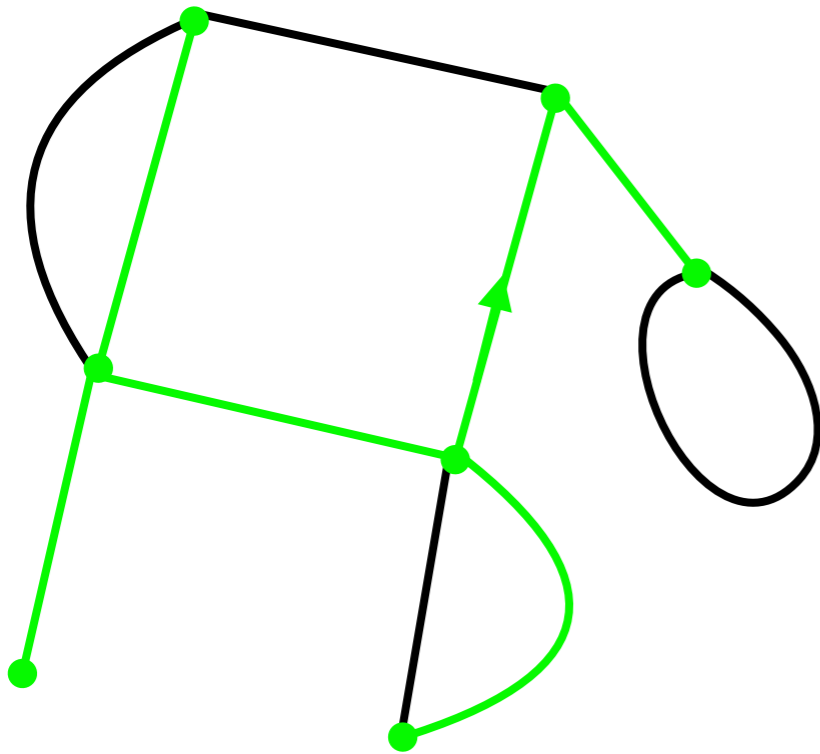
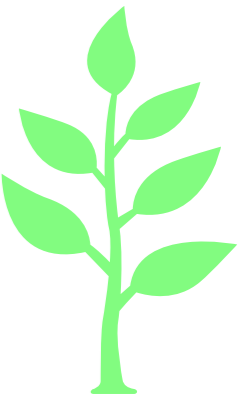
Decorated maps are interesting

Theoretical physics point of view:

- Undecorated maps: “pure gravity” case (nothing happens on the surface);
- Decorated maps: things happen! new asymptotic behaviours! new universality classes! excitement!



Tree-rooted maps



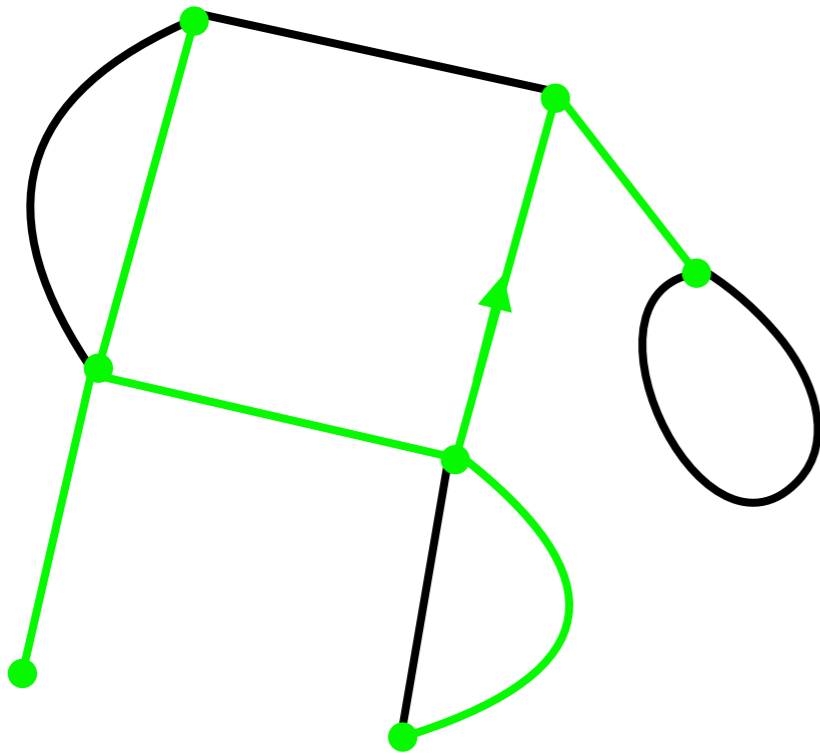
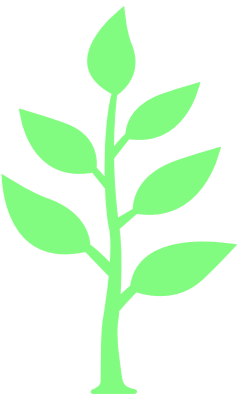
- Combinatorics well understood :
Mullin's bijection;

$$[z^n]M(z) = \text{Cat}_n \text{Cat}_{n+1}$$

[Mullin 67]

- Geometry not so much.

Tree-rooted maps



- Combinatorics well understood :
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We want a phase transition in tree-rooted maps.

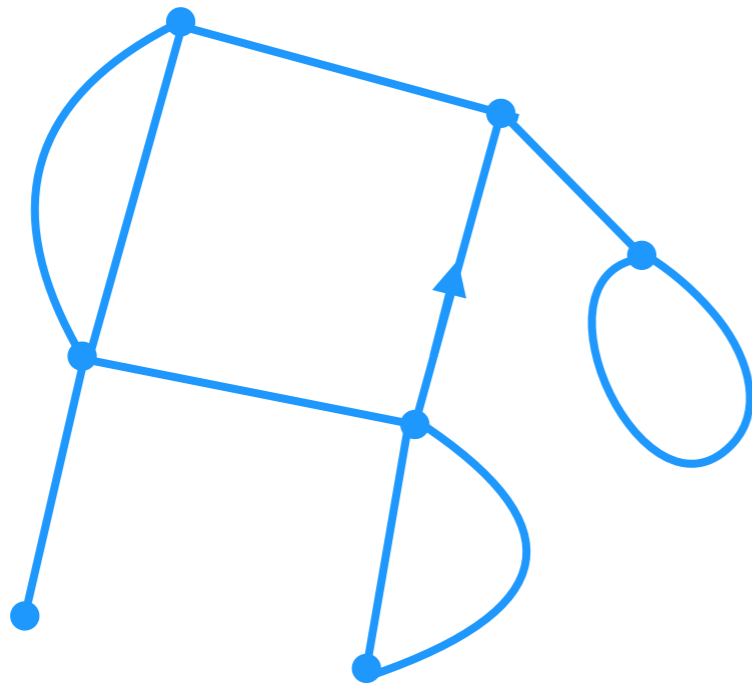
=> Block-weighted tree-rooted maps.

I. “Block-weighted maps”?

Joint work with William Fleurat

Universality results for planar maps

- Enumeration: $\kappa \rho^{-n} n^{-5/2}$ [Tutte 1963];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for quadrangulations [Le Gall 2013, Miermont 2013] and general maps [Bettinelli, Jacob, Miermont 2014];



Brownian Sphere \mathcal{S}_e

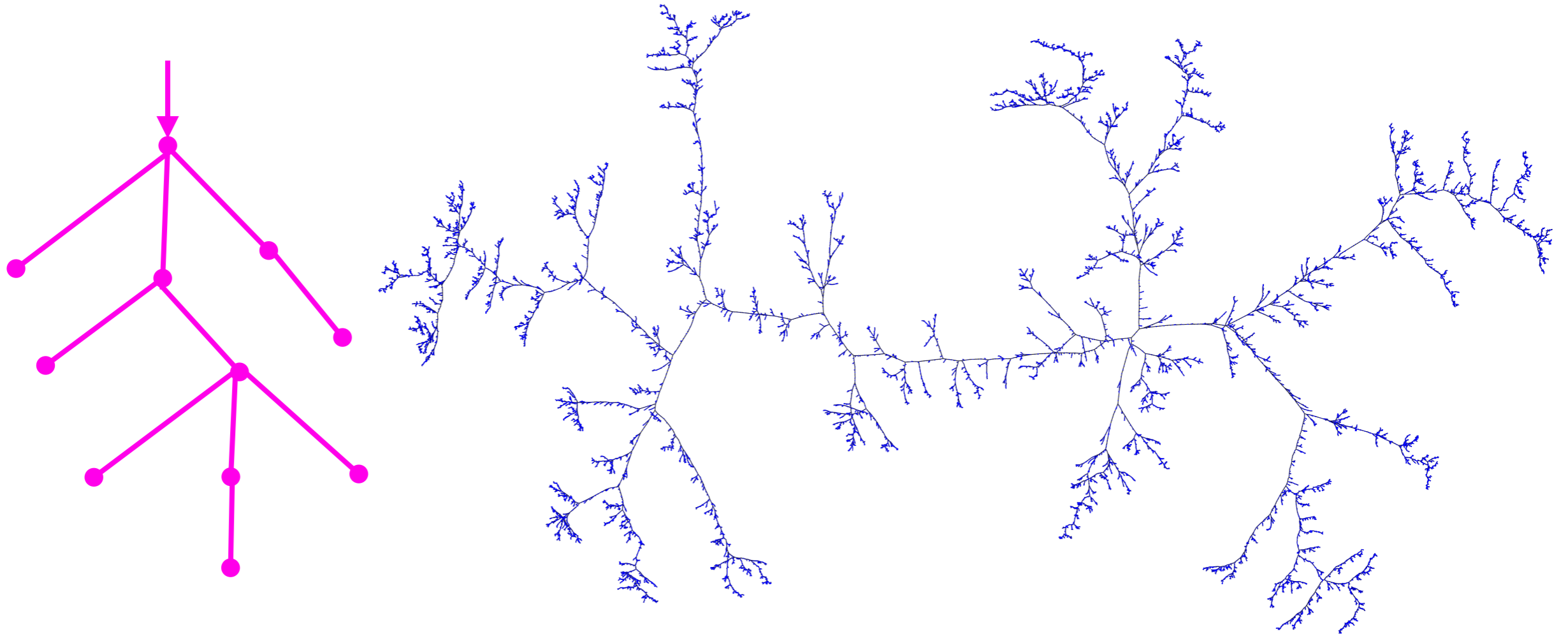


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- Scaling limit: Brownian sphere for quadrangulations [Le Gall 2013, Miermont 2013] and general maps [Bettinelli, Jacob, Miermont 2014];
- Universality:
 - Same enumeration [Drmota, Noy, Yu 2020];
 - Same scaling limit, e.g. for triangulations & $2q$ -angulations [Le Gall 2013], simple quadrangulations [Addario-Berry, Albenque 2017].

Universality results for plane trees

- Enumeration: $\kappa \rho^{-n} n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];



Universality results for plane trees

- Enumeration: $\kappa \rho^{-n} n^{-3/2}$;
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- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];
- Universality:
 - Same enumeration;
 - Same scaling limit;

Even for some classes of **maps**; e.g. outerplanar maps [Caraceni 2016],
maps with a boundary of size $\gg n^{1/2}$ [Bettinelli 2015].

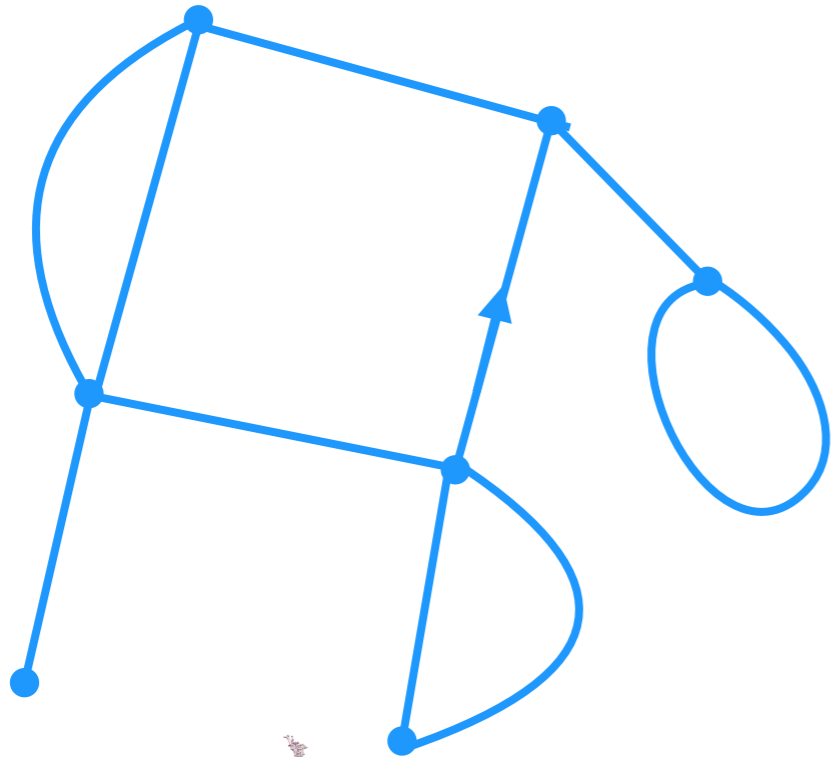
Models with (very) constrained boundaries

Motivation

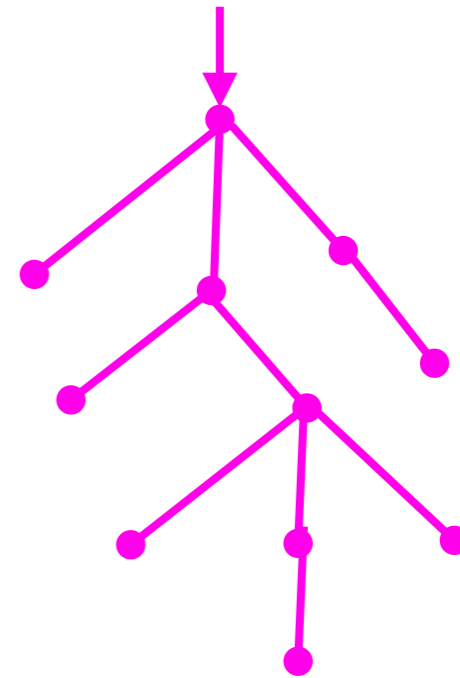
Inspired by [Bonzom 2016].

Two rich situations with universality results:

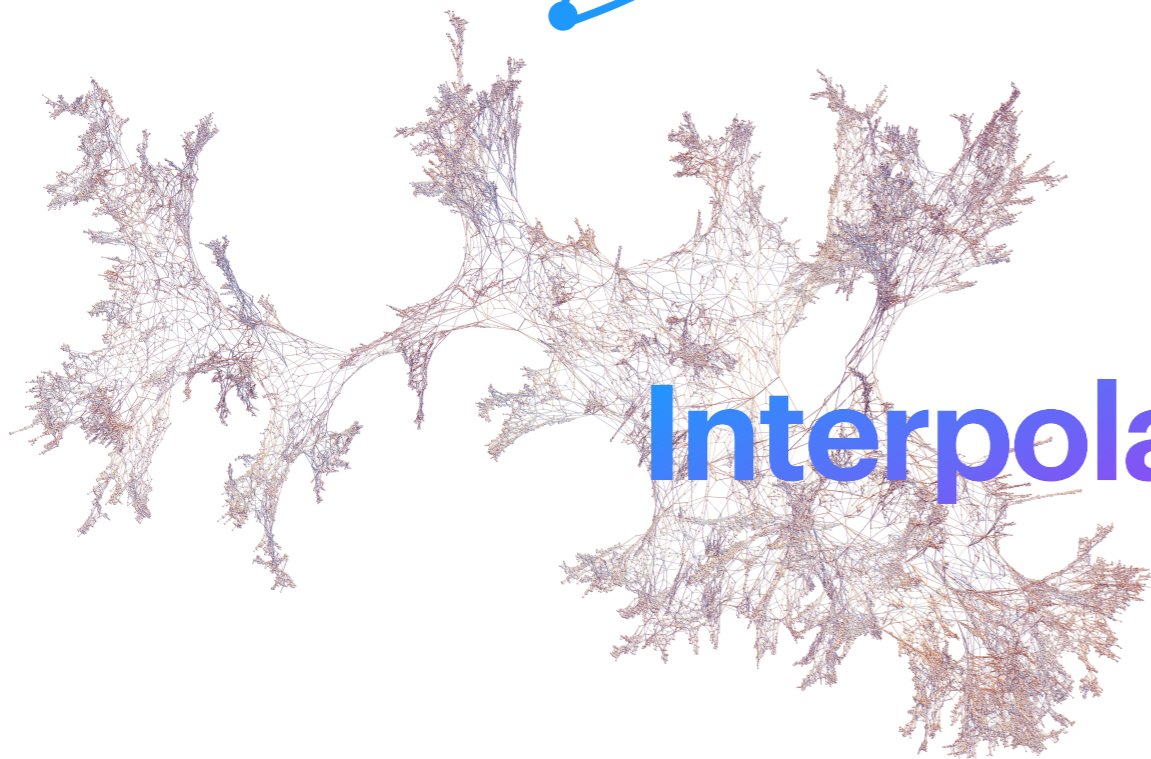
Planar maps



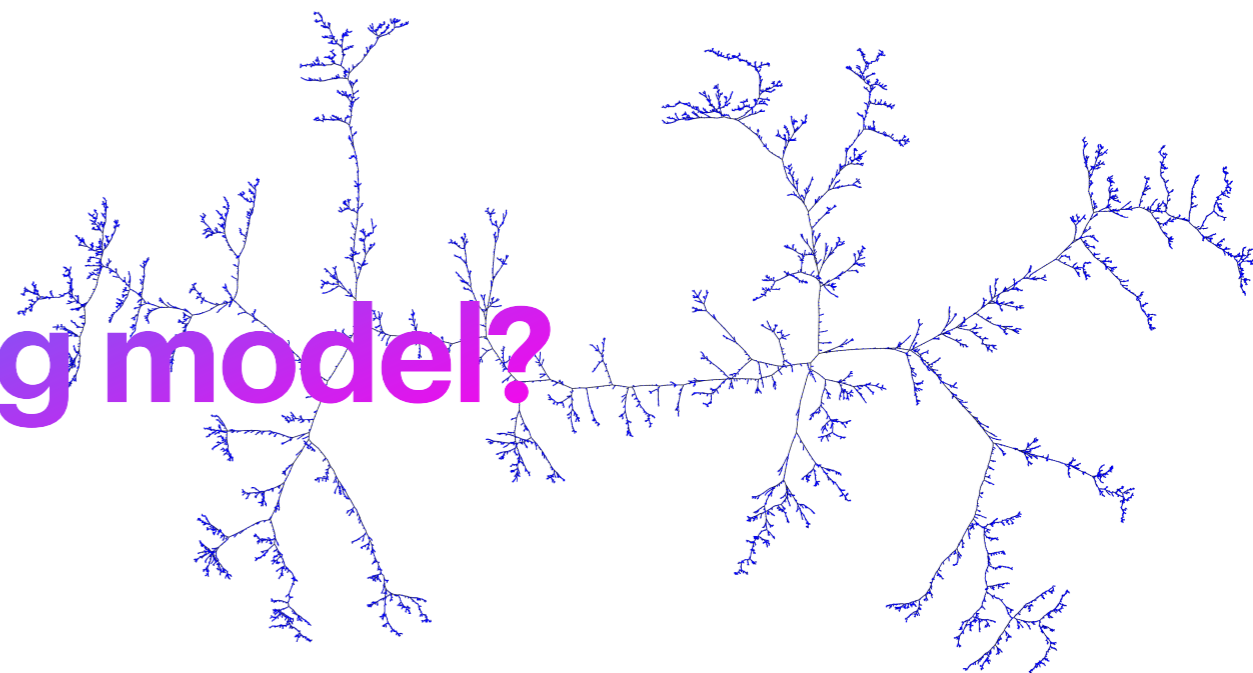
Plane trees



Interpolating model?



Brownian Sphere \mathcal{S}_e

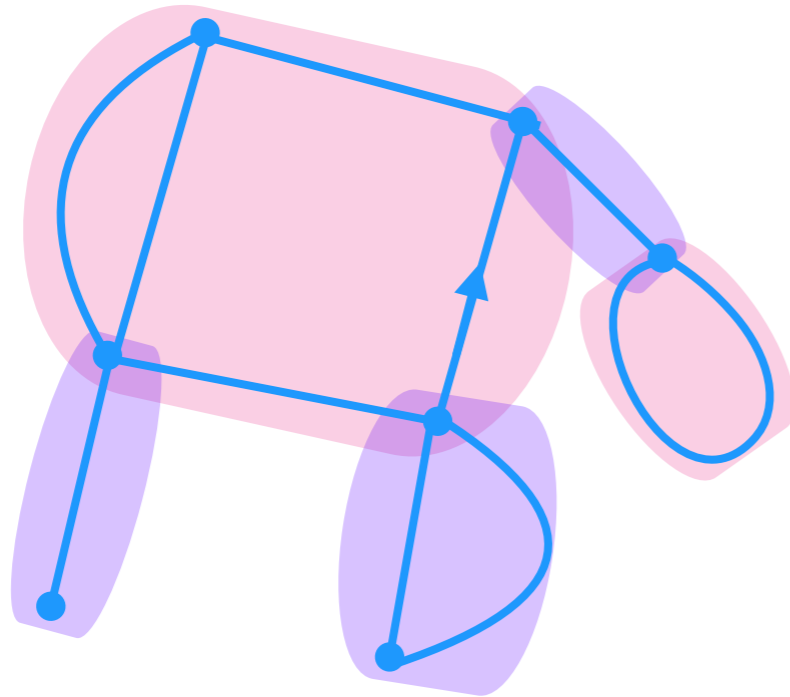


Brownian Tree \mathcal{T}_e

Model definition

2-connected = two vertices must be removed to disconnect.

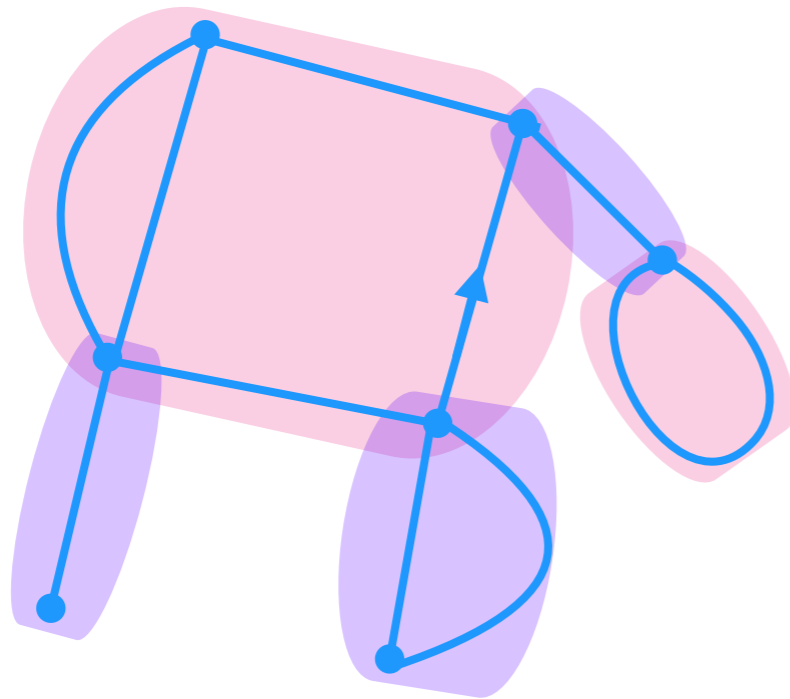
Block = maximal (for inclusion) 2-connected submap.



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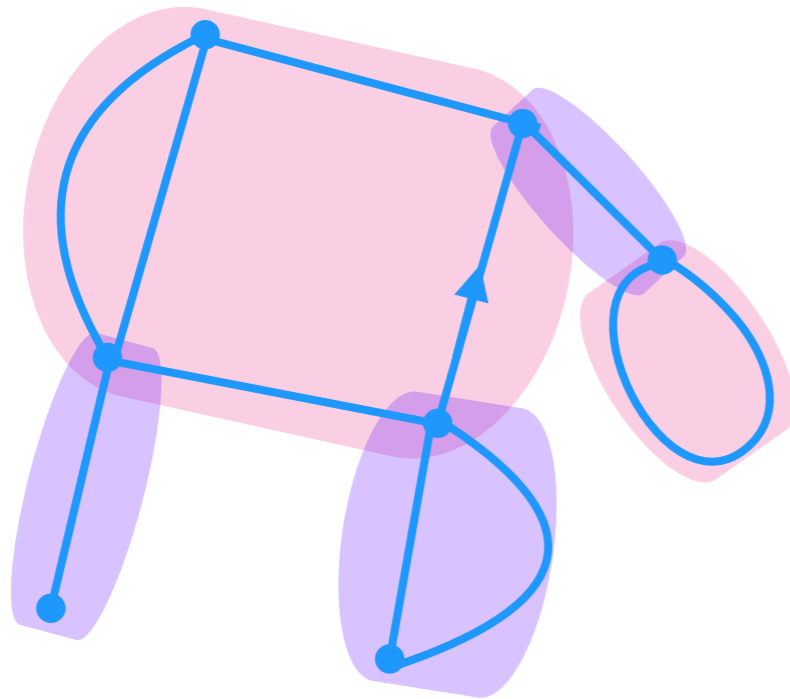


Condensation phenomenon: a large block concentrates a macroscopic part of the mass
[Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

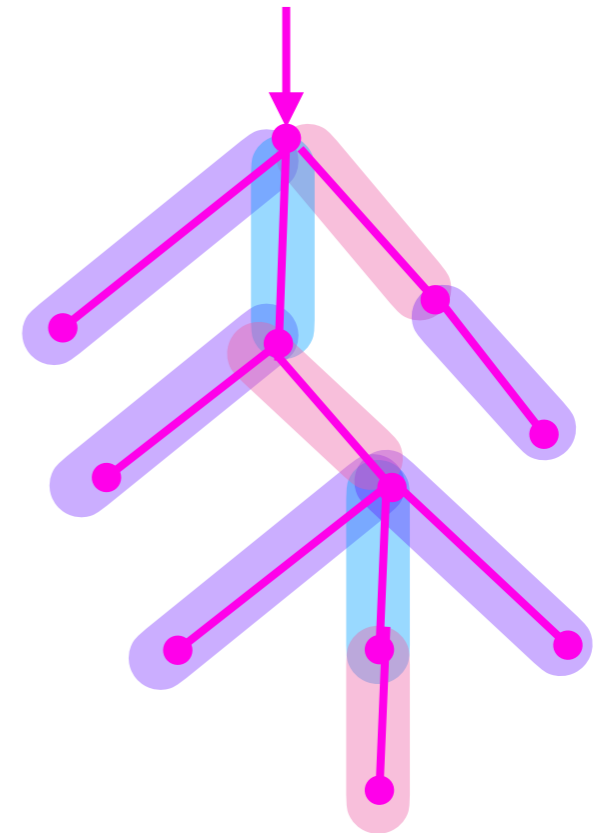
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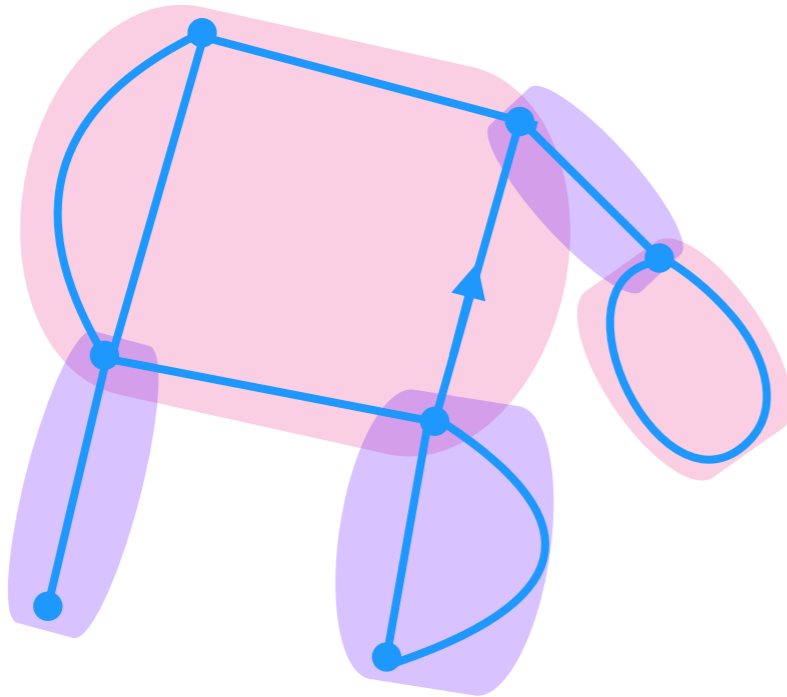


Only small blocks.

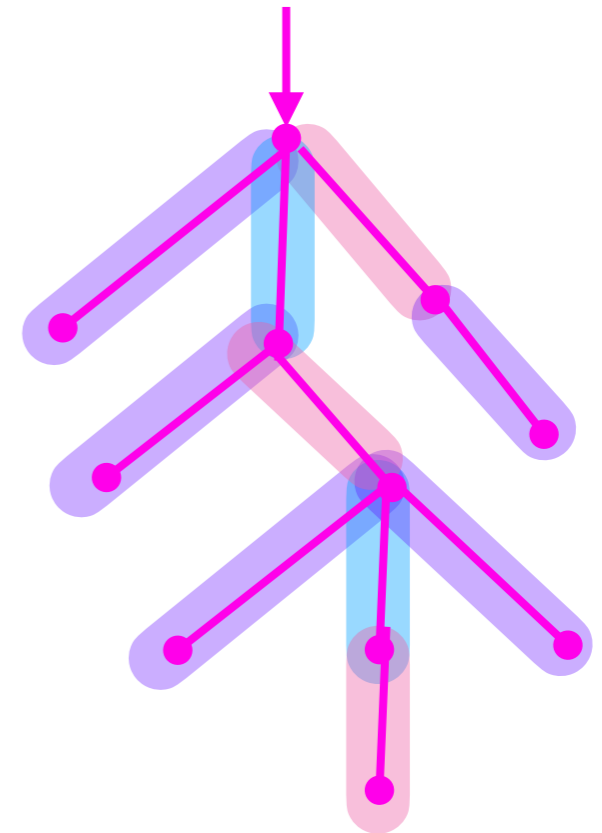
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Only small blocks.

Interpolating model using blocks!

Model

Inspired by [Bonzom 2016];
General setting in [Stufler 2020].

Goal: parameter that affects the typical number of blocks.

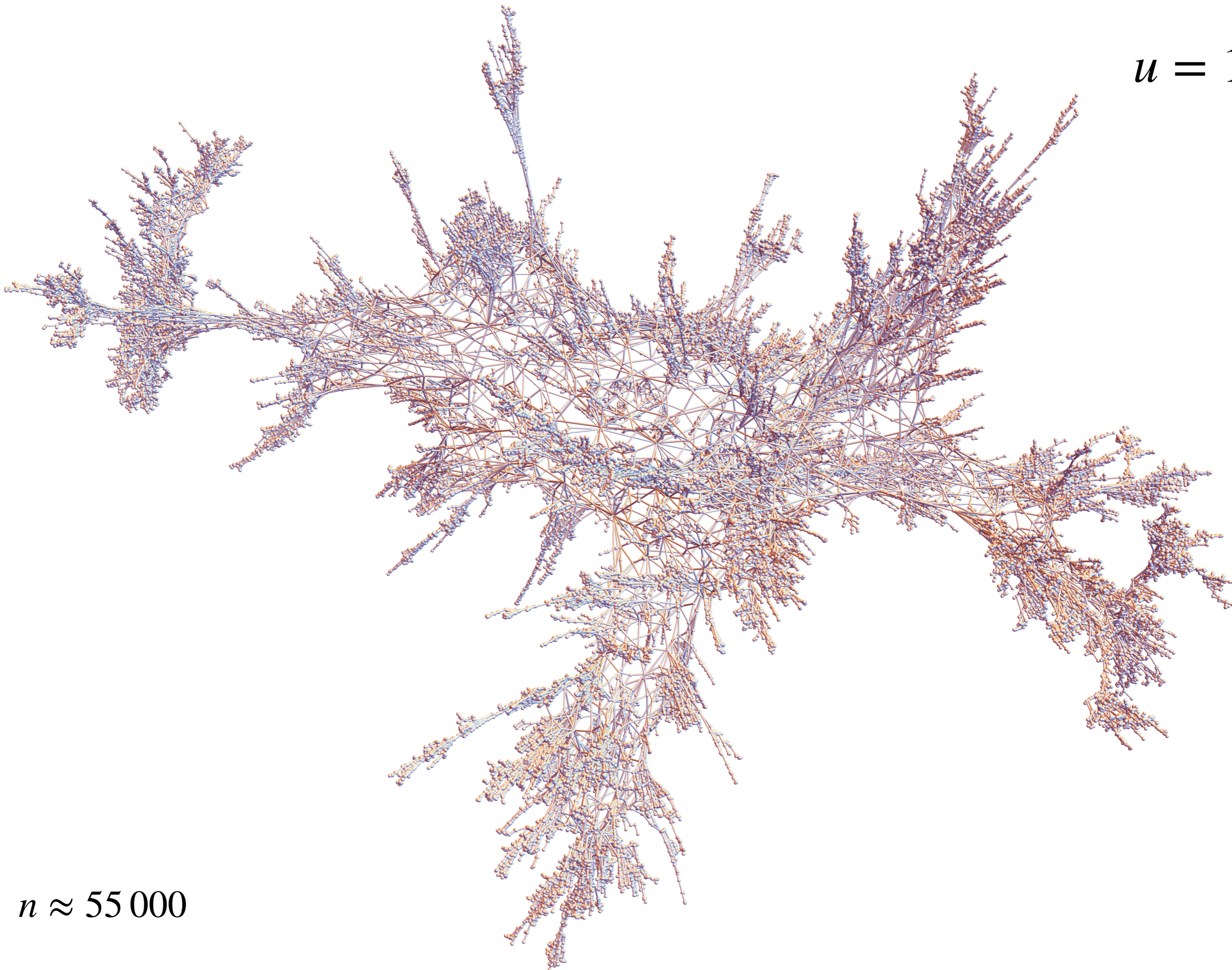
We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,
 $\mathcal{M}_n = \{\text{maps of size } n\}$,
 $\mathfrak{m} \in \mathcal{M}_n$,
 $Z_{n,u} = \text{normalisation.}$

- $u = 1$: uniform distribution on maps of size n ;
- $u \rightarrow 0$: minimising the number of blocks (=2-connected maps);
- $u \rightarrow \infty$: maximising the number of blocks (= trees!).

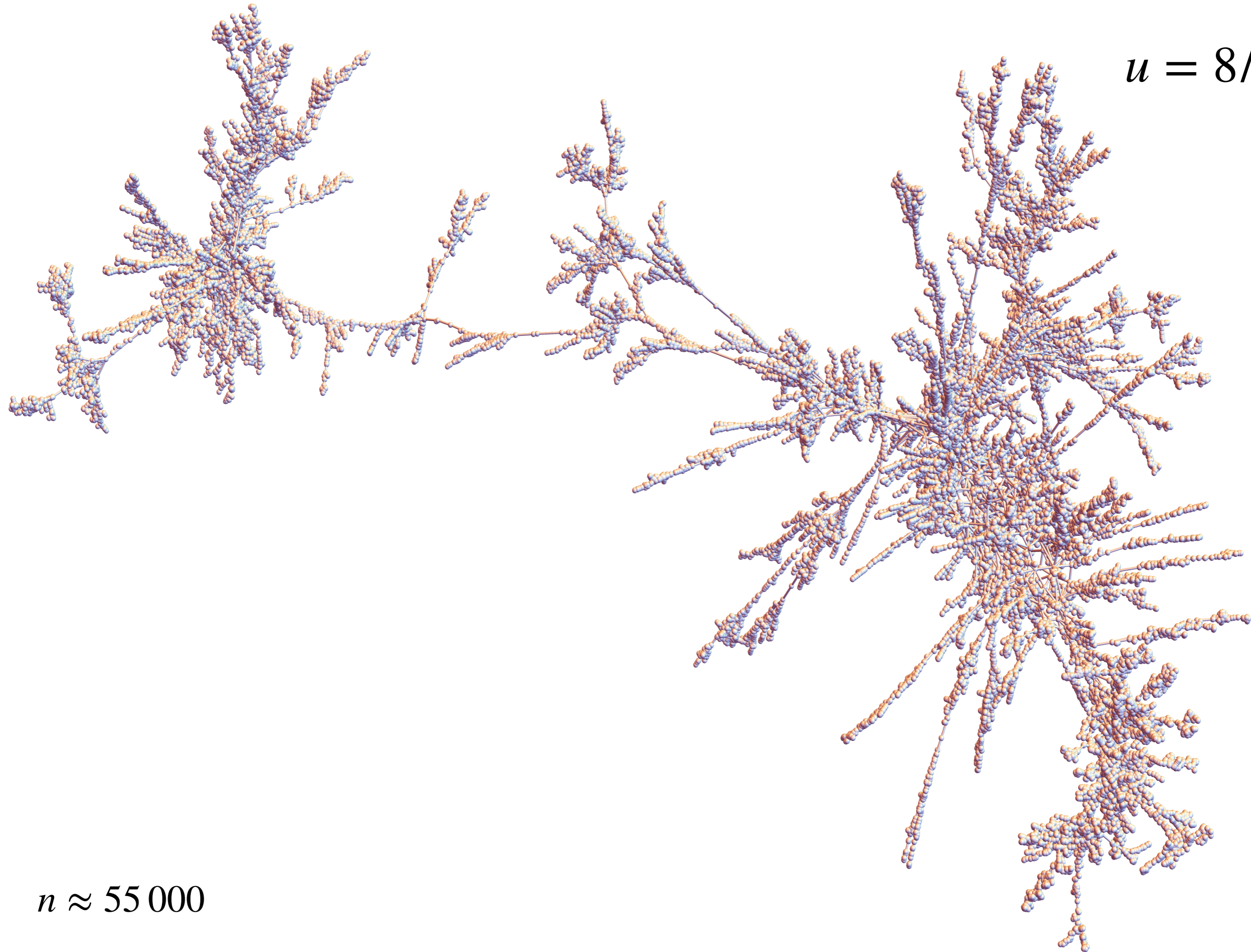
Given u , asymptotic behaviour when $n \rightarrow \infty$?

$$u = 1$$



$$n \approx 55\,000$$

$$u = 8/5$$



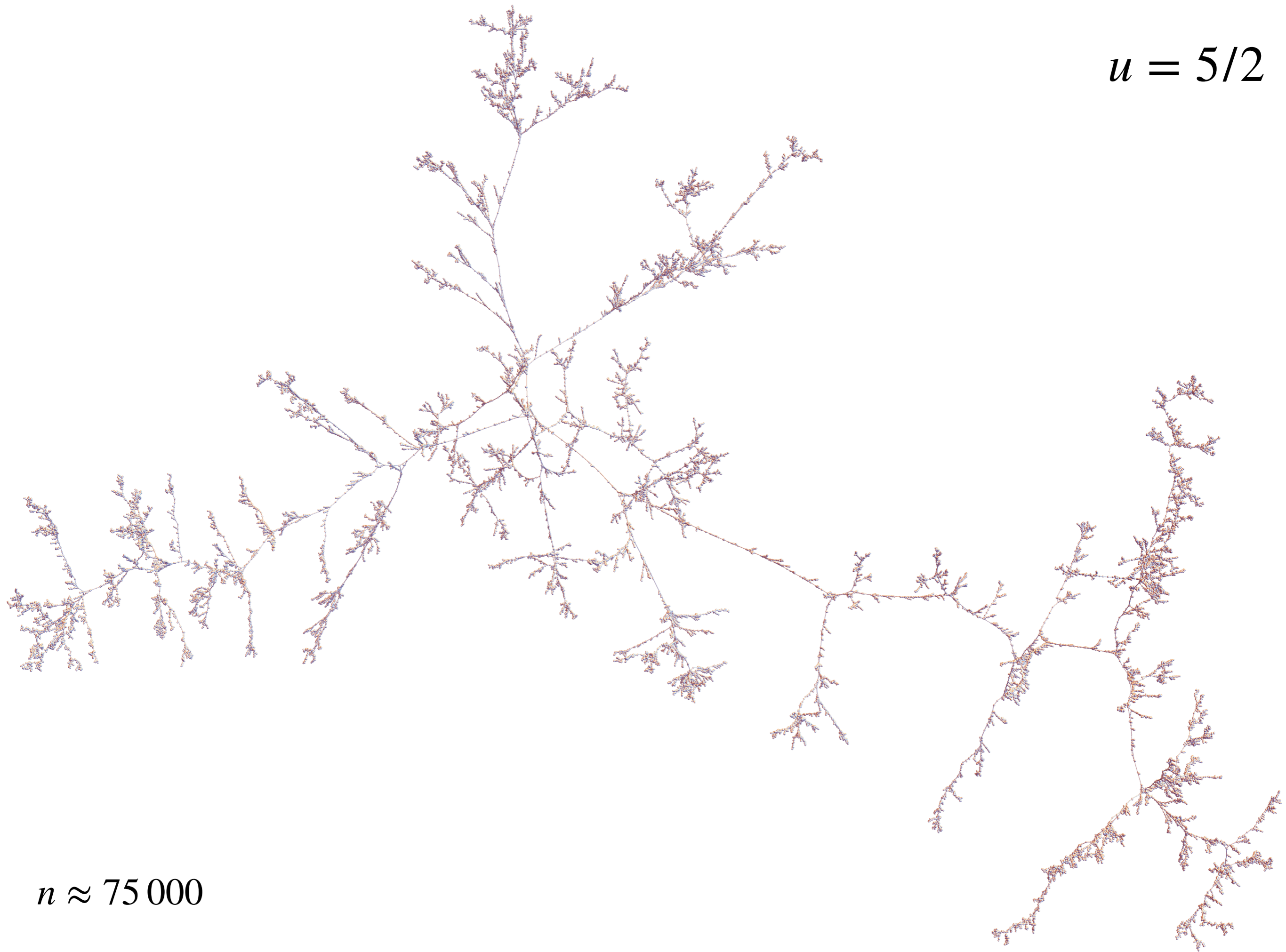
$$n \approx 55\,000$$

$$u = 9/5$$



$$n \approx 80\,000$$

$$u = 5/2$$



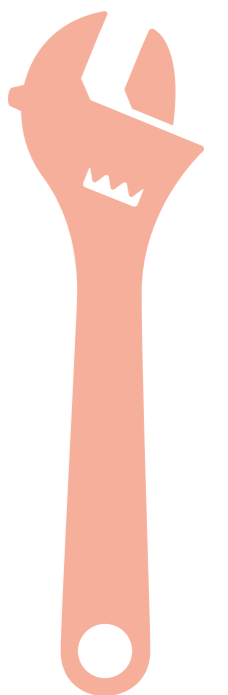
$$n \approx 75\,000$$

$$u = 5$$



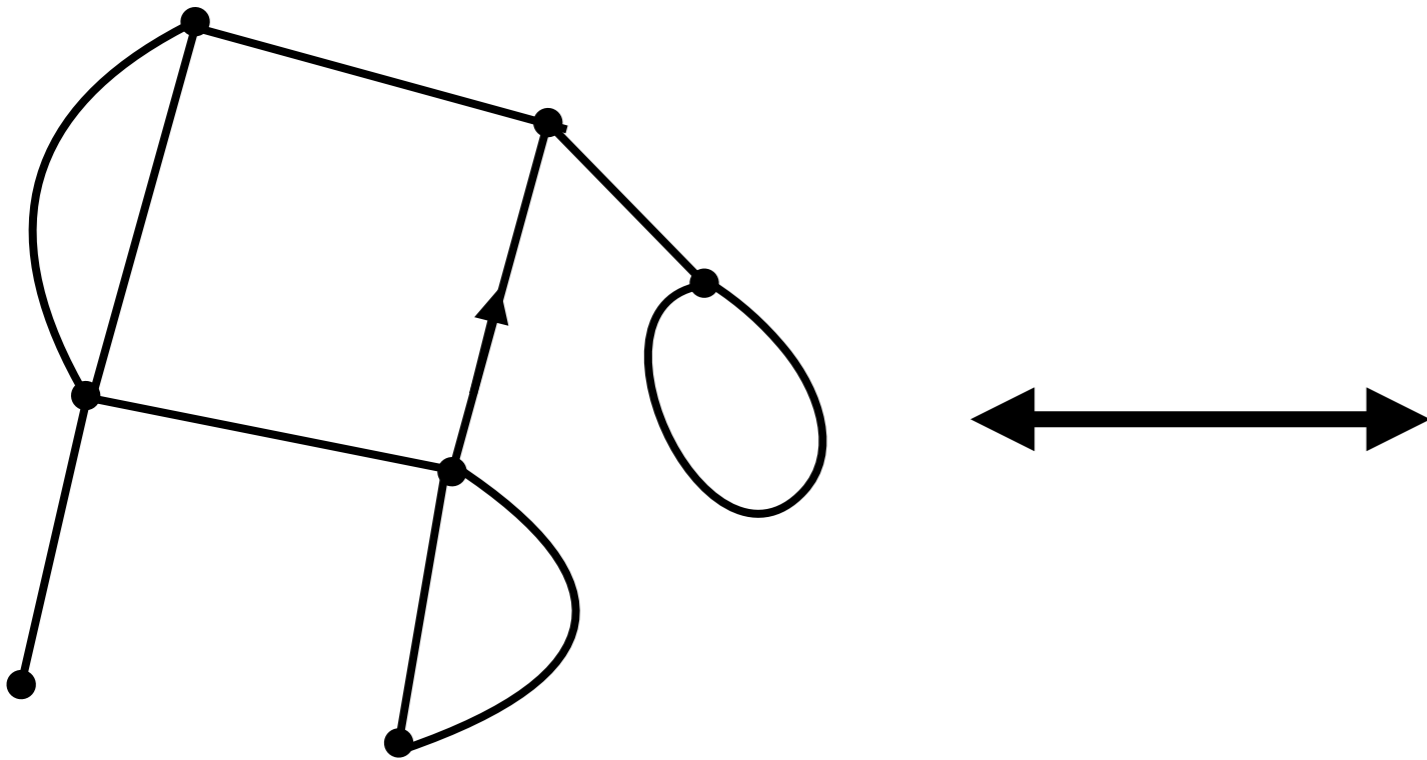
$$n \approx 50\,000$$

II. Block tree of a map



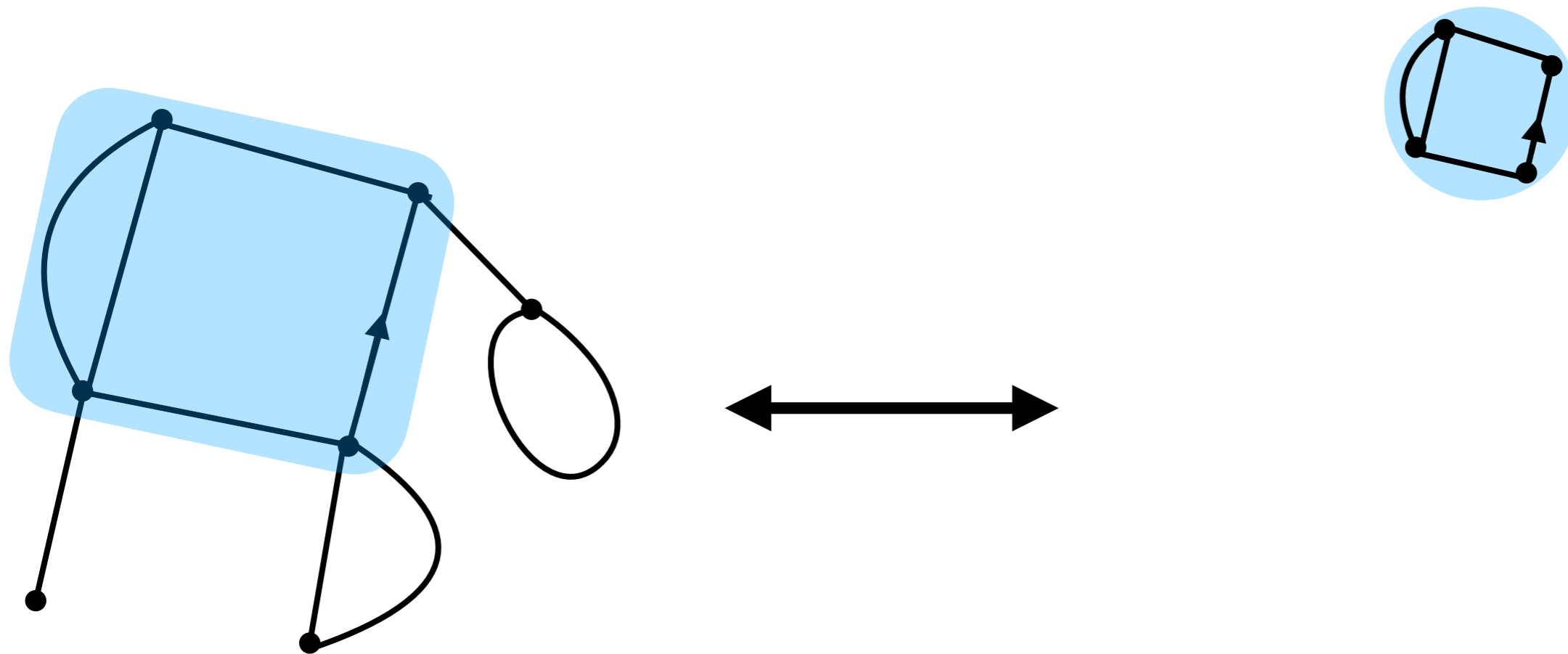
Decomposition of a map into blocks

Inspiration from [Tutte 1963]



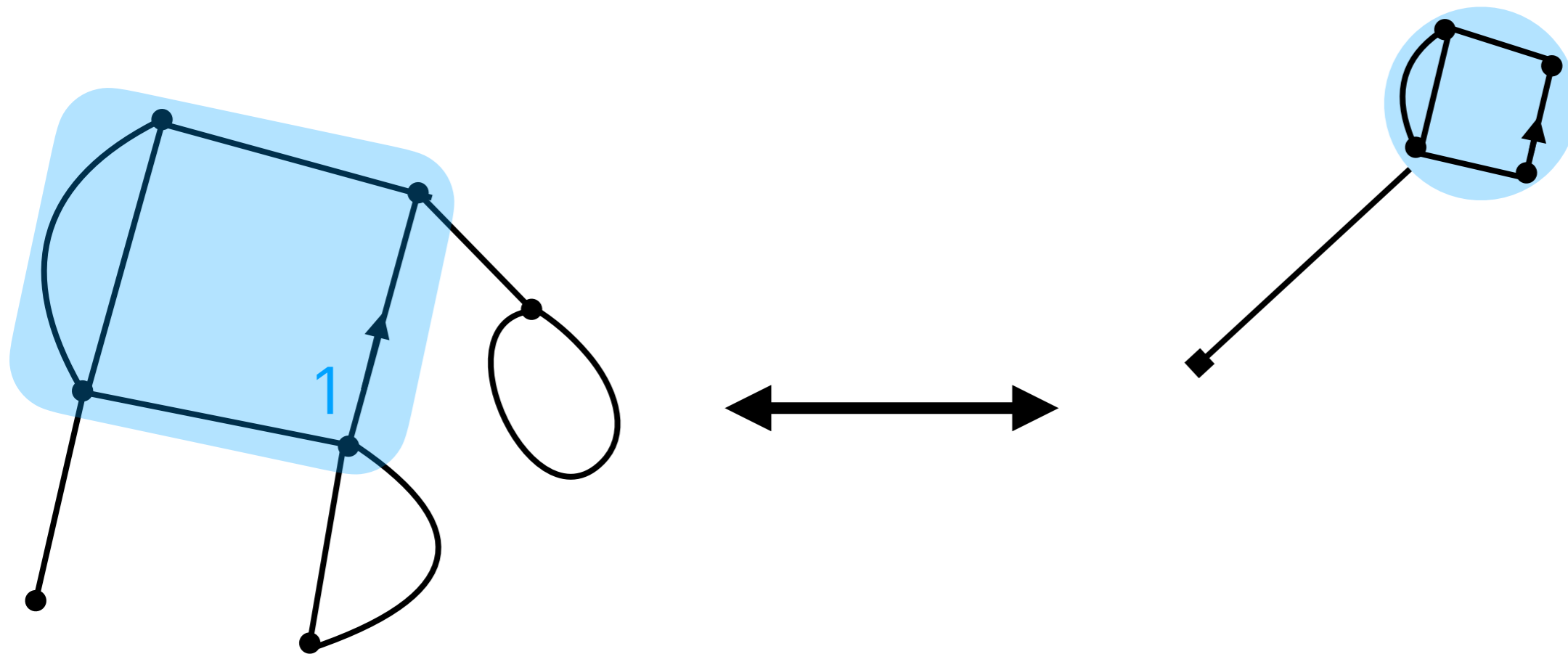
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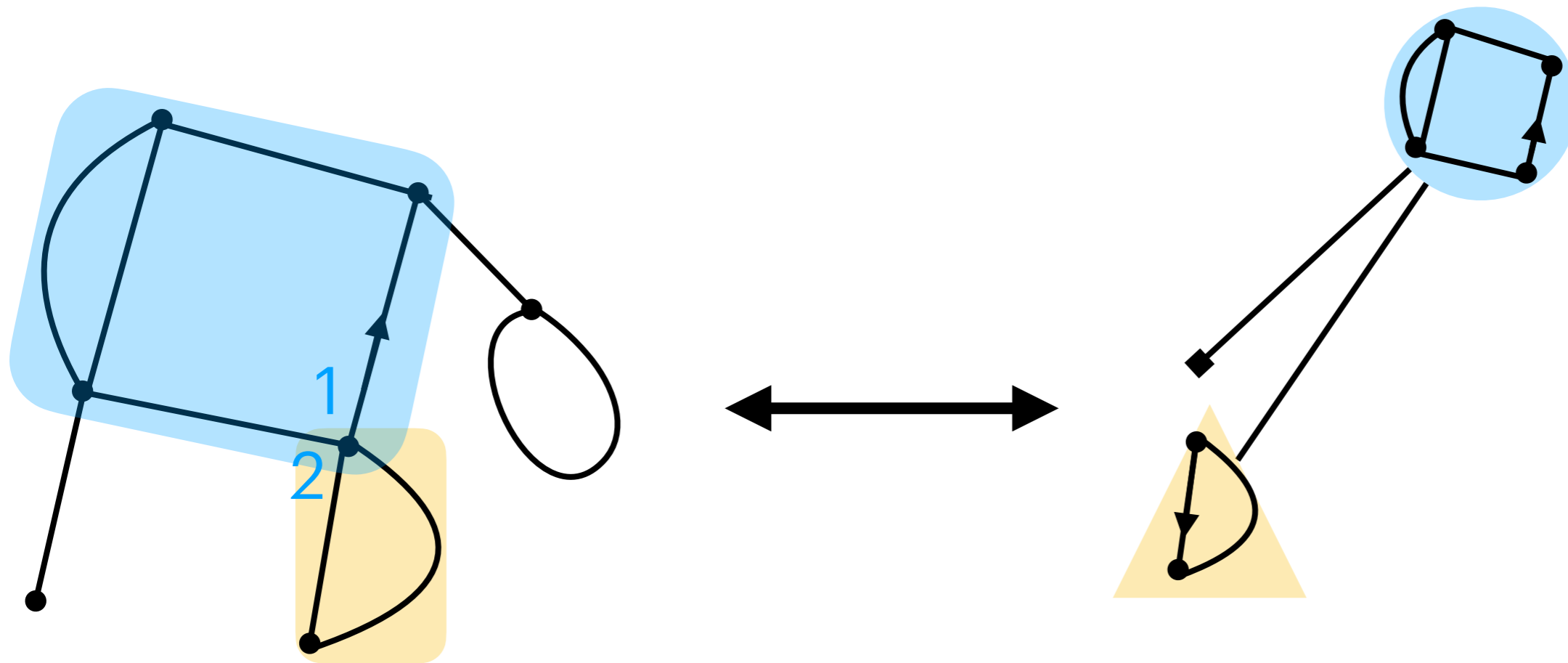
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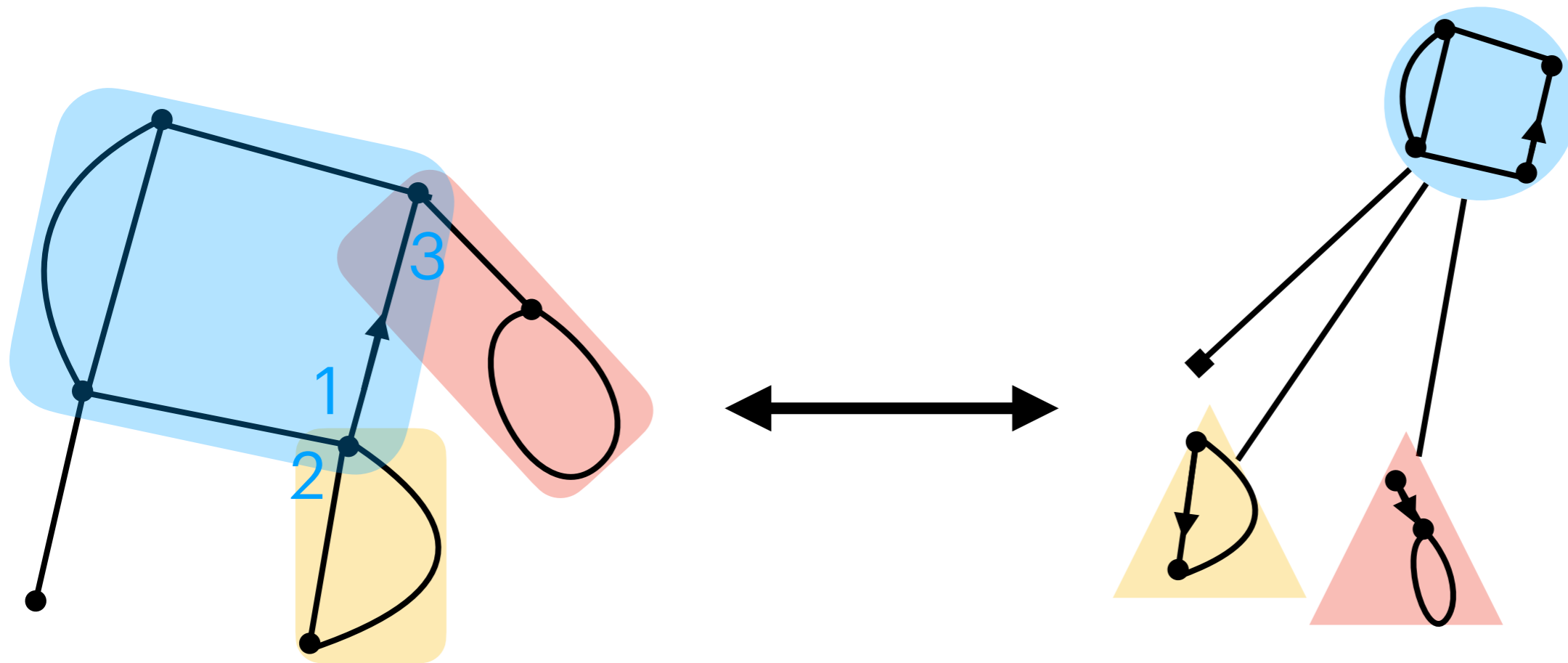
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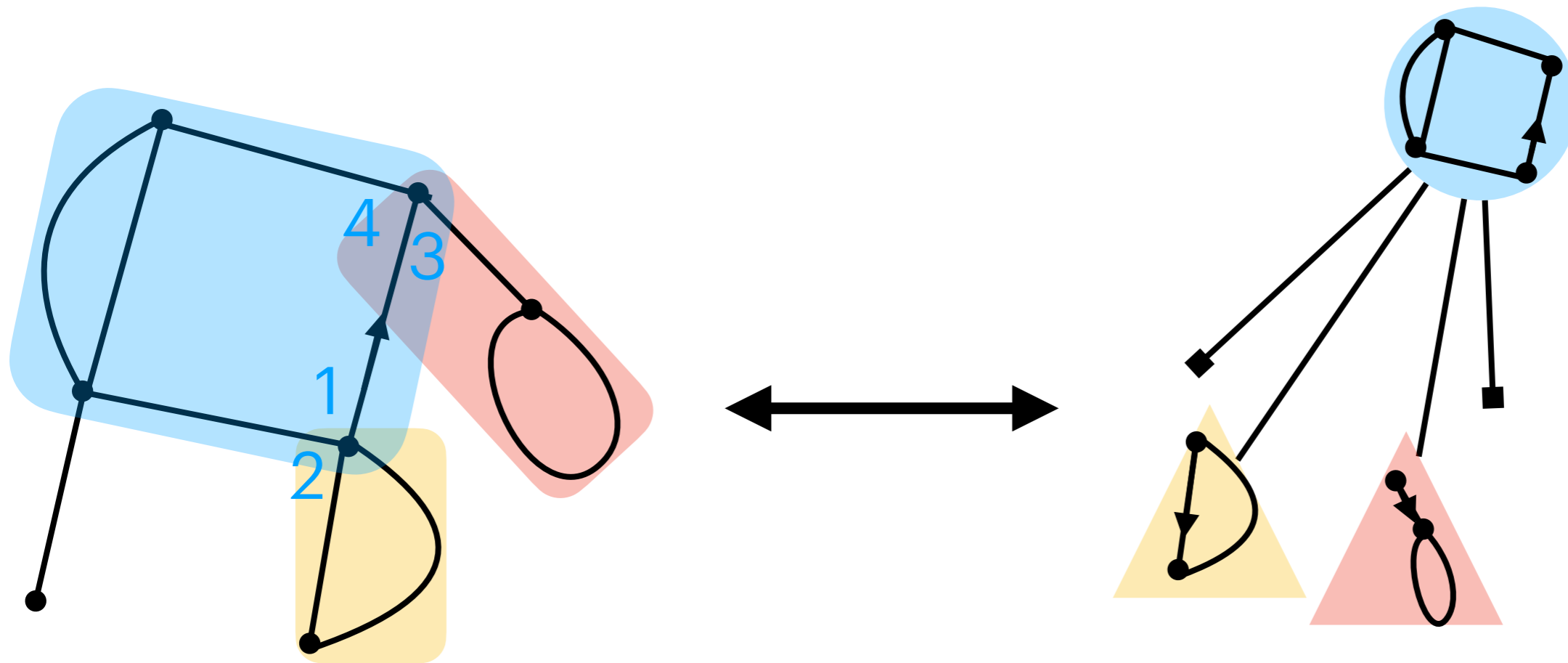
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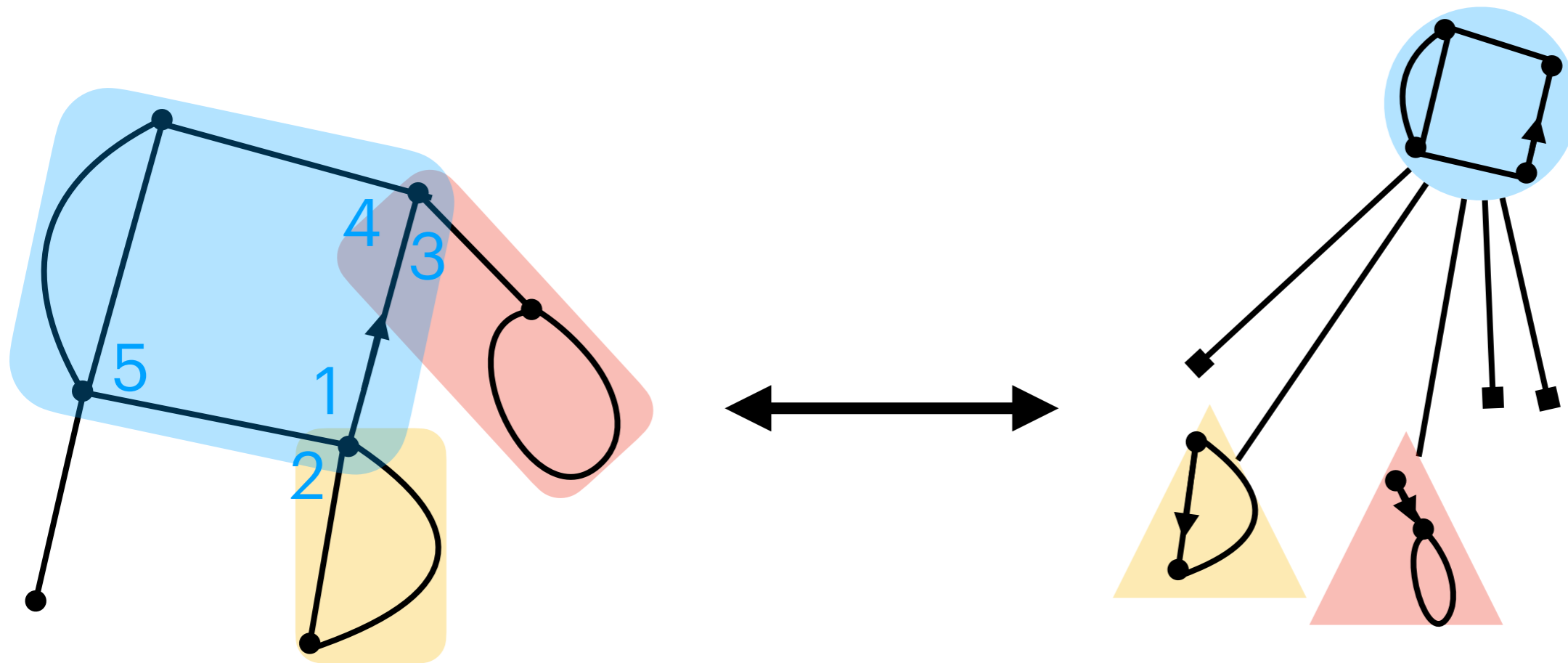
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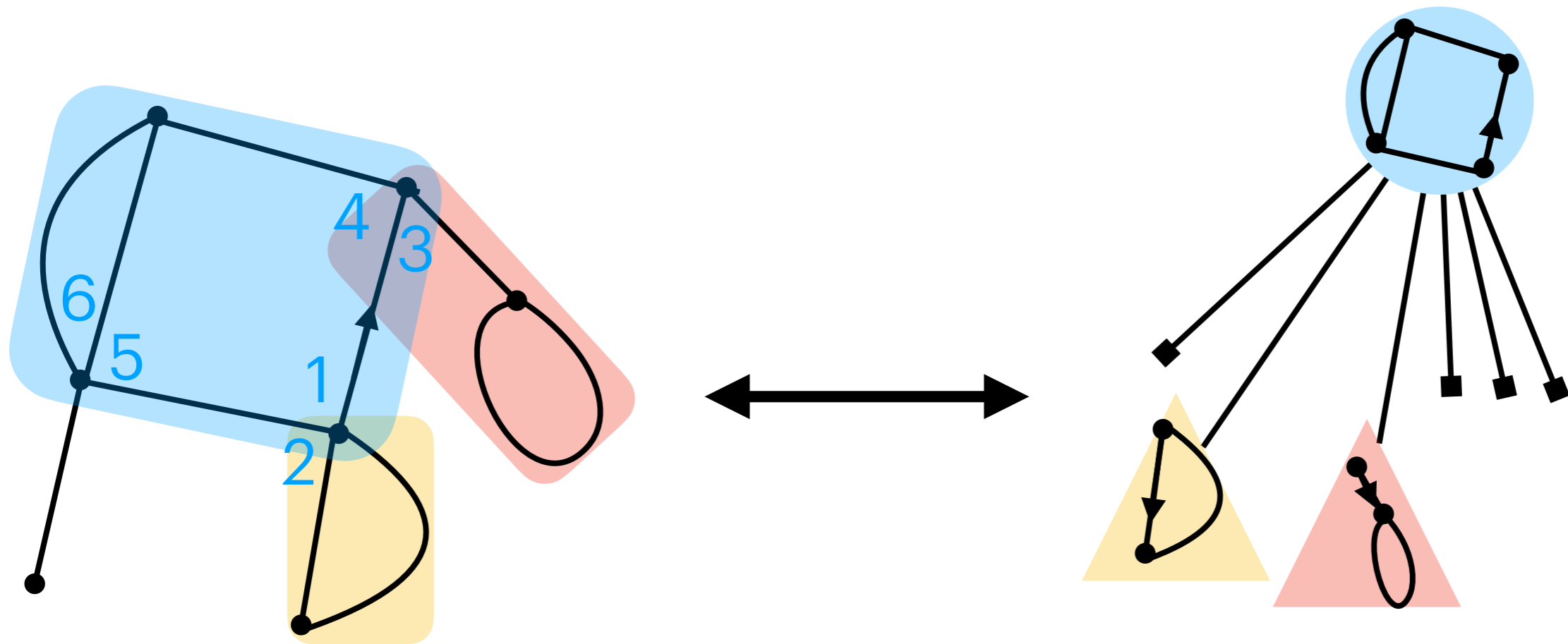
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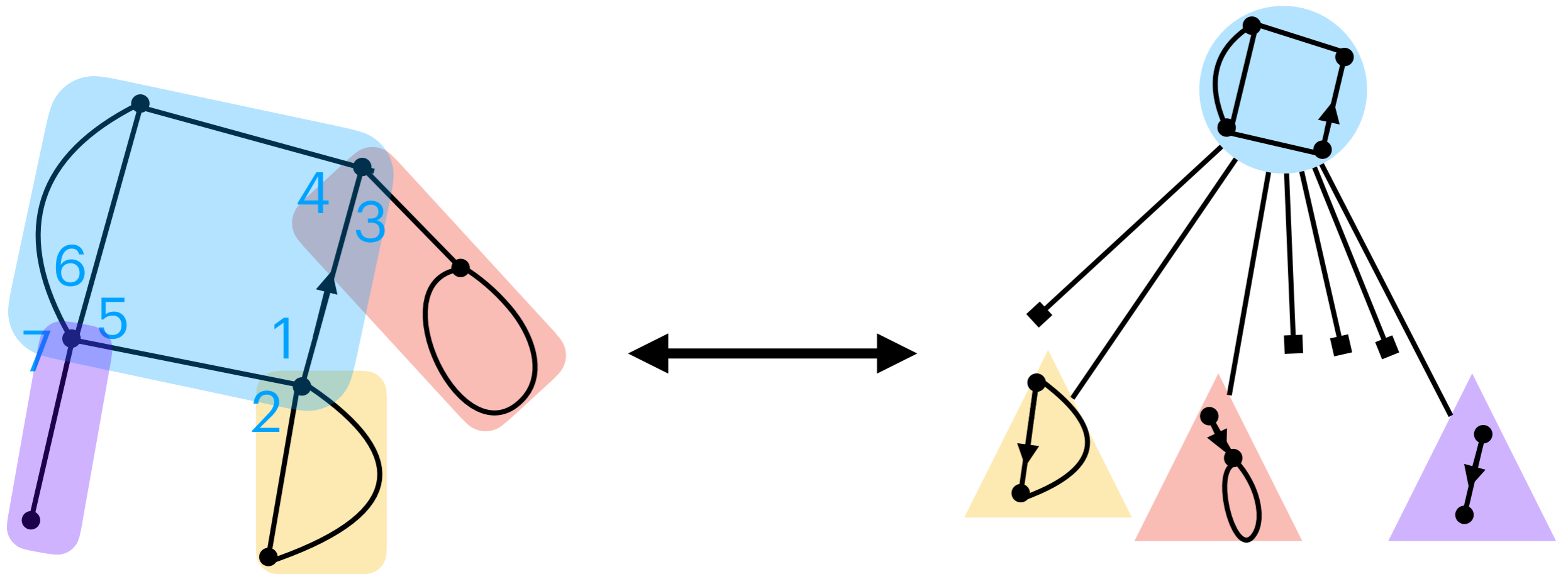
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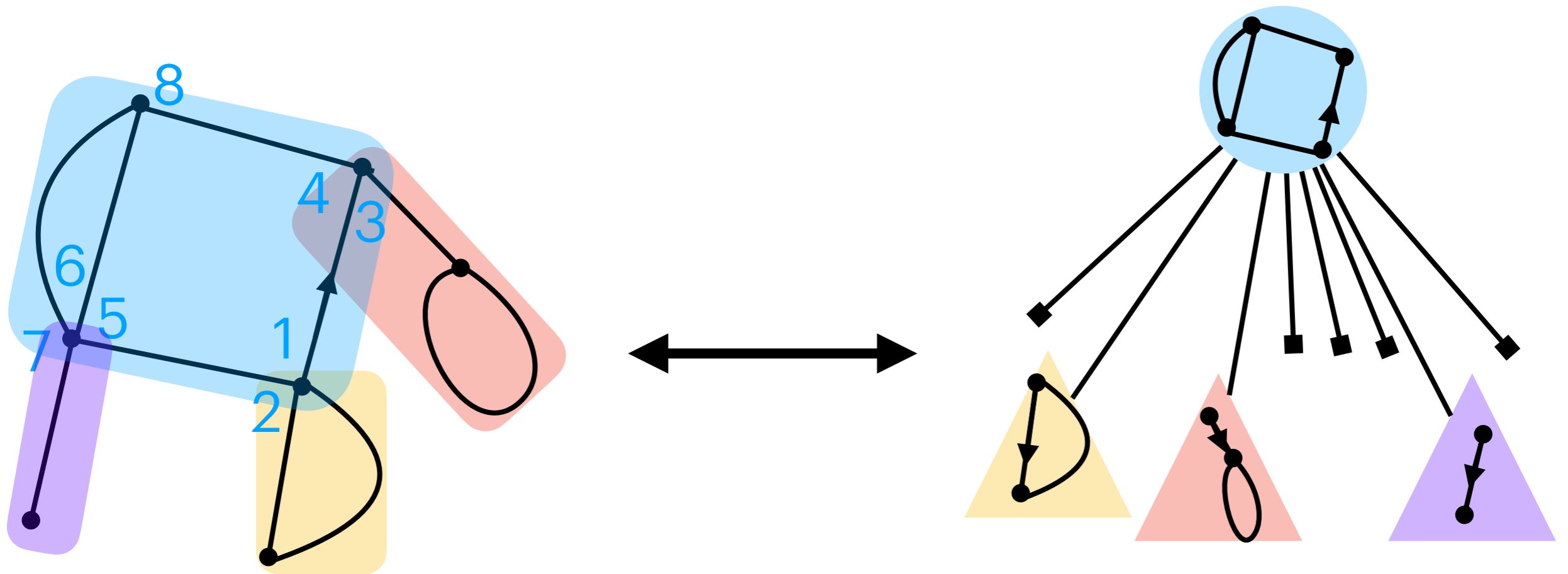
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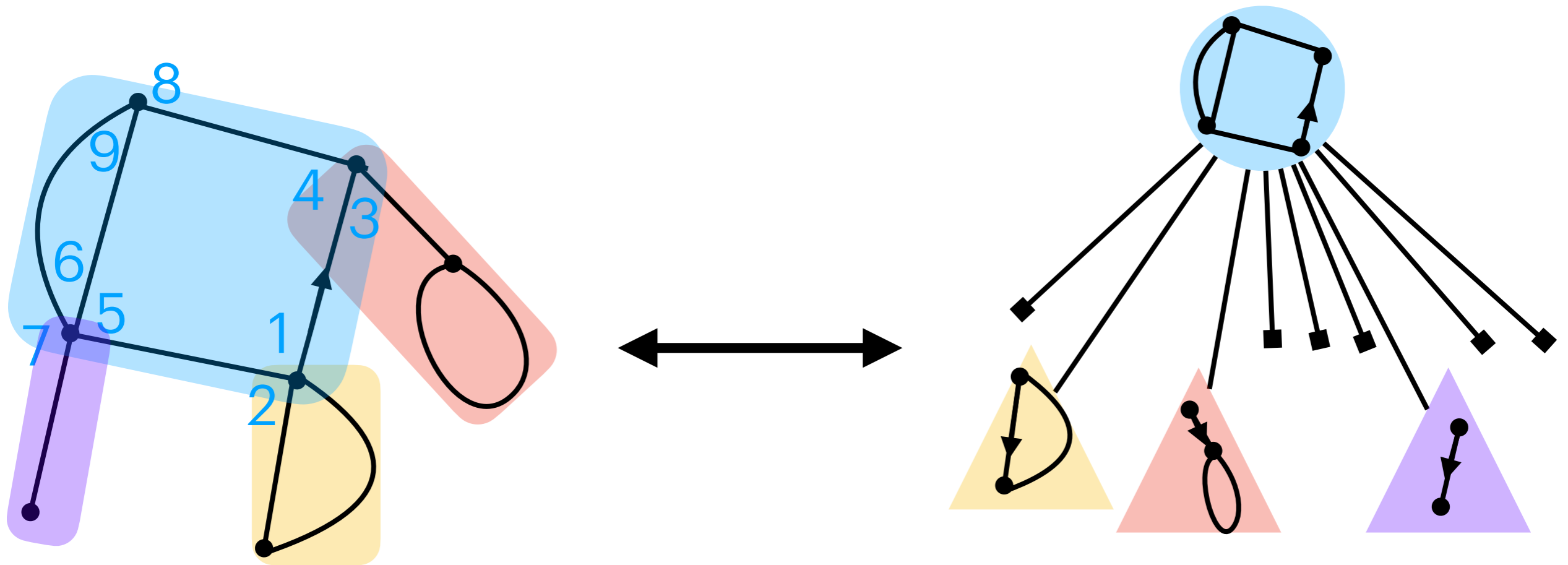
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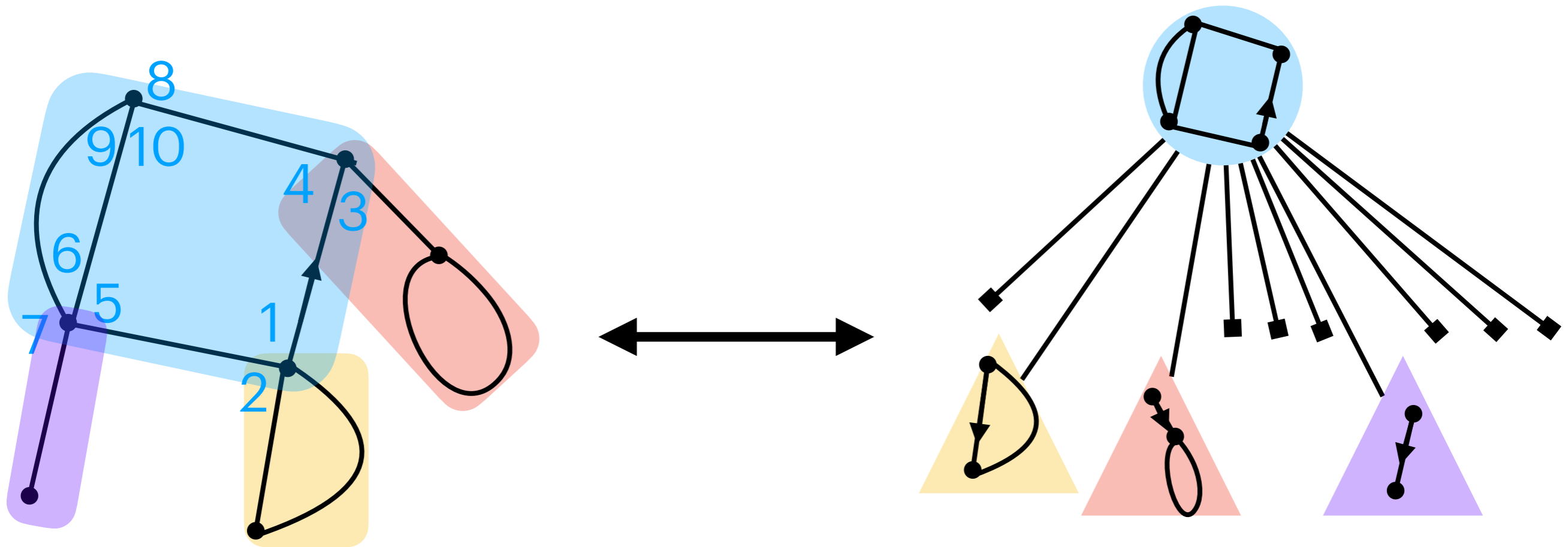
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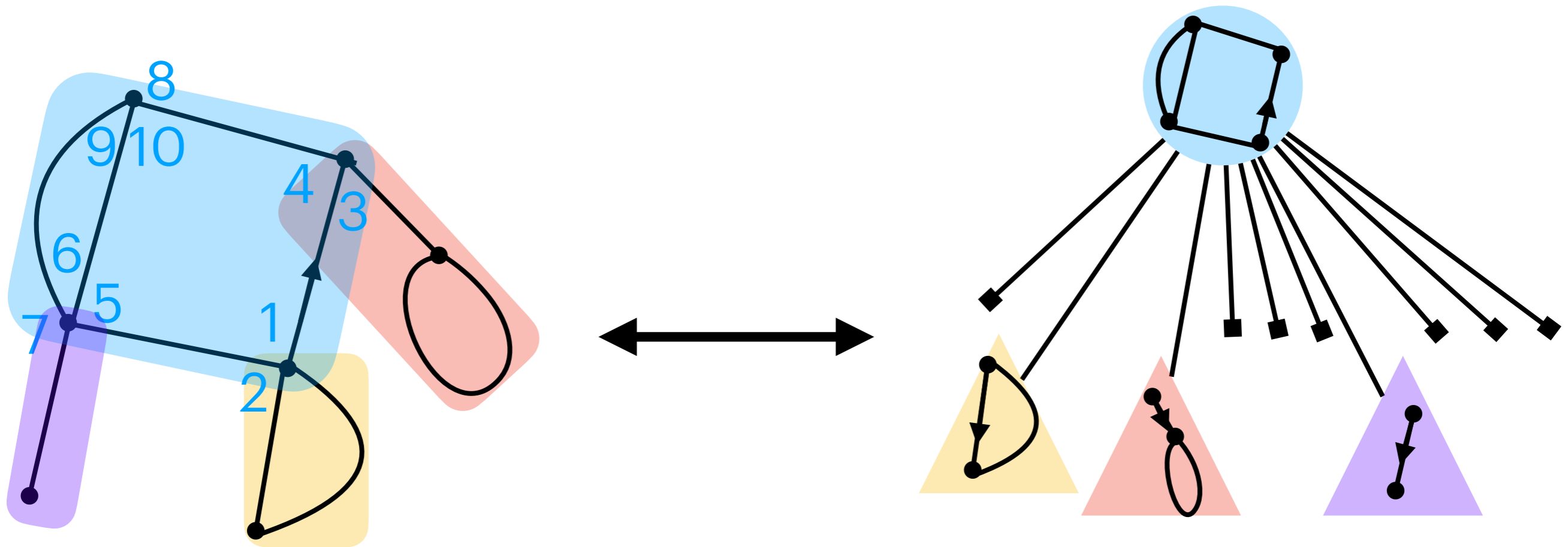
Decomposition of a map into blocks

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$$M(z) = B(zM^2(z))$$

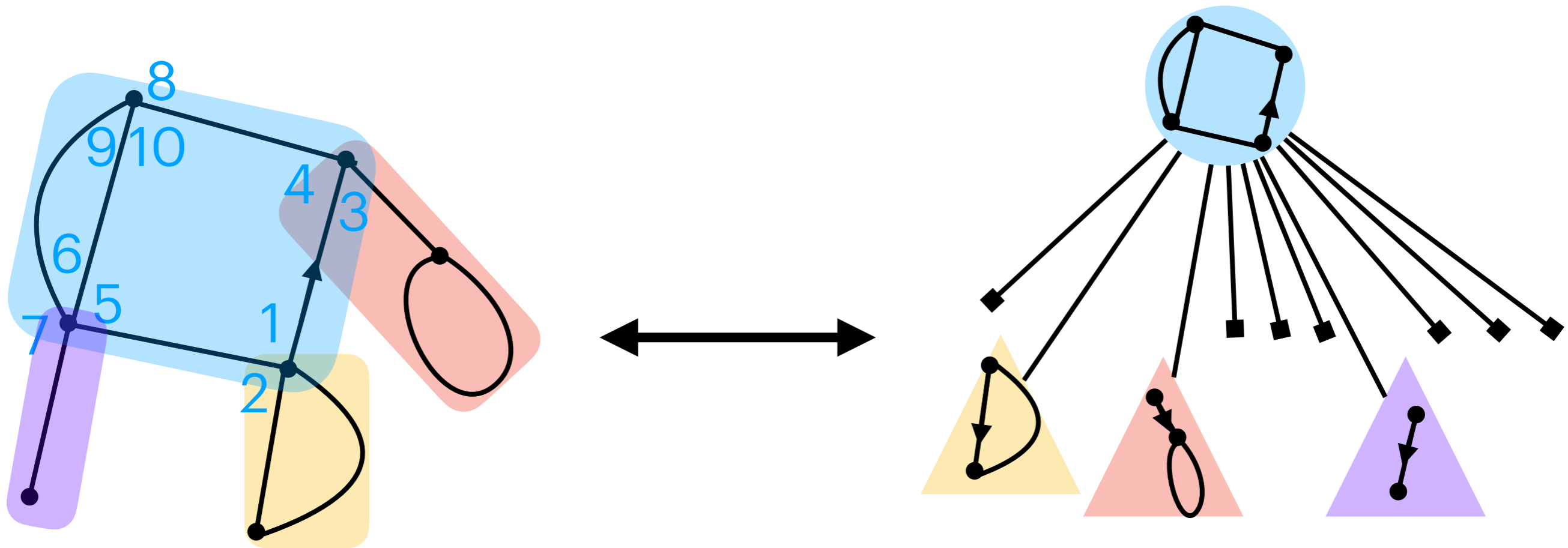
GS of maps

GS of 2-connected maps

Decomposition of a map into blocks

$$M(z, u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$

Inspiration from [Tutte 1963]



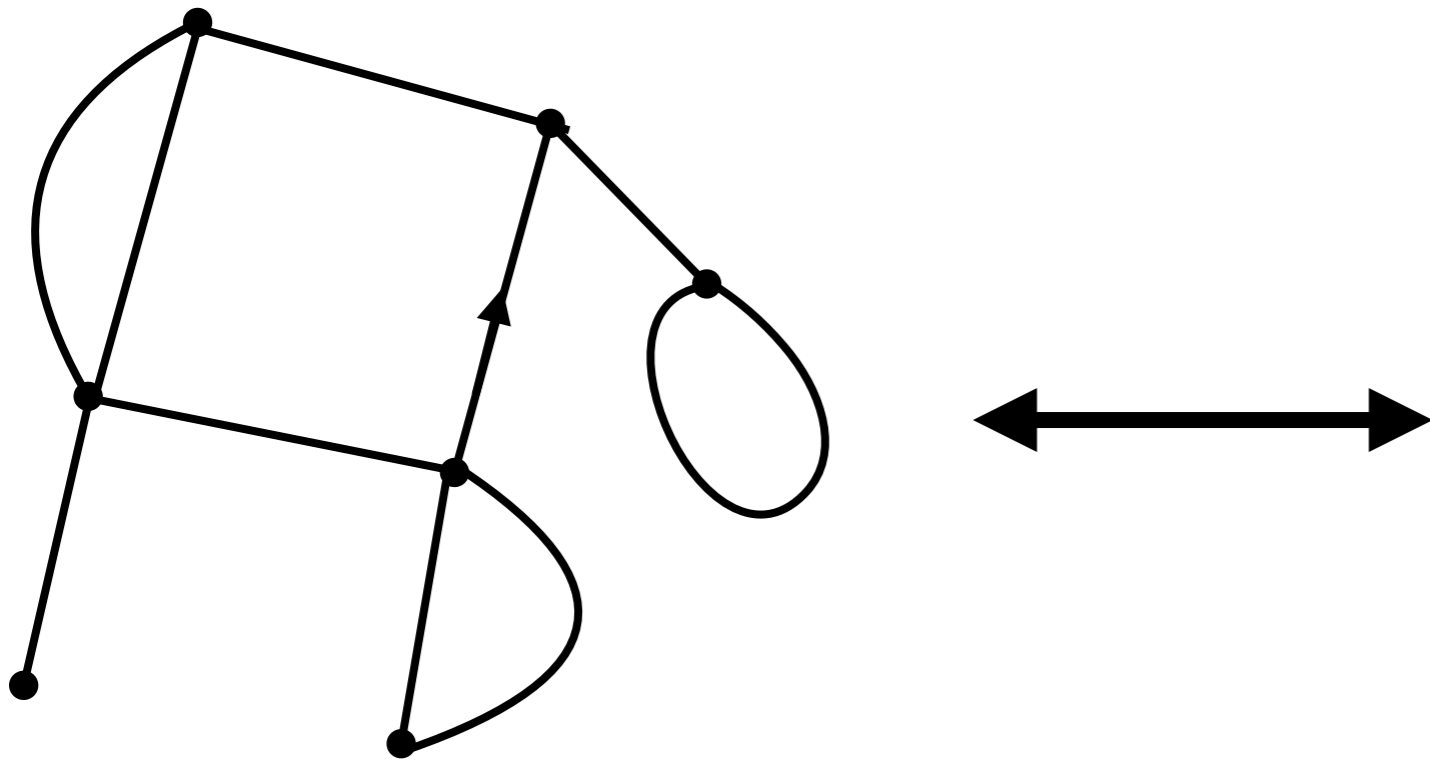
$$M(z) = B(zM^2(z))$$

GS of maps GS of 2-connected maps

With a weight u on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

Decomposition of a map into blocks

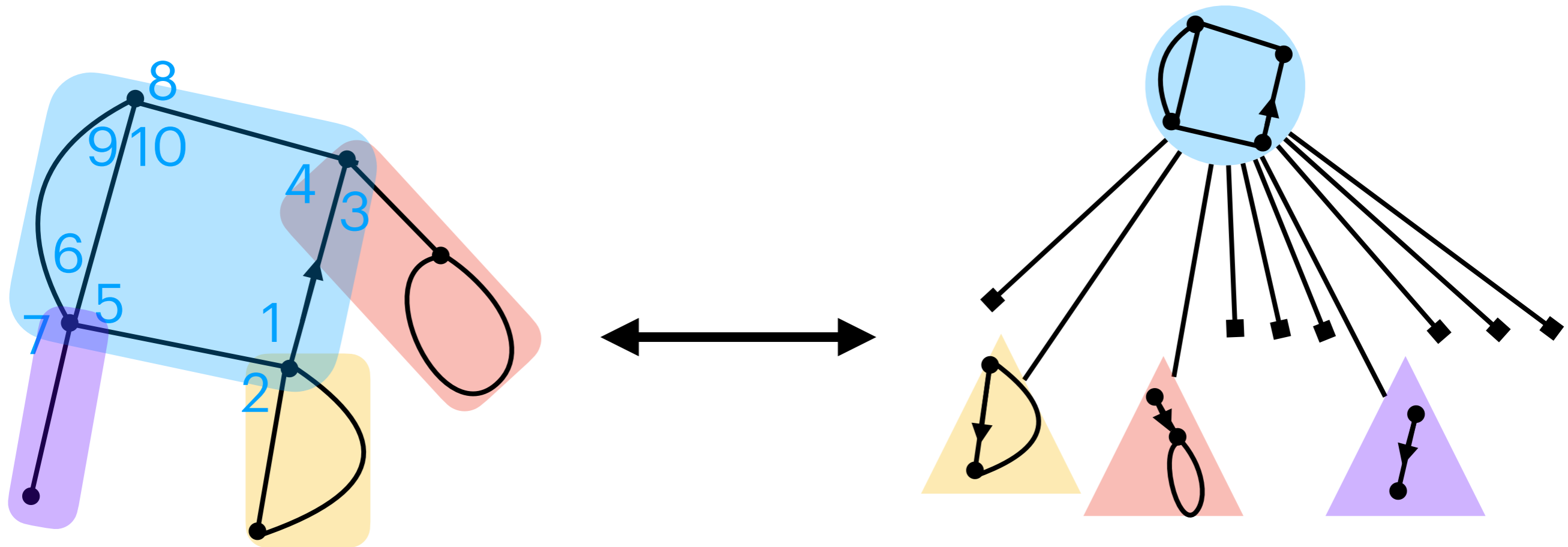
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⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

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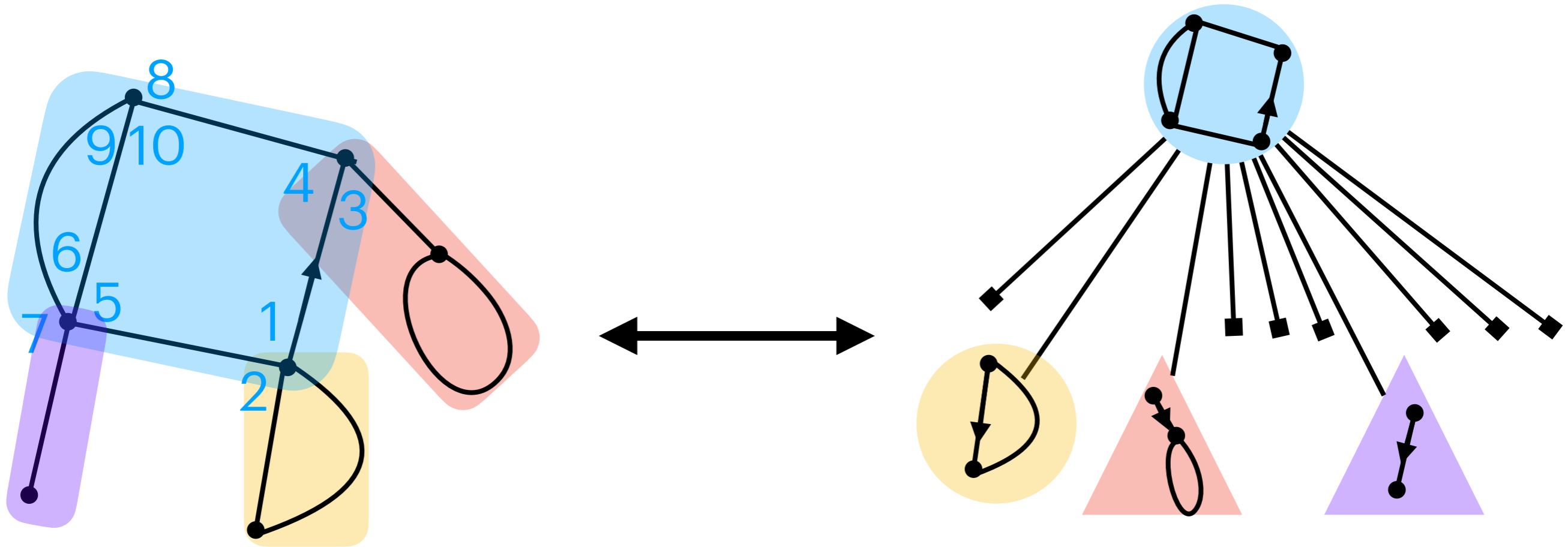
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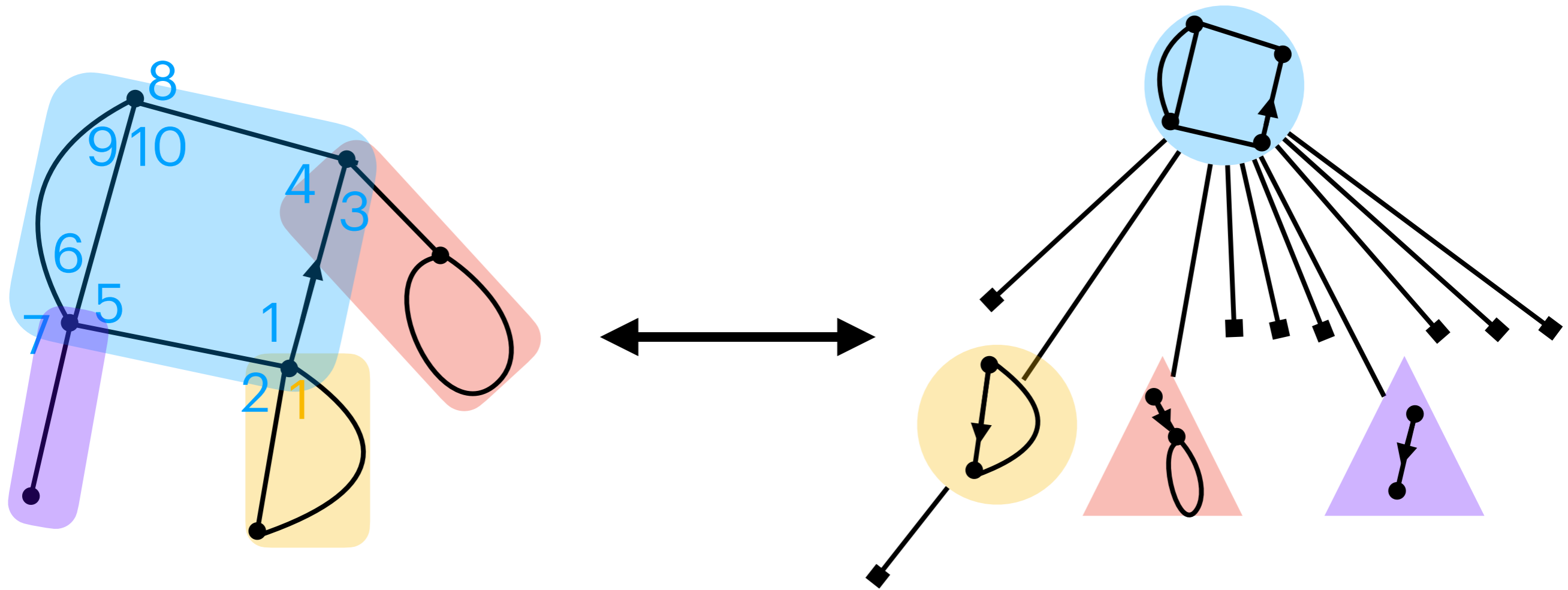
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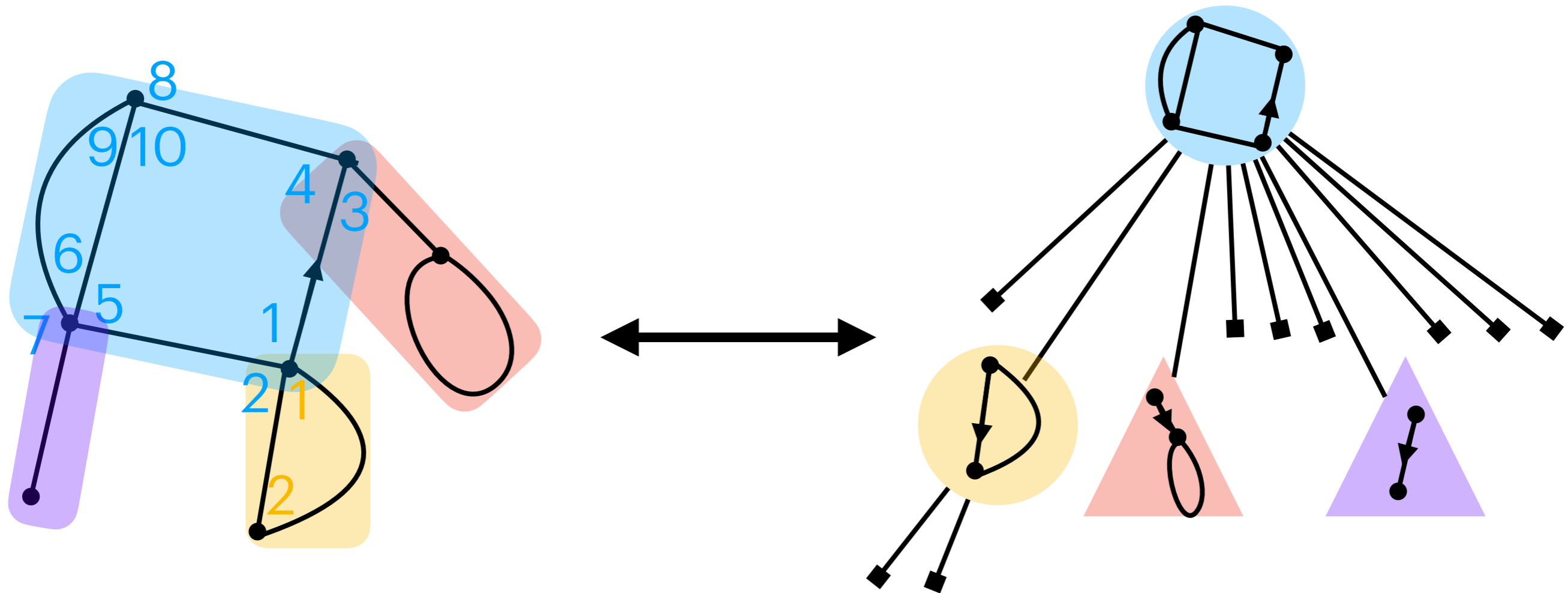
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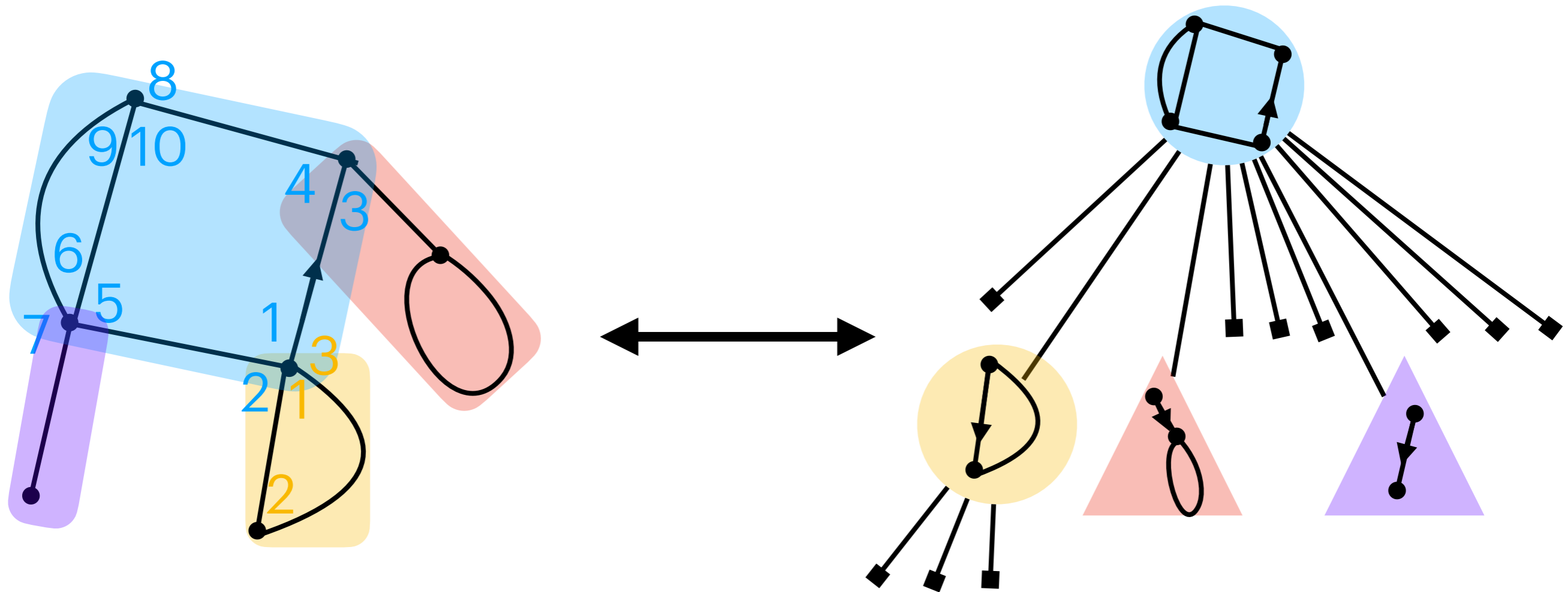
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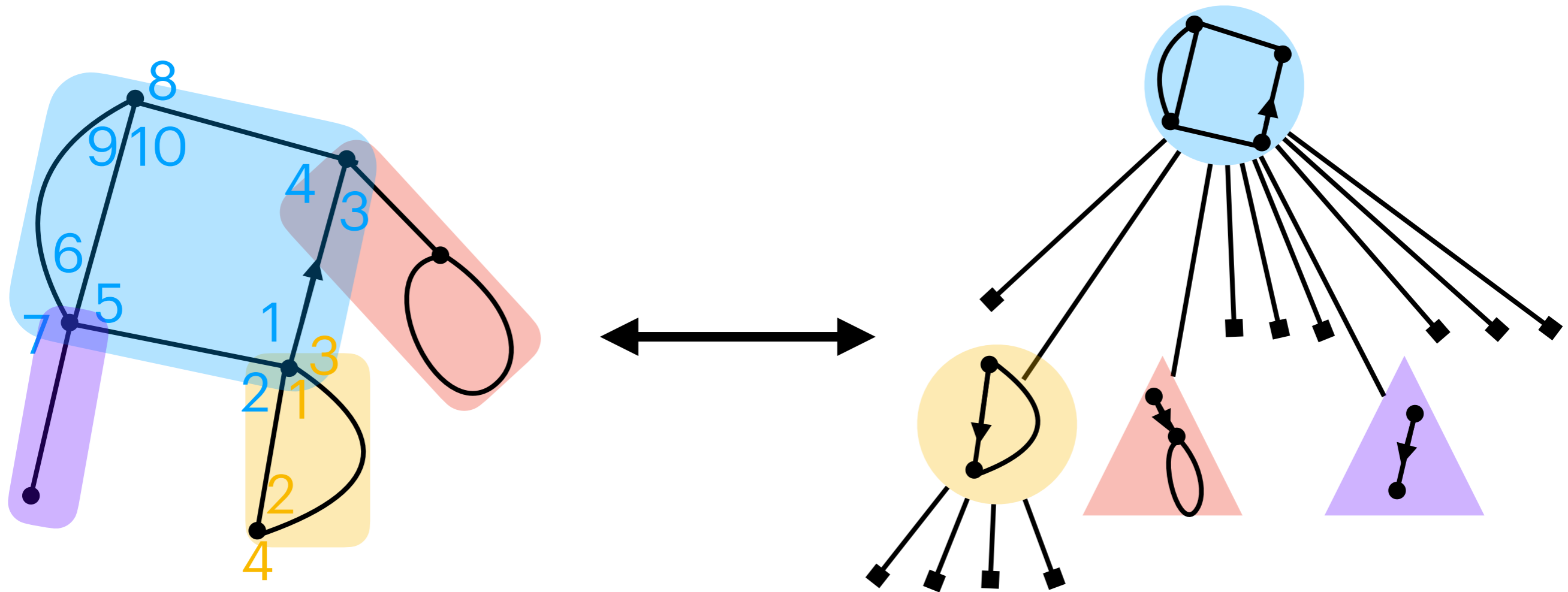
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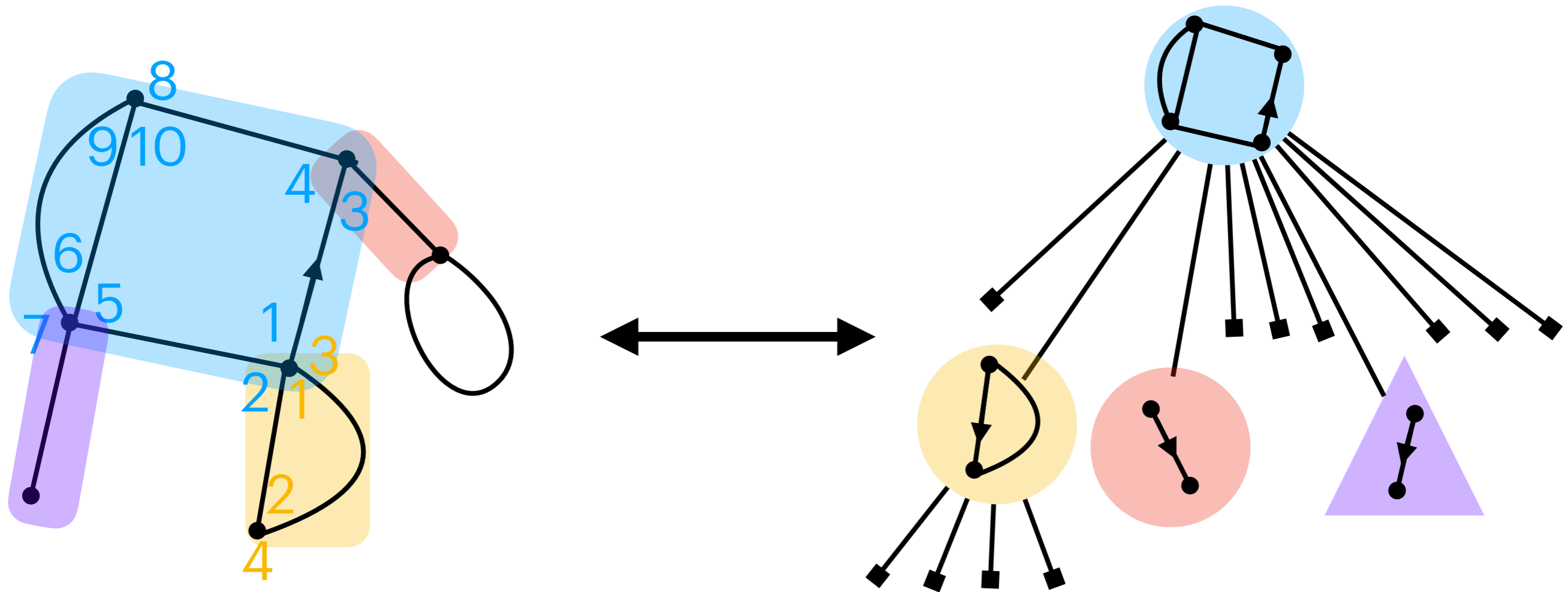
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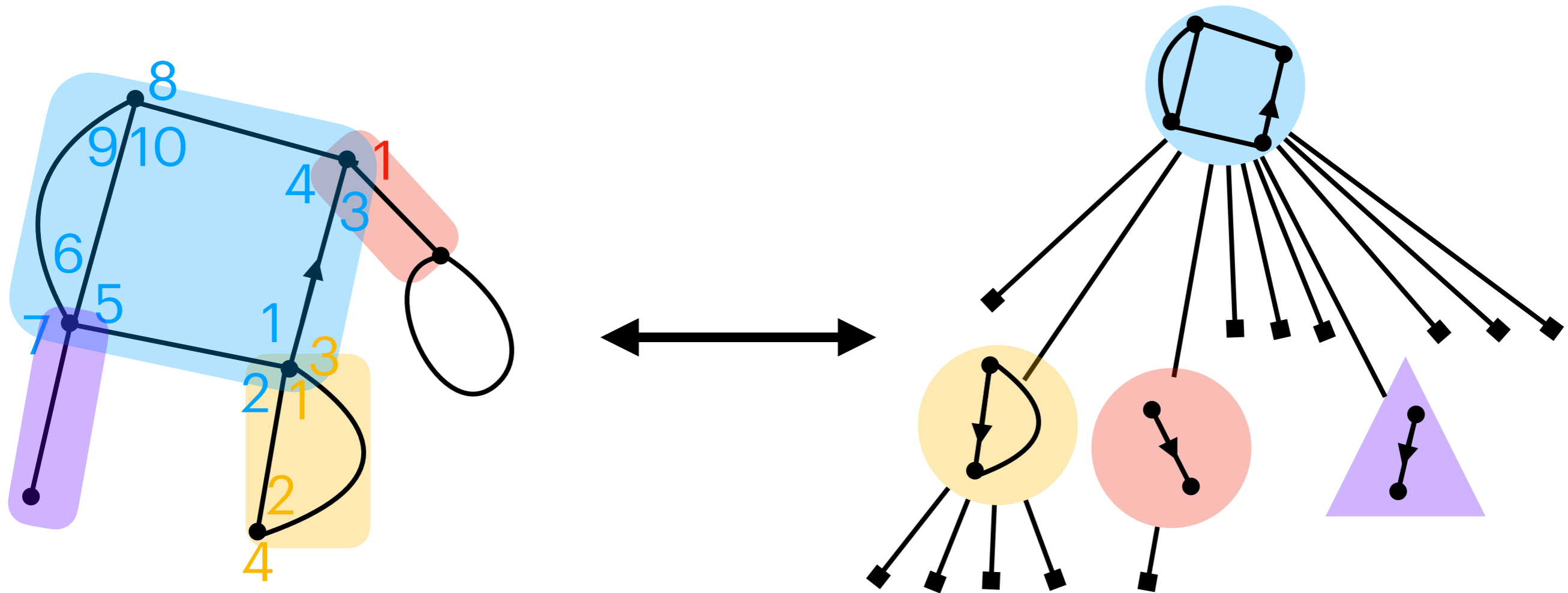
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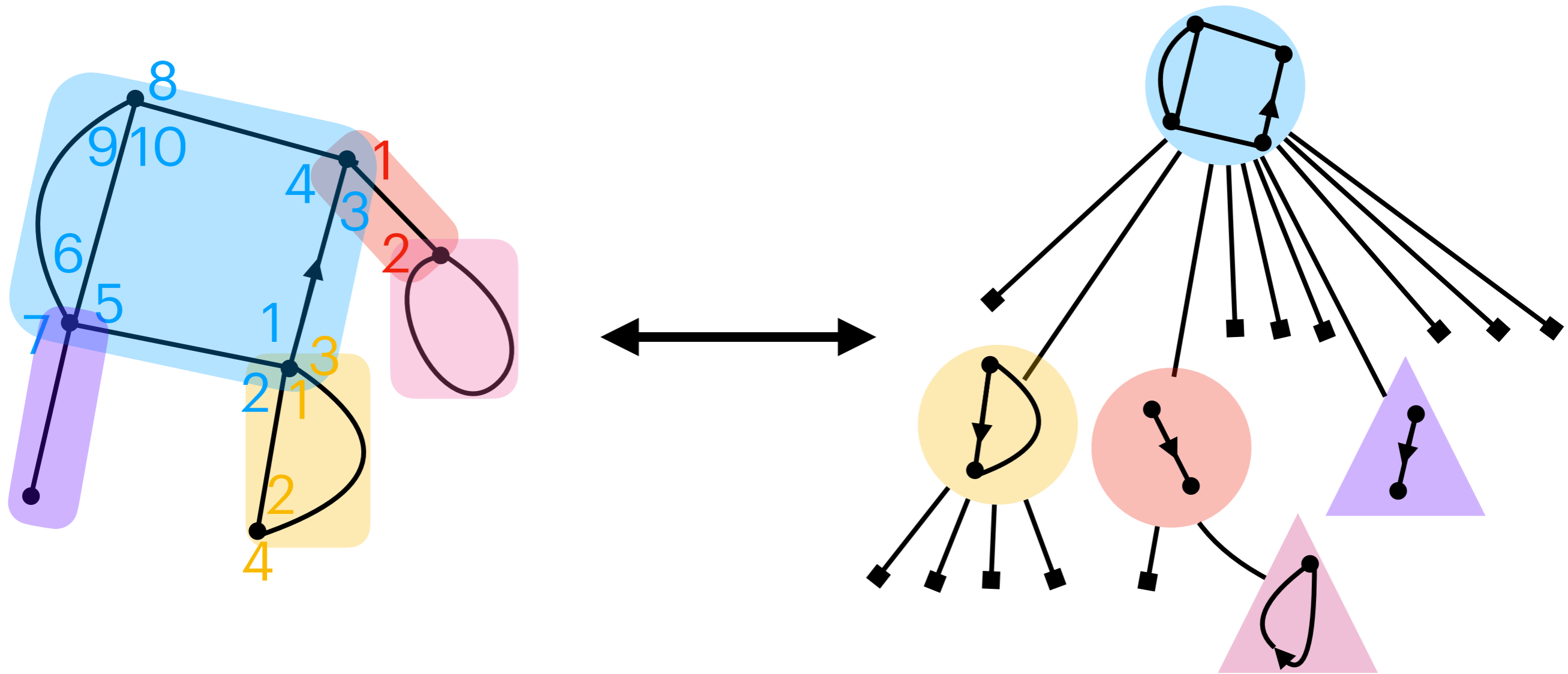
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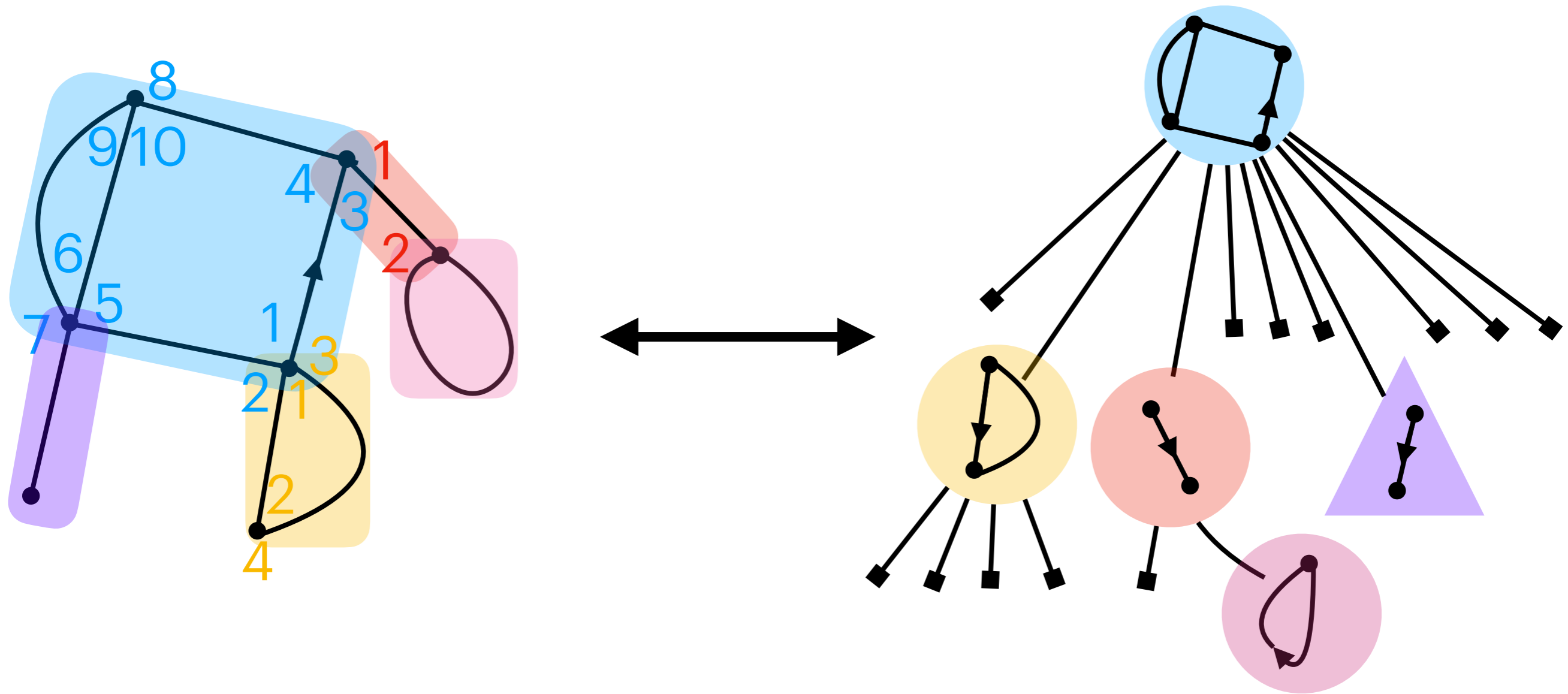
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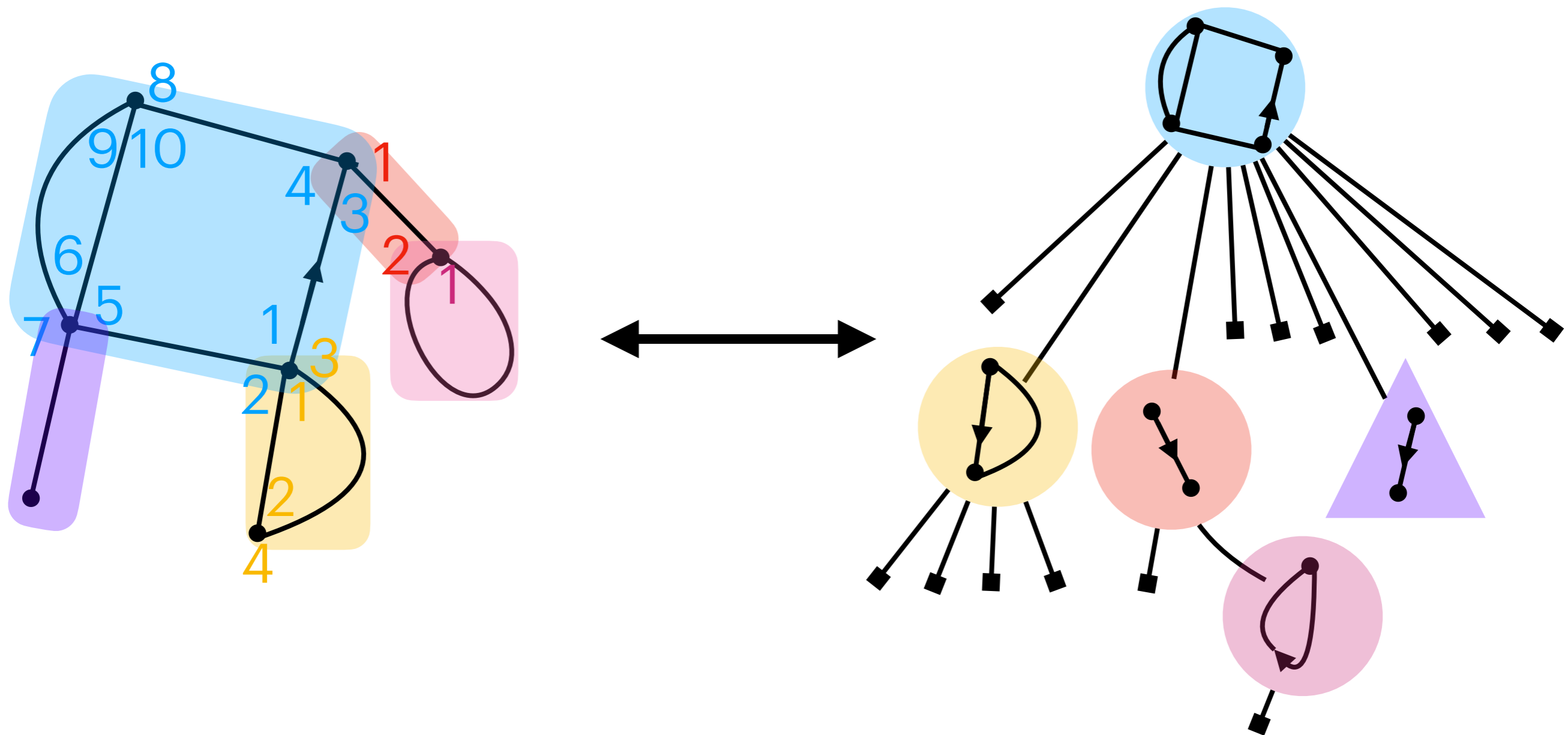
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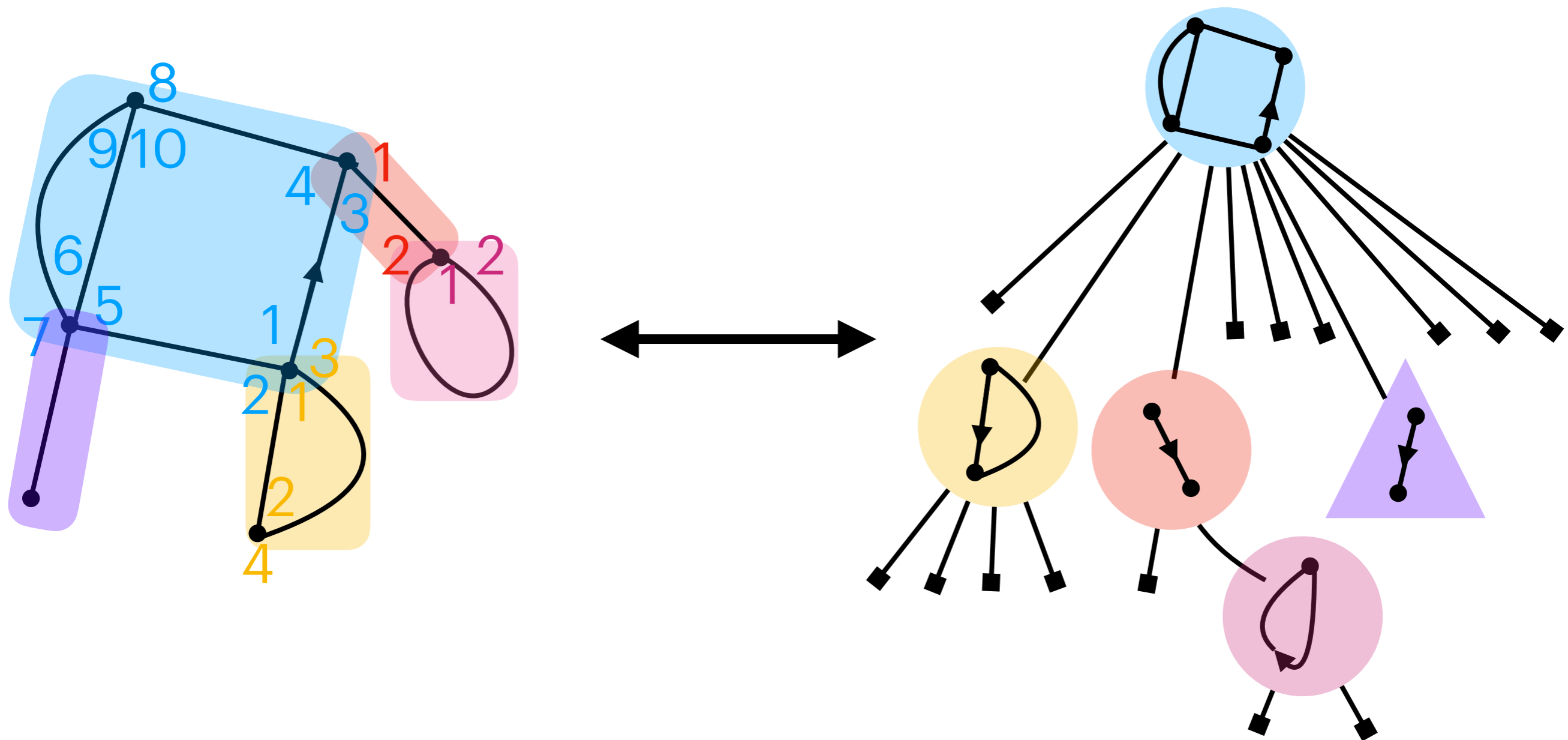
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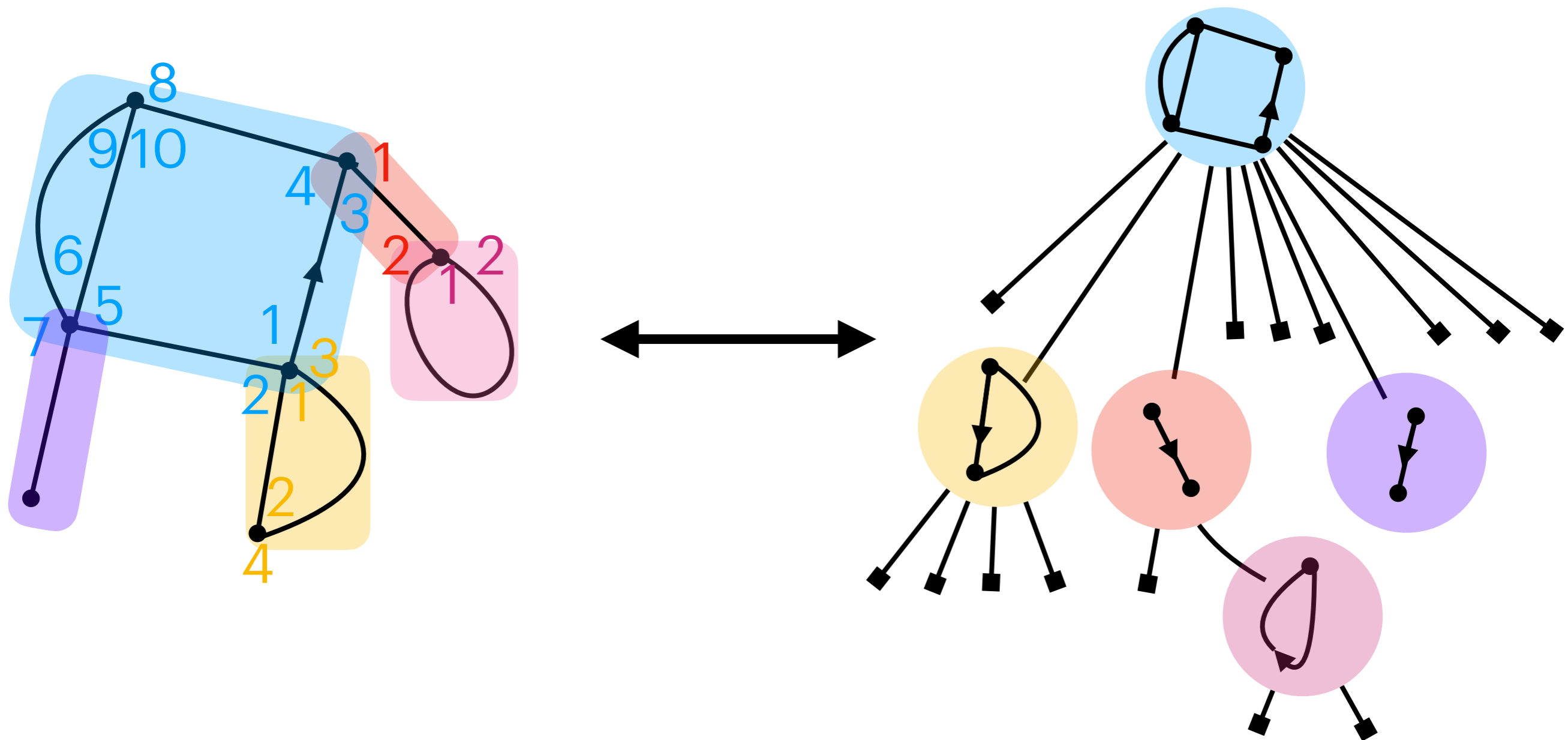
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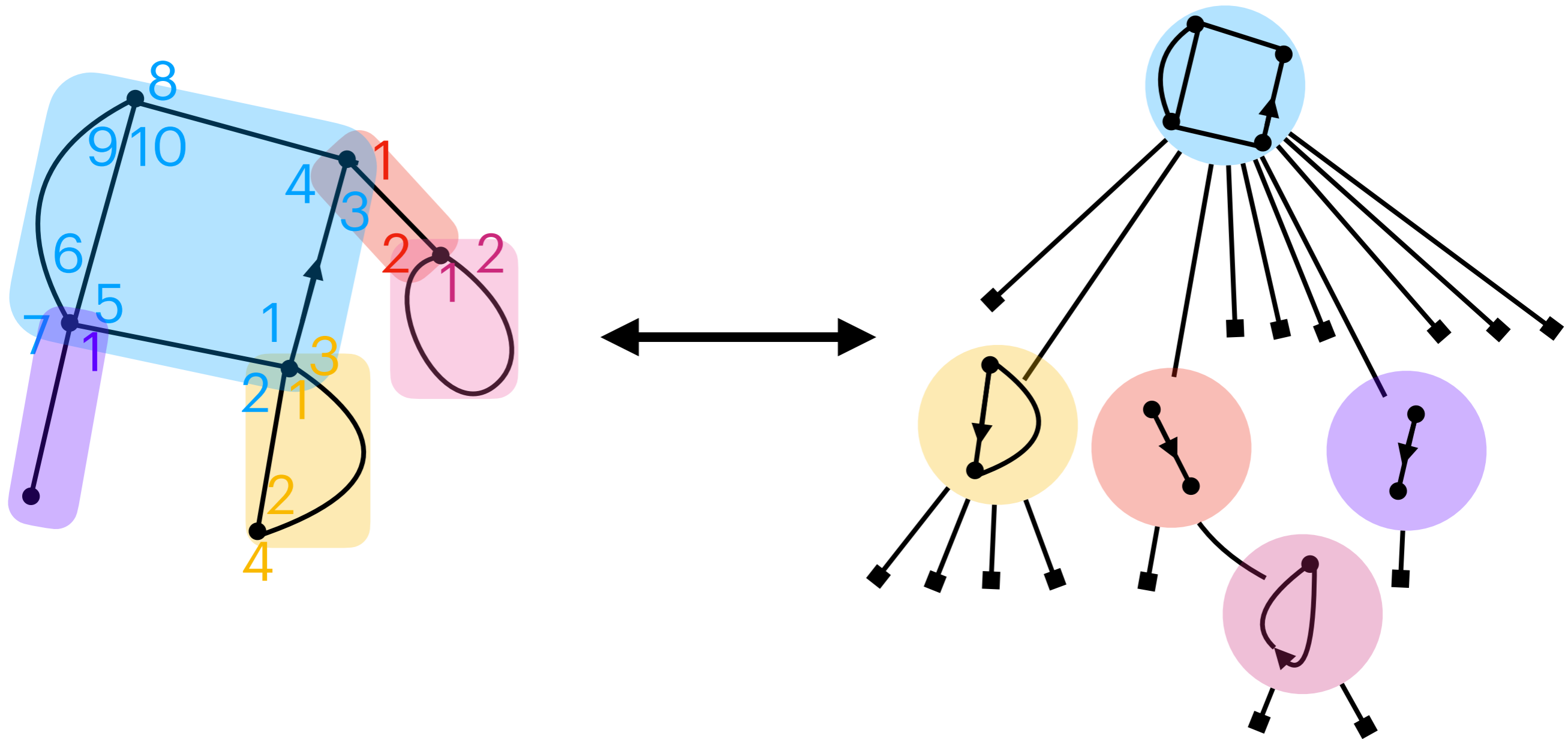
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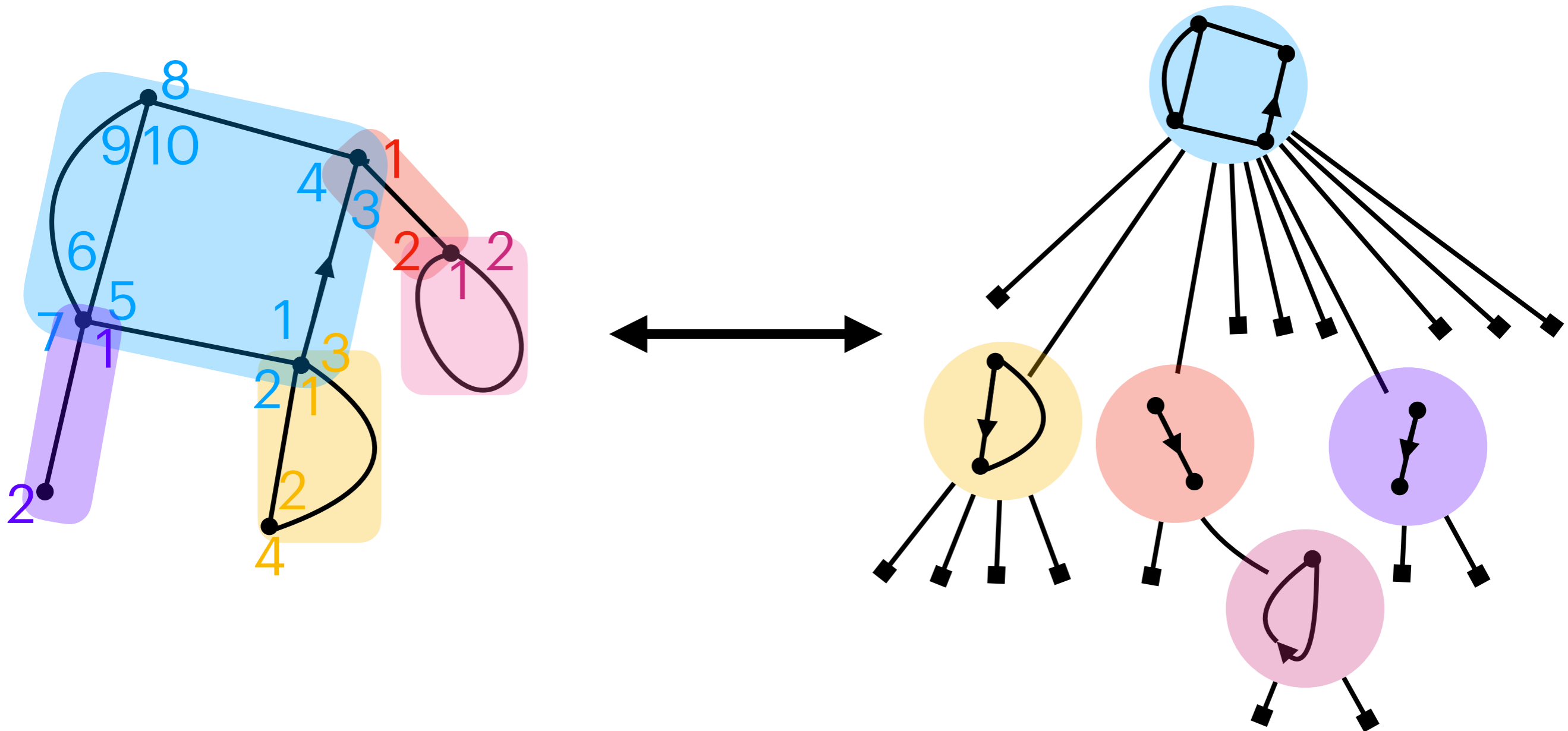
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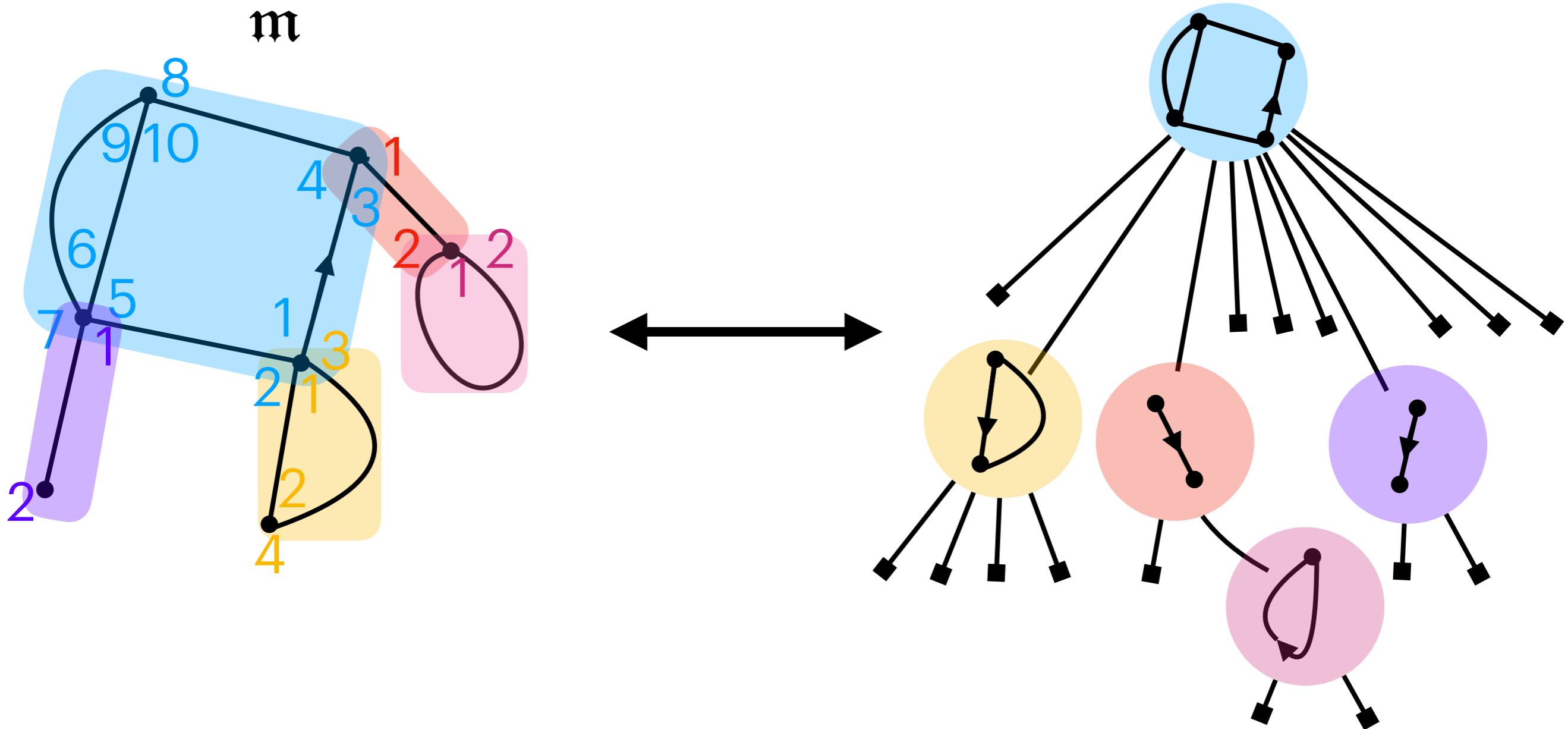
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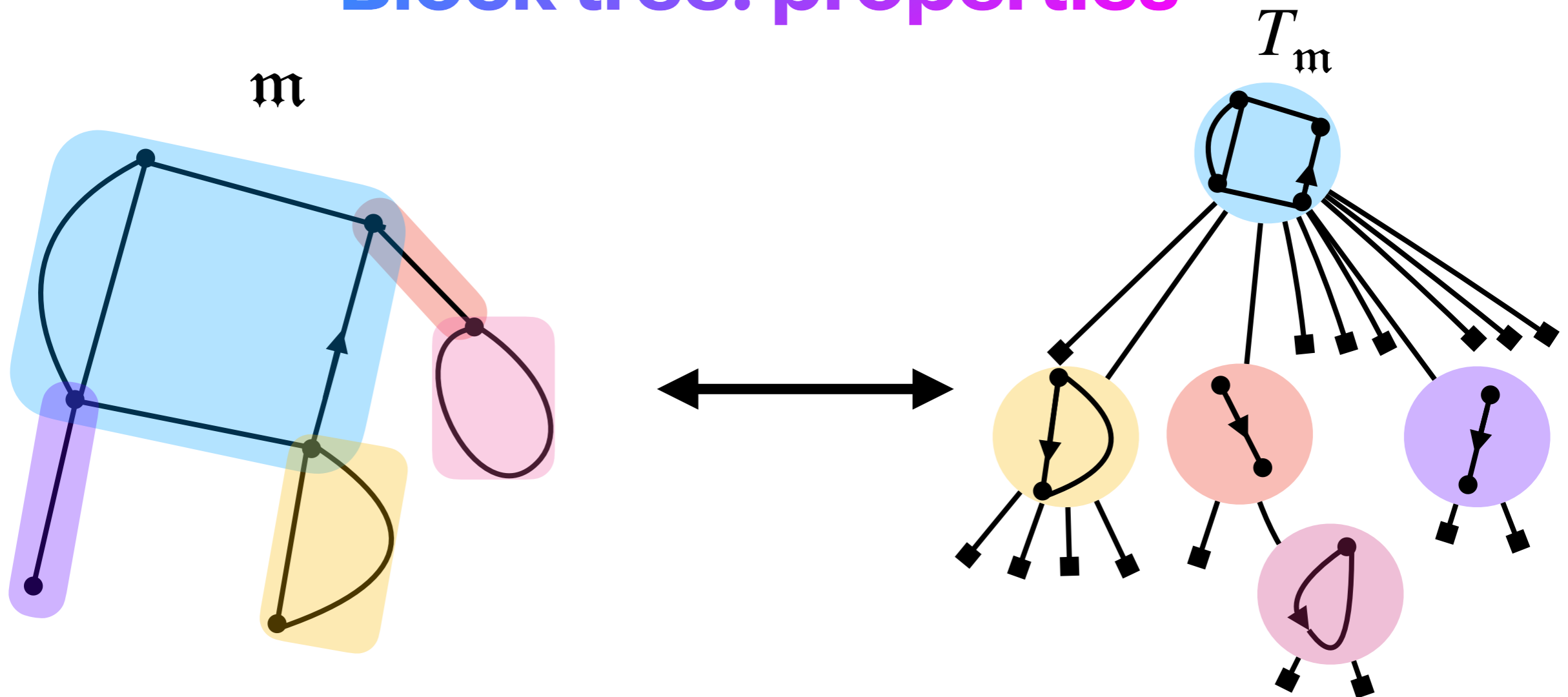
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Block tree: properties



- \mathfrak{m} is entirely determined by $T_{\mathfrak{m}}$ and $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ where \mathfrak{b}_v is the block of \mathfrak{m} represented by v in $T_{\mathfrak{m}}$;
- Internal node (with k children) of $T_{\mathfrak{m}} \leftrightarrow$ block of \mathfrak{m} of size $k/2$.

T_{M_n} gives the block sizes of a random map M_n .

Galton-Watson trees for map blocks

μ -Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on \mathbb{N} .

Galton-Watson trees for map blocks

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Theorem [Fleurat, S. 24]

$u > 0$

If $M_n \hookrightarrow \mathbb{P}_{n,u'}$ then T_{M_n} has the law of a Galton-Watson tree of explicit reproduction law μ^u conditioned to be of size $2n$.

Results for non tree-rooted maps

Theorem [Fleurat, S. 24] Model exhibits a phase transition at $u = 9/5$. When $n \rightarrow \infty$:

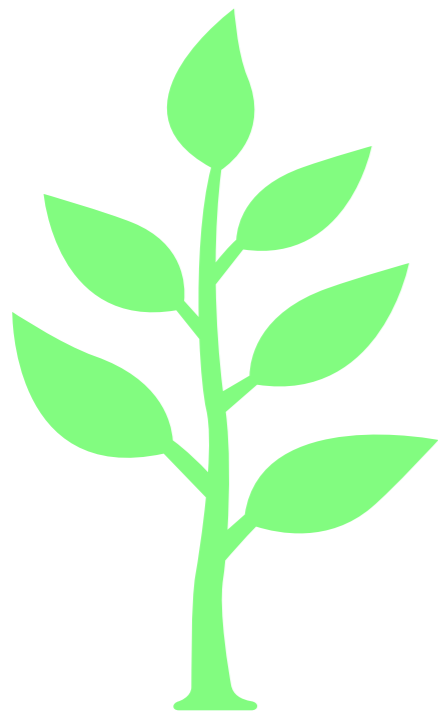
- Subcritical phase $u < 9/5$: “general map phase” one huge block;
- Critical phase $u = 9/5$: a few large blocks;
- Supercritical phase $u > 9/5$: “tree phase” only small blocks.

We obtain explicit results on enumeration, size of blocks and scaling limits in each case.

→ *A phase transition in block-weighted random maps*

W. Fleurat & Z. S., Electronic Journal of Probability, 2024

How can we do the same for tree-rooted maps?



III. Tree-rooted maps

Joint work with Marie Albenque and Éric Fusy

Model

Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

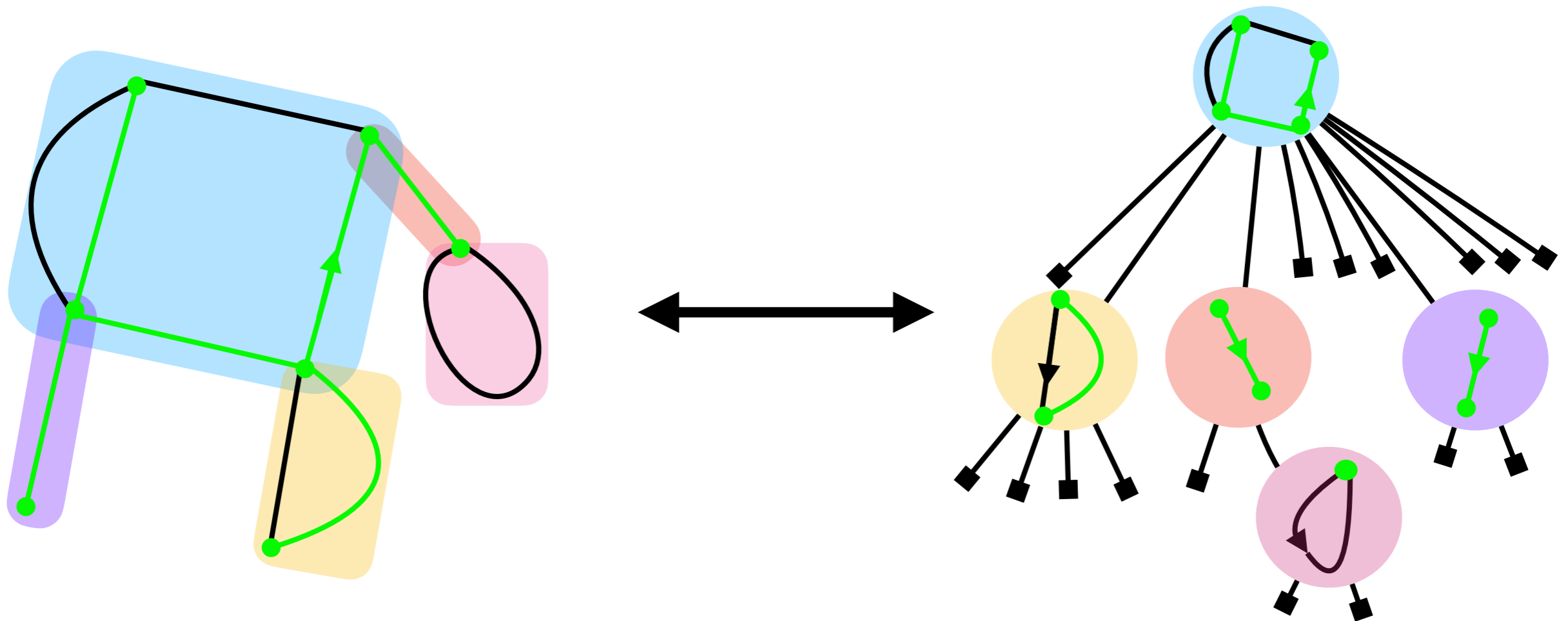
$u > 0,$
 $\mathcal{M}_n = \{\text{tree-rooted maps of size } n\},$
 $\mathfrak{m} \in \mathcal{M}_n,$
 $Z_{n,u} = \text{normalisation.}$

- $u = 1$: uniform distribution on tree-rooted maps of size n ;
- $u \rightarrow 0$: minimising the number of blocks (=2-connected tree-rooted maps);
- $u \rightarrow \infty$: maximising the number of blocks (= tree-rooted trees!).

Given u , asymptotic behaviour when $n \rightarrow \infty$?

Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



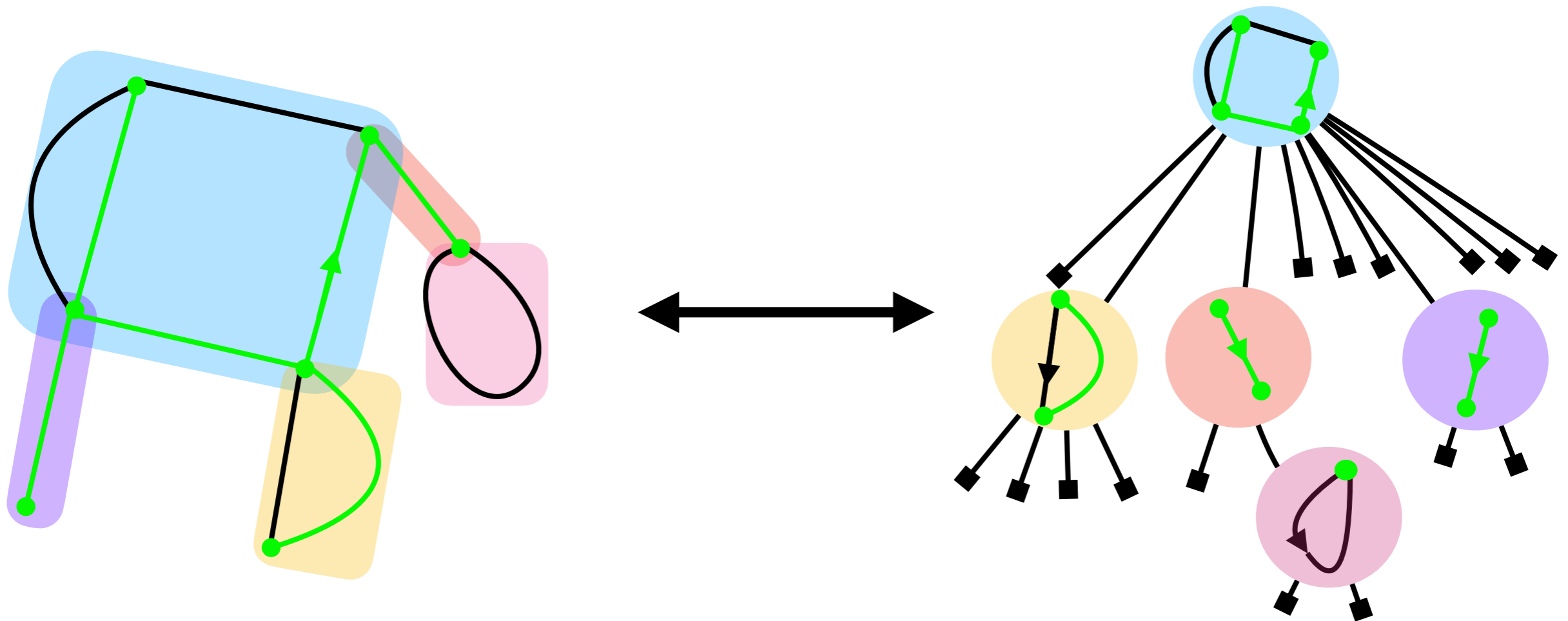
$$M(z) = B(zM^2(z))$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



$$M(z, u) = uB(zM^2(z, u)) + 1 - u$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

So everything should be easy, right?

Tree-rooted maps are not so nice

$$M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \text{ so}$$

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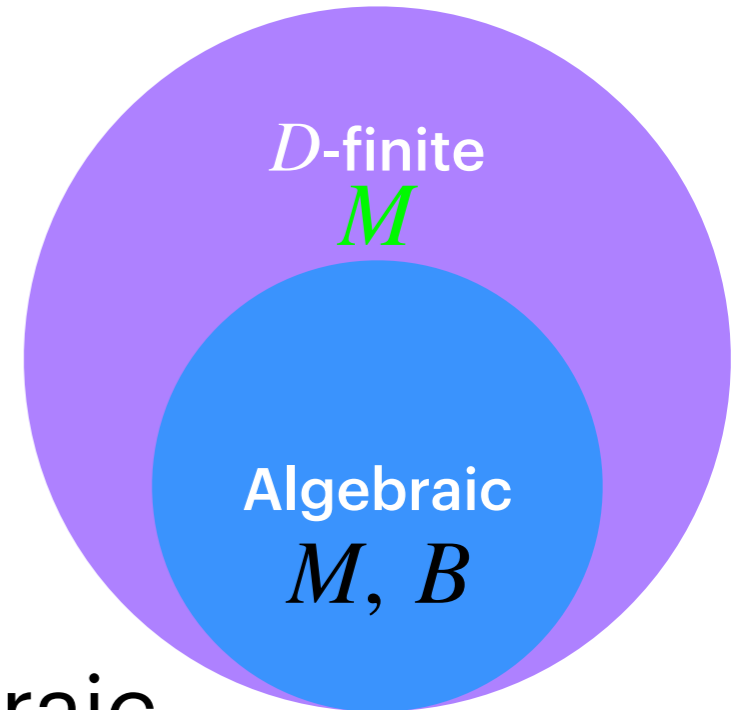

$$P(z, M(z)) = 0$$

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$$P(z, M(z)) = 0$$

An arrow points from the text "so M is not algebraic..." to this equation. A large pink 'X' is drawn over the equation, indicating it is false.

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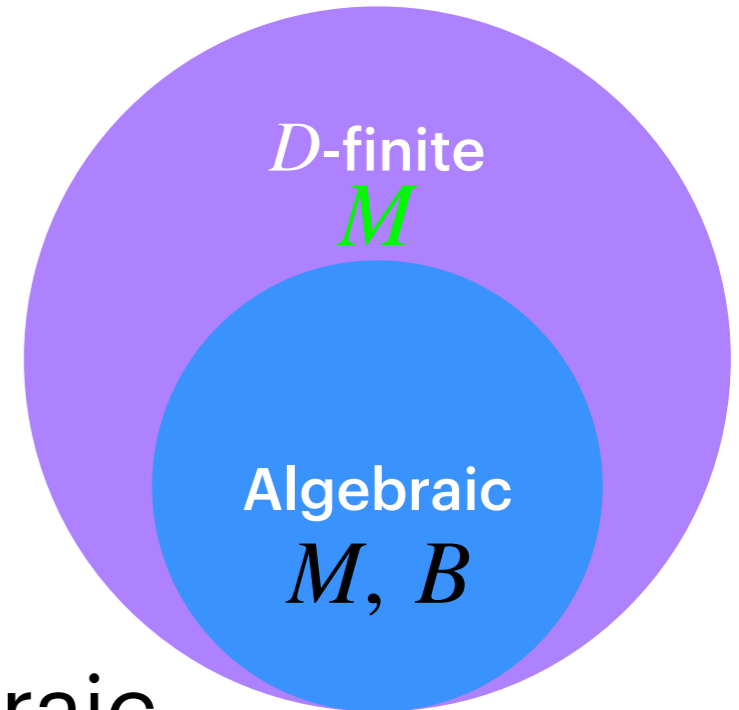
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- Fortunately, it is still *D-finite*

$$P(z, M(z)) = 0$$

$$P_0(z) \frac{\partial^2 M}{\partial z^2}(z) + P_1(z) \frac{\partial M}{\partial z}(z) + P_2(z) M(z) + P_3(z) = 0.$$



2-connected tree-rooted maps are naughty

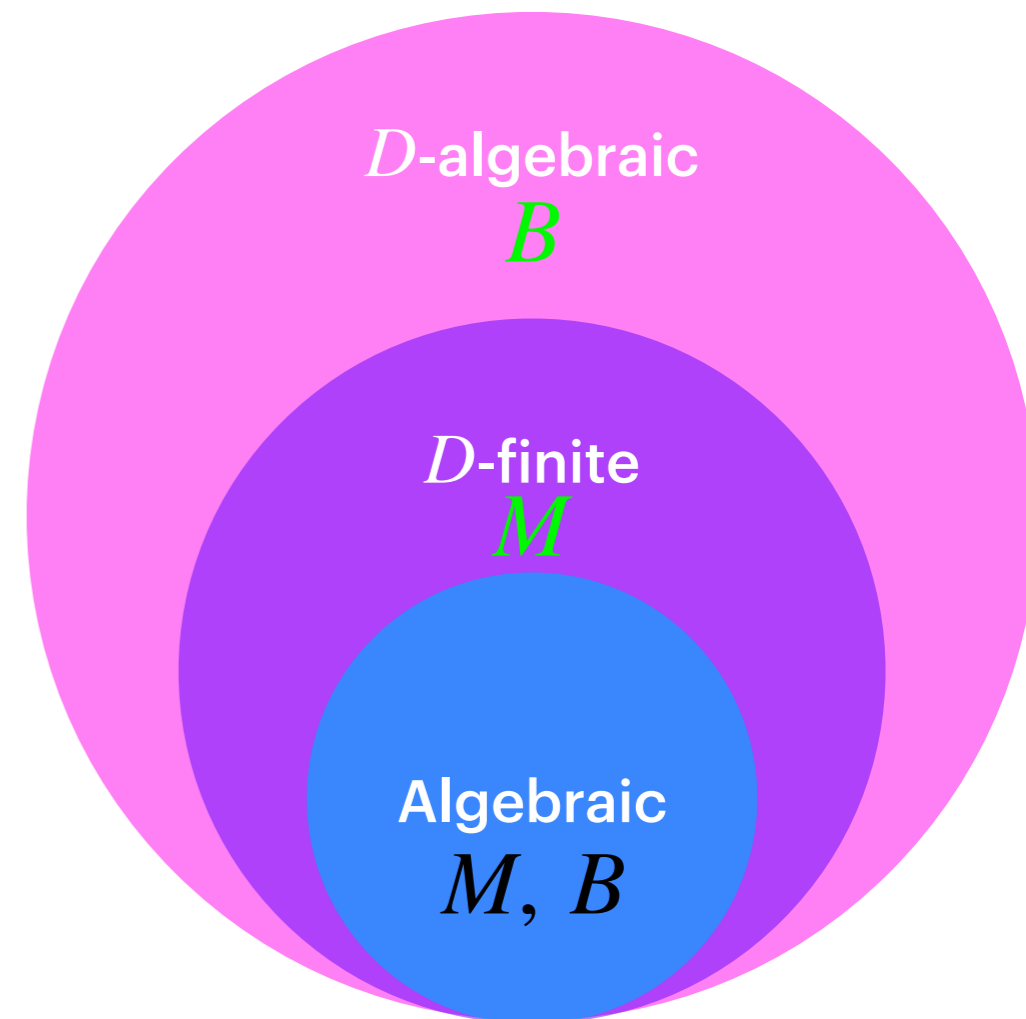
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is not algebraic so B is not D -finite



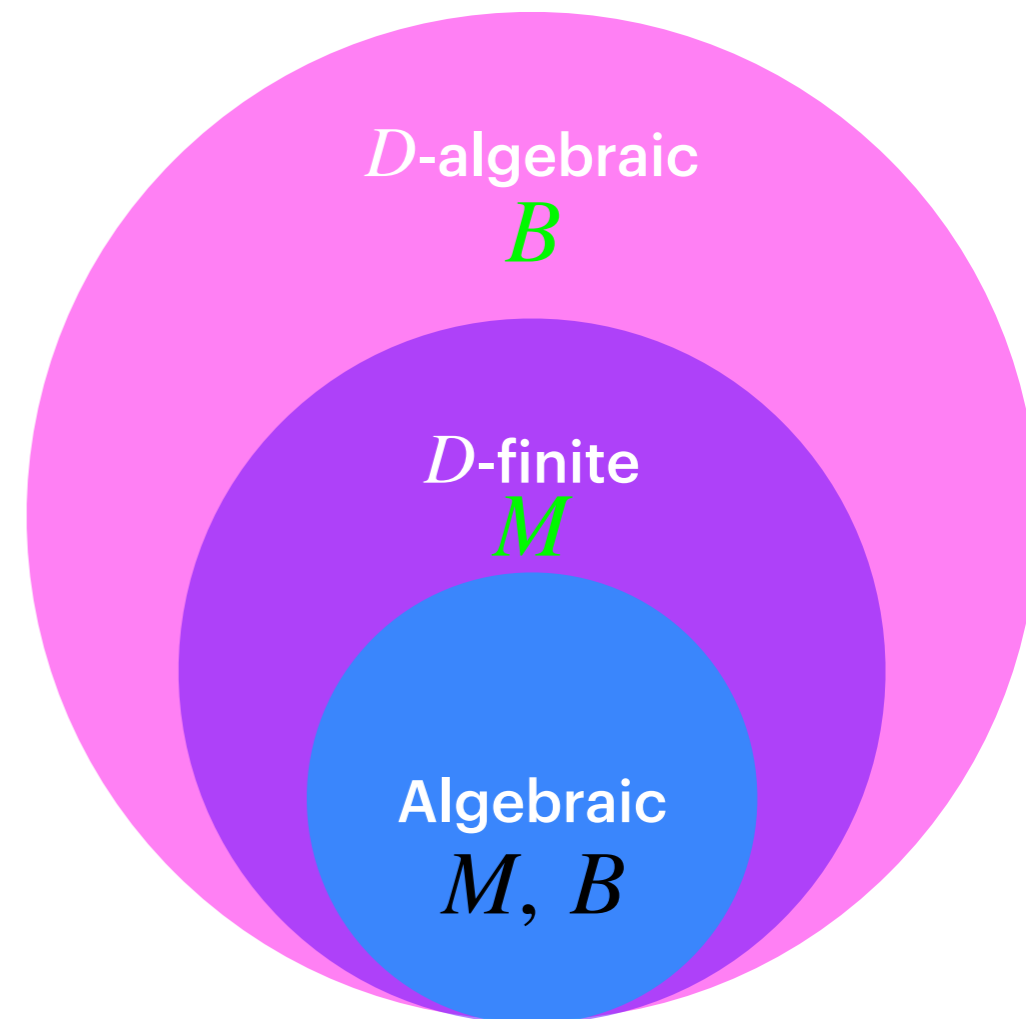
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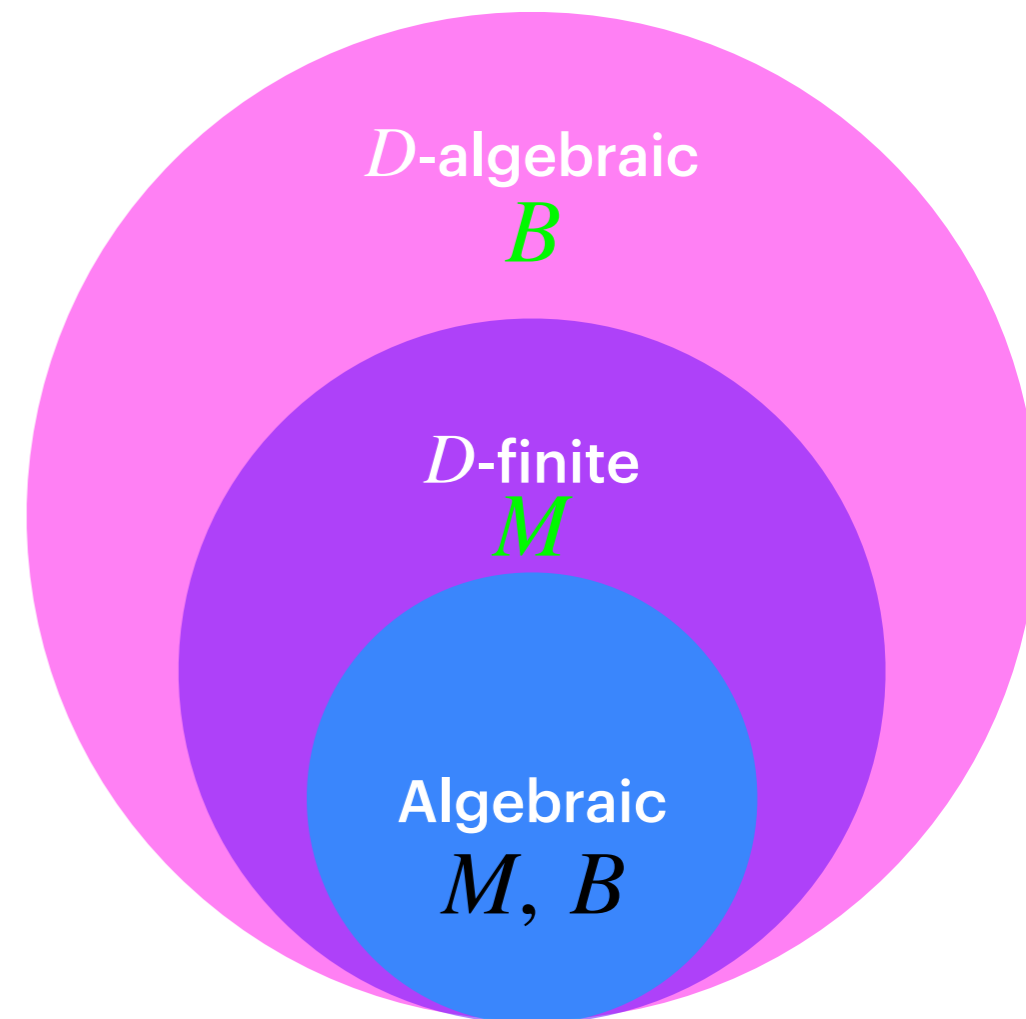
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$$P\left(\frac{\partial^2 B}{\partial y^2}(y), \frac{\partial B}{\partial y}(y), B(y), y\right) = 0.$$



Enumeration of 2-connected tree-rooted maps

Using $M(z) = B(zM^2(z))$ and the properties of M , we show

Theorem [Albenque, Fusy, S. 24]

$$[y^n]B(y) \sim \frac{4(3\pi - 8)^3}{27\pi(4 - \pi)^3} \times \rho_B^{-n} \times n^{-3}.$$

Phase transition

Theorem [Albenque, Fusy, S. 24] Model exhibits a phase

transition at $u_C = \frac{9\pi(4 - \pi)}{420\pi - 81\pi^2 - 512} \simeq 3.02$.

When $n \rightarrow \infty$:

- Subcritical phase $u < u_C$: “general tree-rooted map phase” one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: “tree phase” only small blocks.

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration			
Size of <ul style="list-style-type: none"> - the largest block - the second one 			
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of <ul style="list-style-type: none"> - the largest block - the second one 			
Scaling limit of M_n			

Results

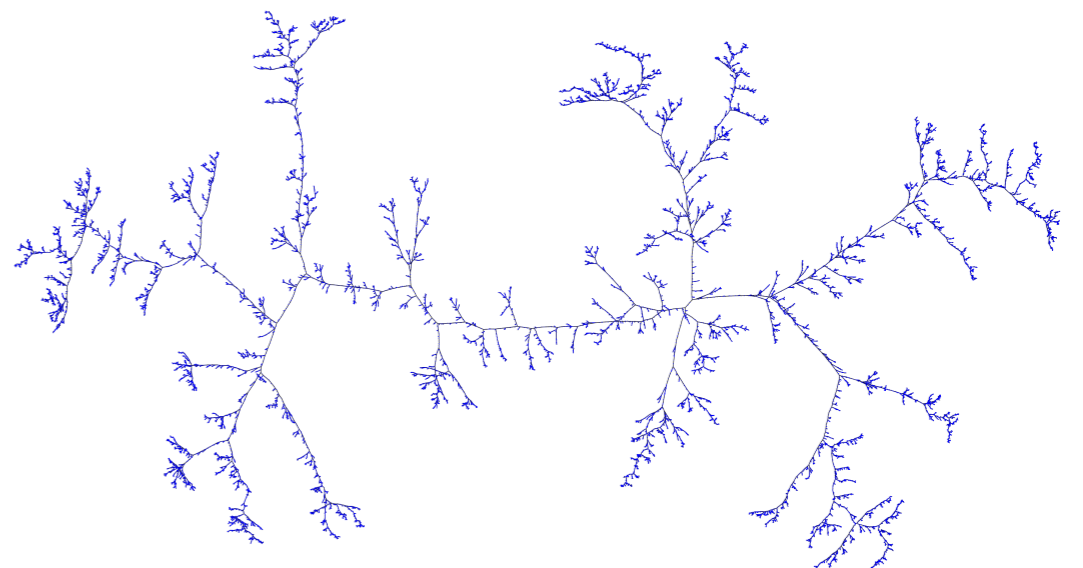
For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3 \ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
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Scaling limit of M_n	Ordered atoms of a Poisson Point Process		

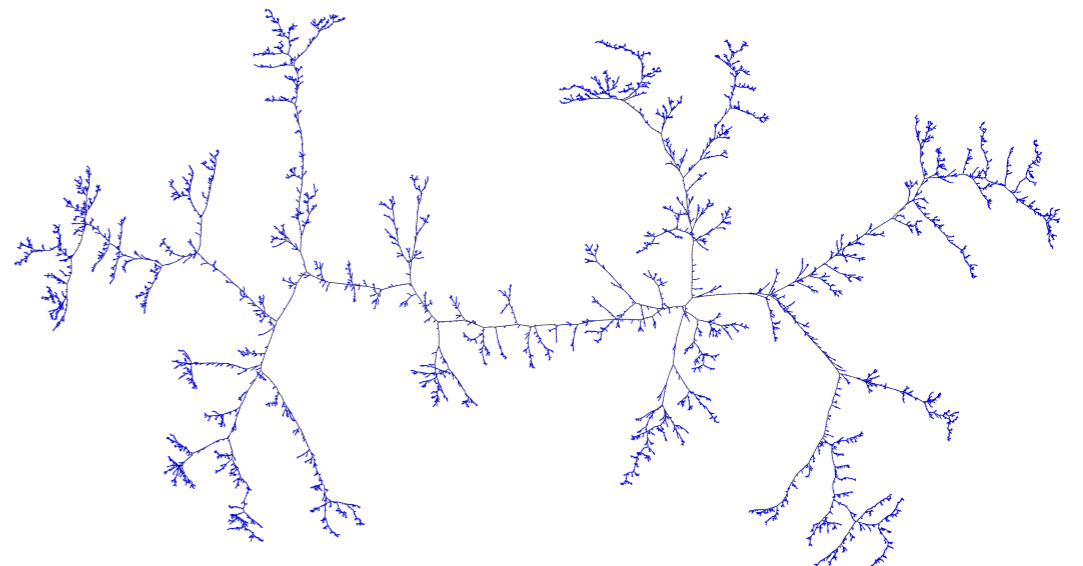
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
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Scaling limit of M_n	?	$\frac{C_2 \ln(n)^{1/2}}{n^{1/2}} M_n \rightarrow \mathcal{T}_e$	$\frac{C_3(u)}{n^{1/2}} M_n \rightarrow \mathcal{T}_e$ [Stufler 2020]



Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
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IV. Perspectives

Extensions to more involved decompositions

Block-weighted

- Tree-rooted quadrangulations;
- Forested maps;
- Maps endowed with a Potts model / Ising model;
- 2-oriented quadrangulations (resp. 3-oriented triangulations) decomposed into irreducible blocks...

Thank you!