A phase transition in block-weighted tree-rooted random maps

Journées Aléa
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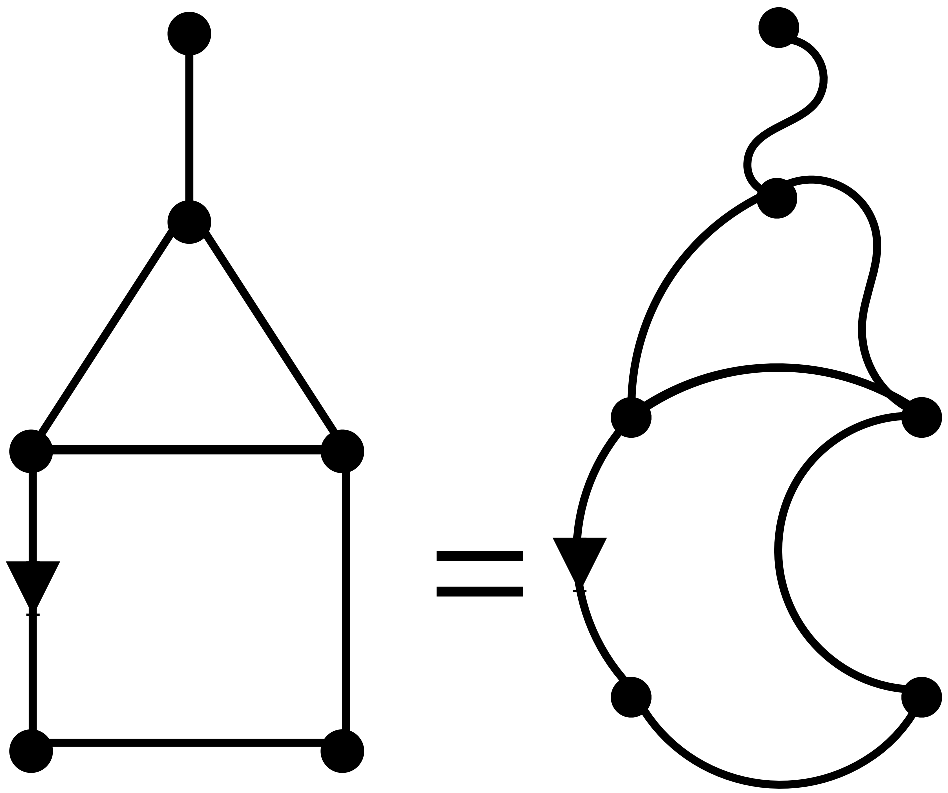
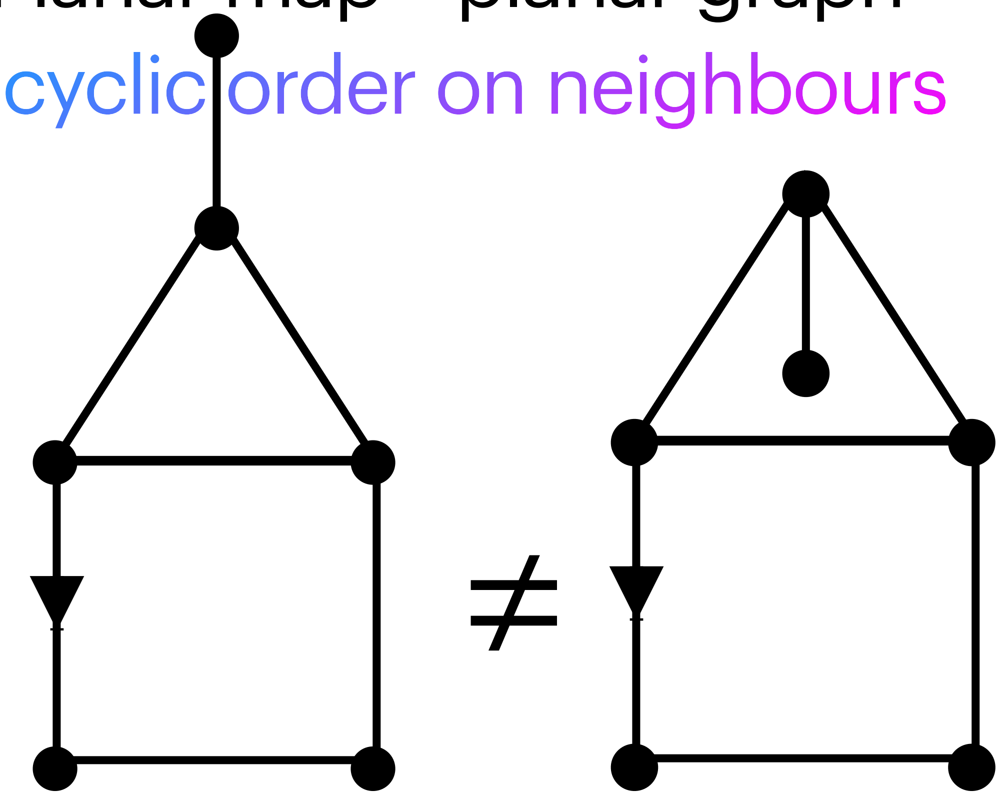
LIGM, Université Gustave Eiffel

Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms



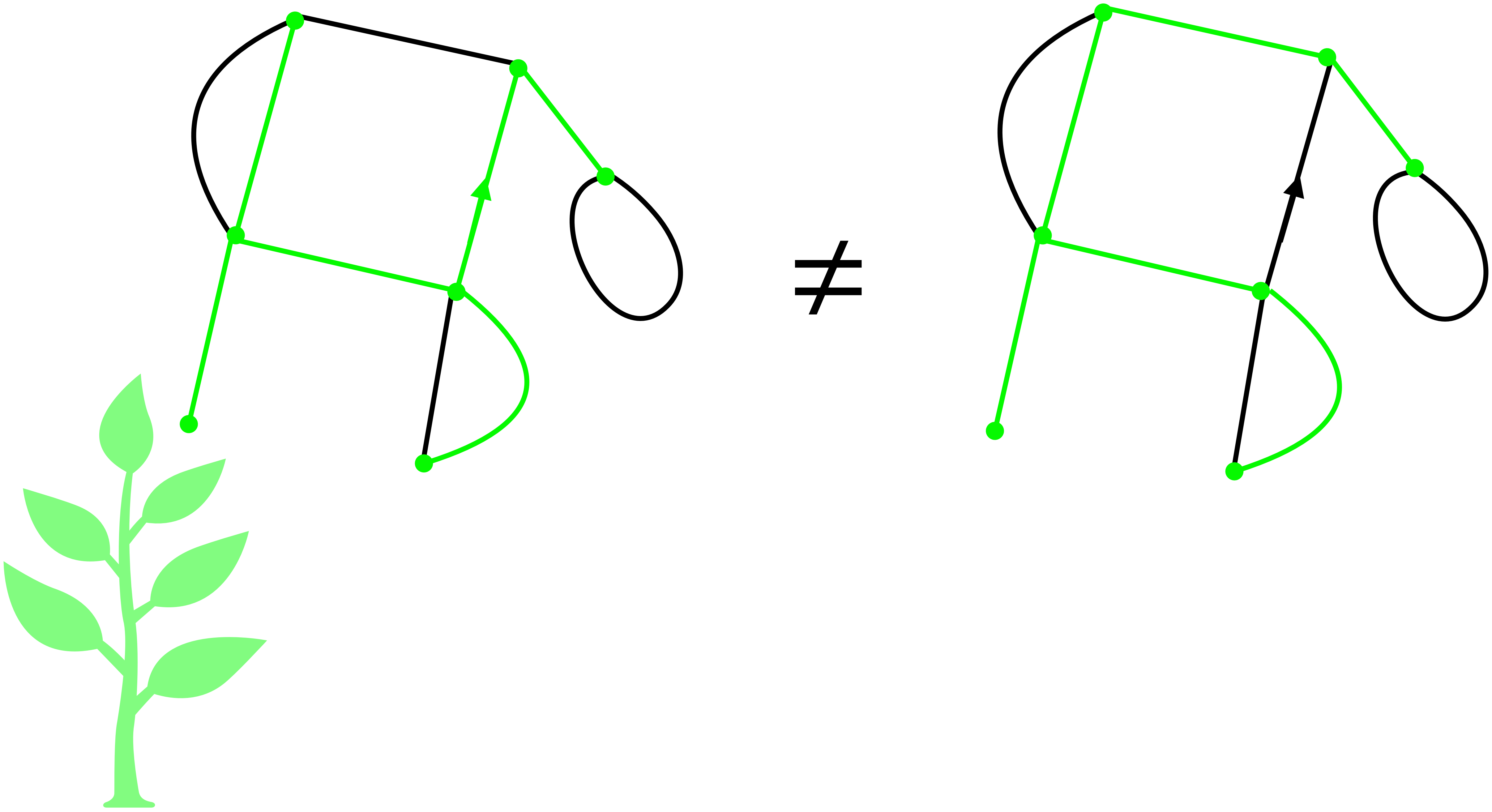
Planar map = planar graph + cyclic order on neighbours



- Rooted planar map = map endowed with a marked oriented edge (represented by an arrow);
- Size $|\mathfrak{m}|$ = number of edges;
- Corner (does not exist for graphs !) = space between two consecutive edges around a vertex (trigonometric order).

Tree-rooted maps

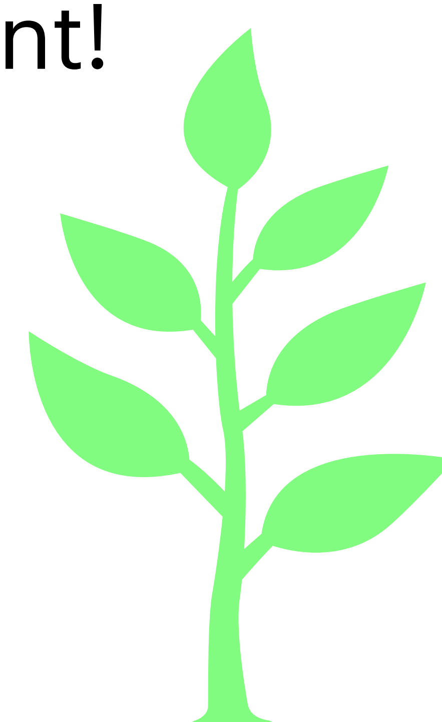
= (rooted planar) maps endowed with a spanning tree.



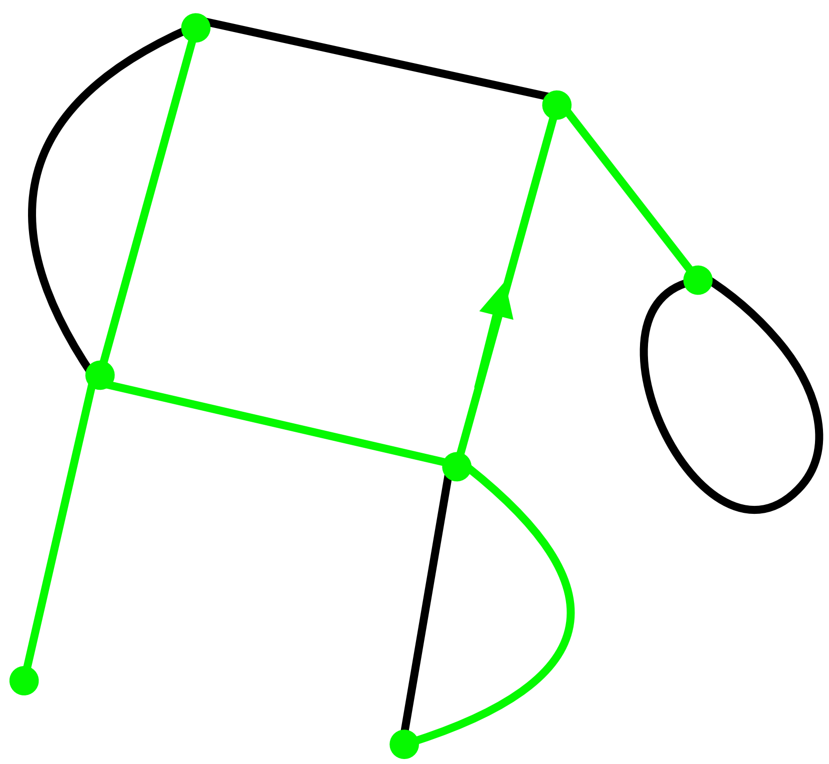
Decorated maps are interesting

Theoretical physics point of view:

- Undecorated maps: “pure gravity” case (nothing happens on the surface);
- Decorated maps: things happen! new asymptotic behaviours! new universality classes! excitement!



Tree-rooted maps

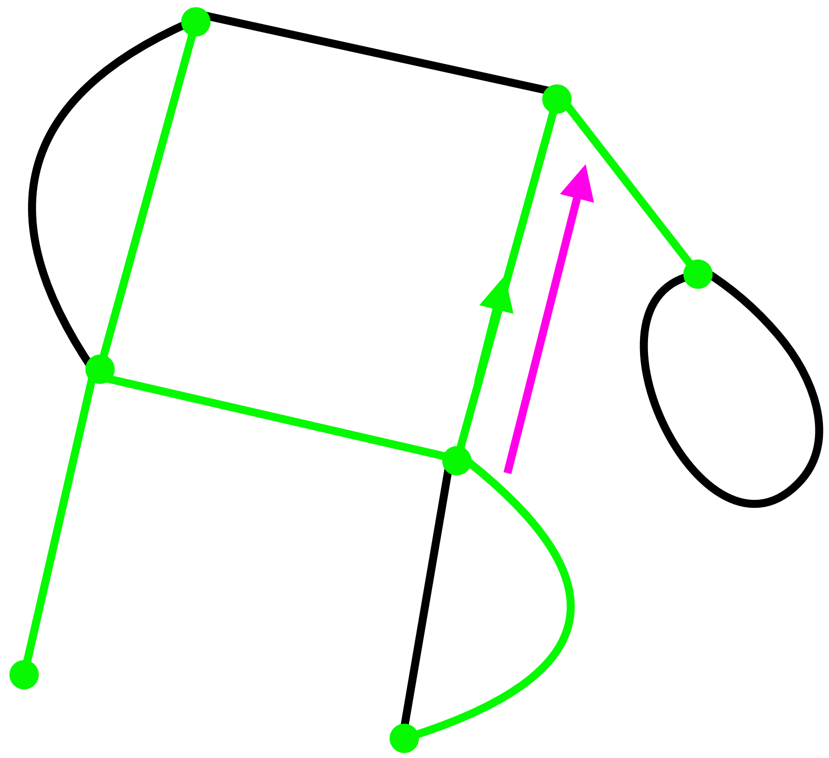


- Combinatorics well understood : Mullin's bijection;
- Geometry not so much.

$$[z^n]M(z) = \mathrm{Cat}_n \mathrm{Cat}_{n+1}$$

[Mullin 67]

Tree-rooted maps



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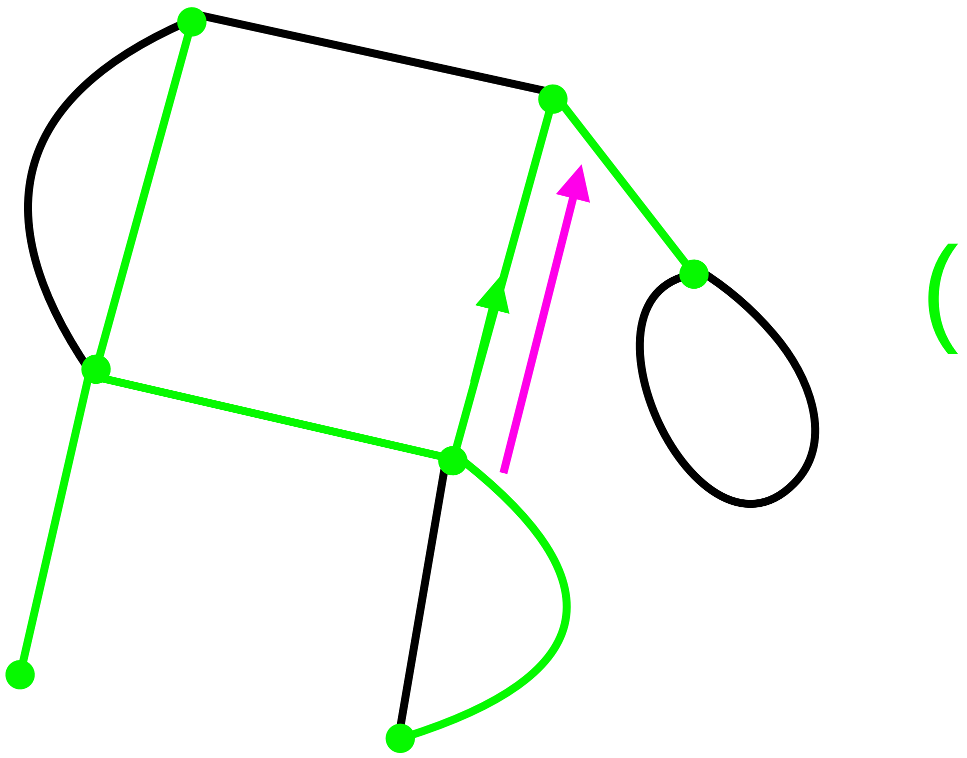

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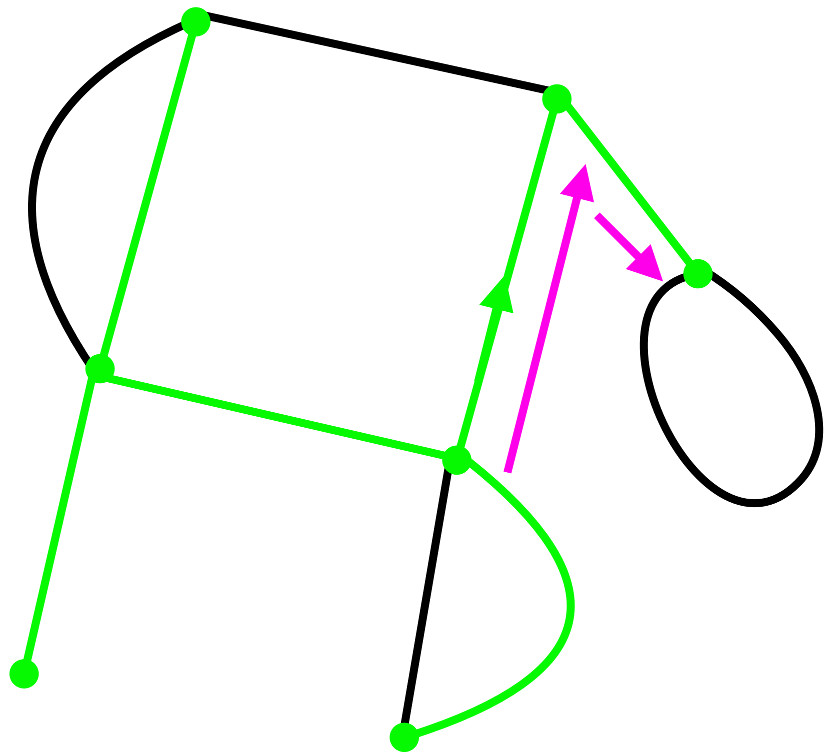

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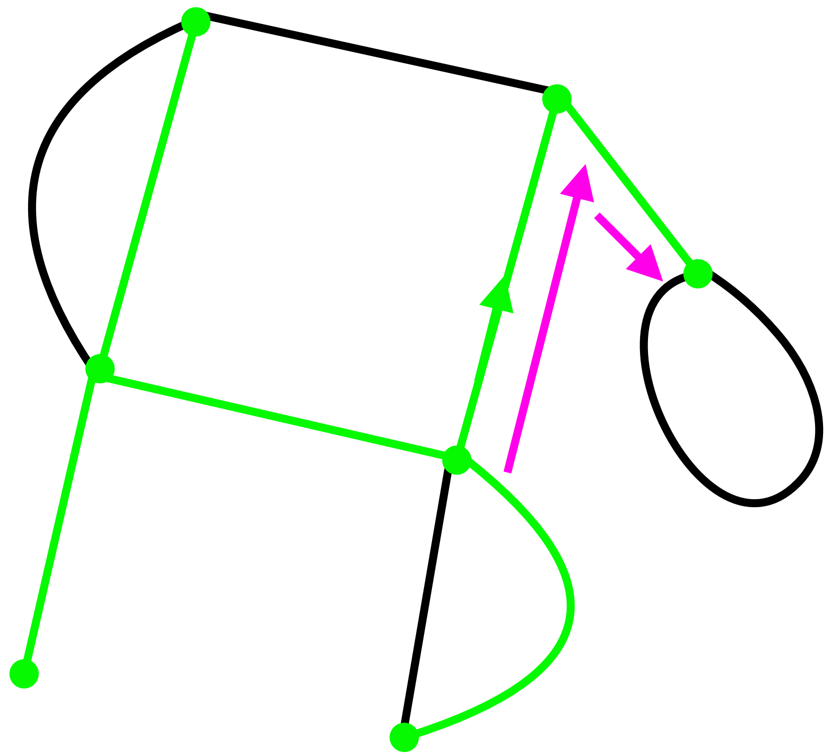

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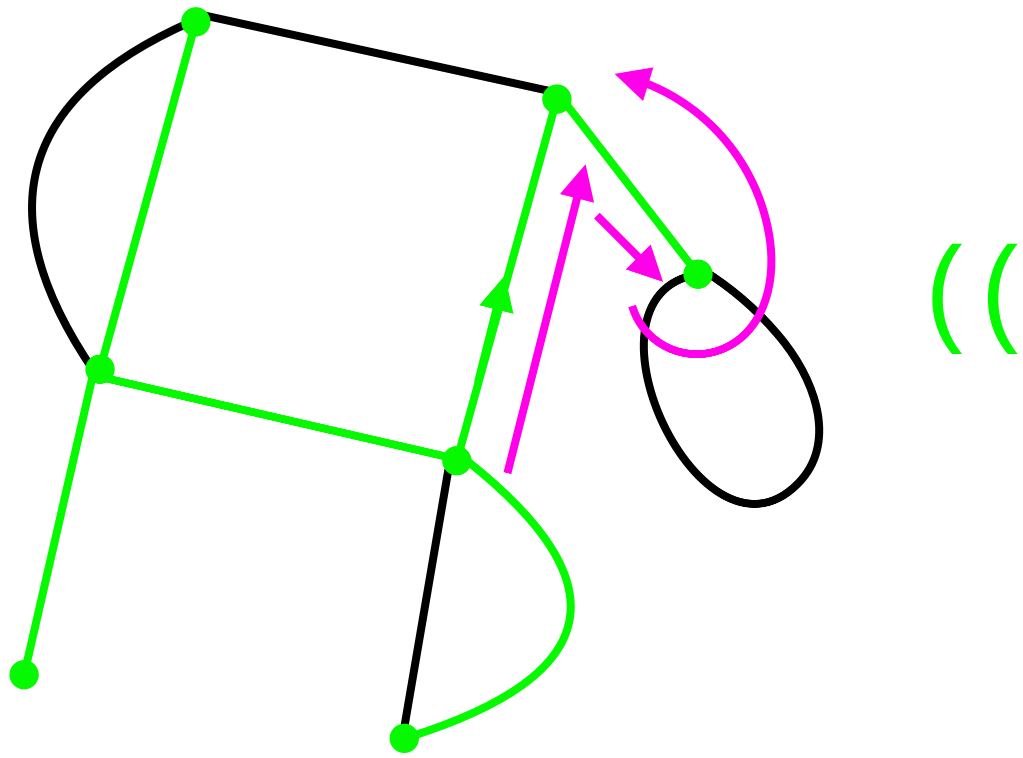

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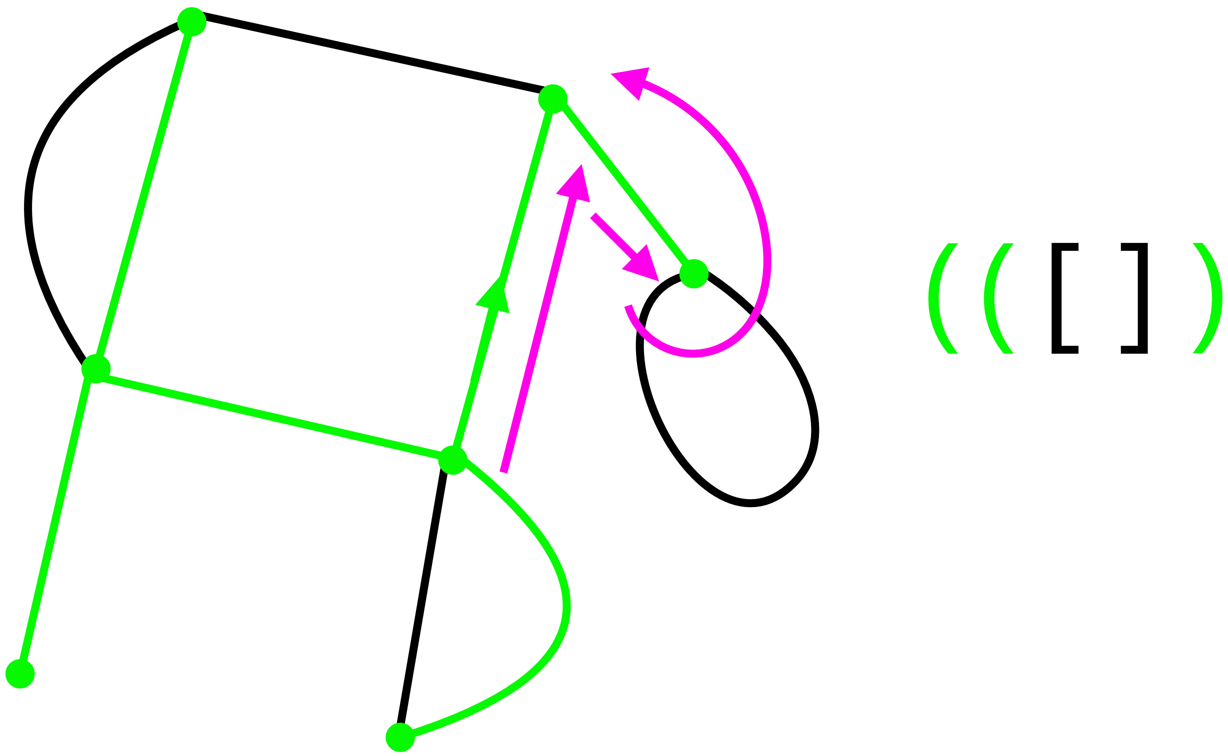

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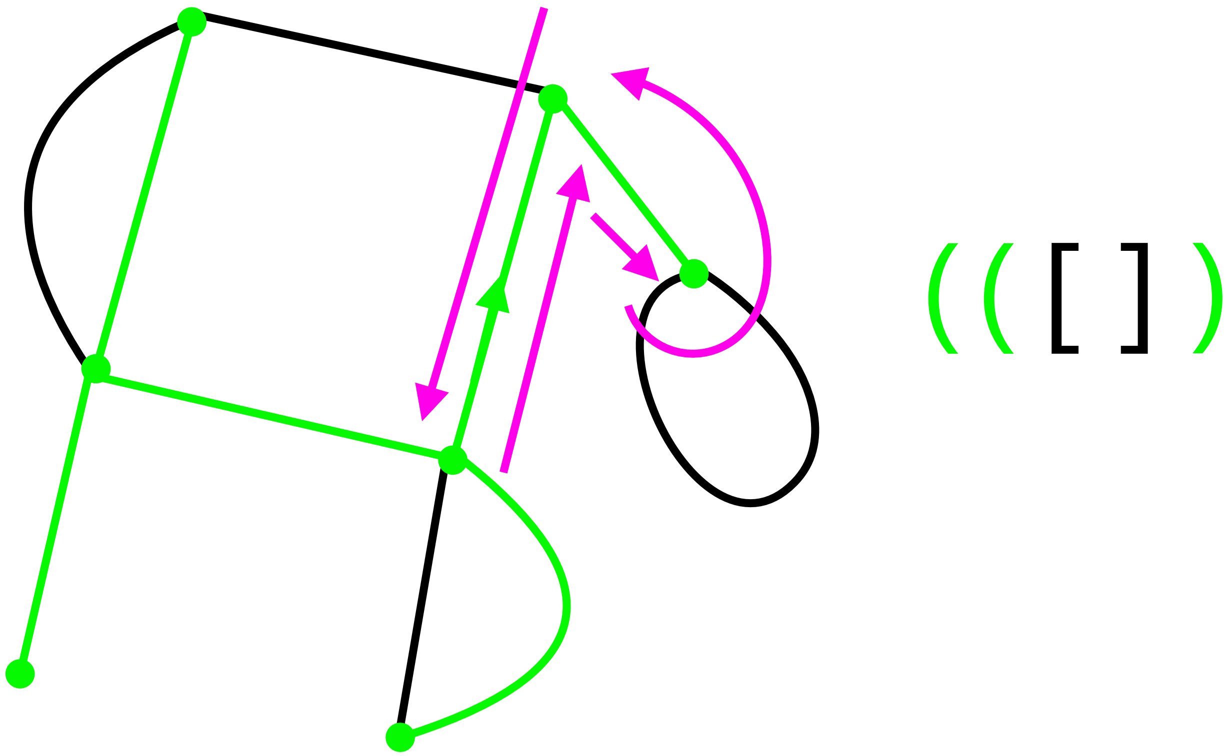

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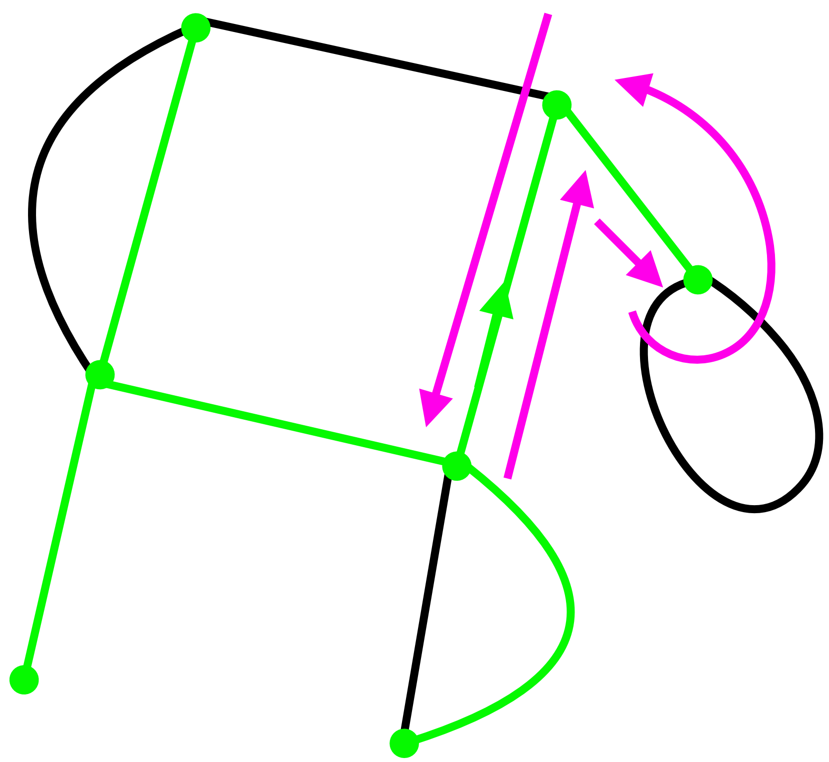

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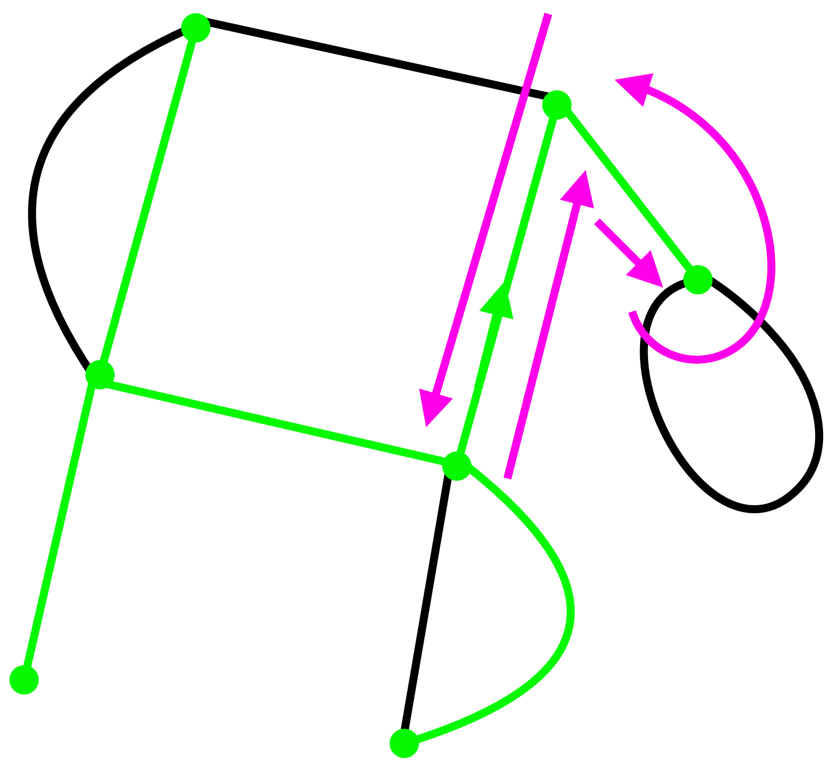

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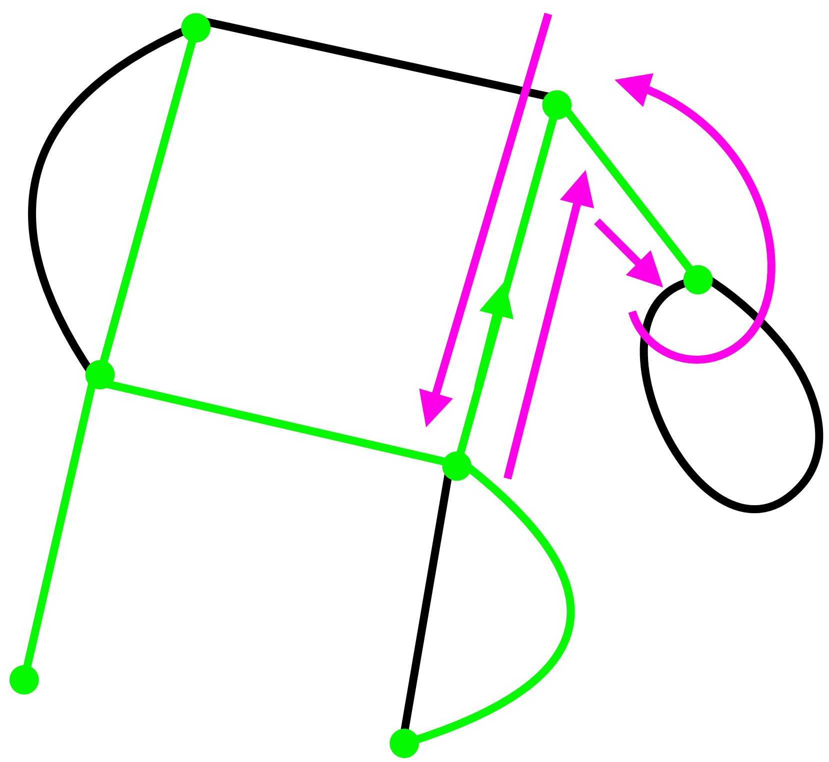

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Dyck word of size $|\mathfrak{t}|$

Dyck word of size $|\mathfrak{m}| - |\mathfrak{t}|$

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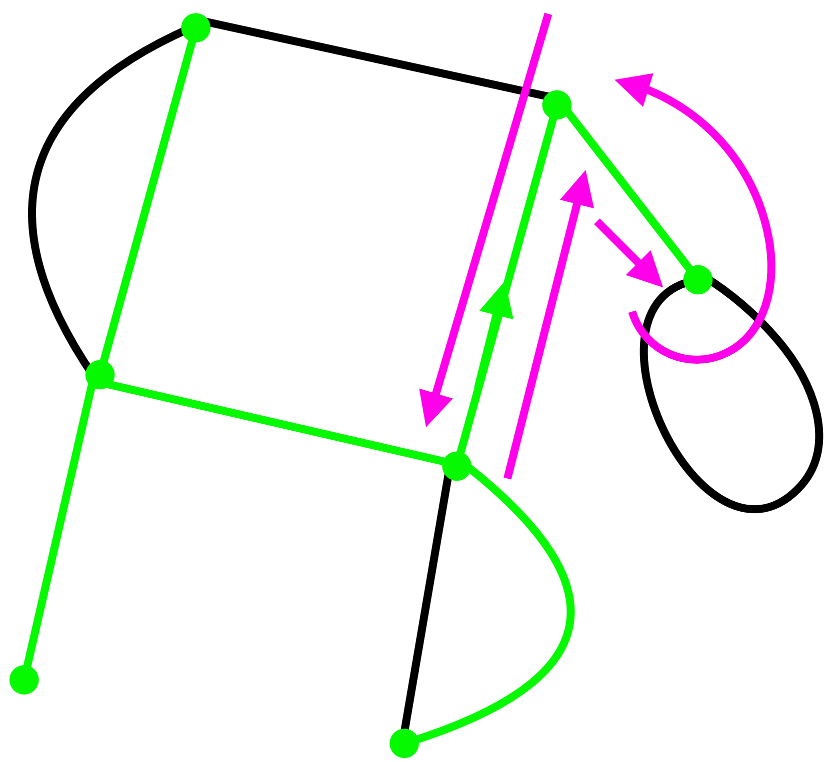

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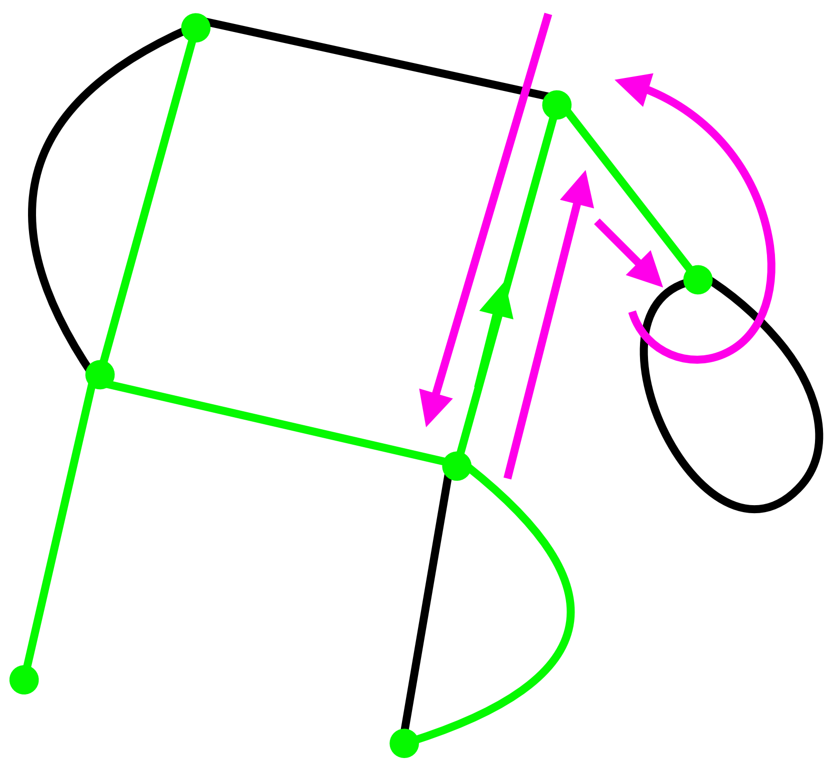

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Vandermonde identity

$$[z^n]M(z) = \mathrm{Cat}_n \mathrm{Cat}_{n+1}$$

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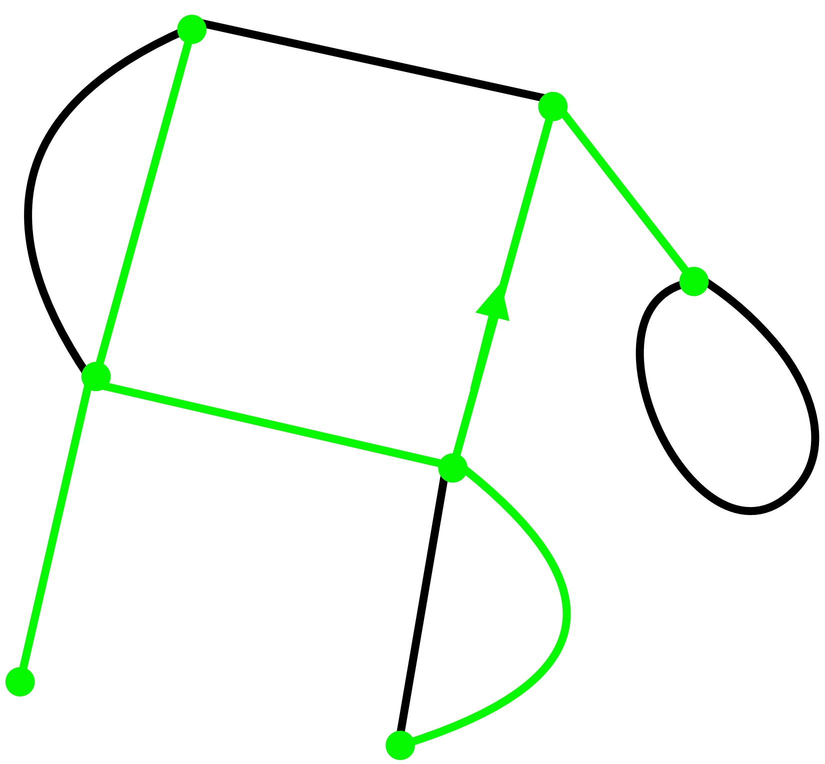
Tree-rooted maps



$$M(z) = \sum_{n\geq 0} \mathrm{Cat}_n \mathrm{Cat}_{n+1} z^n$$ [Mullin 67]

We want a phase transition in tree-rooted maps.

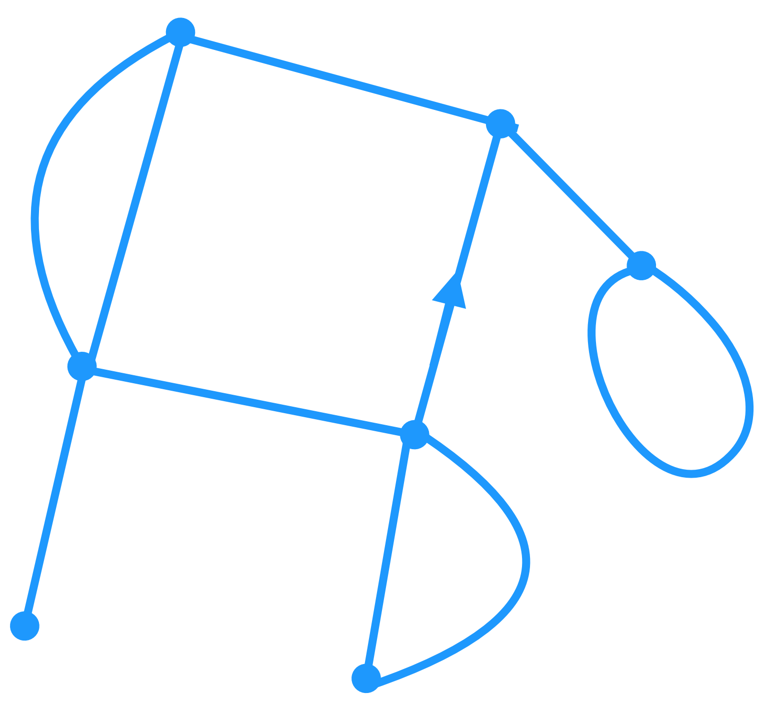
=> Block-weighted tree-rooted maps.

I. “Block-weighted maps“?

Joint work with William Fleurat

Universality results for planar maps

- Enumeration: $\kappa\rho^{-n}n^{-5/2}$ [Tutte 1963];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for quadrangulations [Le Gall 2013, Miermont 2013] and general maps [Bettinelli, Jacob, Miermont 2014];



Brownian Sphere $\mathcal{S}_e$

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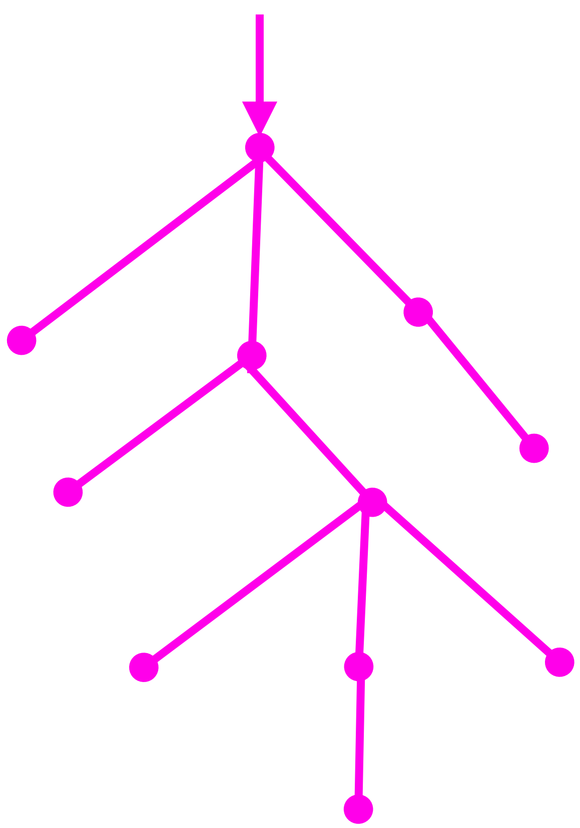
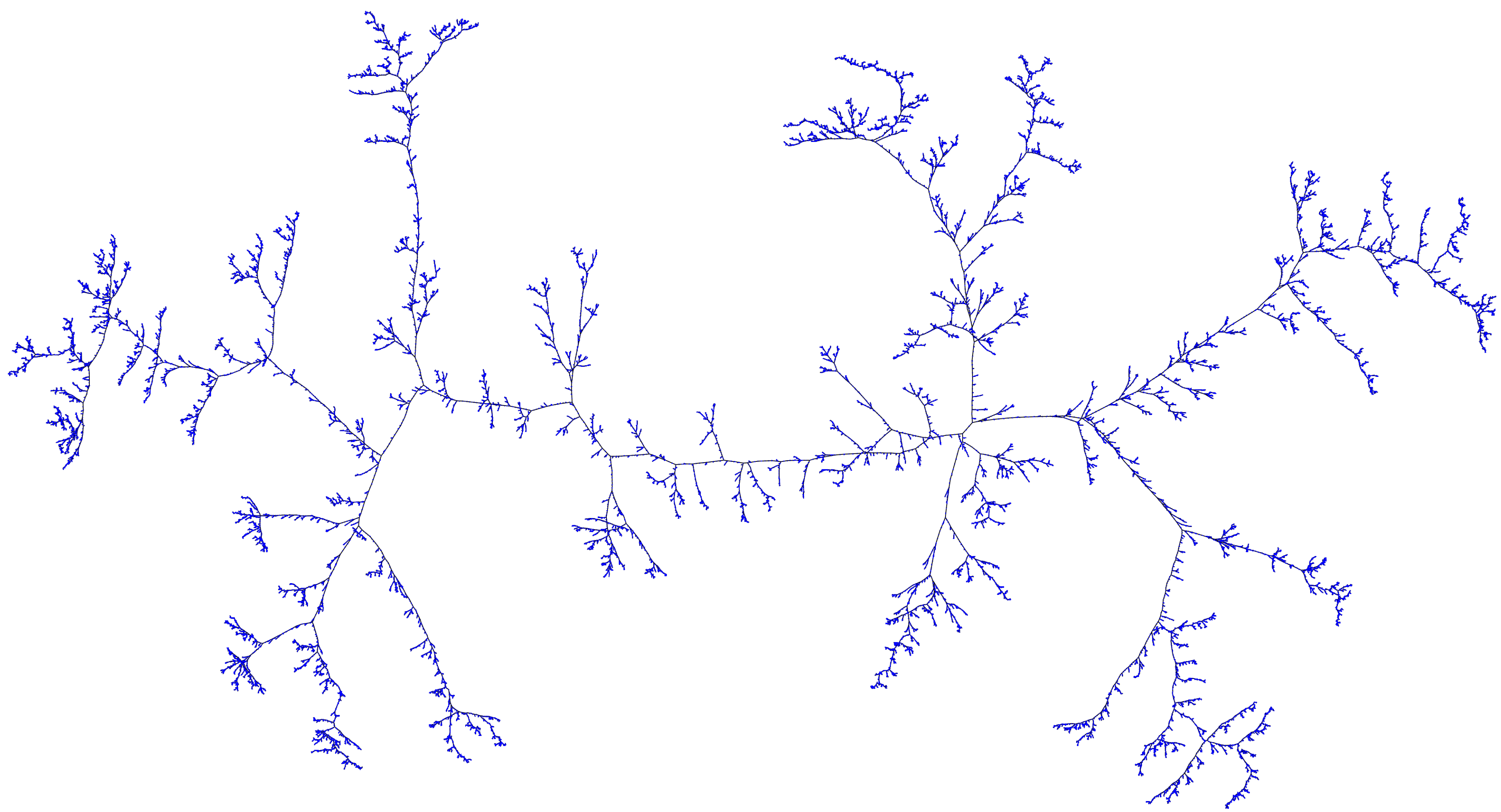
- Universality:
  - Same enumeration [Drmota, Noy, Yu 2020];
  - Same scaling limit, *e.g.* for triangulations & $2q$-angulations [Le Gall 2013], simple quadrangulations [Addario-Berry, Albenque 2017].

Universality results for plane trees

- Enumeration: $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];





Brownian Tree $\mathscr{T}_e$

Universality results for plane trees

- Enumeration: $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];

- Universality:
  - Same enumeration,
  - Same scaling limit, even for some classes of maps; e.g. outerplanar maps [Caraceni 2016], maps with a boundary of size >> $n^{1/2}$ [Bettinelli 2015].

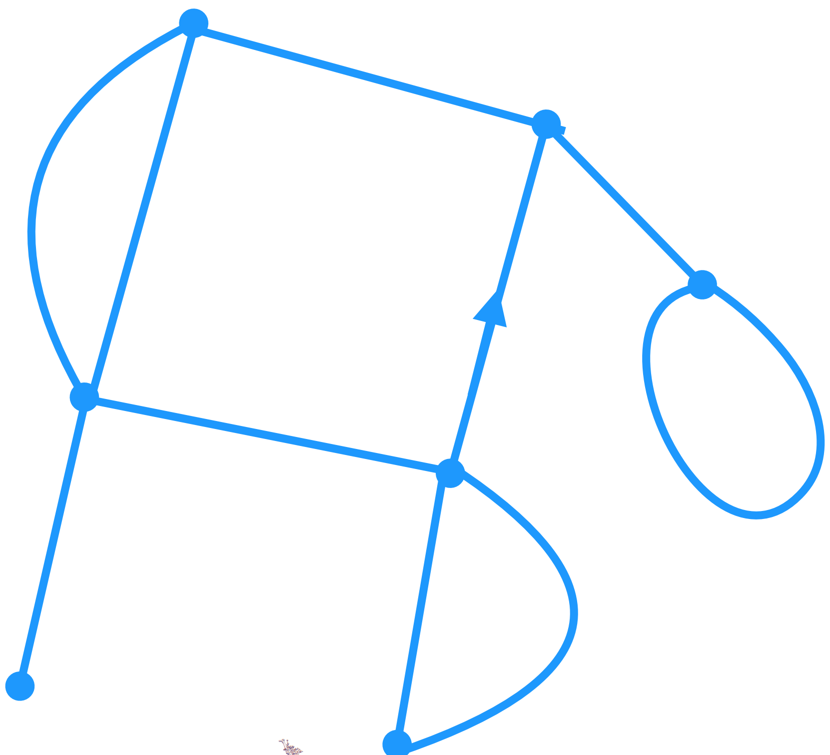
Models with (very) constrained boundaries

Motivation

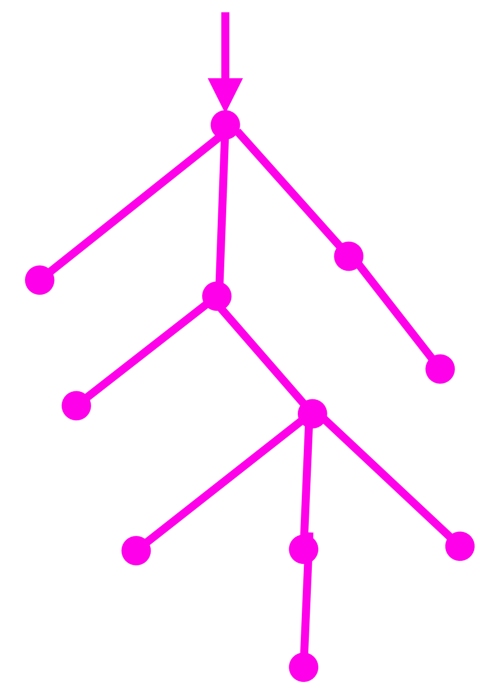
Inspired by [Bonzom 2016].

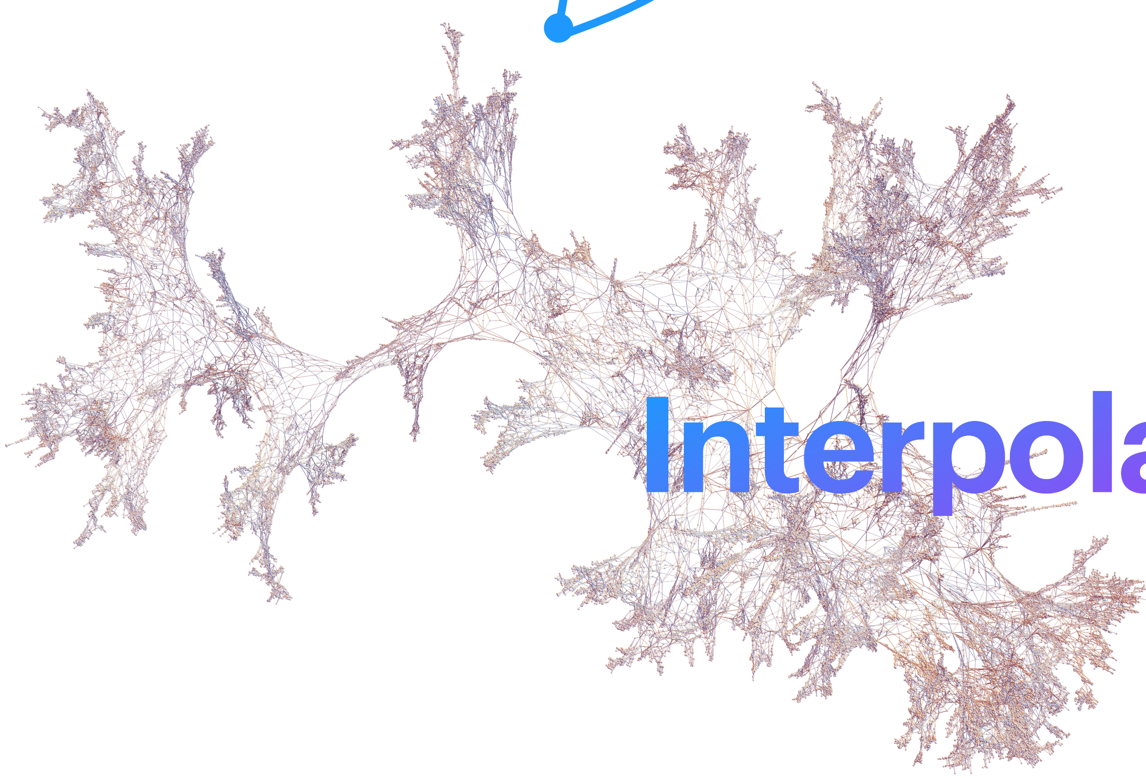
Two rich situations with universality results:

Planar maps

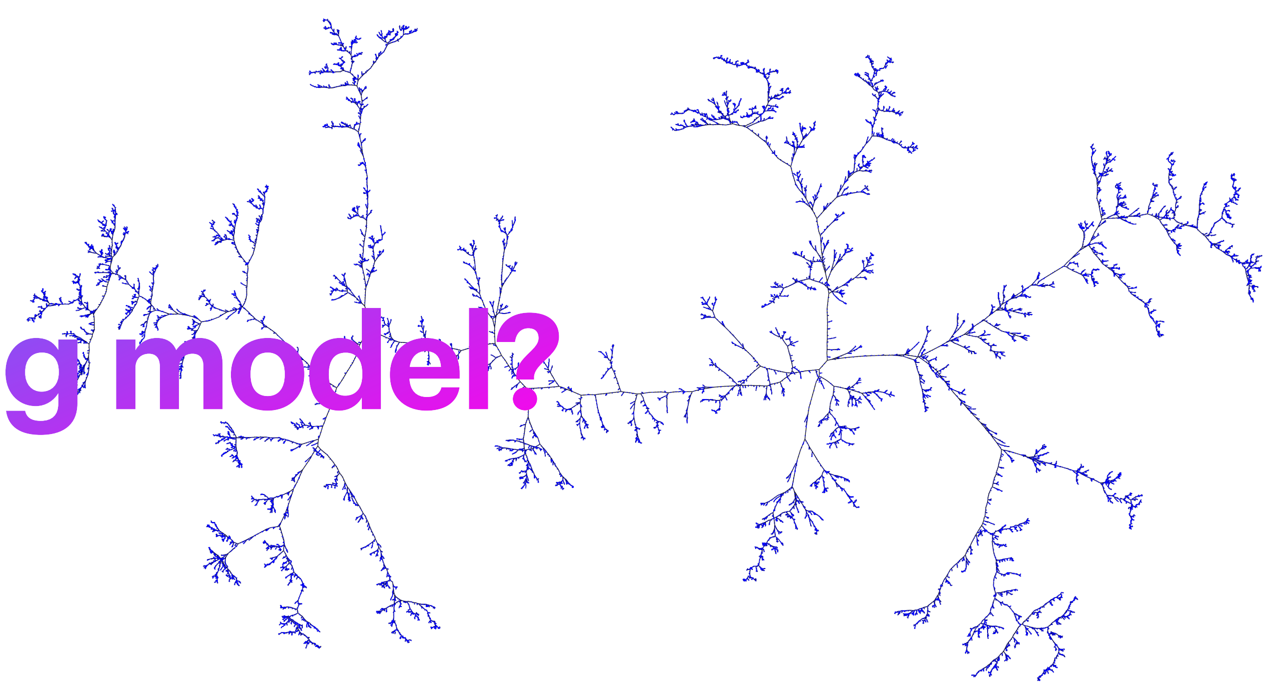


Plane trees





Interpolating model?



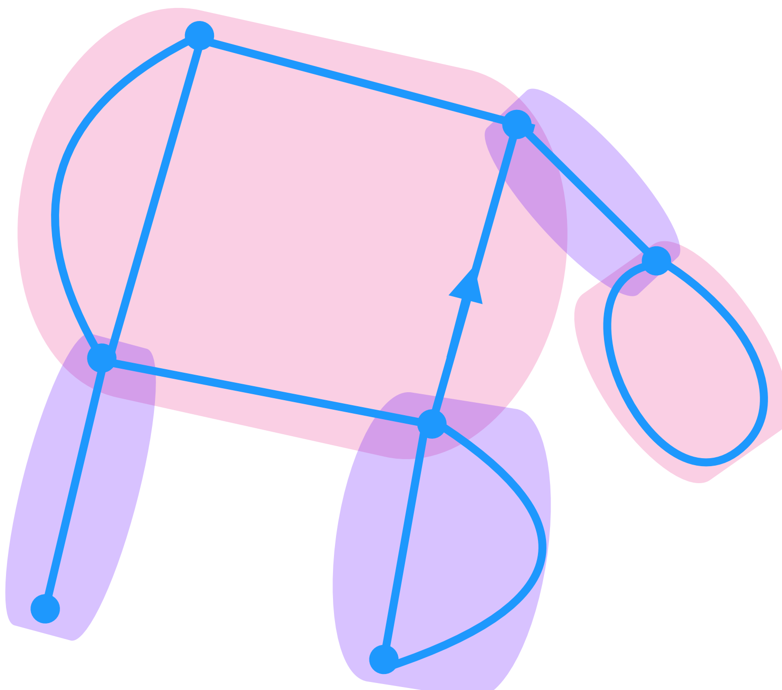
Brownian Sphere $\mathcal{S}_e$

Brownian Tree $\mathcal{T}_e$

Model definition

2-connected = two vertices must be removed to disconnect.

Block = maximal (for inclusion) 2-connected submap.



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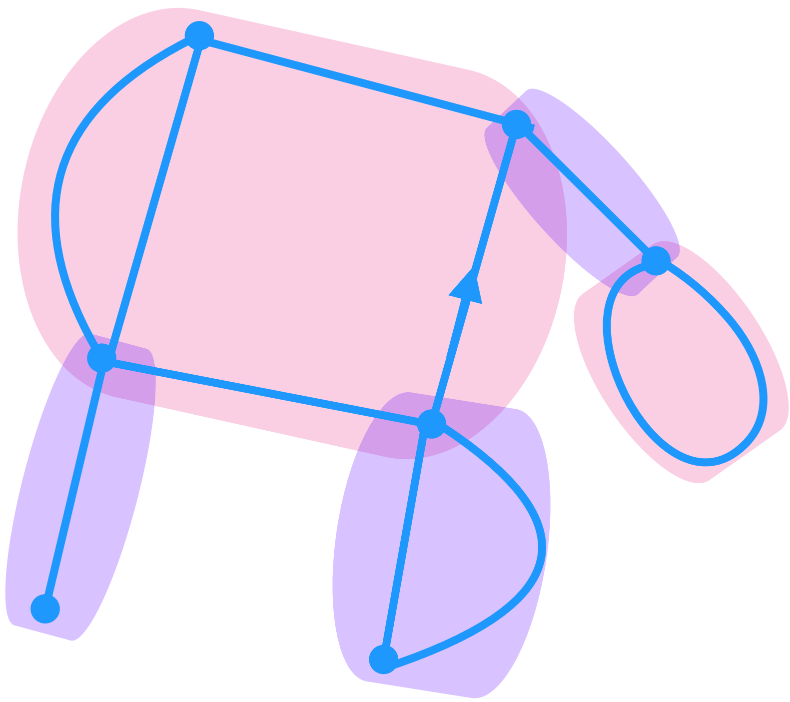

Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

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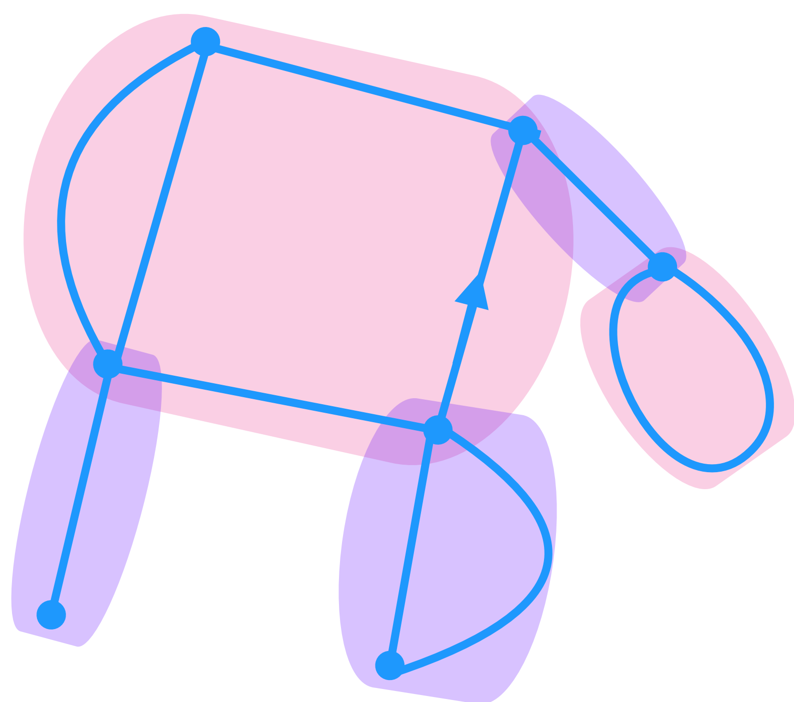

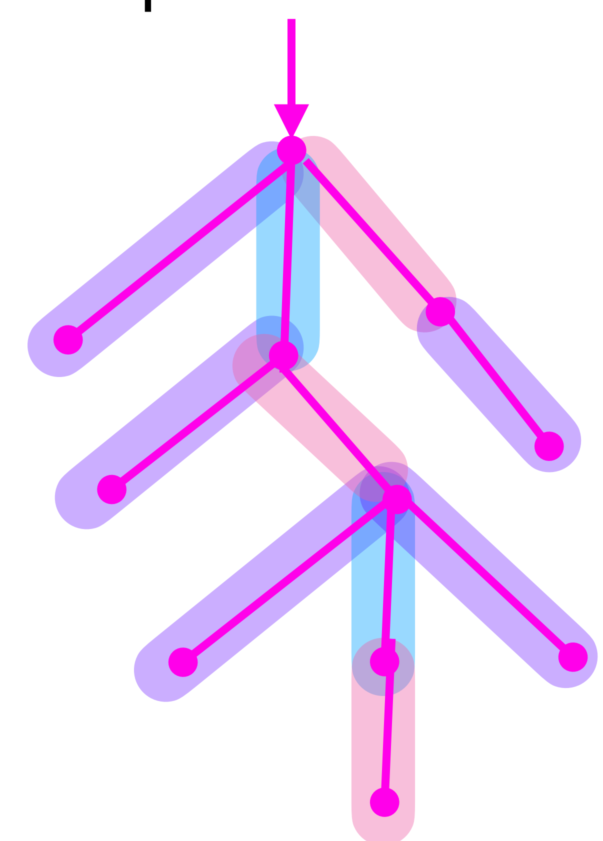

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Only small blocks.

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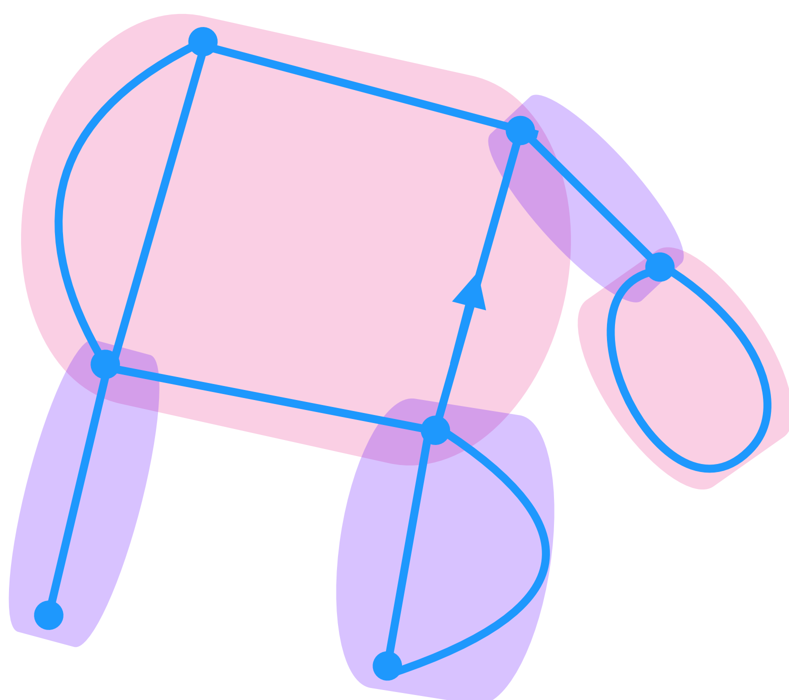

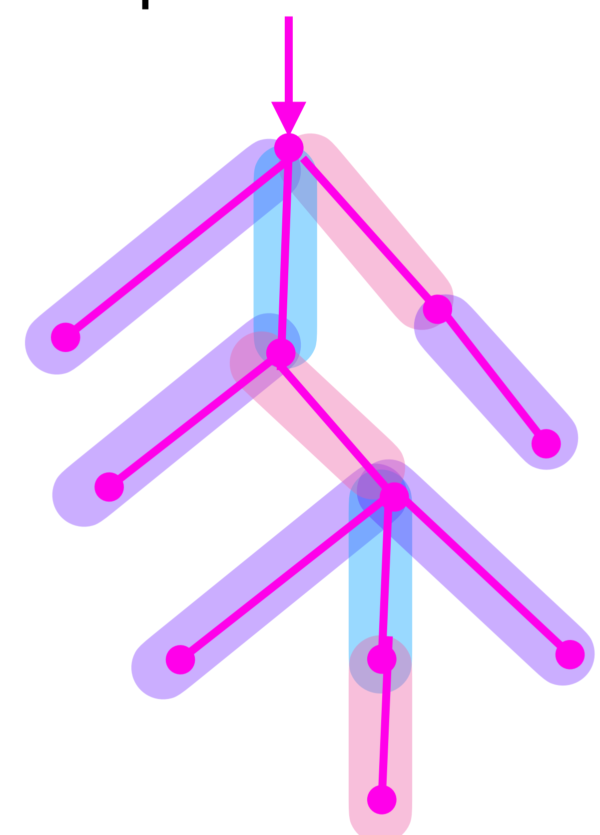

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Interpolating model using blocks!

Model

Inspired by [Bonzom 2016].

Goal: parameter that affects the typical number of blocks.

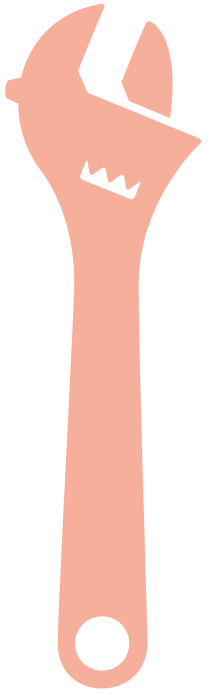
We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,
$\mathcal{M}_n$ = {maps of size n},
$\mathfrak{m} \in \mathcal{M}_n$,
$Z_{n,u}$ = normalisation.

- $u = 1$: uniform distribution on maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected maps);
- $u \to \infty$: maximising the number of blocks (= trees!).

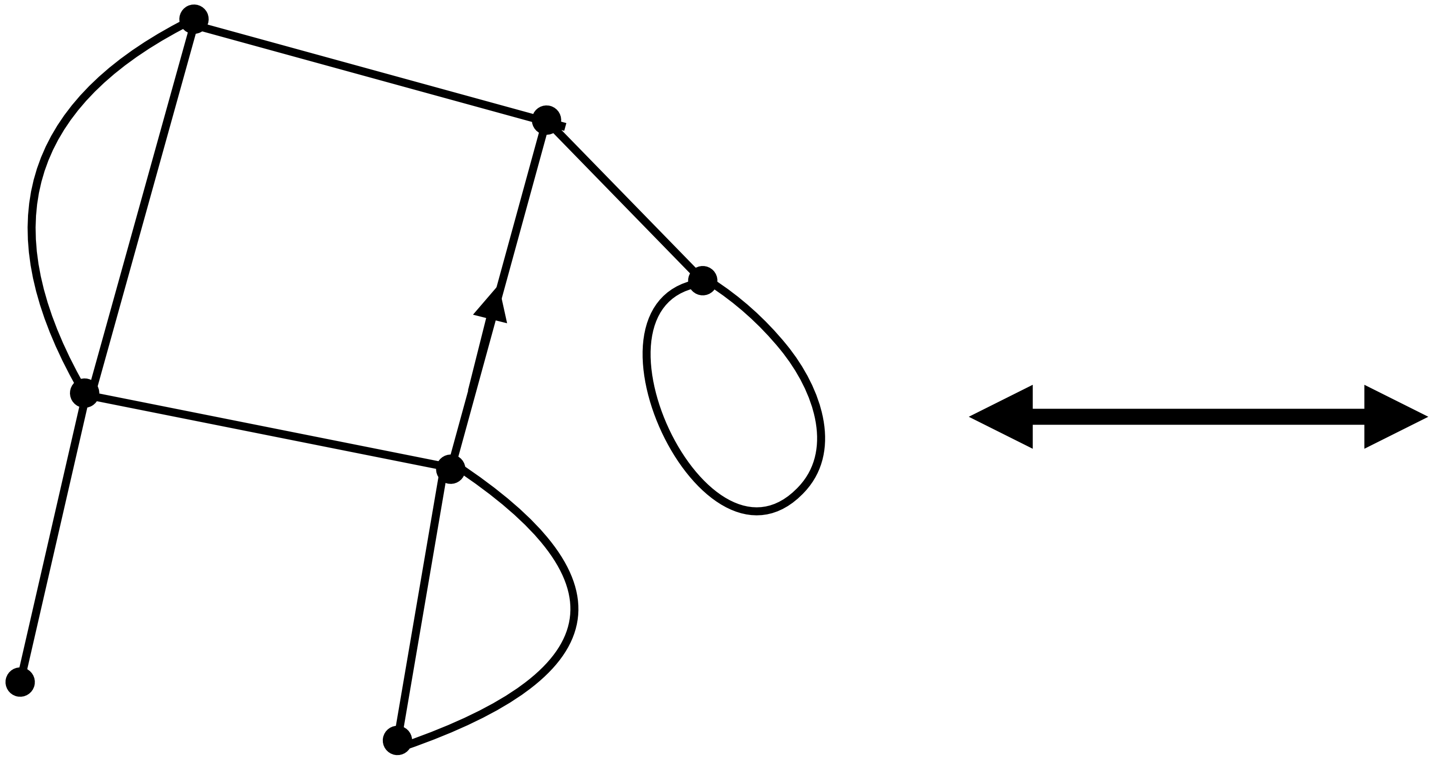
Given u, asymptotic behaviour when $n \to \infty$?

II. Block tree of a map



Decomposition of a map into blocks

Inspiration from [Tutte 1963]



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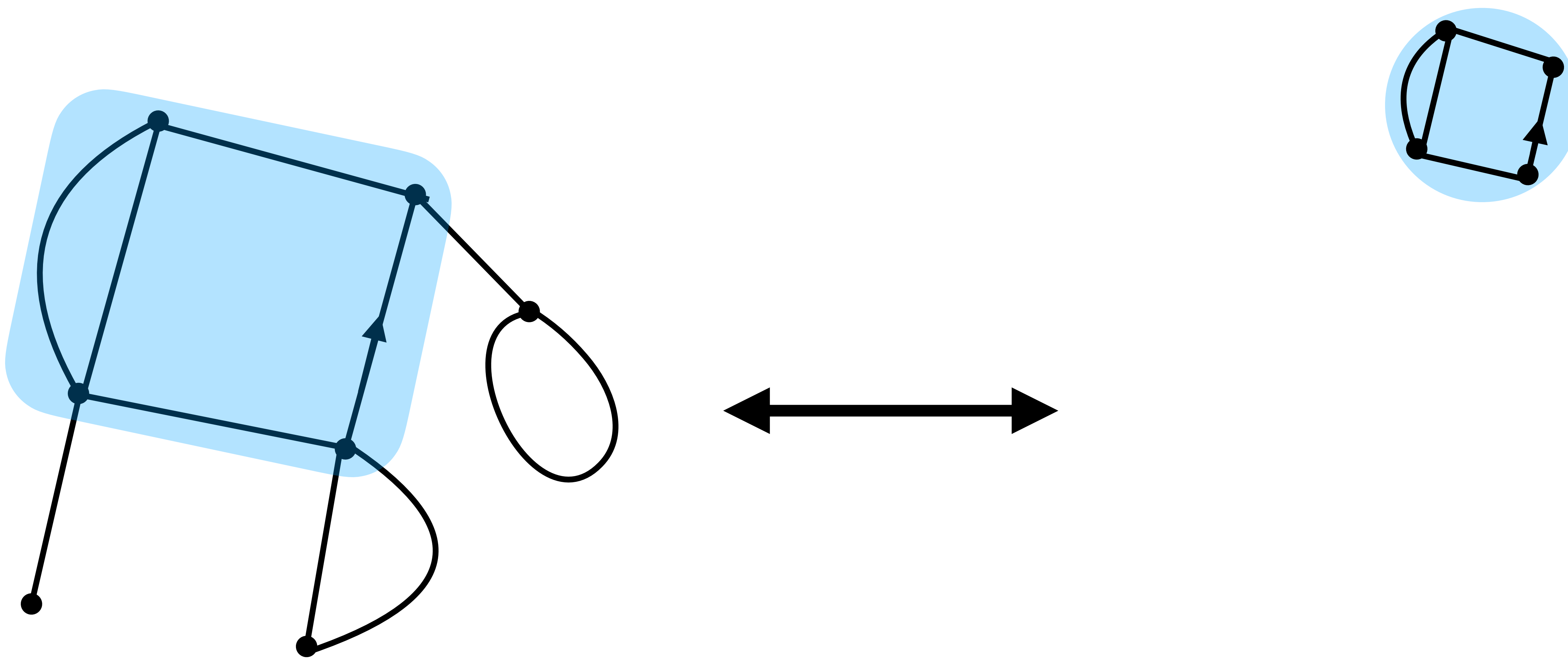

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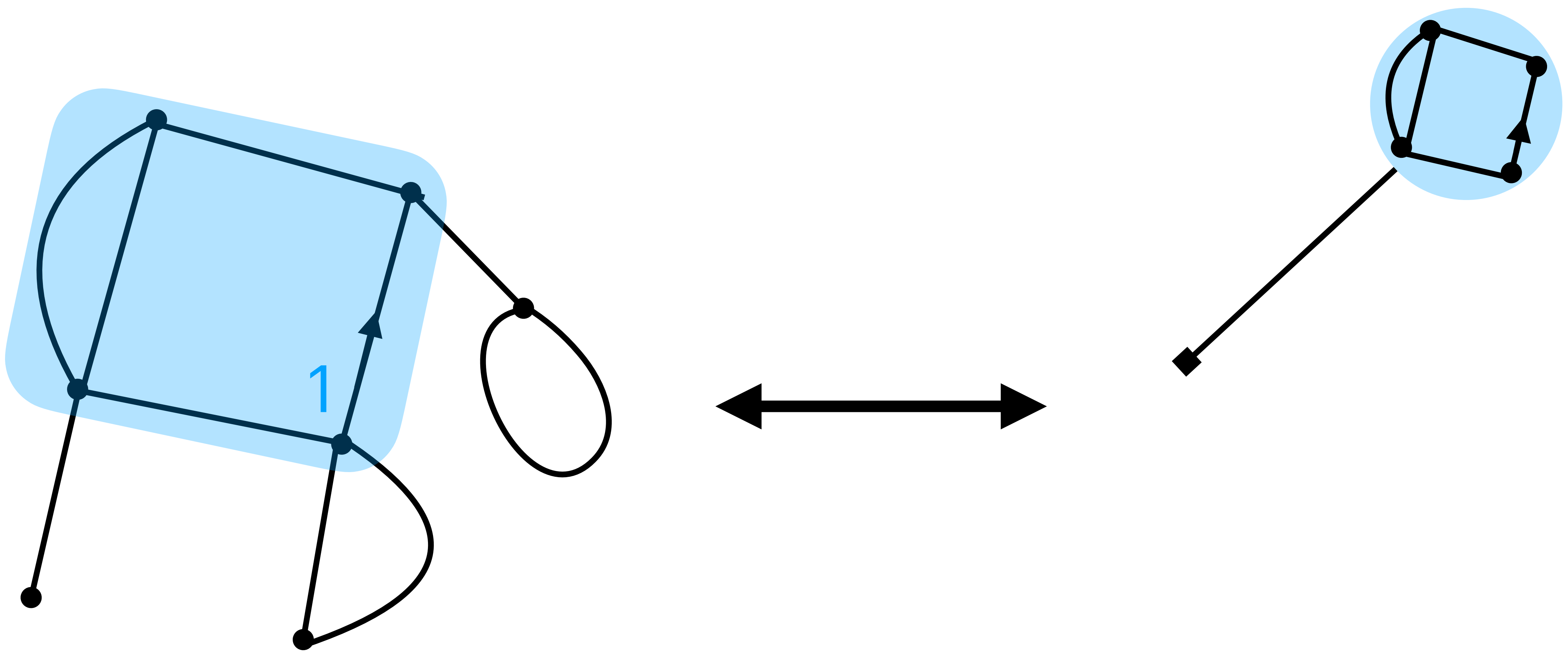

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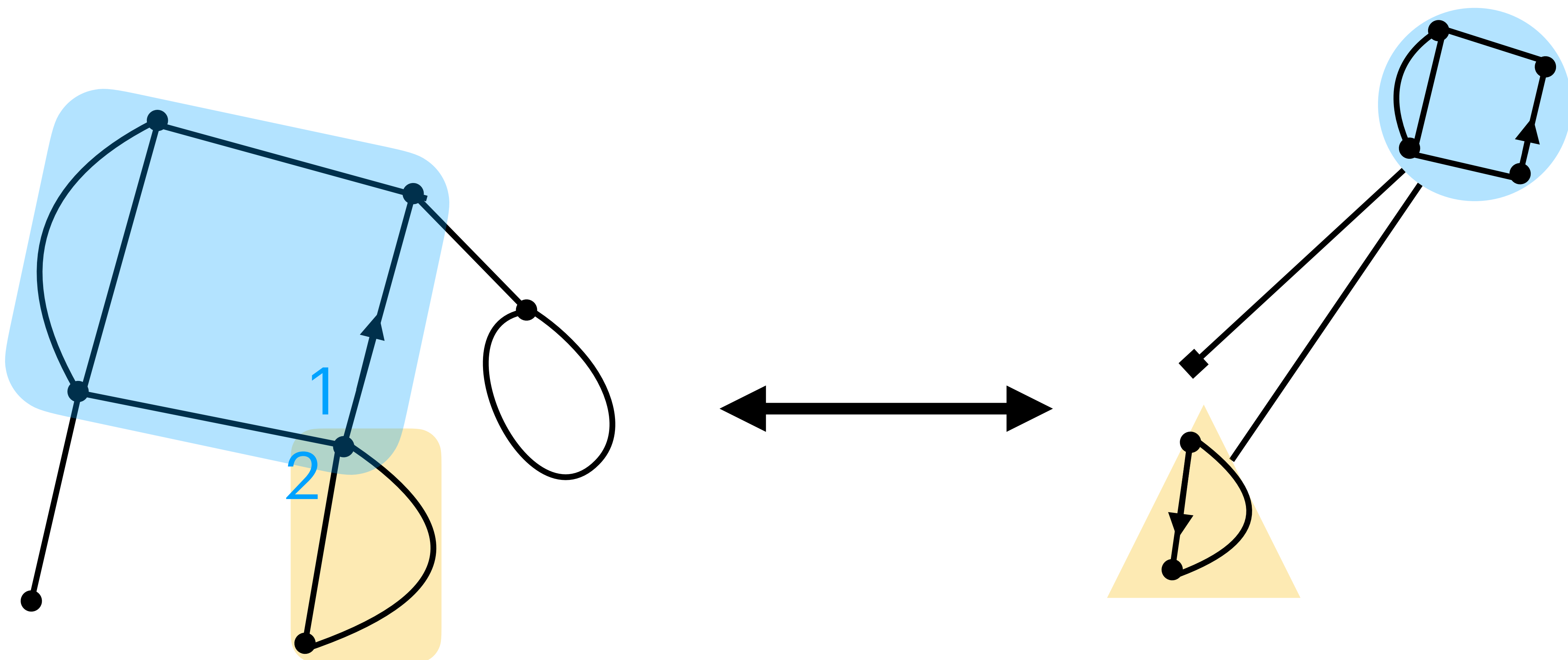

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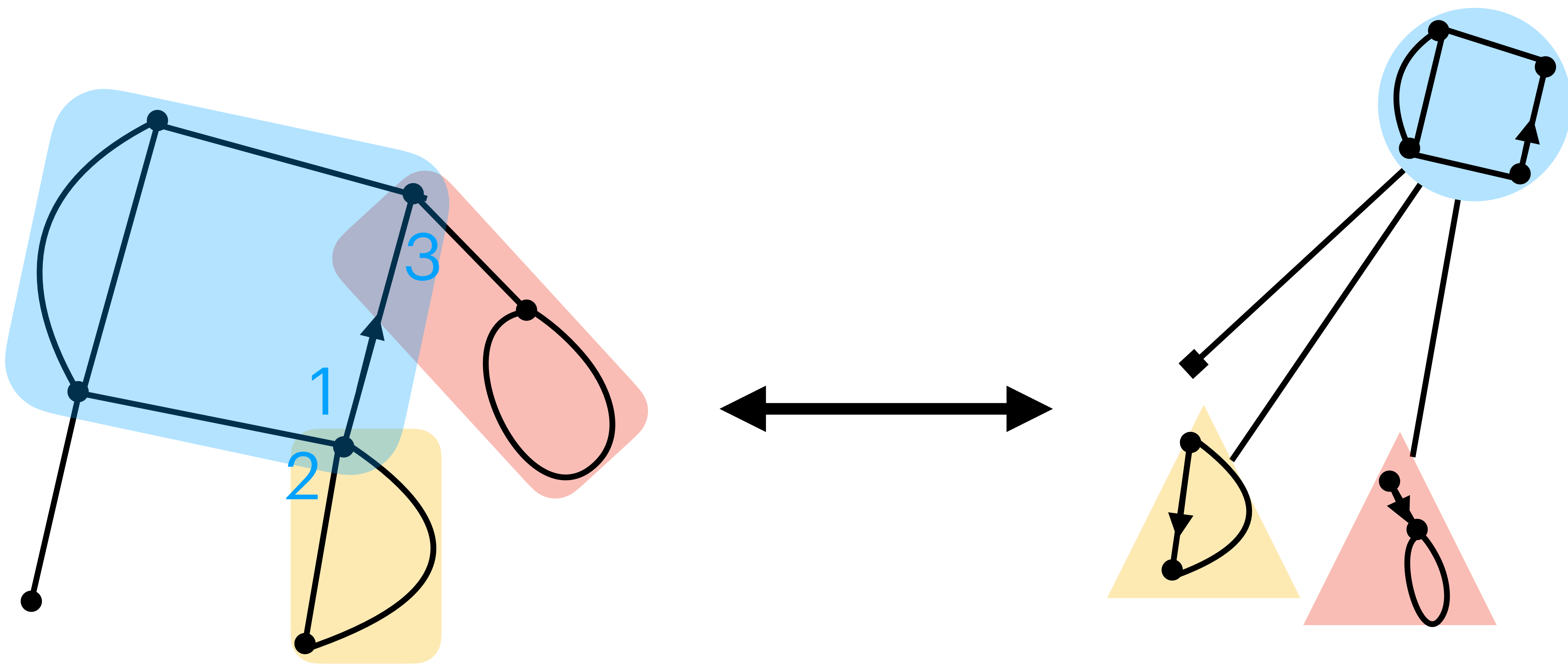

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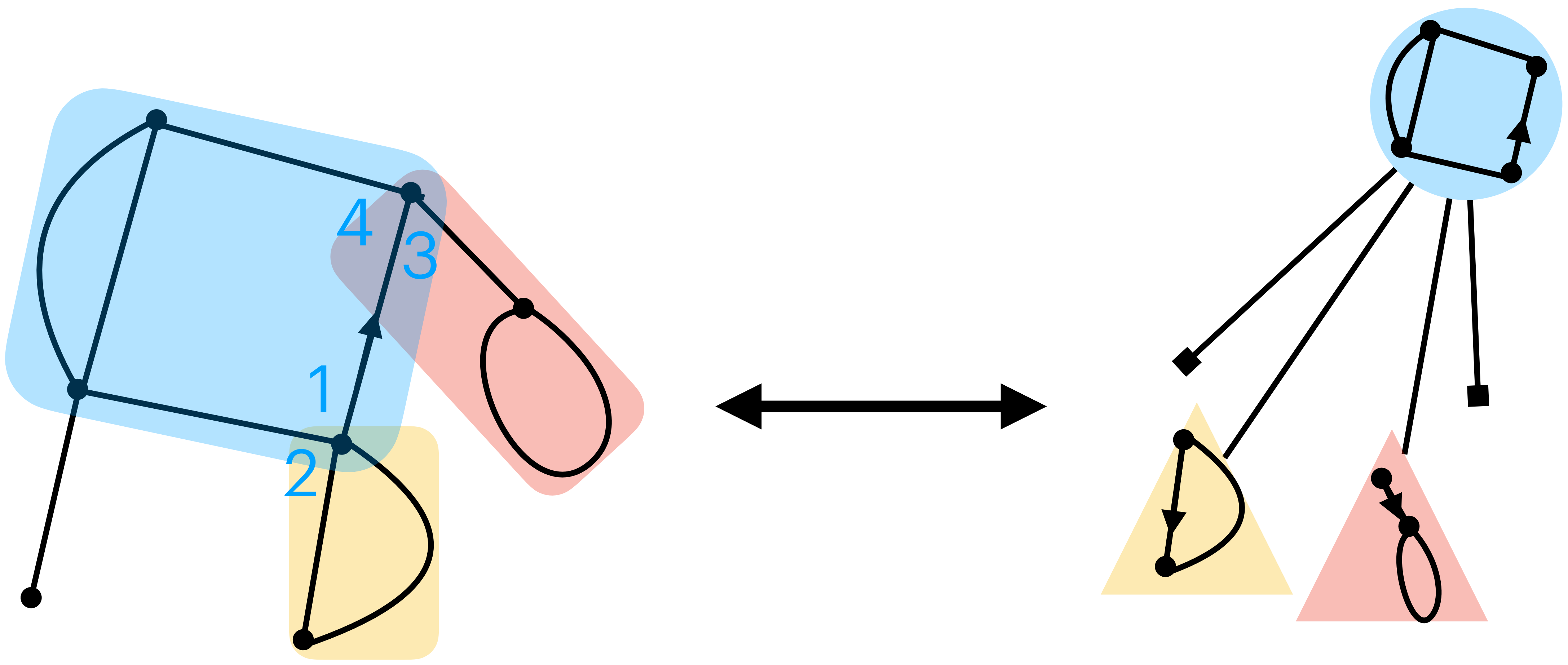

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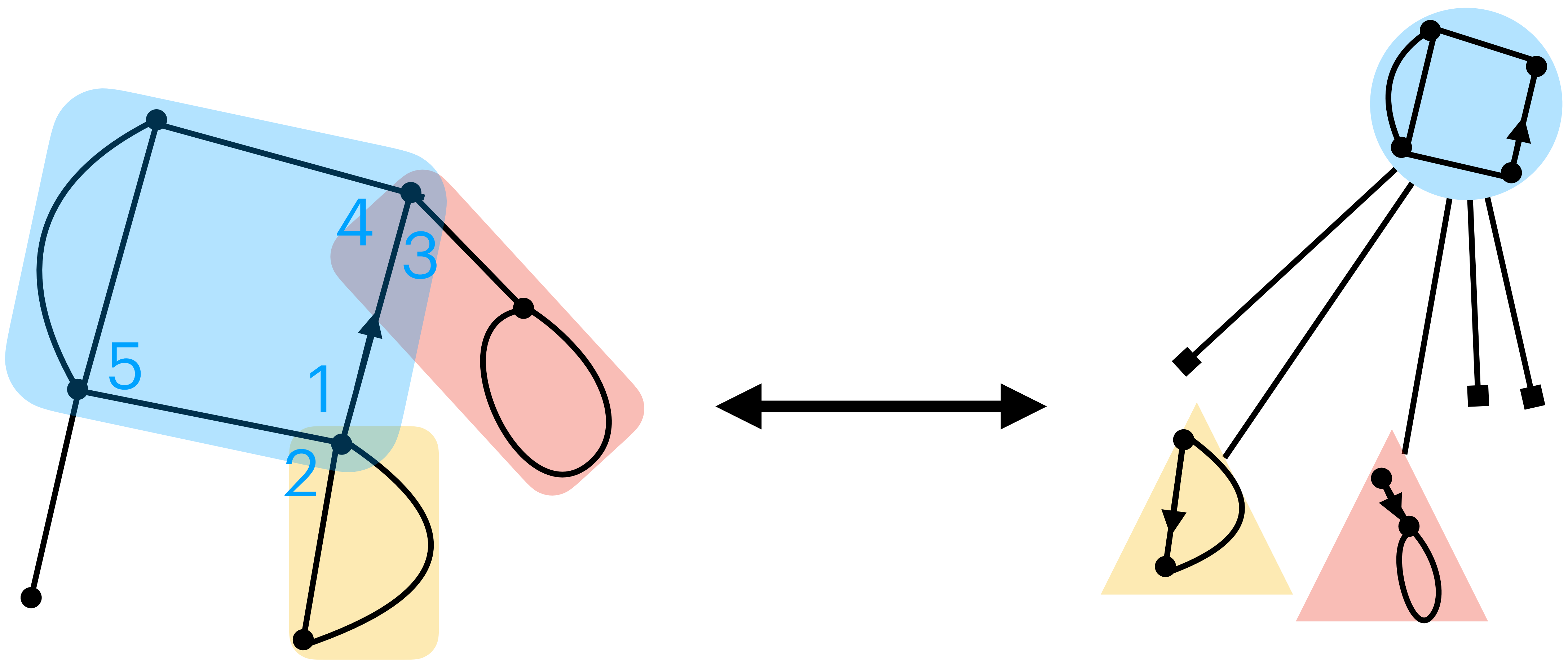

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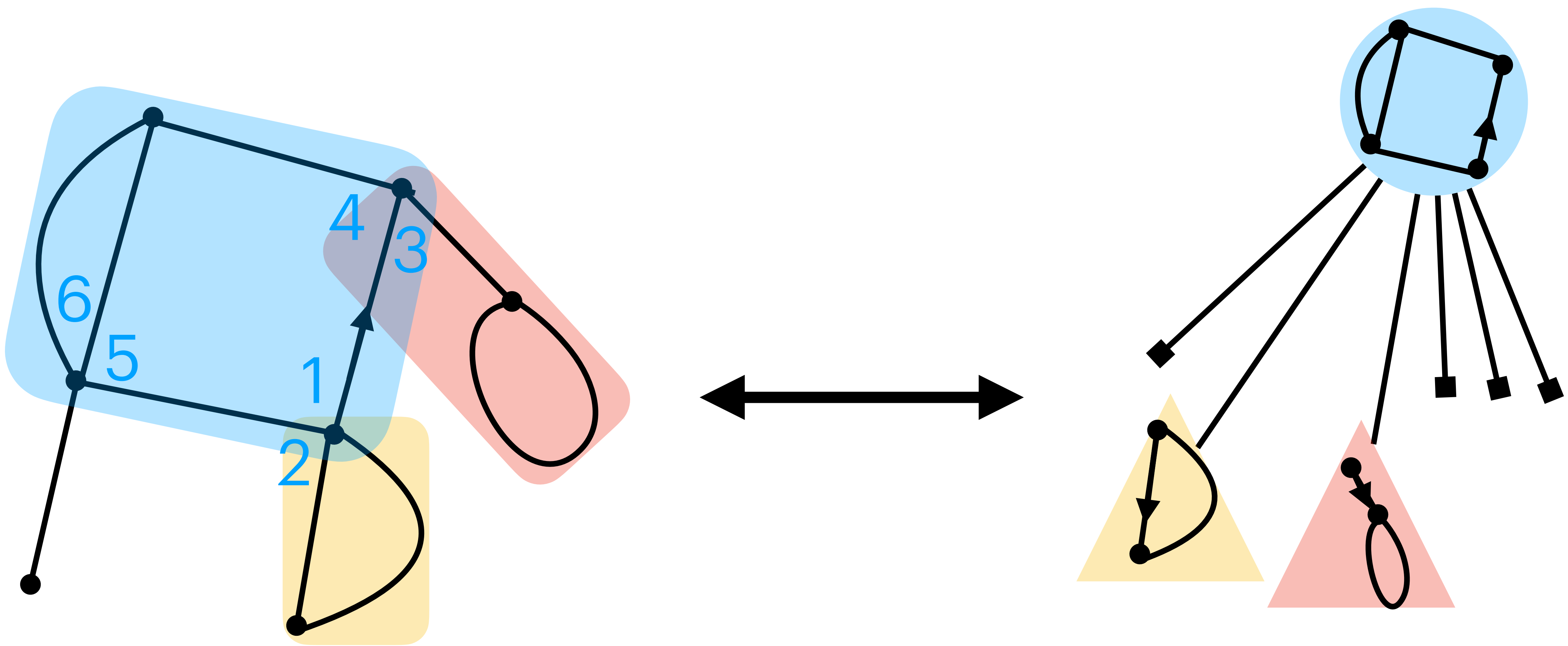

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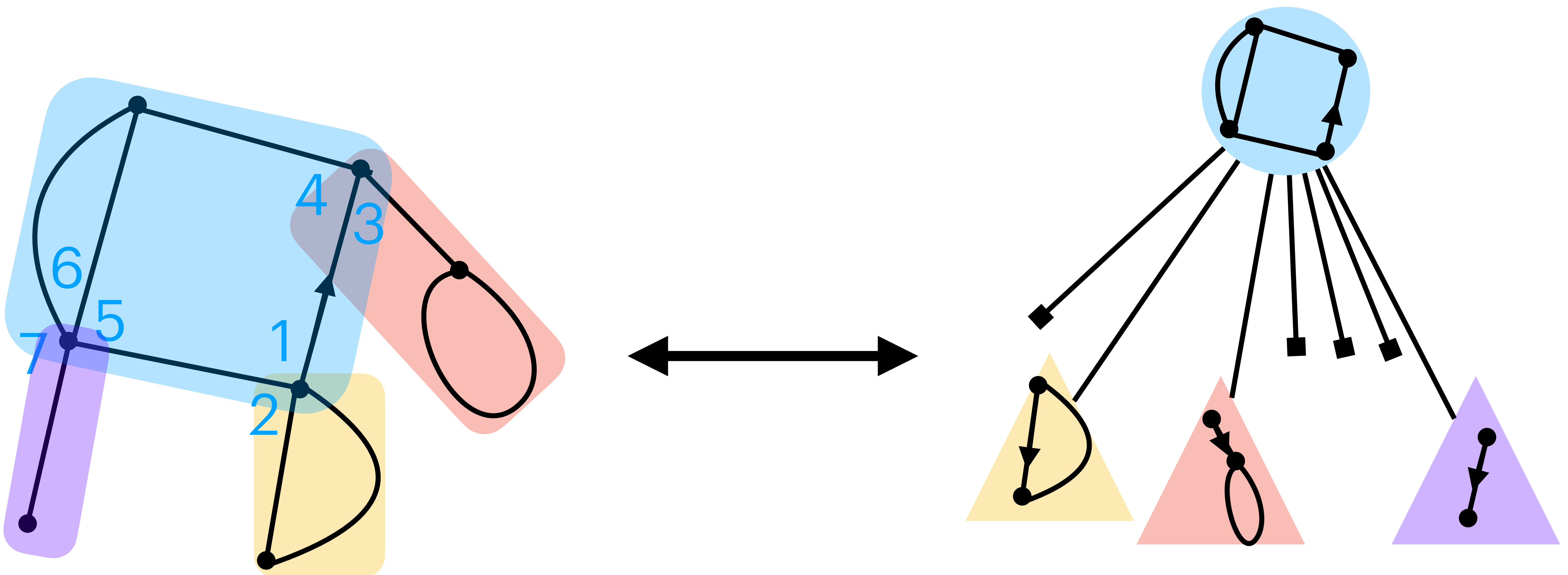

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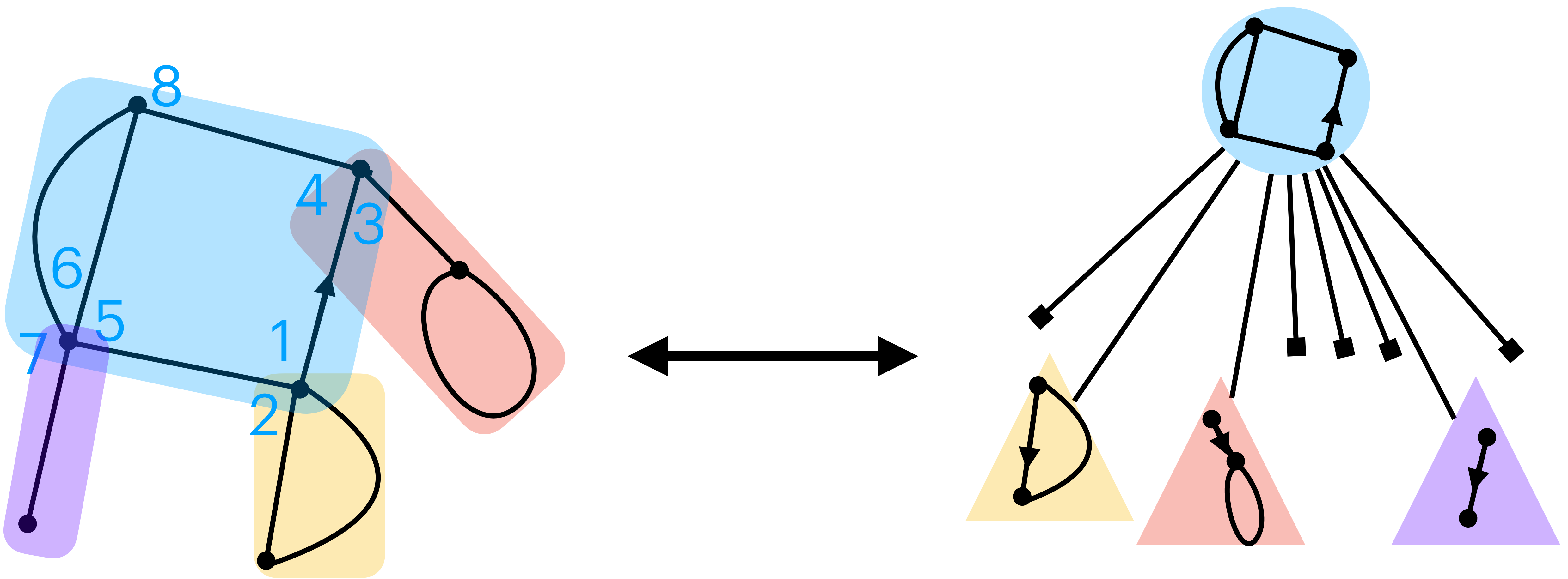

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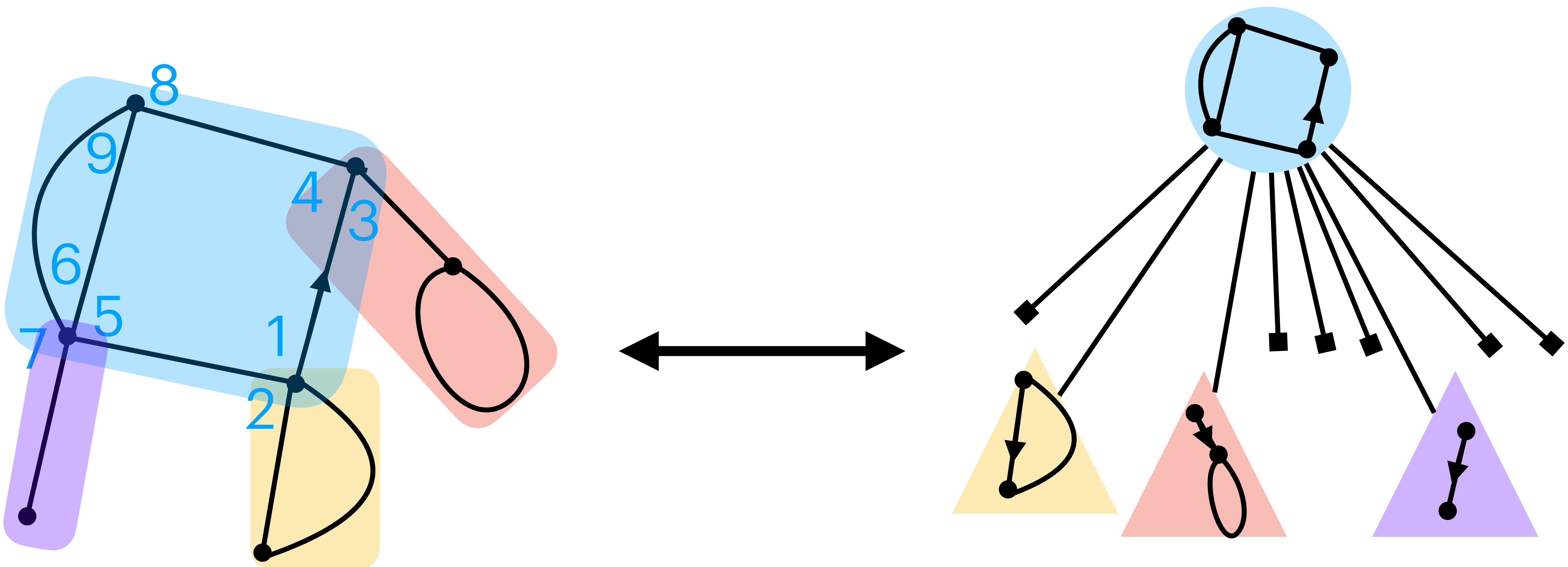

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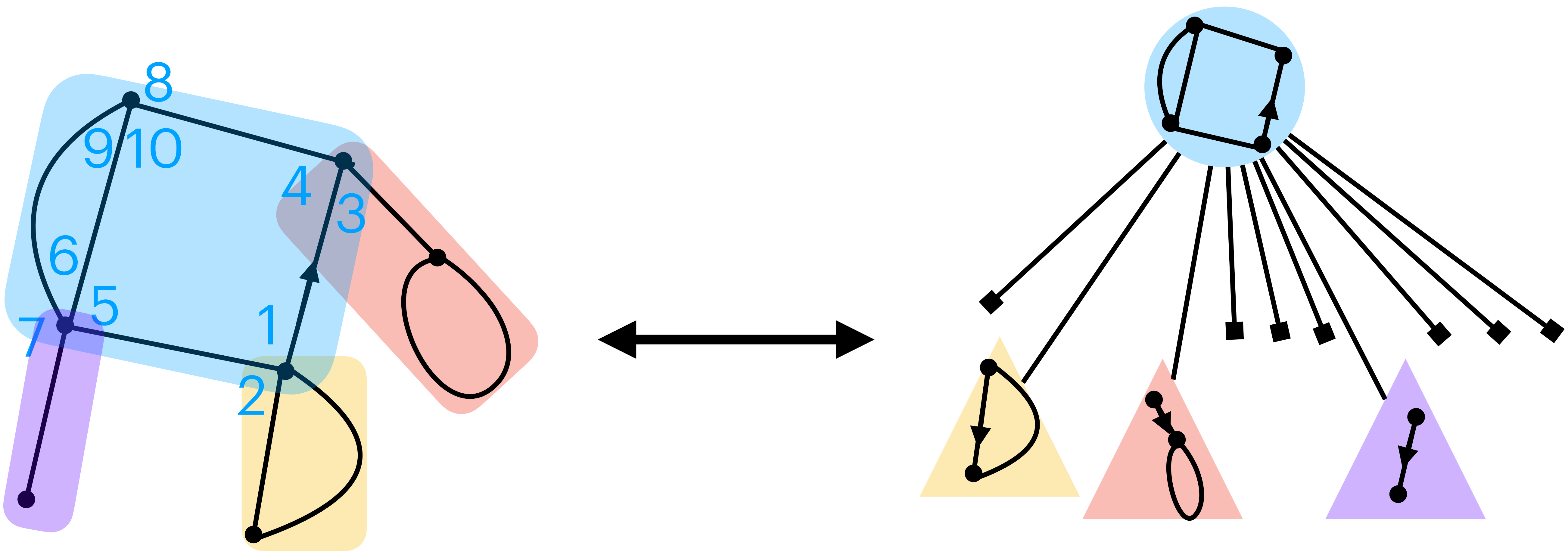

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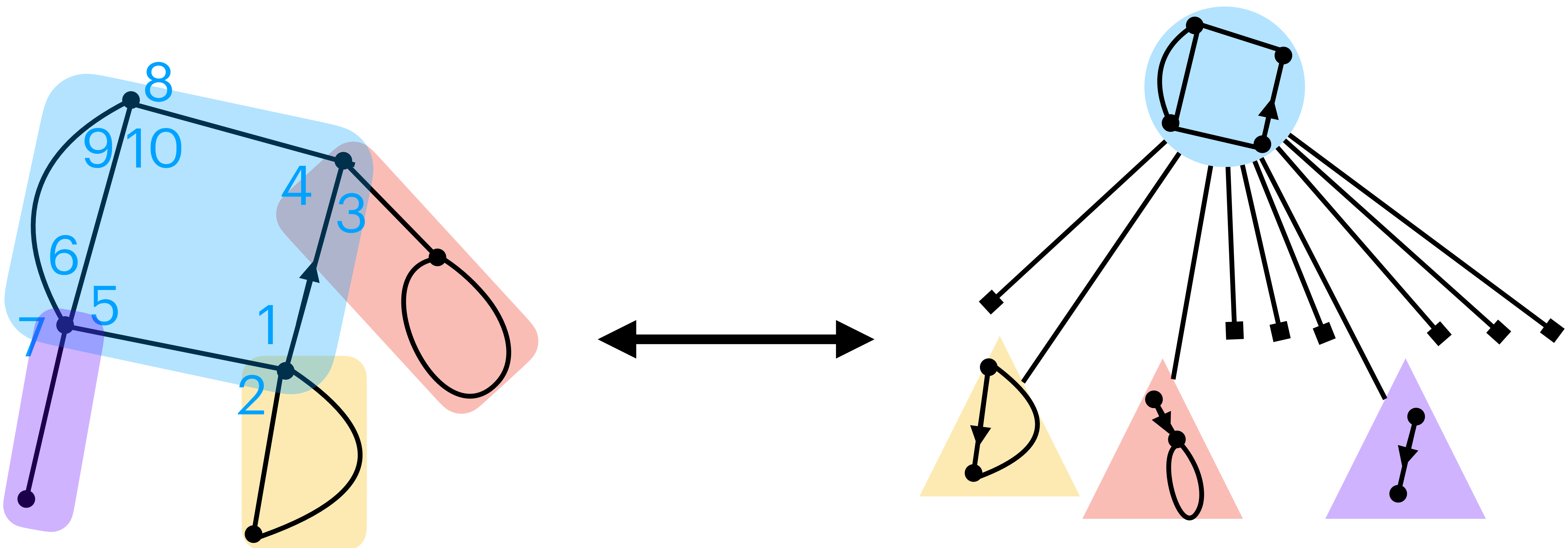


$$M(z) = B(zM^2(z))$$

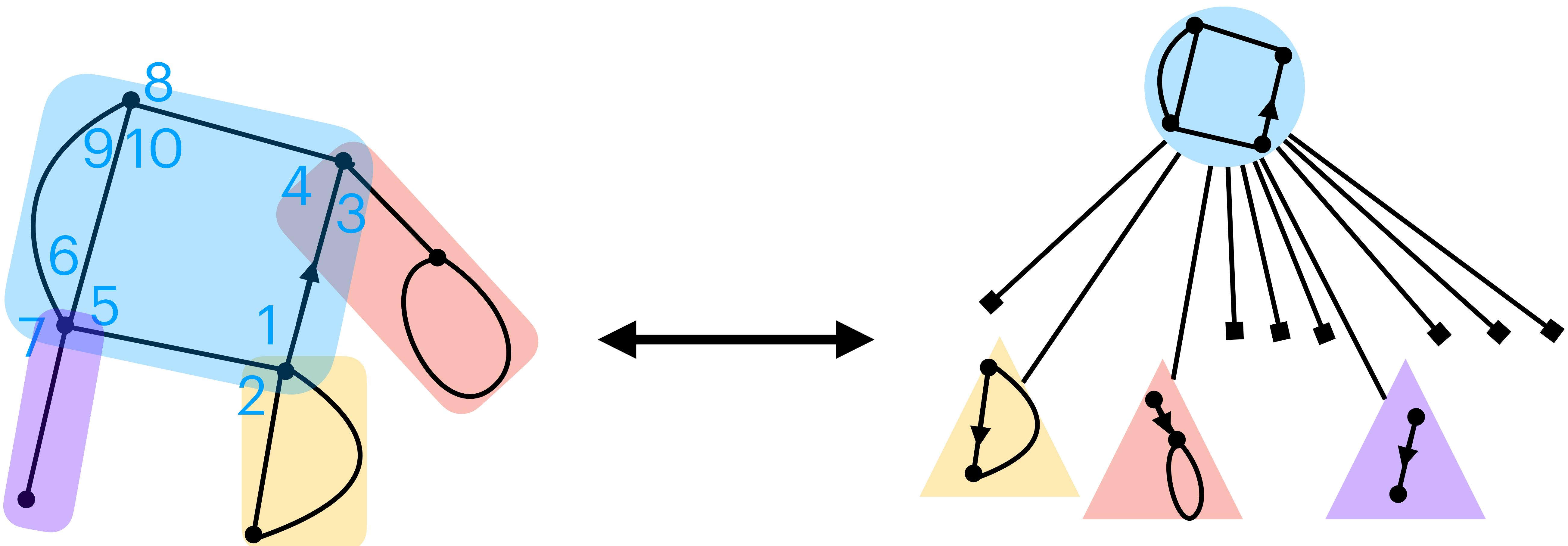
GS of maps

GS of 2-connected maps

Decomposition of a map into blocks

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$

Inspiration from [Tutte 1963]



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GS of maps

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With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

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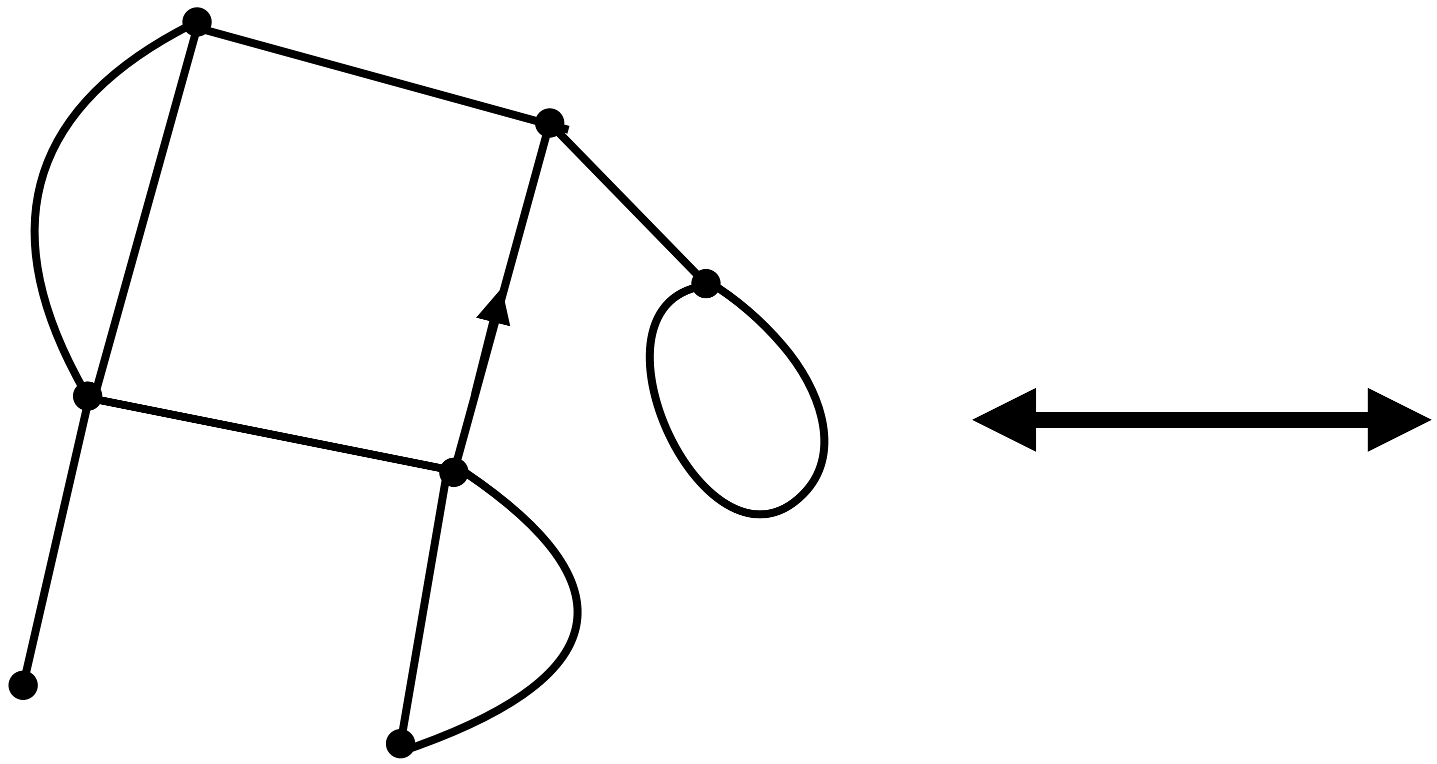

⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

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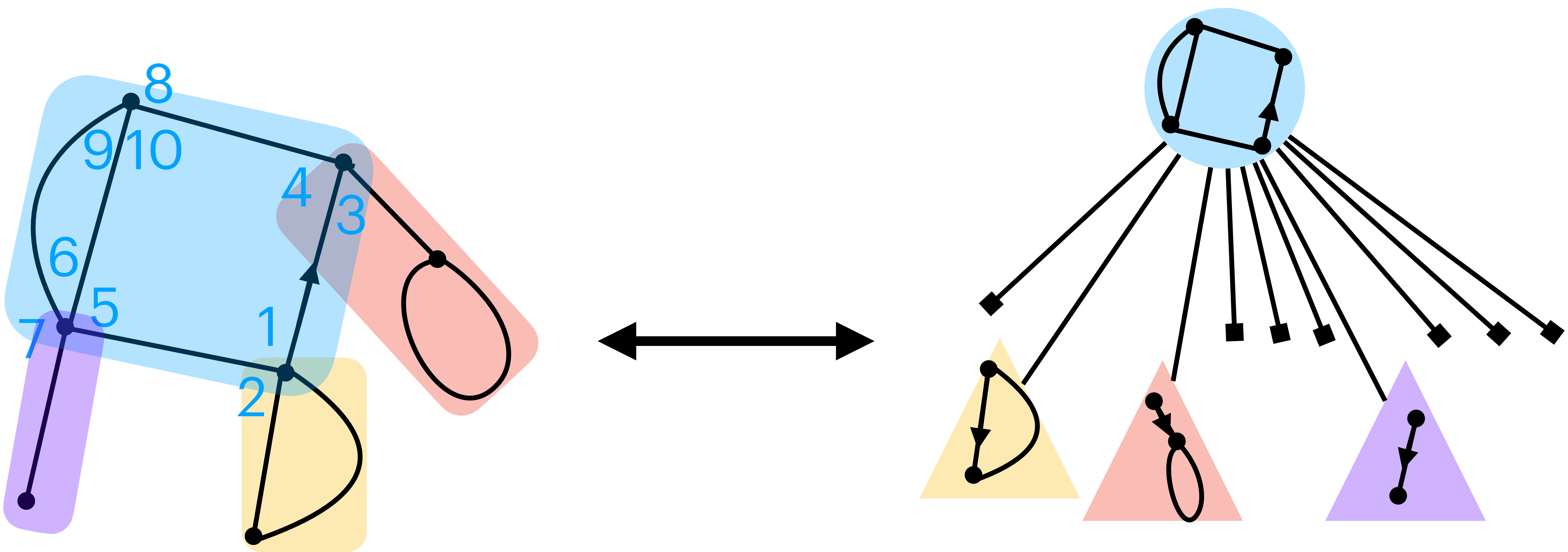

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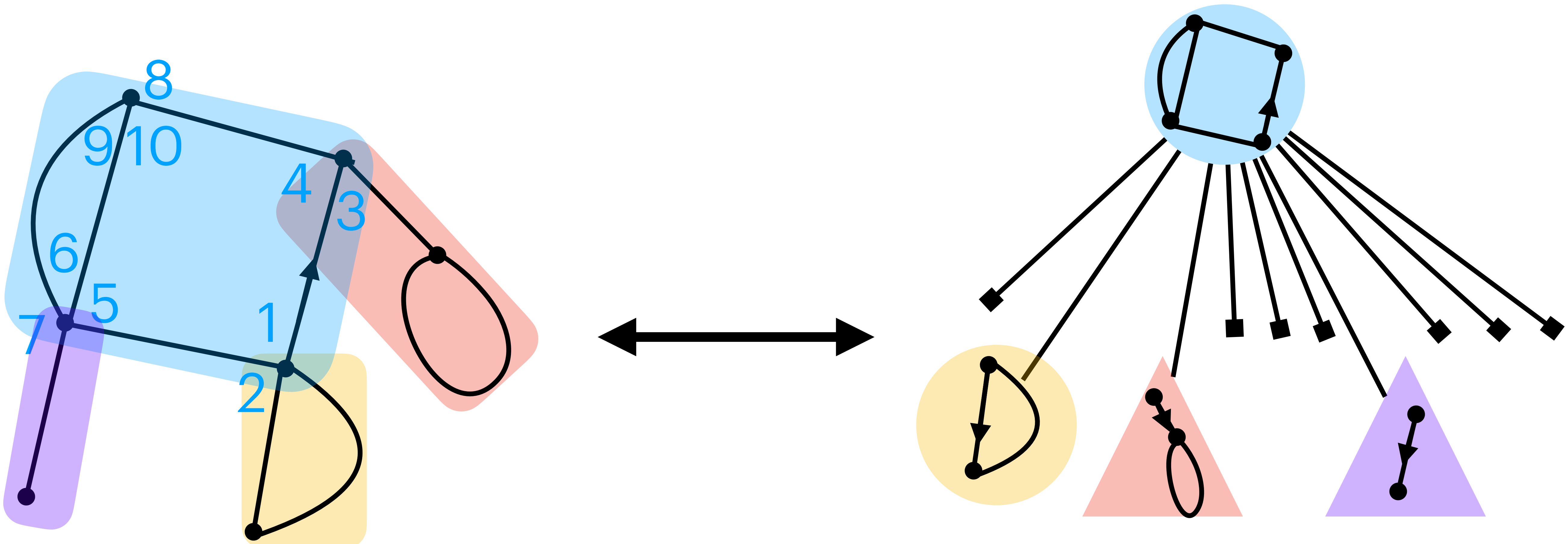

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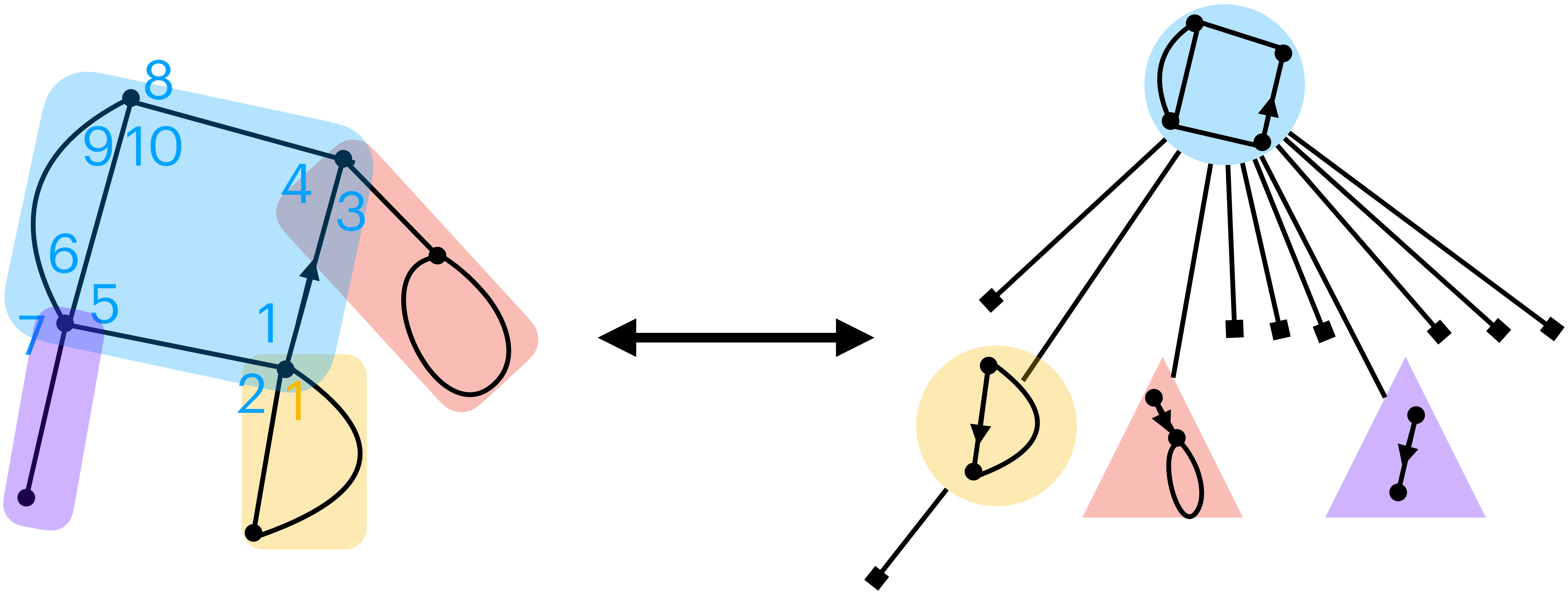

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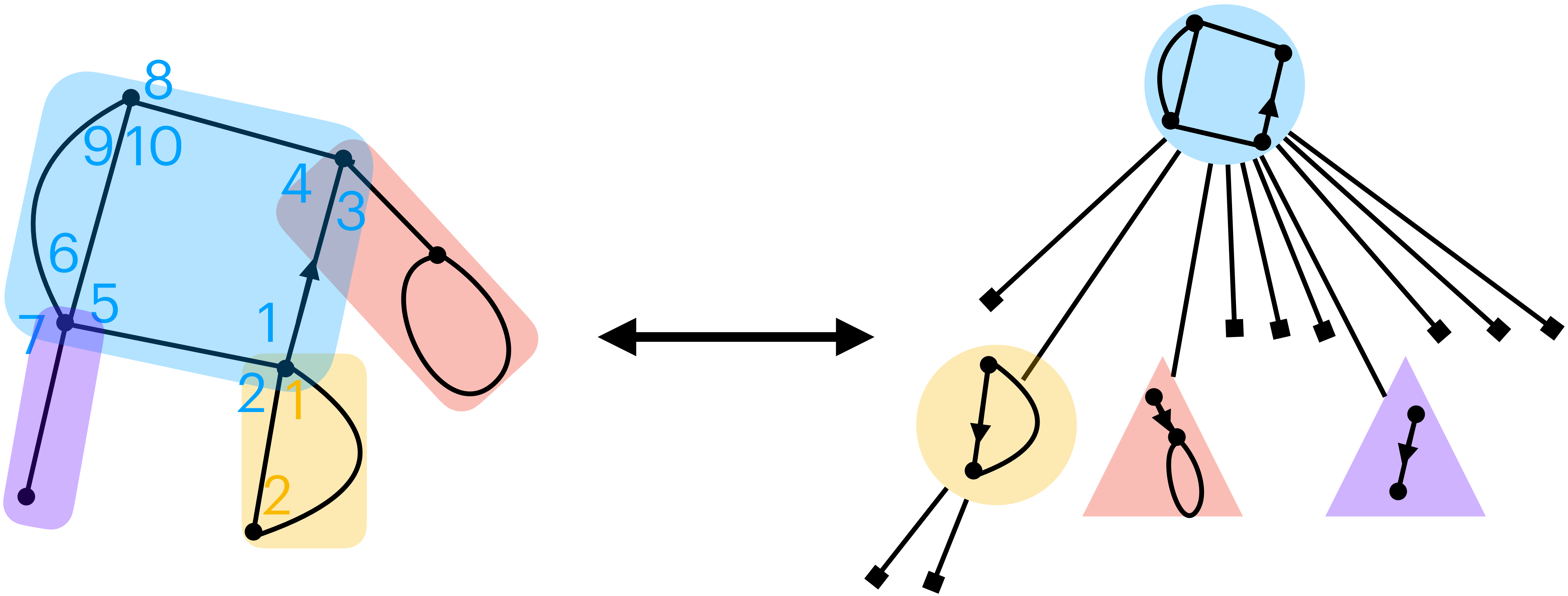

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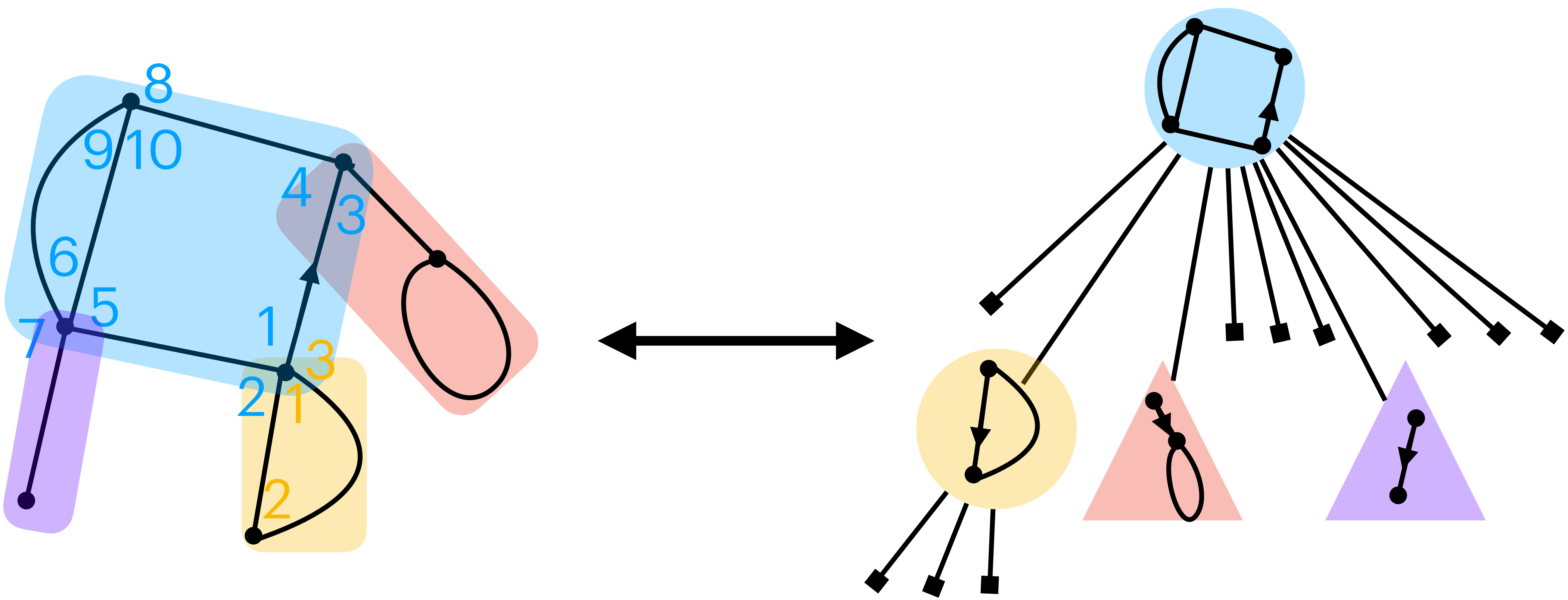

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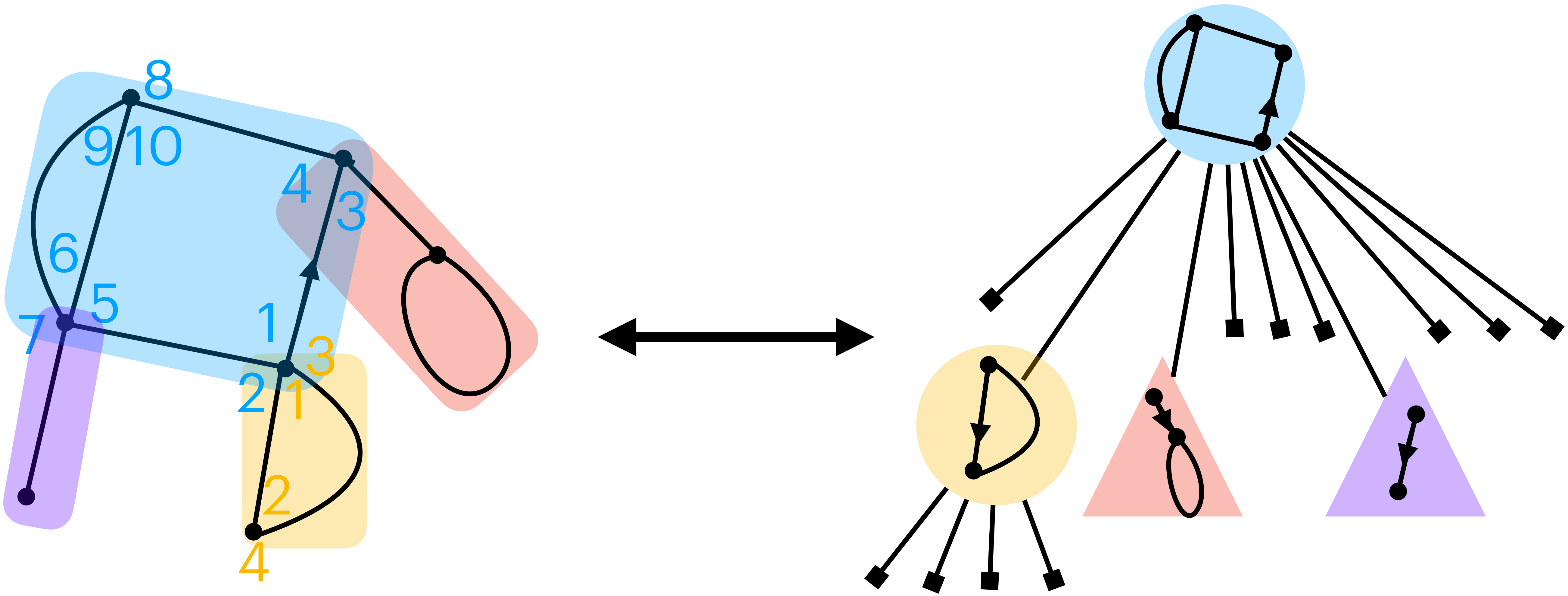

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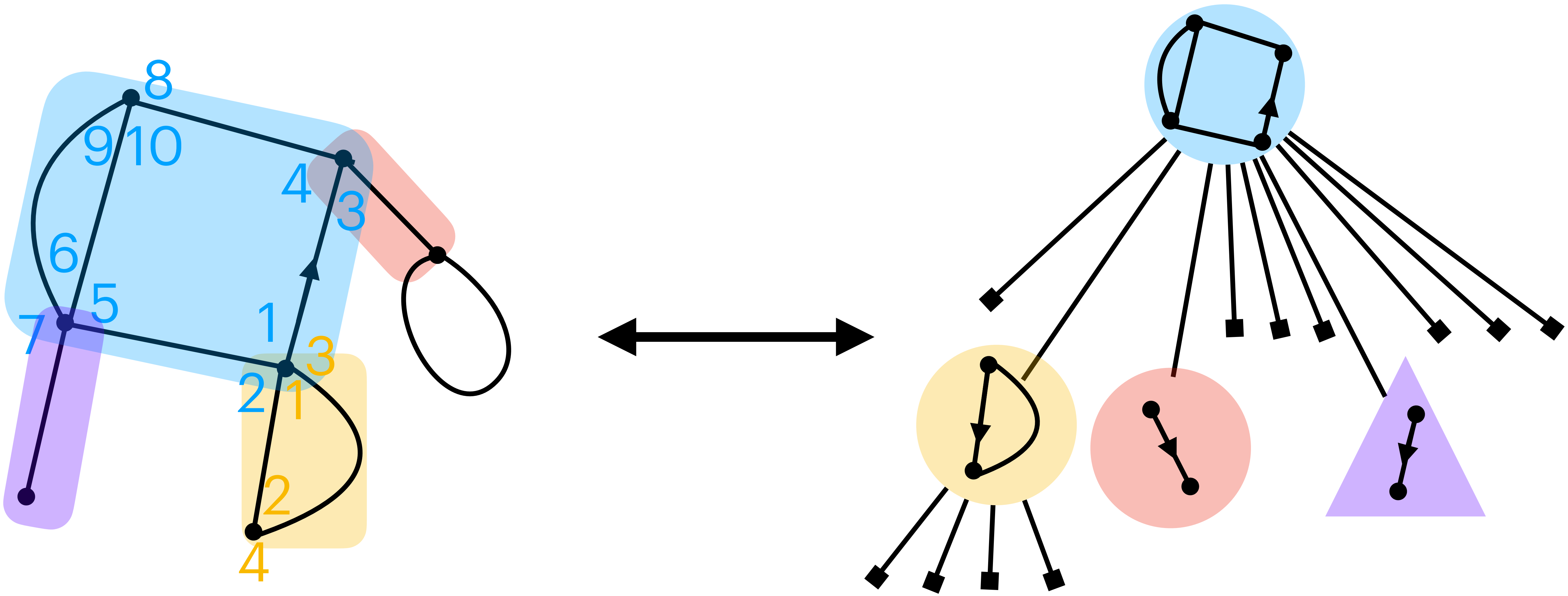

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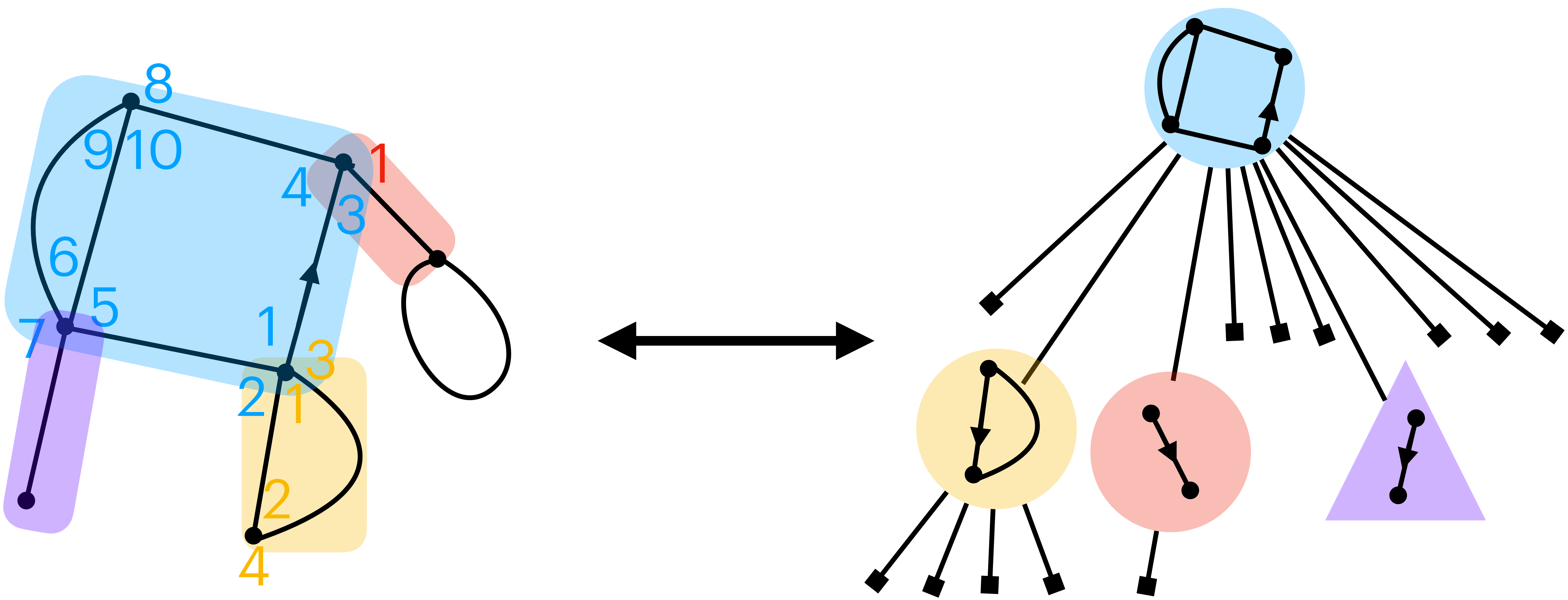

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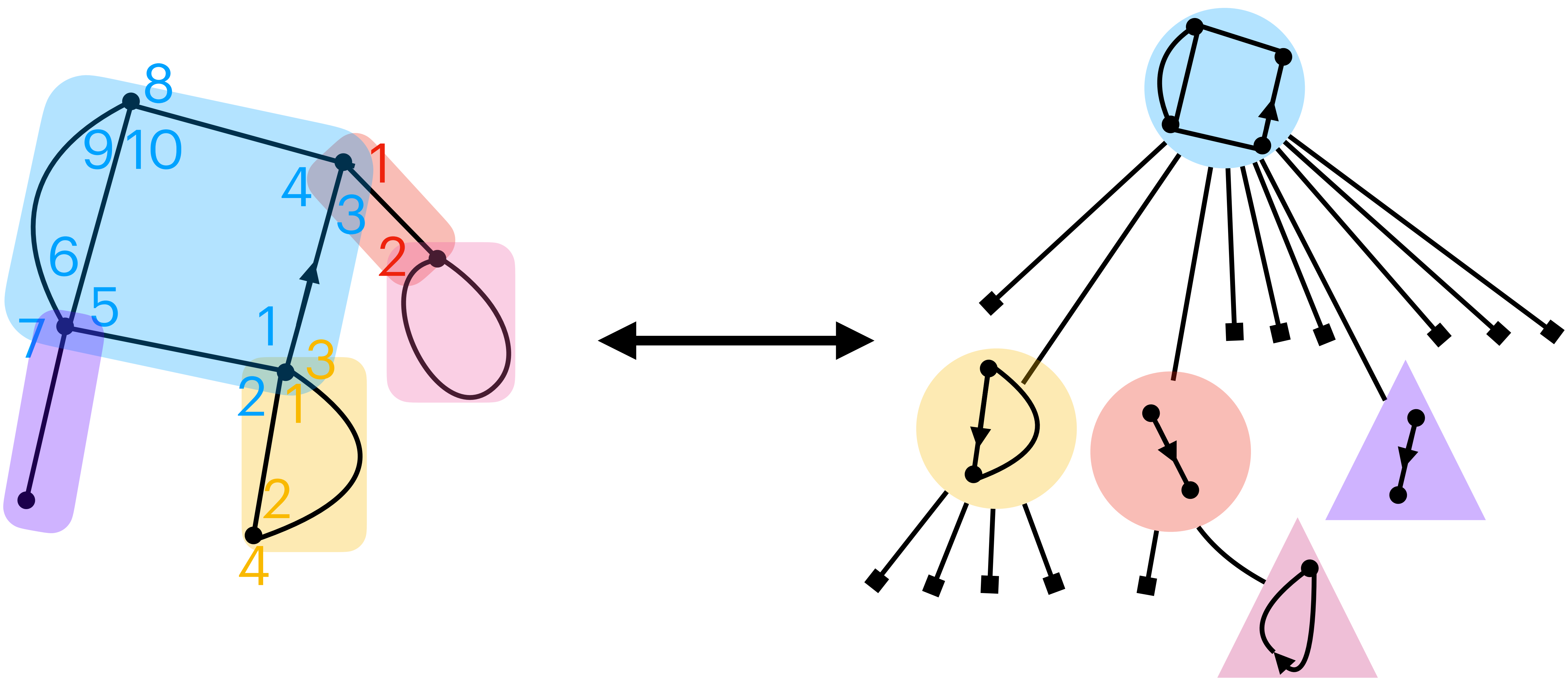

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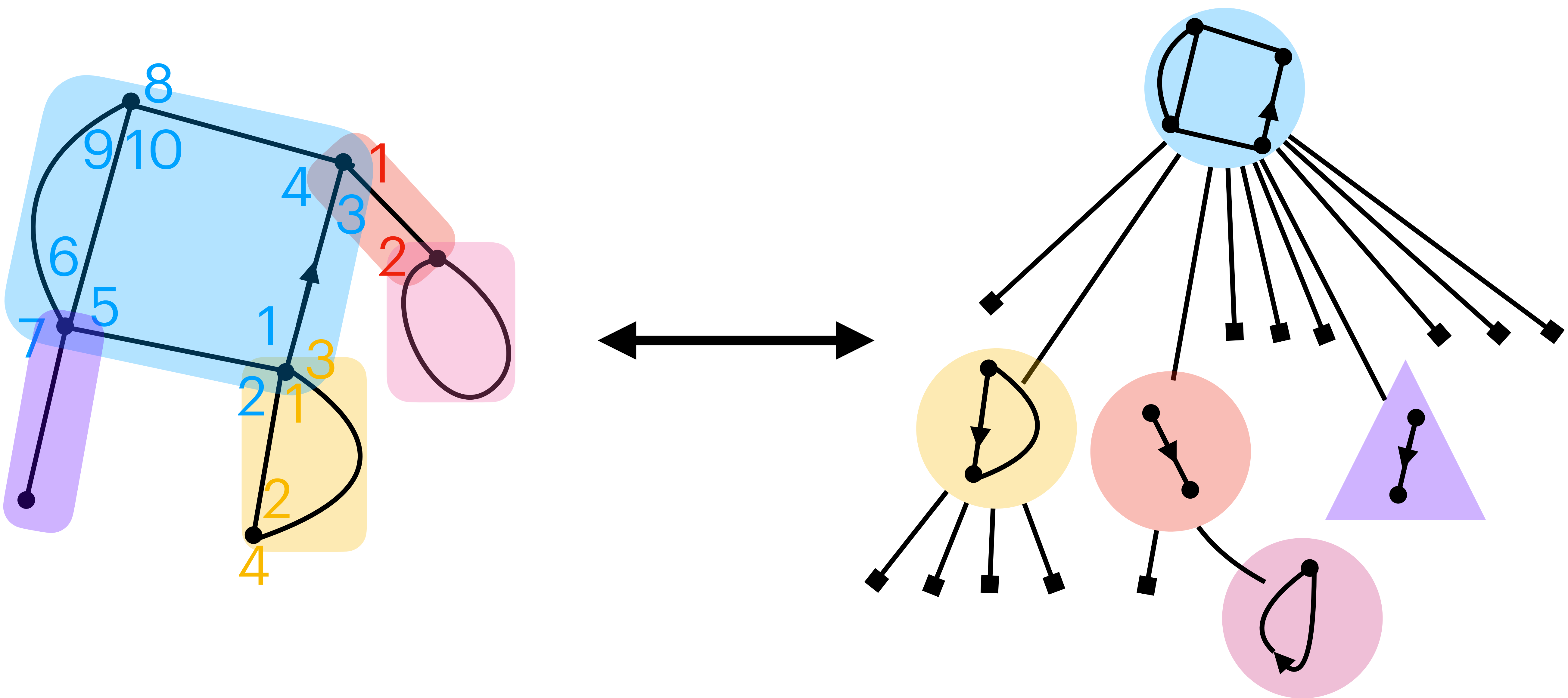

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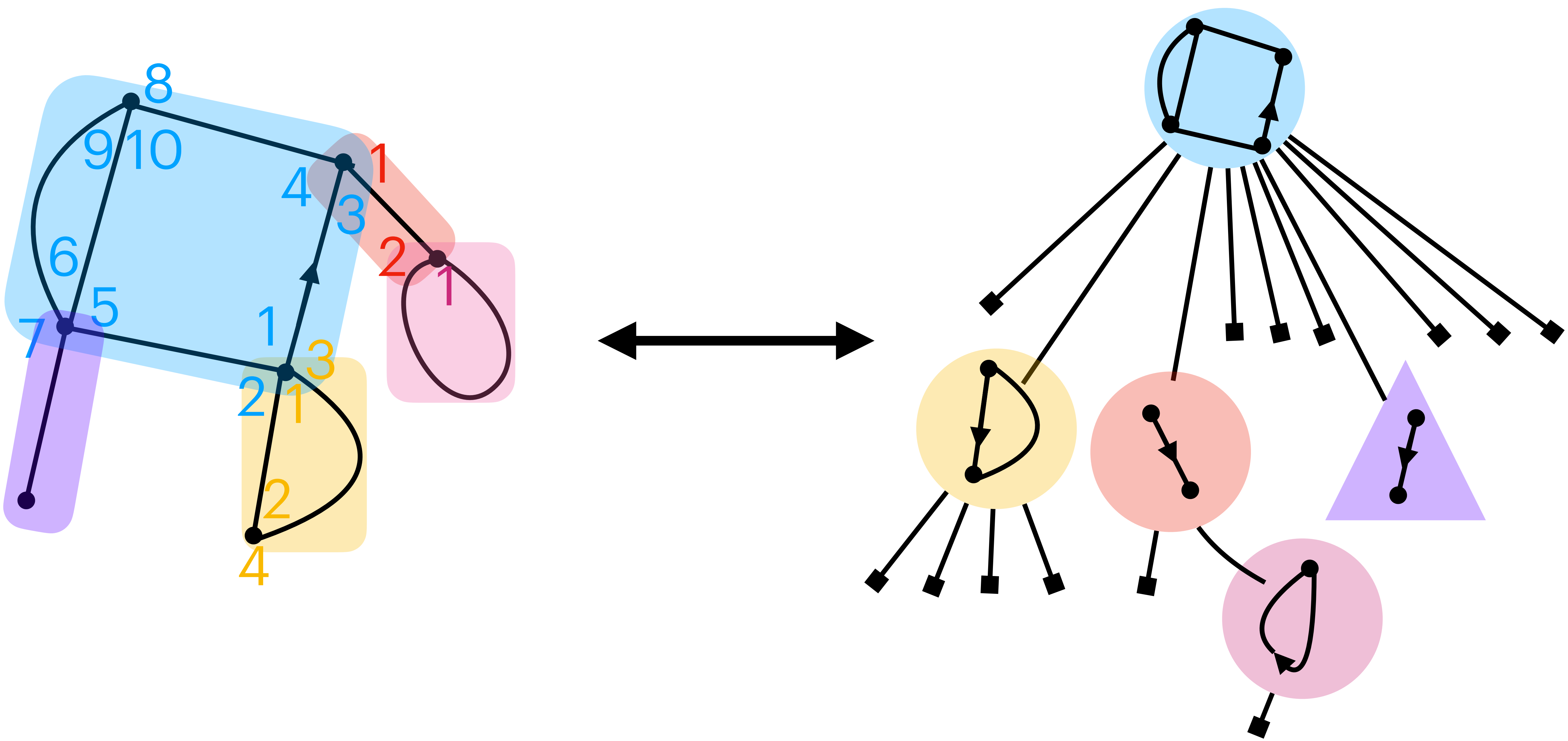

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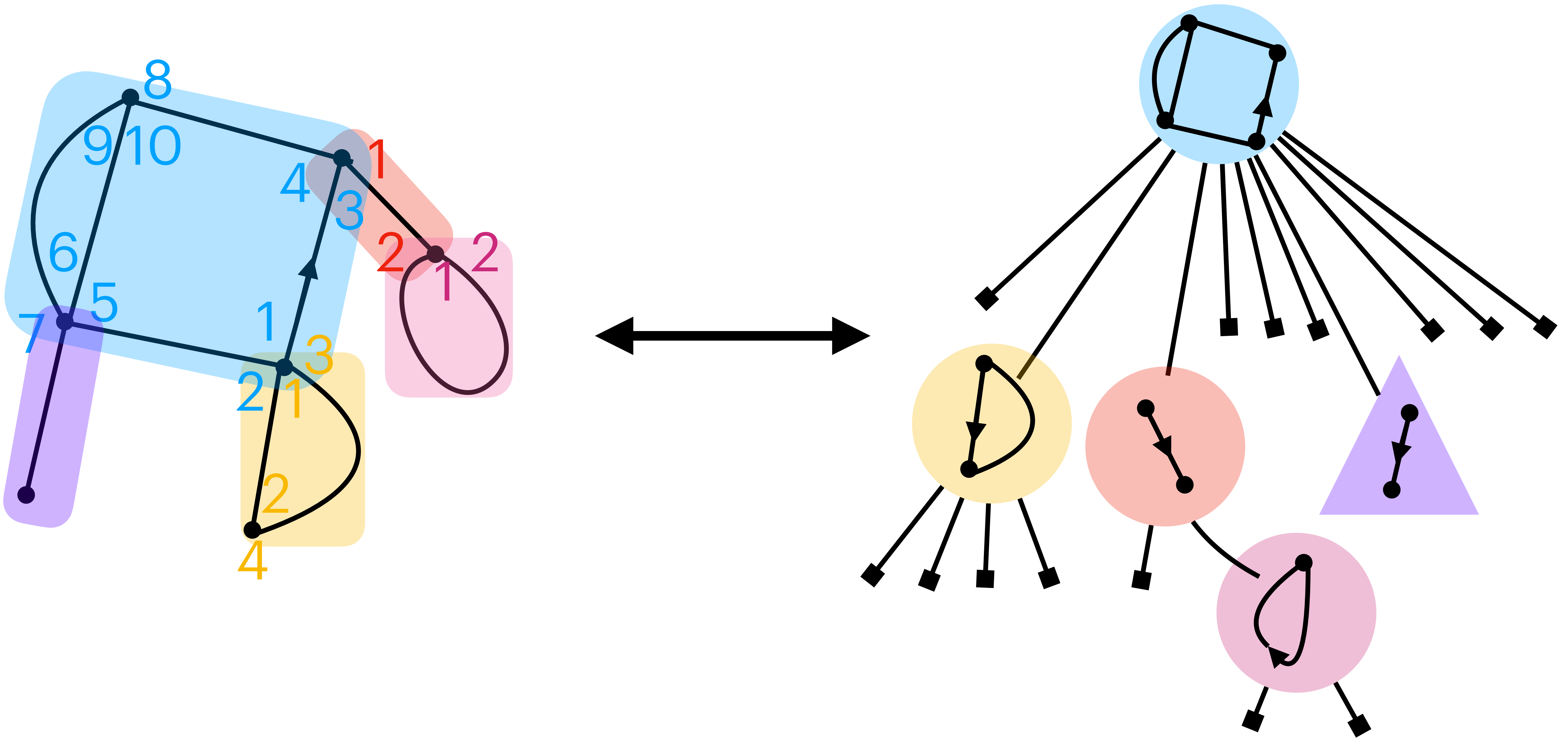

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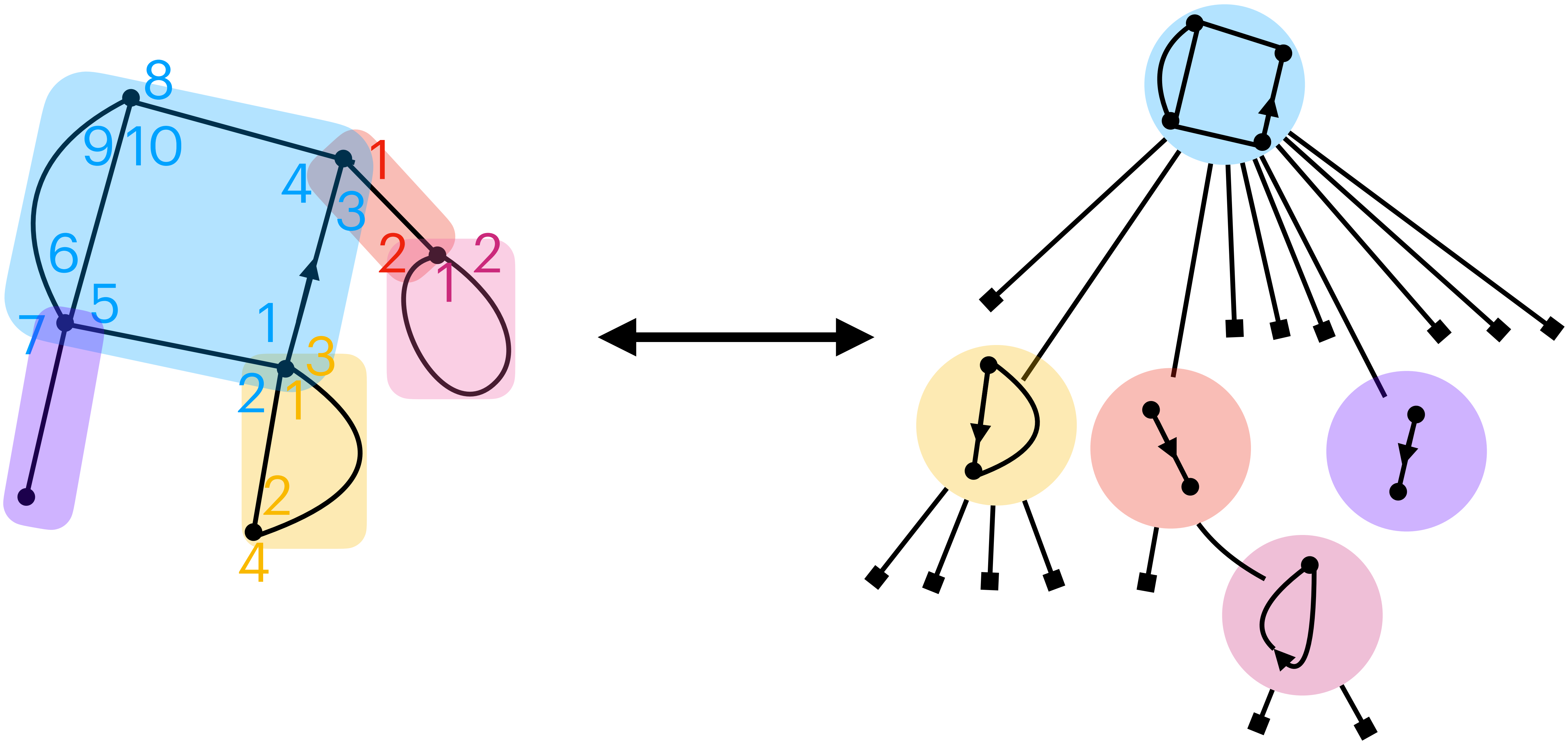

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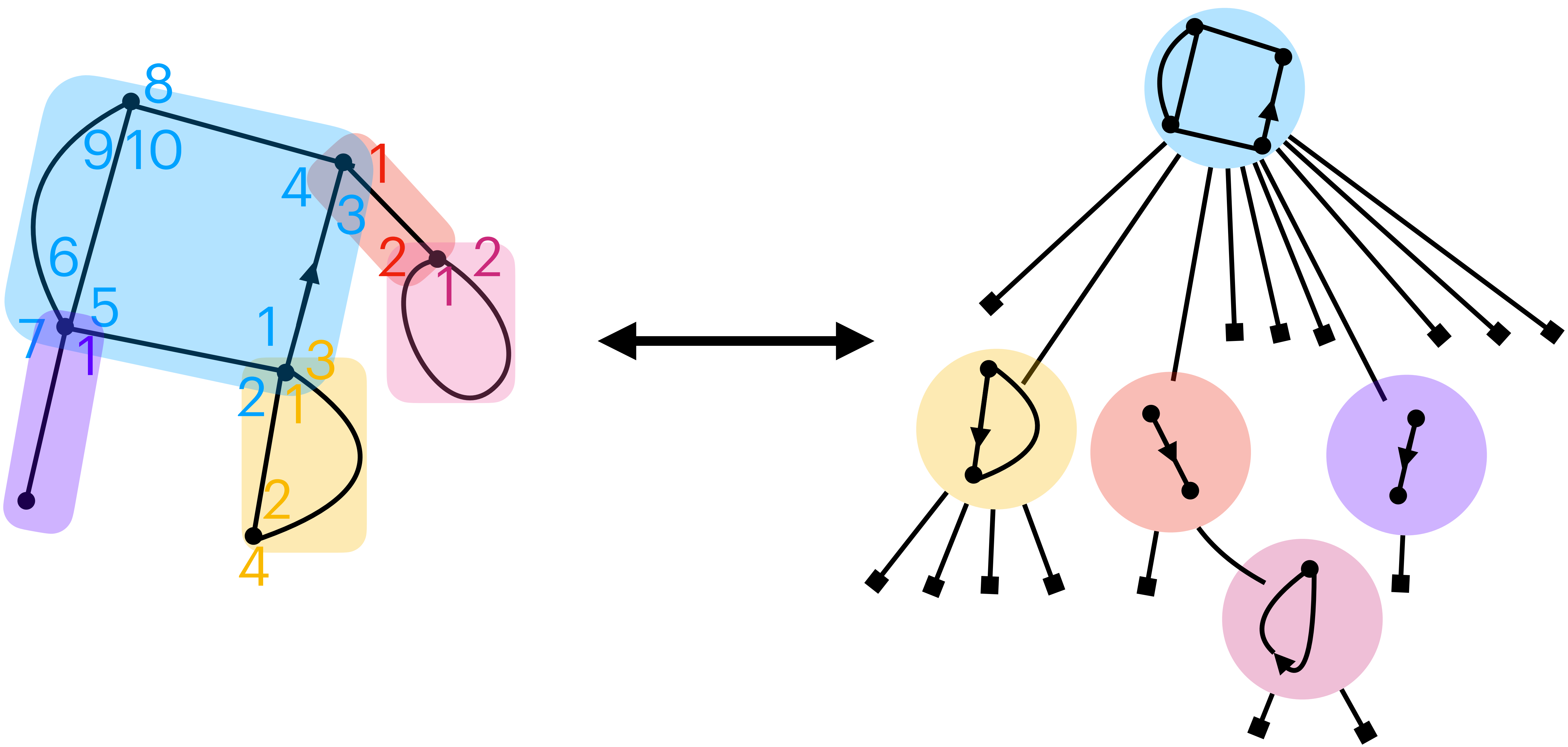

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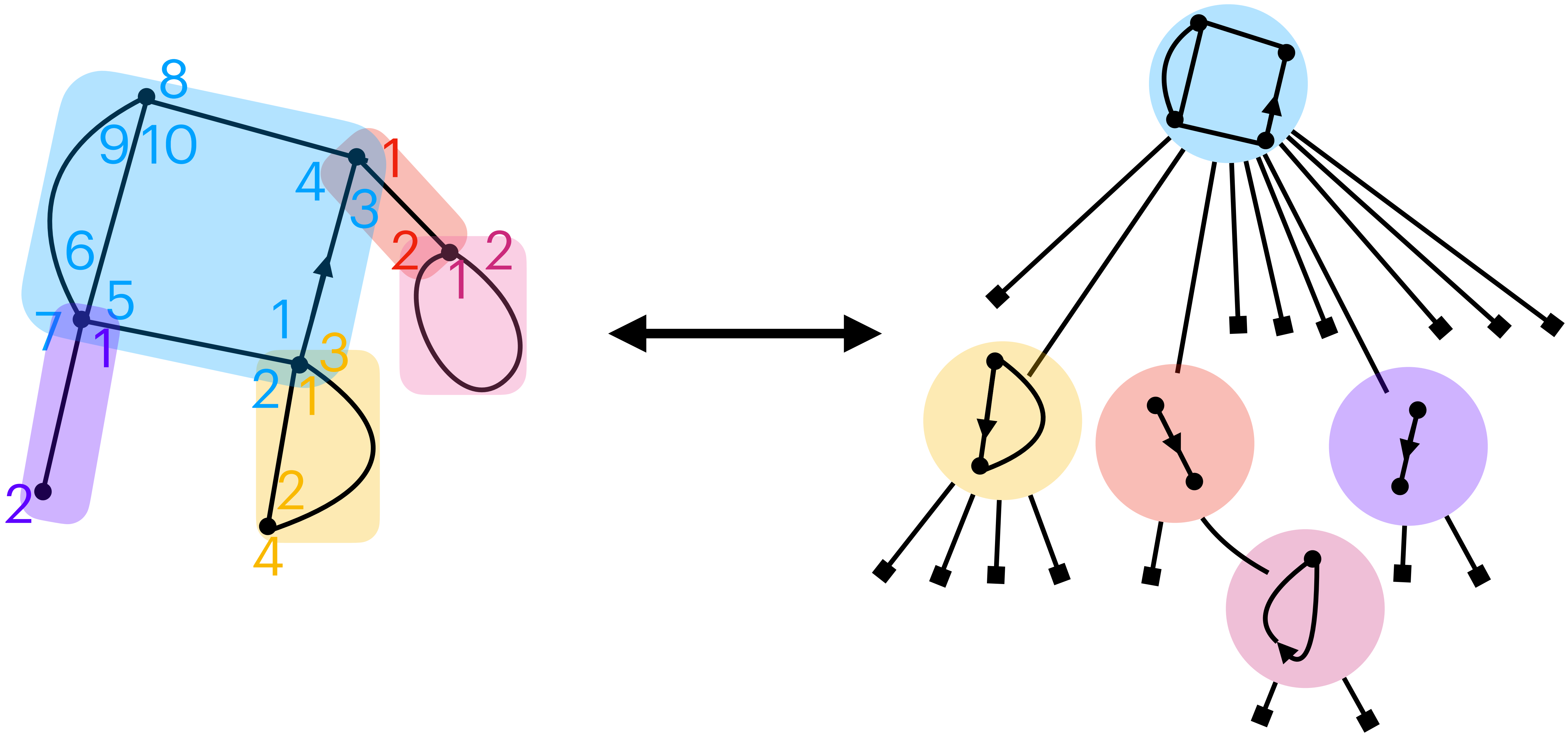

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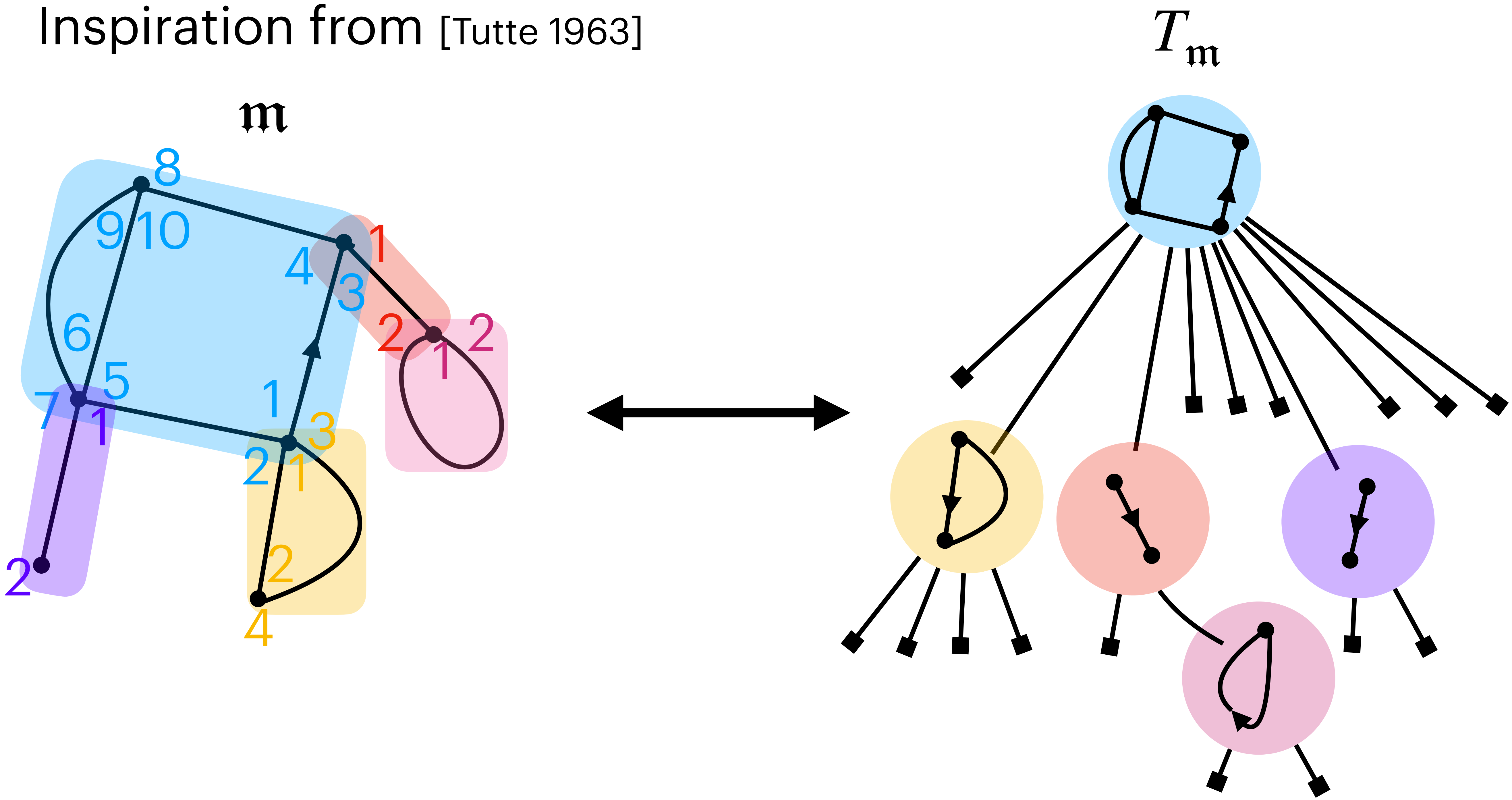

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Galton-Watson trees for map blocks

μ-Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

Galton-Watson trees for map blocks

μ-Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

<u>Theorem</u> [Fleurat, S. 24] $u > 0$

If $M_n \hookrightarrow \mathbb{P}_{n,u'}$, then T_{M_n} has the law of a Galton-Watson tree of explicit reproduction law μ^u conditioned to be of size $2n$.

III. Results for non tree-rooted maps

Joint work with William Fleurat

Phase transition

<u>Theorem</u> [Fleurat, S. 24] Model exhibits a phase transition at $u = 9/5$. When $n \to \infty$:

- Subcritical phase $u < 9/5$: "general map phase" one huge block;
- Critical phase $u = 9/5$: a few large blocks;
- Supercritical phase $u > 9/5$: "tree phase" only small blocks.

We obtain explicit results on enumeration, size of blocks and scaling limits in each case.

→ *A phase transition in block-weighted random maps*
W. Fleurat & Z. S., Electronic Journal of Probability, 2024

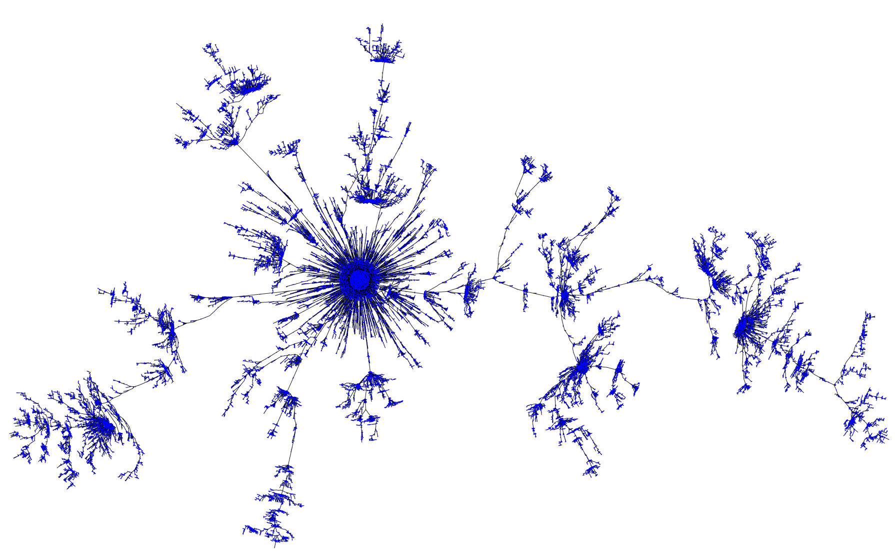
Scaling limits

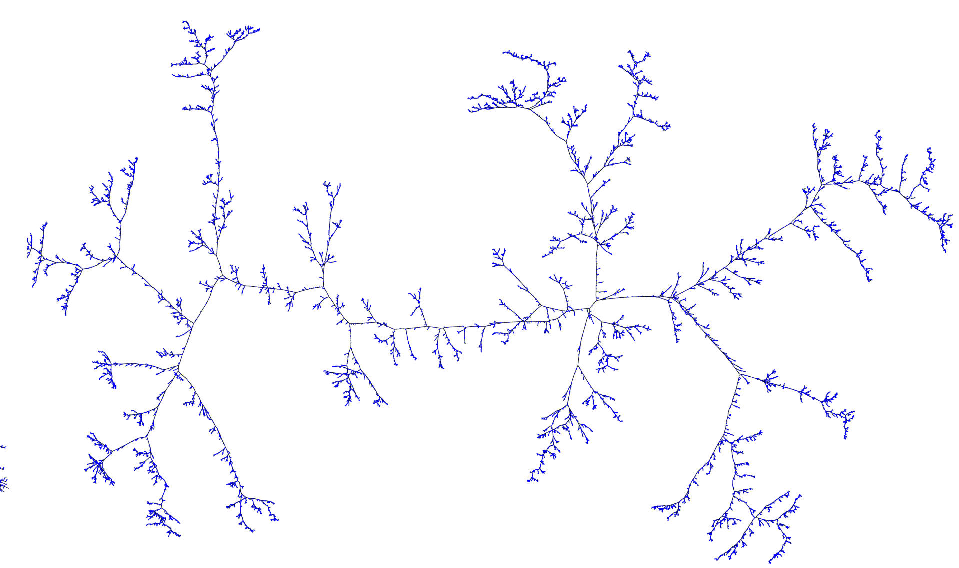
Theorem [Fleurat, S. 24] Scaling limits:

- Subcritical phase $u < 9/5$: $\frac{C_1(u)}{n^{1/4}} M_n \to \mathscr{S}_e$; (assuming the convergence of 2-connected maps towards the Brownian sphere)
- Critical phase $u = 9/5$: $\frac{C_2}{n^{1/3}} M_n \to \mathscr{T}_{3/2}$;
- Supercritical phase $u > 9/5$: $\frac{C_3(u)}{n^{1/2}} M_n \to \mathscr{T}_e$ [Stufler 2020].

Brownian Sphere $\mathscr{S}_e$

Stable Tree $\mathscr{T}_{3/2}$





Brownian Tree $\mathscr{T}_e$

Proof for the supercritical and critical cases

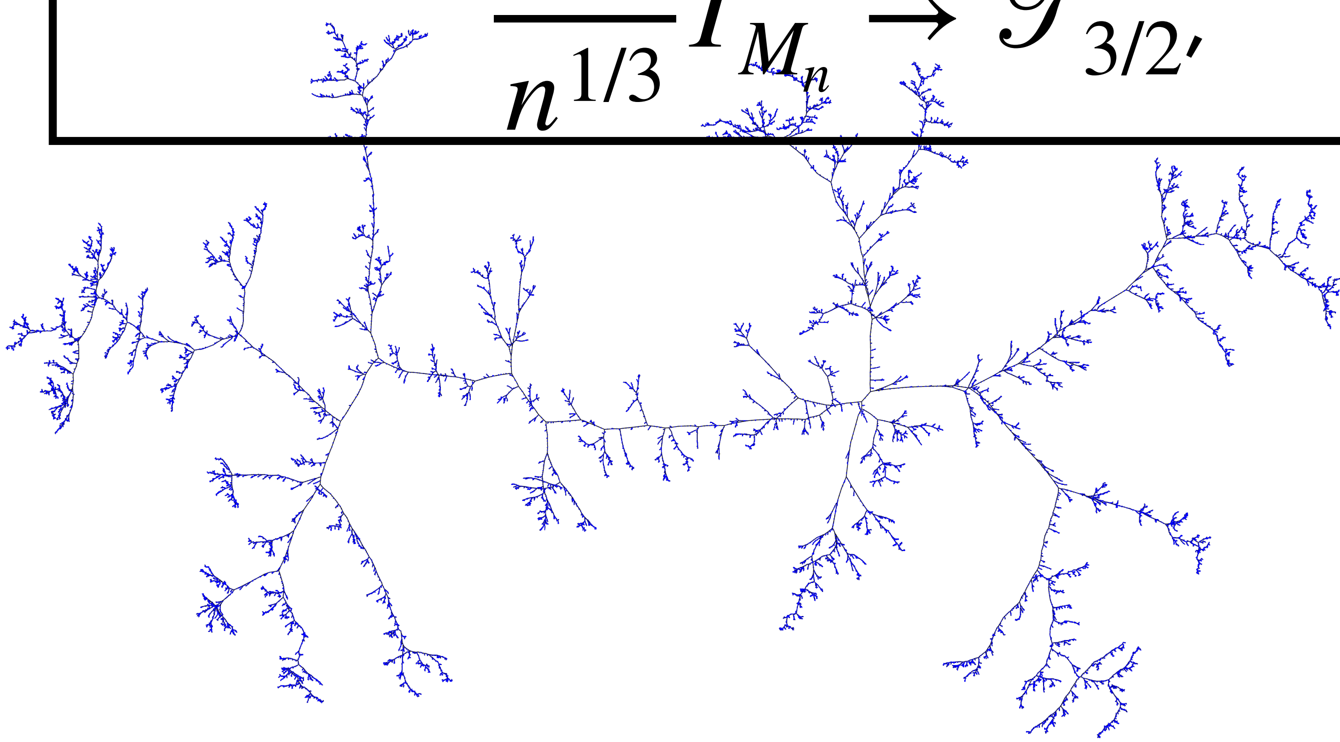
Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

- [Stufler 2020] If $u > 9/5$,

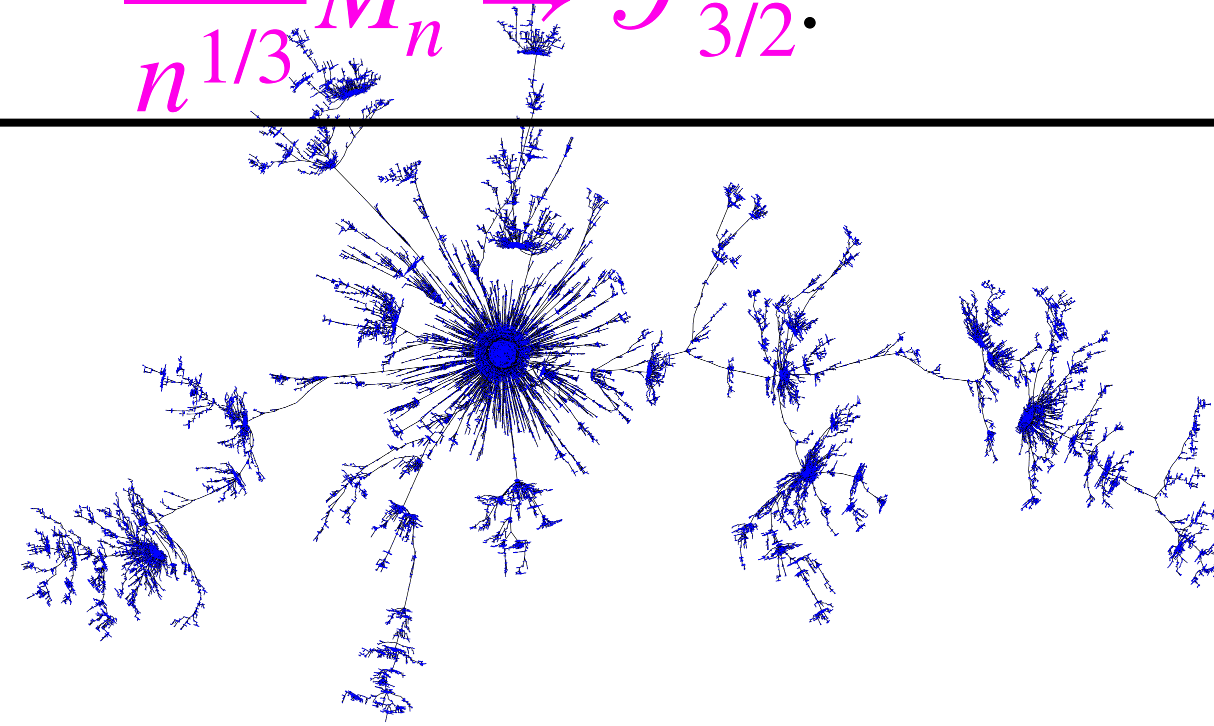
$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e, \qquad \frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e.$$

- [Fleurat, S. 24] If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}, \qquad \frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}.$$



Brownian Tree $\mathcal{T}_e$



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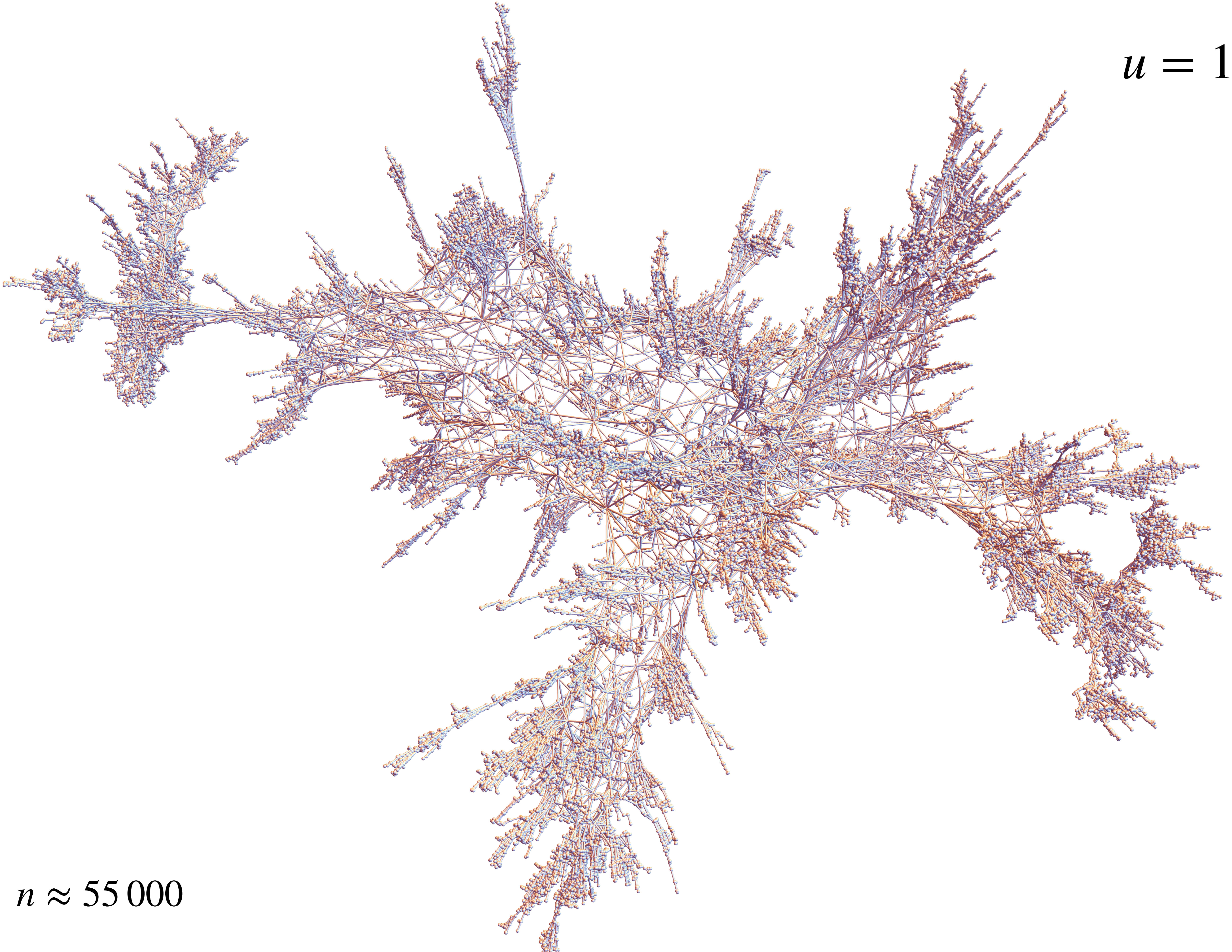
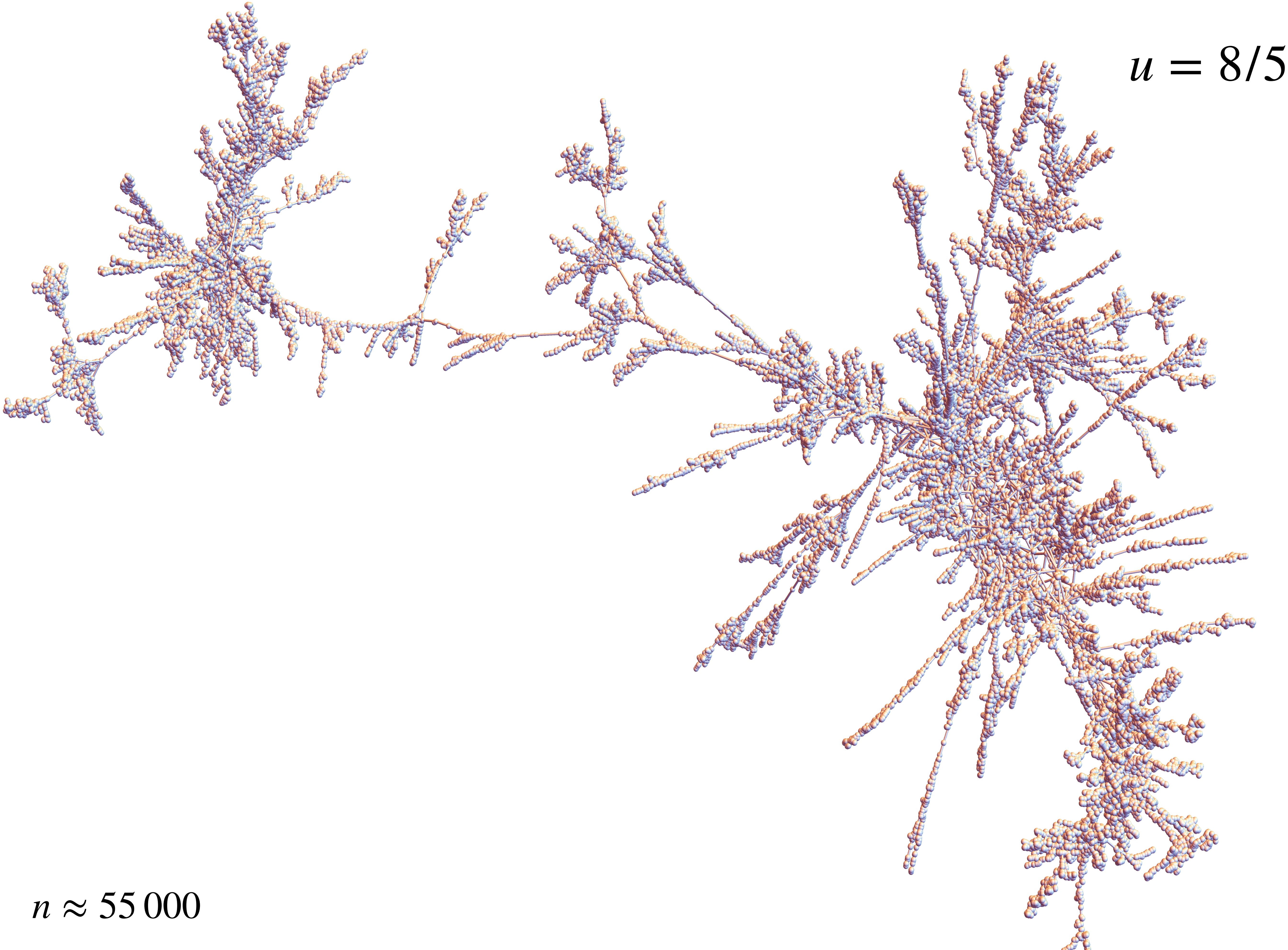
- [Fleurat, S. 24] If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}, \qquad \frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}.$$

Proof

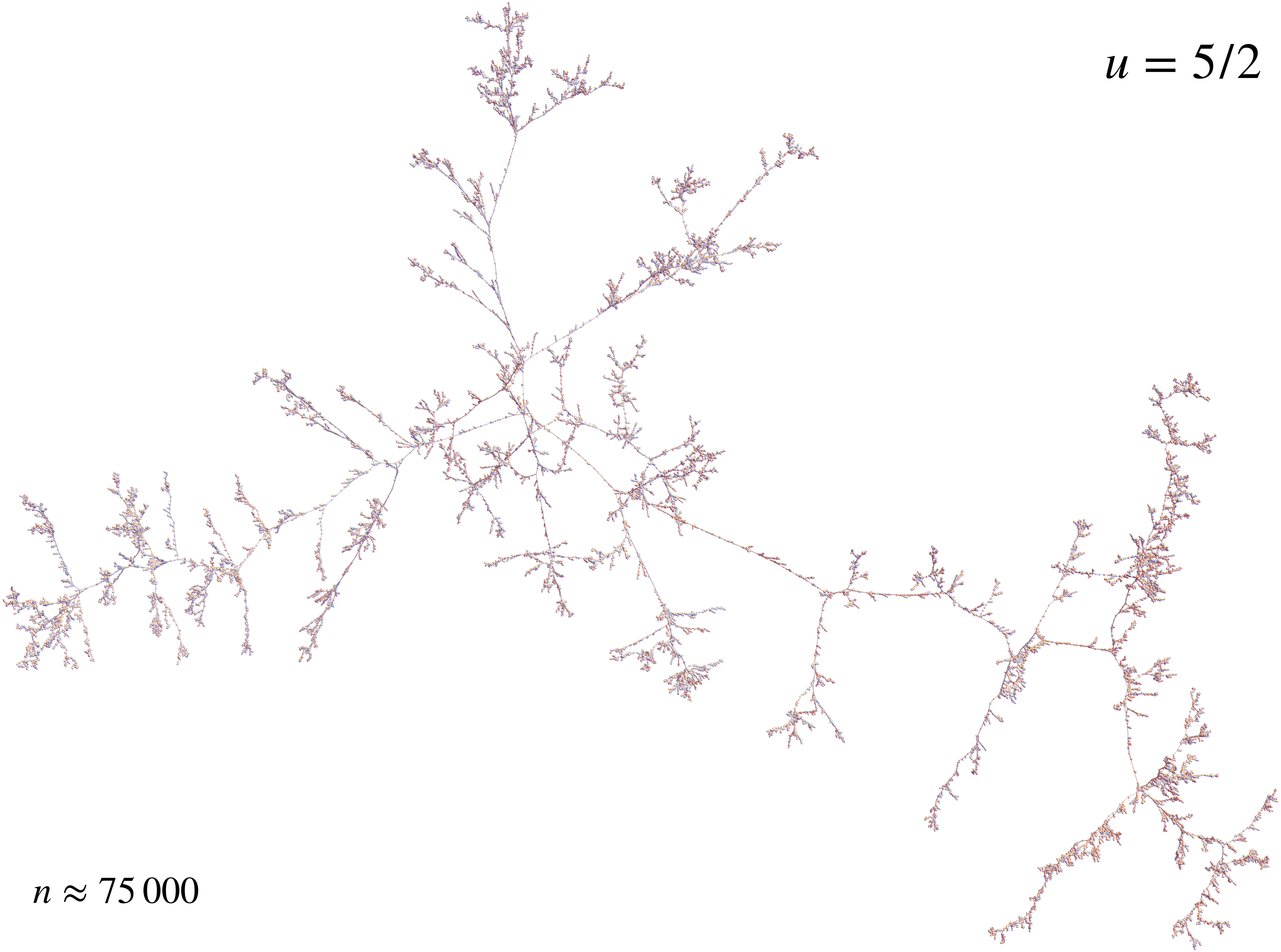
- Known scaling limits of critical Galton-Watson trees
  - with finite variance [Aldous 1993, Le Gall 2006];
  - infinite variance and polynomial tails [Duquesne 2003].
- Distances in M_n behave like distances in T_{M_n}!


u = 1
n ≈ 55 000



$u = 8/5$
$n \approx 55\,000$


$u = 9/5$
$n \approx 80\,000$


$u = 5/2$
$n \approx 75\,000$


$u = 5$
$n \approx 50\,000$

How can we do the same for tree-rooted maps?

IV. Tree-rooted maps

Joint work with Marie Albenque and Éric Fusy

Model

Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,

$\mathcal{M}_n$ = {tree-rooted maps of size n},

$\mathfrak{m} \in \mathcal{M}_n$,

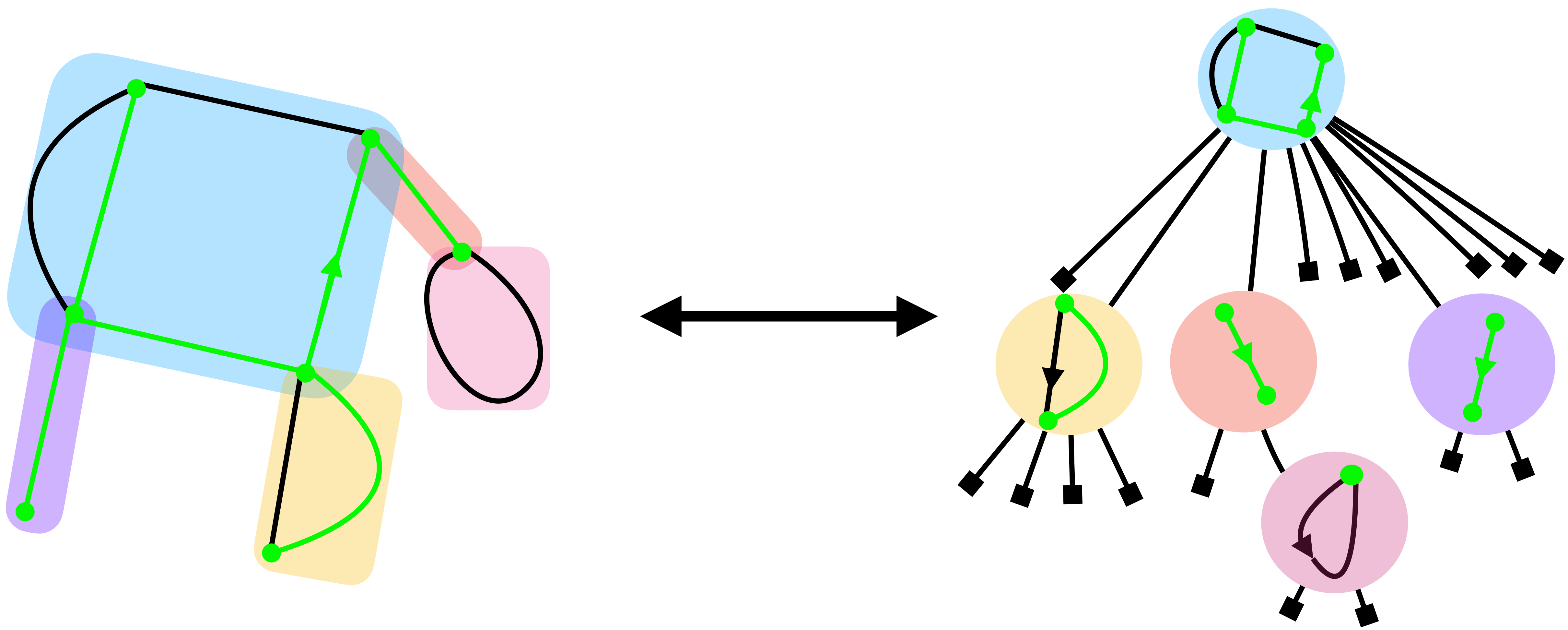
$Z_{n,u}$ = normalisation.

- $u = 1$: uniform distribution on tree-rooted maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected tree-rooted maps);
- $u \to \infty$: maximising the number of blocks (= tree-rooted trees!).

Given u, asymptotic behaviour when $n \to \infty$?

Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



$$M(z) = B(zM^2(z))$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

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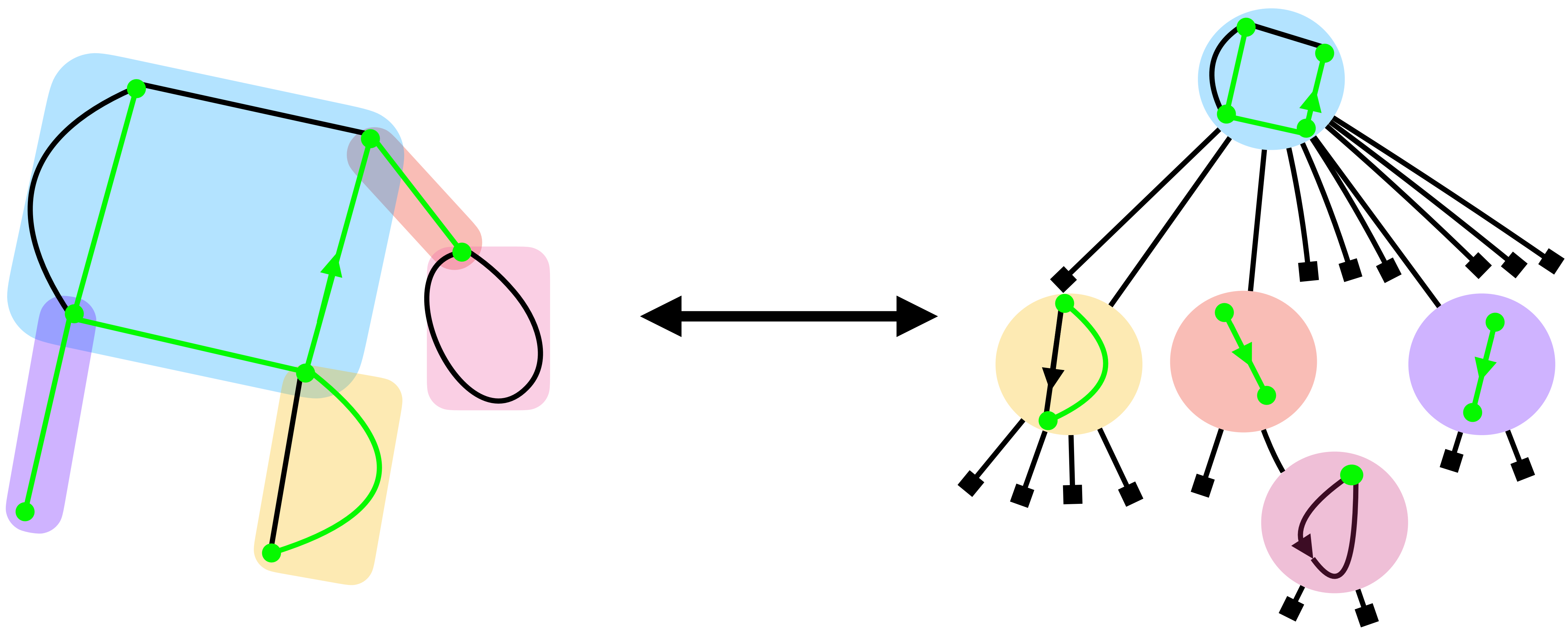
The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



$$M(z,u) = uB(zM^2(z,u)) + 1 - u$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

So everything should be easy, right?

Tree-rooted maps are not so nice

$$M(z) = \sum_{n \geq 0} \mathrm{Cat}_n \mathrm{Cat}_{n+1} z^n$$ so

Tree-rooted maps are not so nice

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- $[z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3};$

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- $M(\rho_M) = 8 - \frac{64}{3\pi} \simeq 1.2$ so M is not algebraic...

$$P\left(z, M(z)\right) = 0$$

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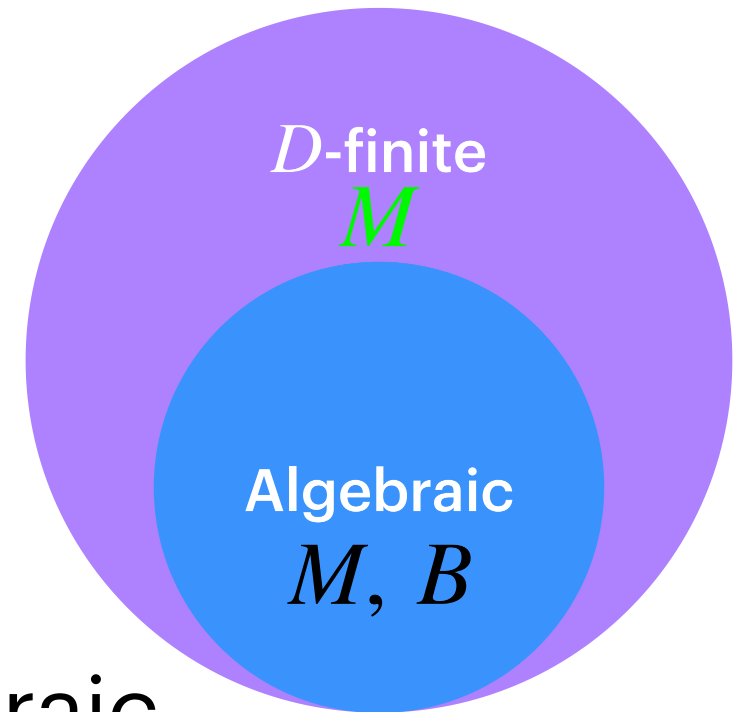

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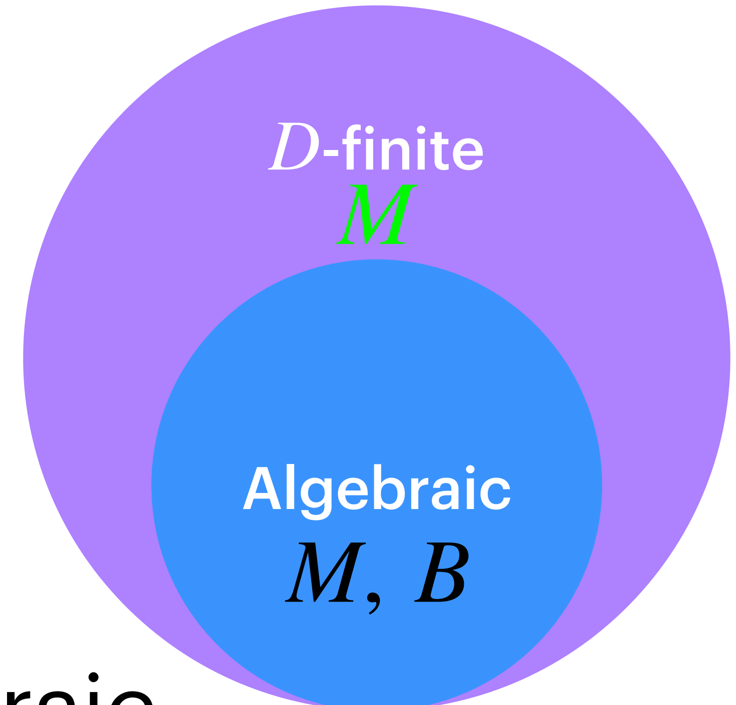

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$$P\left(z, M(z)\right) = 0$$

- Fortunately, it is still D-finite

$$P_0(z)\frac{\partial^2 M}{\partial z^2}(z) + P_1(z)\frac{\partial M}{\partial z}(z) + P_2(z)M(z) + P_3(z) = 0.$$

2-connected tree-rooted maps are naughty

Using $M(z) = B(zM^2(z))$ and the properties of M, we show

2-connected tree-rooted maps are naughty

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- $\rho_B = \rho_M M^2(\rho_M) = \dfrac{4(3\pi - 8)^2}{9\pi^2} \approx 0.091$

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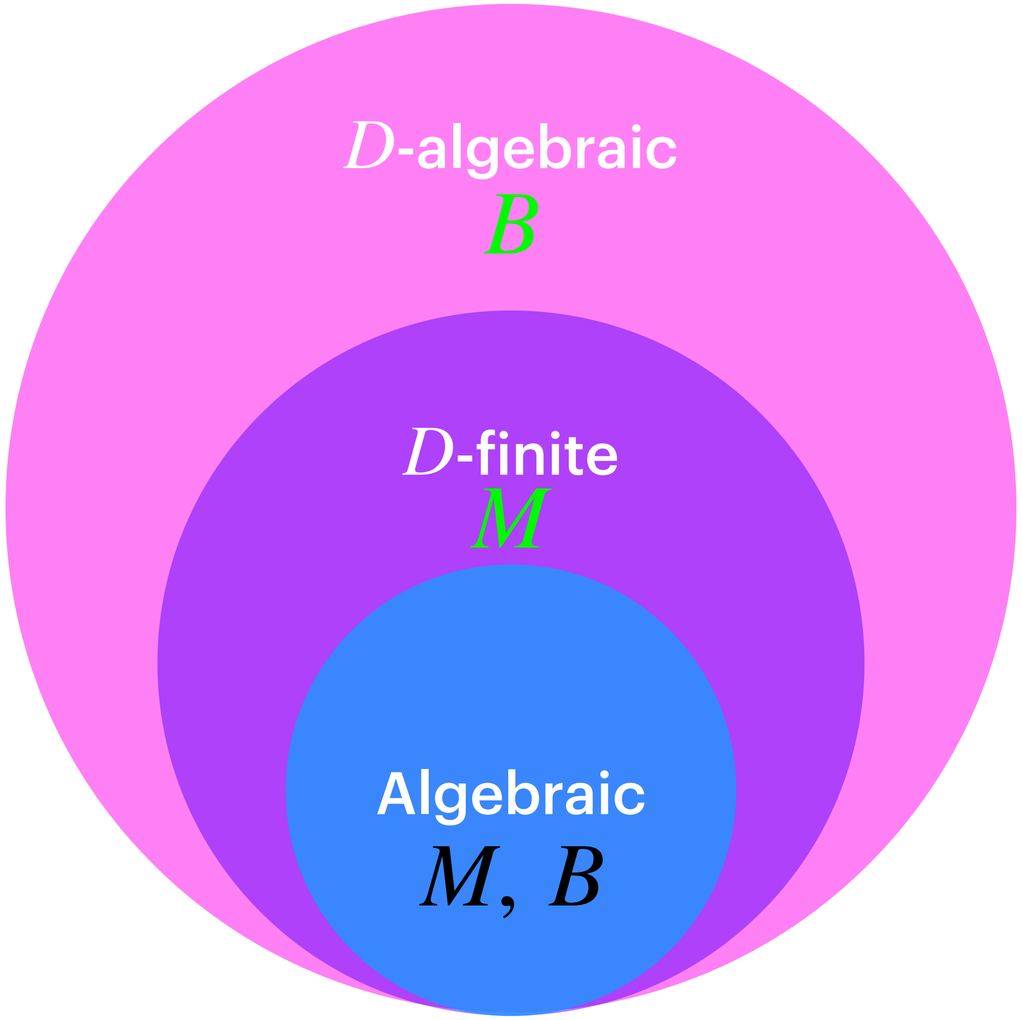

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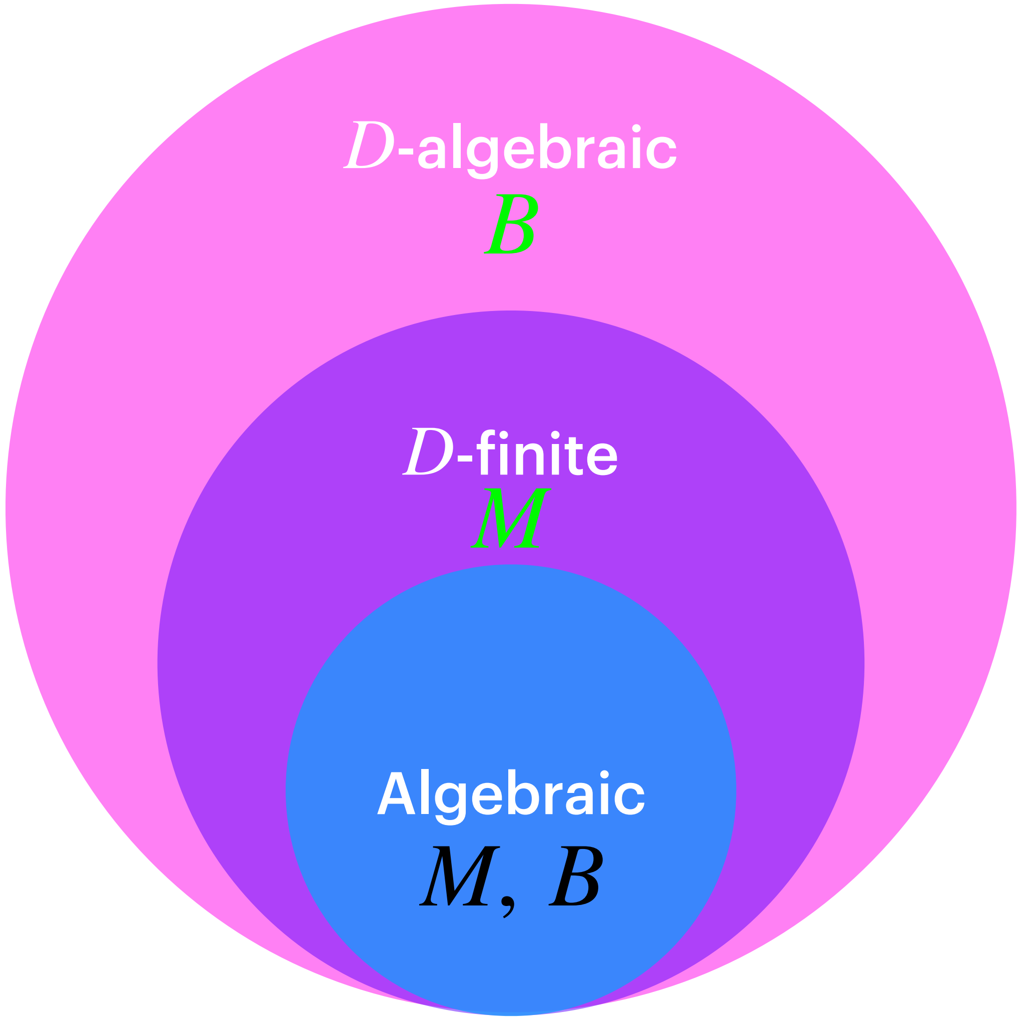
- B is D-algebraic



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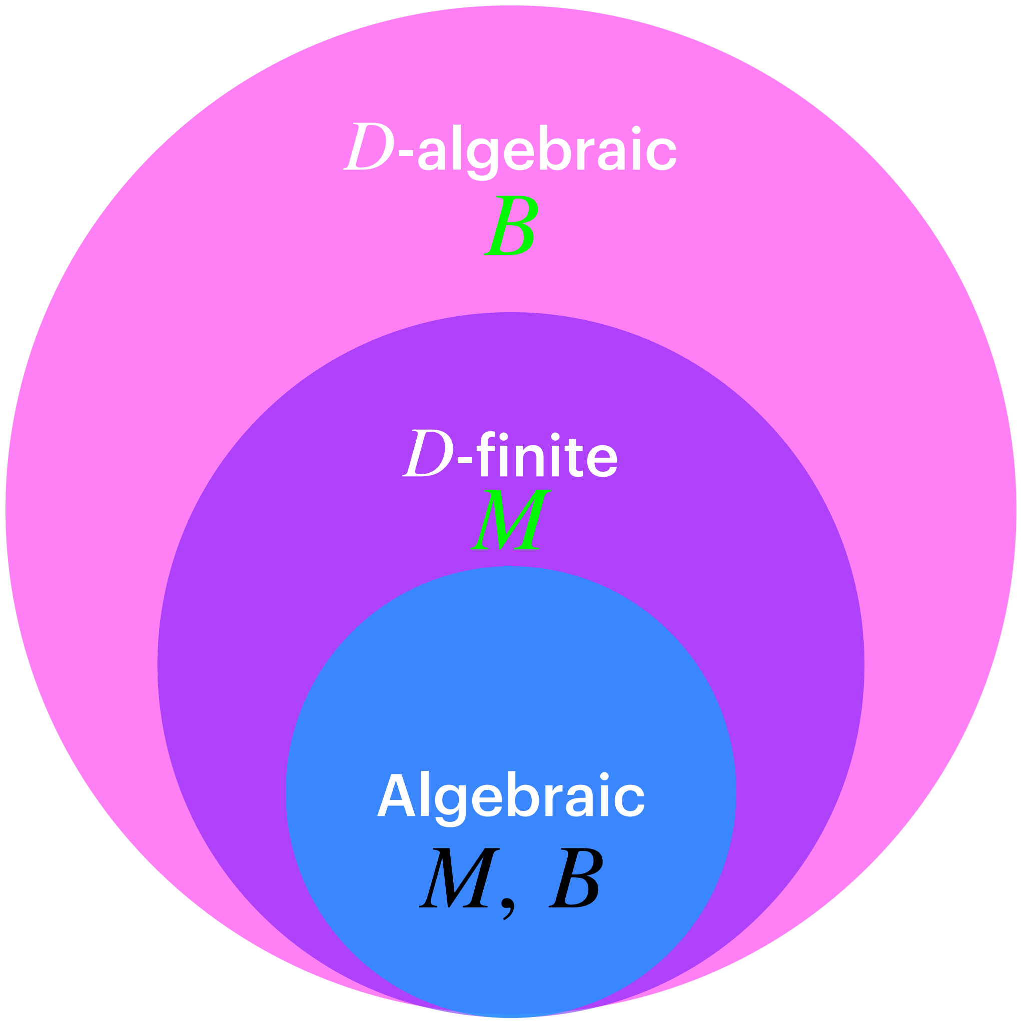

- B is D-algebraic

$$P\left(\frac{\partial^2 B}{\partial y^2}(y), \frac{\partial B}{\partial y}(y), B(y), y\right) = 0.$$

Enumeration of 2-connected tree-rooted maps

Using $M(z) = B(zM^2(z))$ and the properties of M, we show

Theorem [Albenque, Fusy, S. 24+]

$$[y^n]B(y) \sim \frac{4(3\pi - 8)^3}{27\pi(4-\pi)^3} \times \rho_B^{-n} \times n^{-3}.$$

Phase transition

Theorem [Albenque, Fusy, S. 24+] Model exhibits a phase transition at $u_C = \dfrac{9\pi(4-\pi)}{420\pi - 81\pi^2 - 512} \simeq 3.02.$

When $n \to \infty$:

- Subcritical phase $u < u_C$: "general tree-rooted map phase" one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: "tree phase" only small blocks.

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration			
Size of - the largest block - the second one			
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one			
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Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3\ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
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Ordered atoms of a Poisson Point Process

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Results

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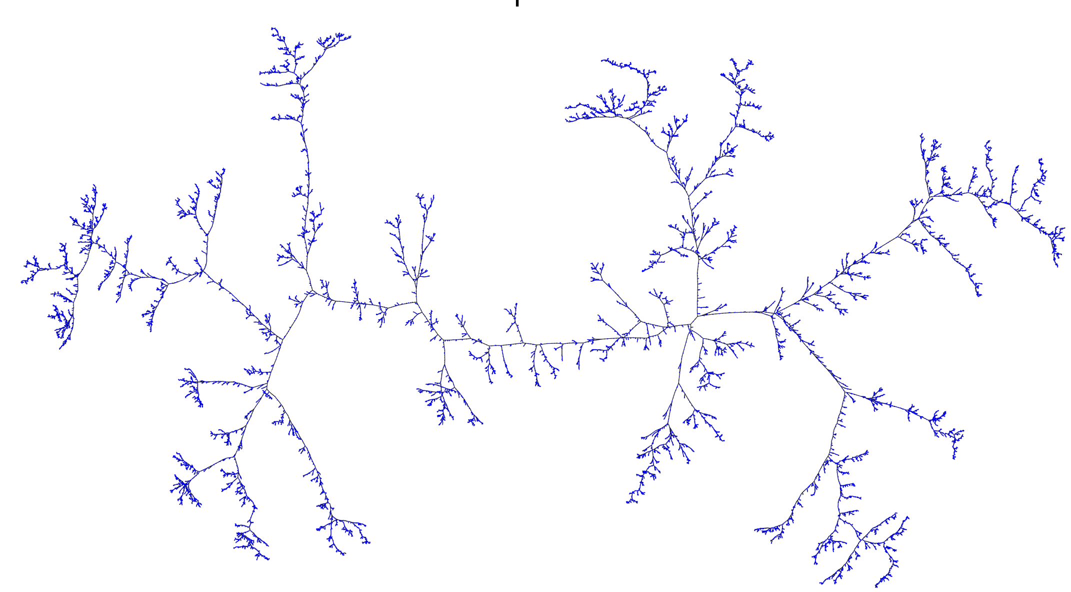
Scaling limit of M_n	?	$\frac{C_2 \ln(n)^{1/2}}{n^{1/2}} M_n \to \mathcal{T}_e$	$\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e$



Results

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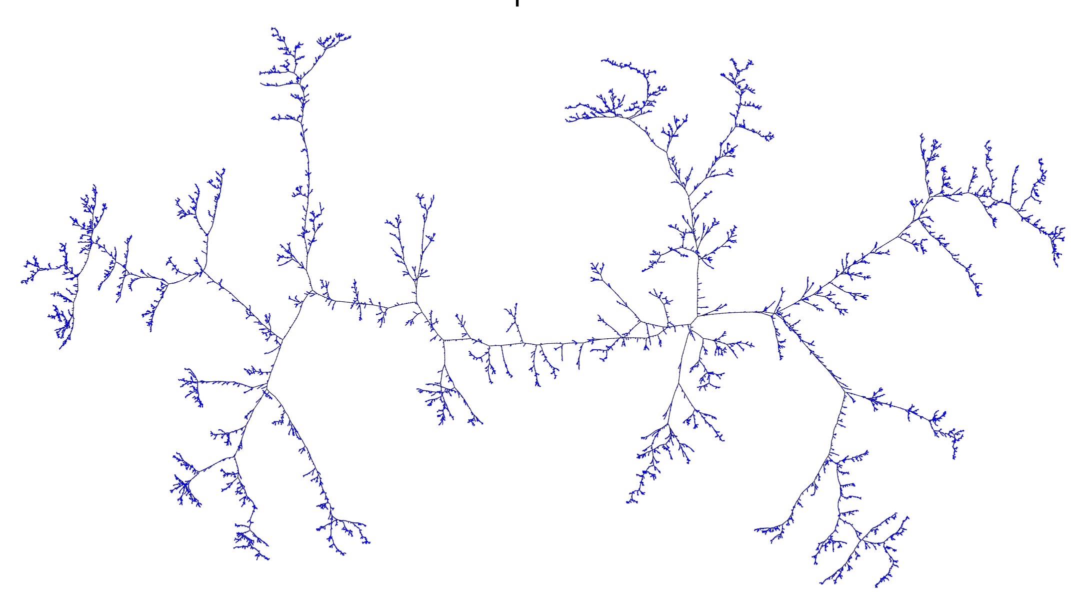

V. Perspectives

Extensions to more involved decompositions

Block-weighted

- Tree-rooted quadrangulations;
- Forested maps;
- Maps endowed with a Potts model / Ising model;
- 2-oriented quadrangulations (resp. 3-oriented triangulations) decomposed into irreducible blocks;
- Schnyder woods...

Thank you!