A phase transition in block-weighted random maps

LACIM Seminar, UQAM, Montréal
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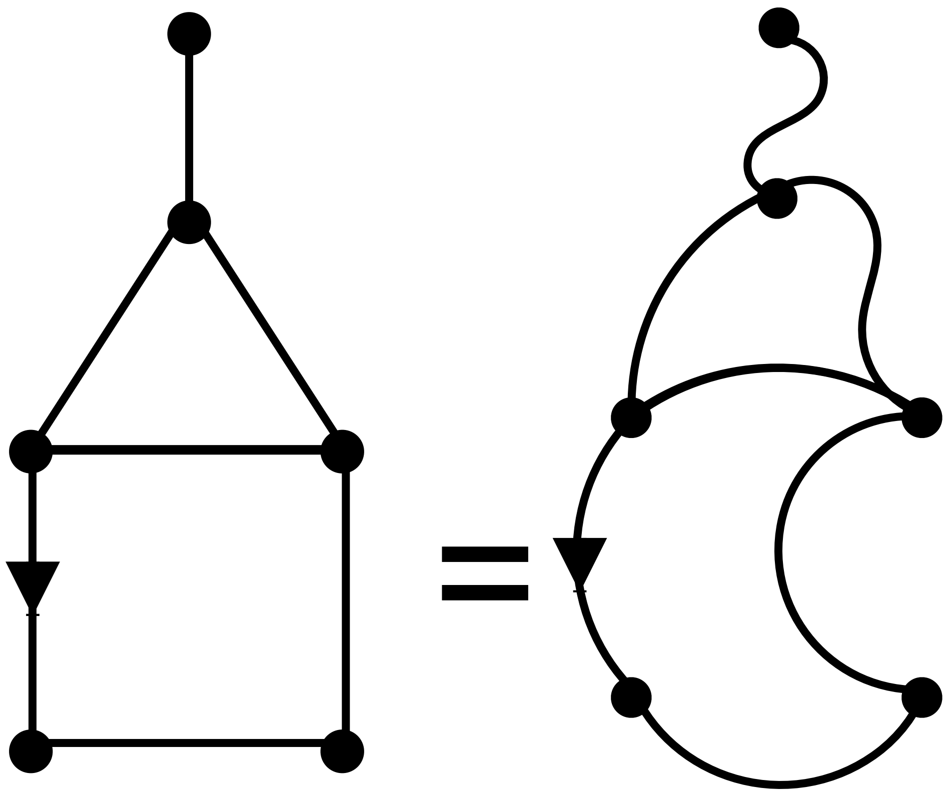
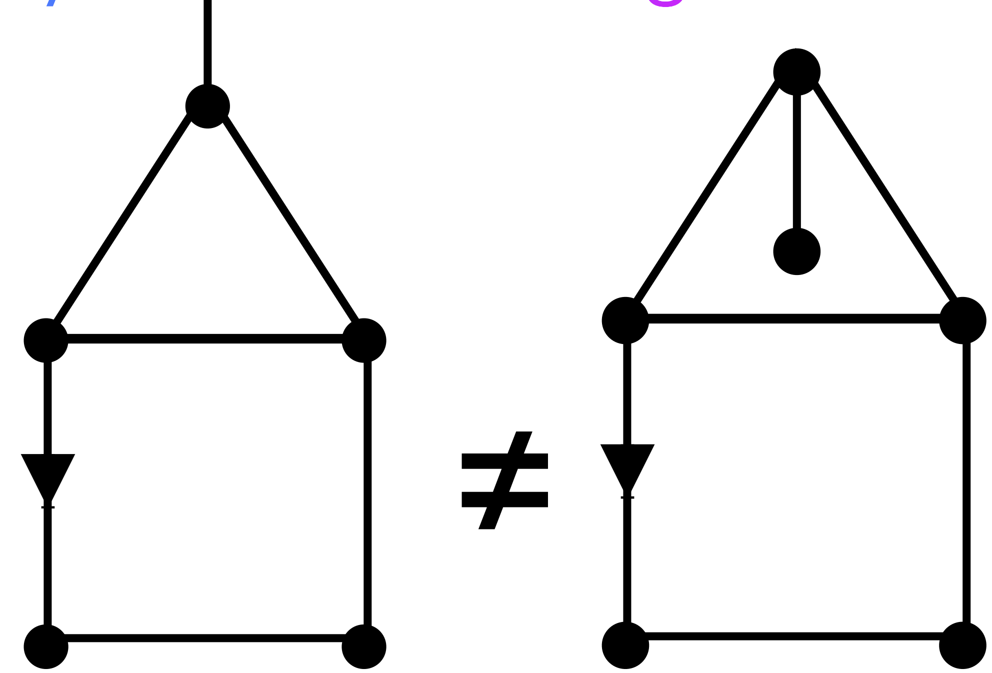
LIGM, Université Gustave Eiffel

Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms



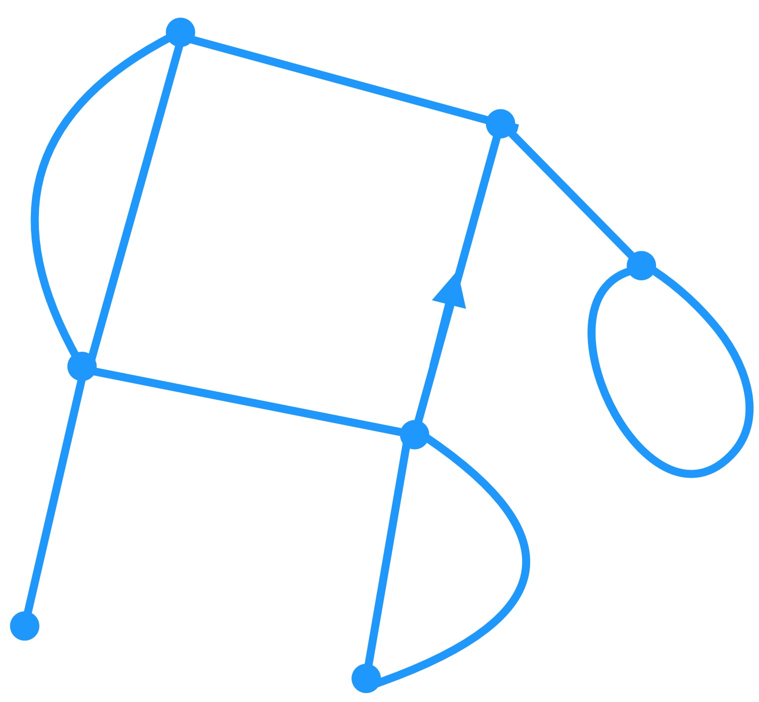
Planar map = planar graph + cyclic order on neighbours



- Rooted planar map = map endowed with a marked oriented edge (represented by an arrow);
- Size $\mathfrak{m}$ = number of edges;
- Corner (does not exist for graphs !) = space between an oriented edge and the next one for the trigonometric order.

Universality results for planar maps

- Enumeration: $\kappa\rho^{-n}n^{-5/2}$ [Tutte 1963];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for quadrangulations [Le Gall 2013, Miermont 2013] and uniform maps [Bettinelli, Jacob, Miermont 2014];



Brownian Sphere $\mathcal{S}_e$

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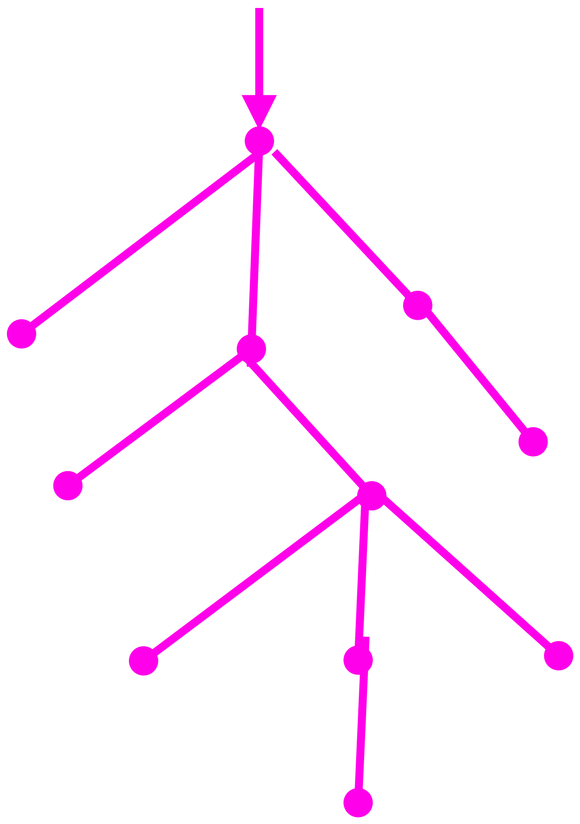
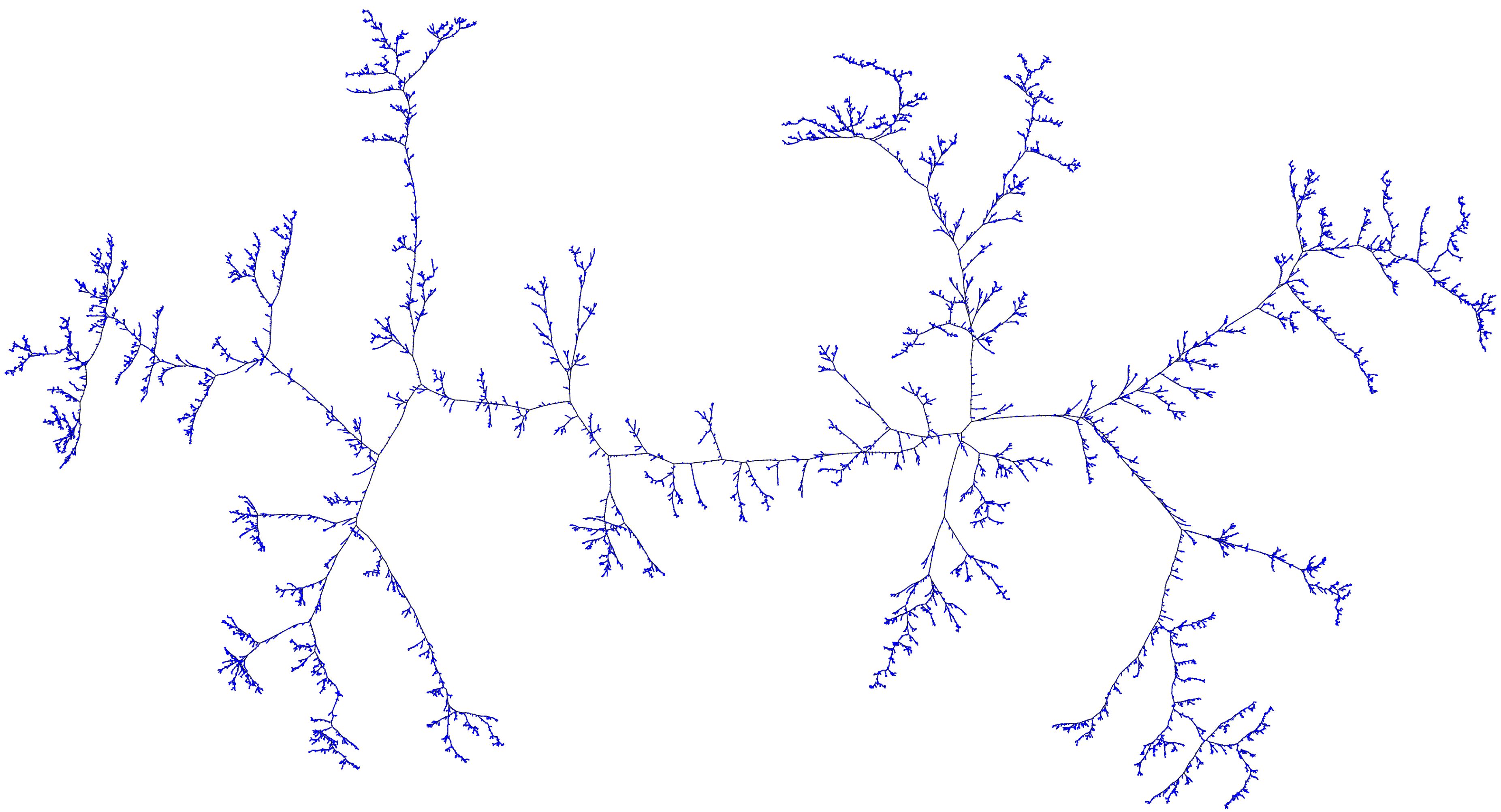
- Universality:
  - Same enumeration [Drmota, Noy, Yu 2020];
  - Same scaling limit, *e.g.* for triangulations & $2q$-angulations [Le Gall 2013], simple quadrangulations [Addario-Berry, Albenque 2017].

Universality results for plane trees

- Enumeration: $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];





Brownian Tree $\mathscr{T}_e$

Universality results for plane trees

- Enumeration: $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];

- Universality:
  - Same enumeration,
  - Same scaling limit, even for some classes of maps; e.g. outerplanar maps [Caraceni 2016], maps with a boundary of size >> $n^{1/2}$ [Bettinelli 2015].

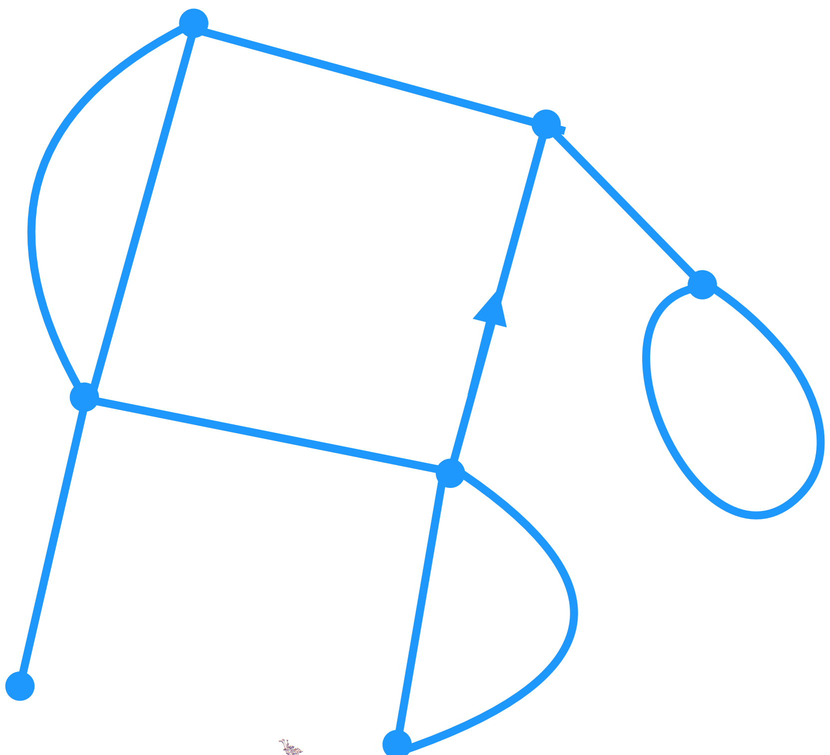
Models with (very) constrained boundaries

Motivation

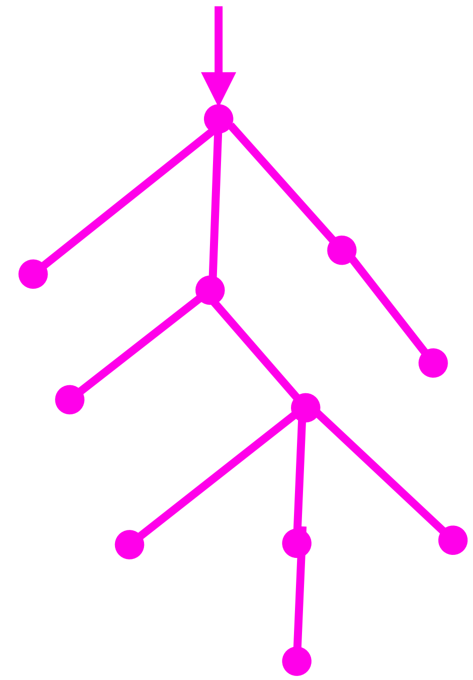
Inspired by [Bonzom 2016].

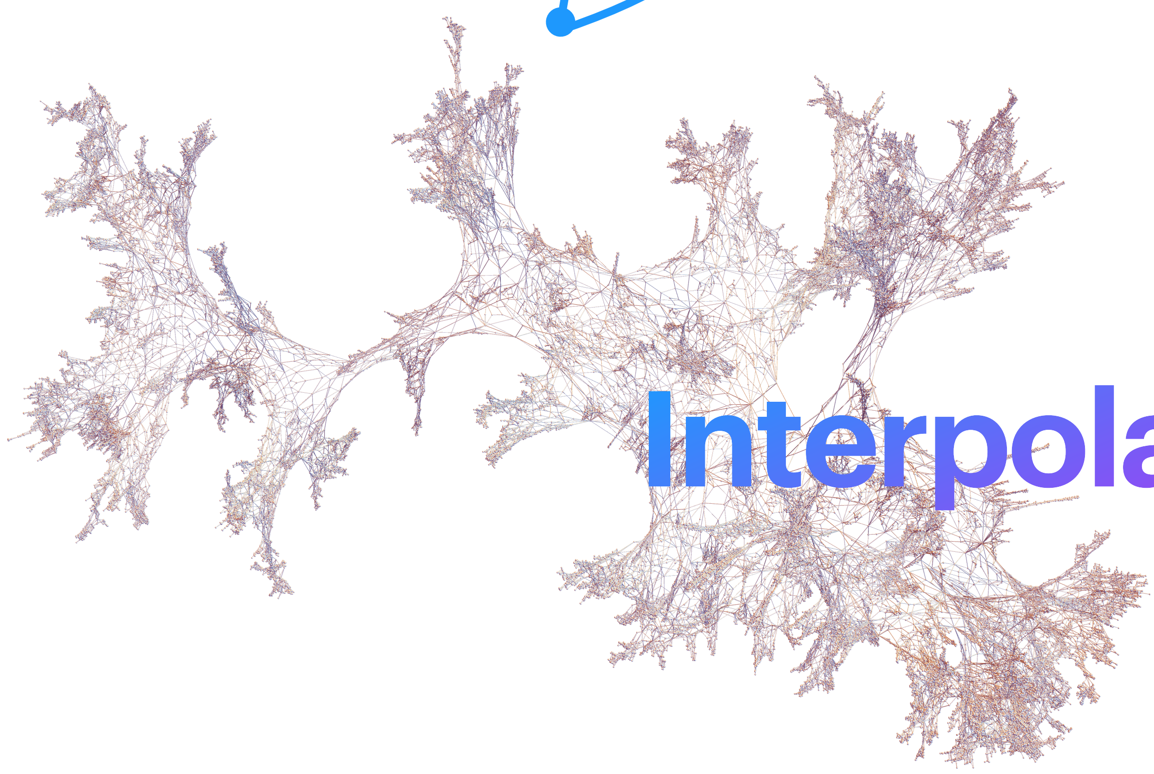
Two rich situations with universality results:

Planar maps

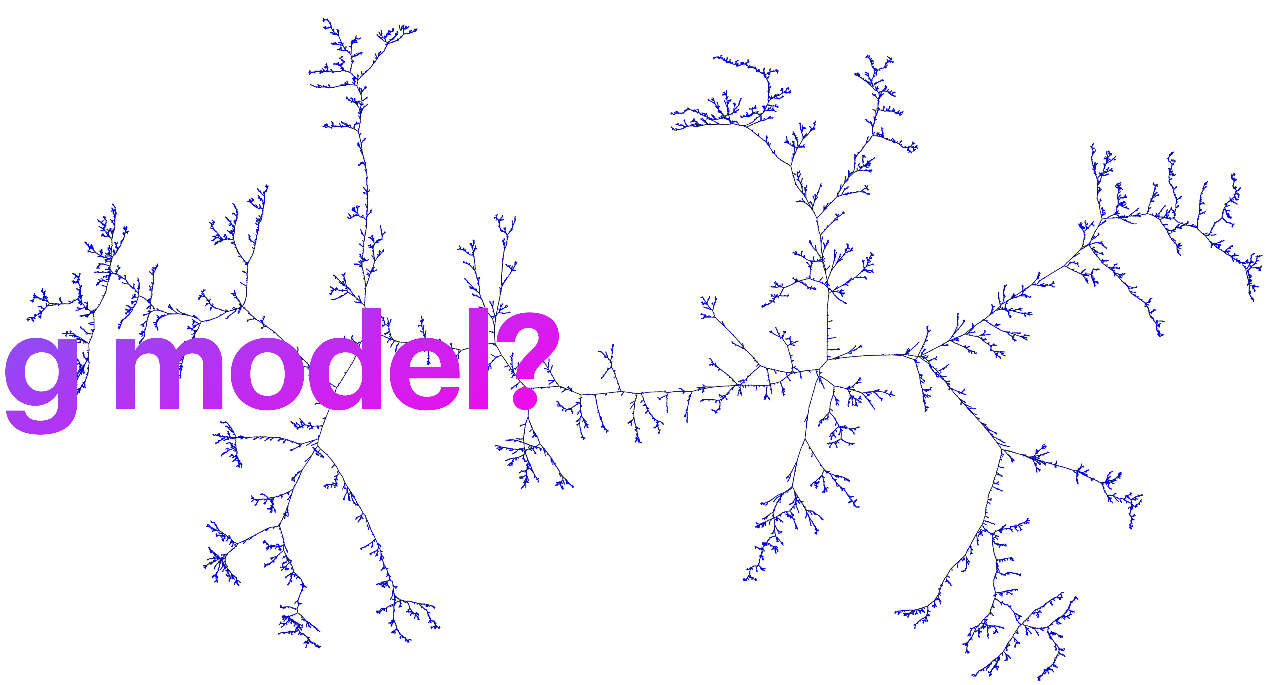


Plane trees





Interpolating model?



Brownian Sphere $\mathcal{S}_e$

Brownian Tree $\mathcal{T}_e$

Model definition

2-connected = two vertices must be removed to disconnect.

Block = maximal (for inclusion) 2-connected submap.

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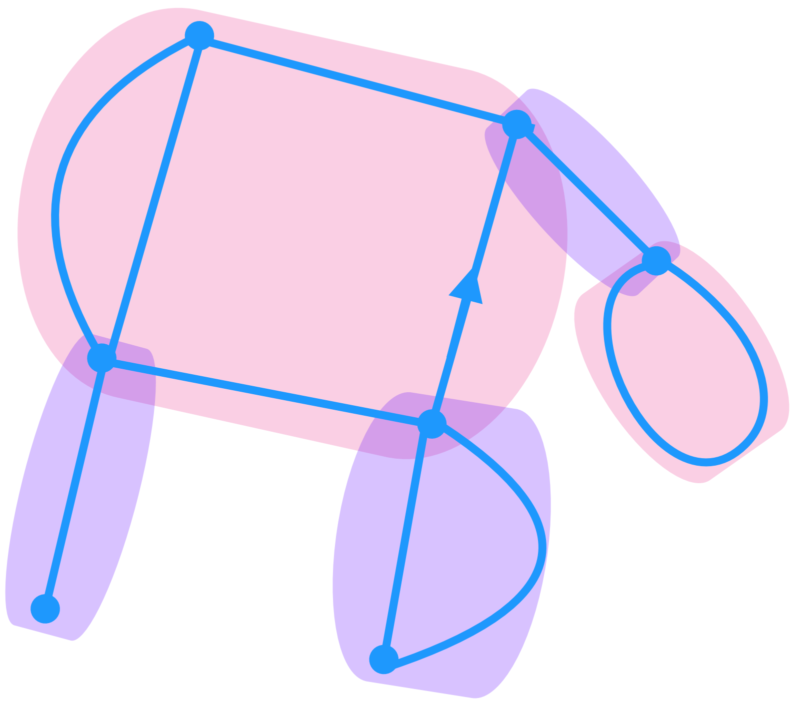

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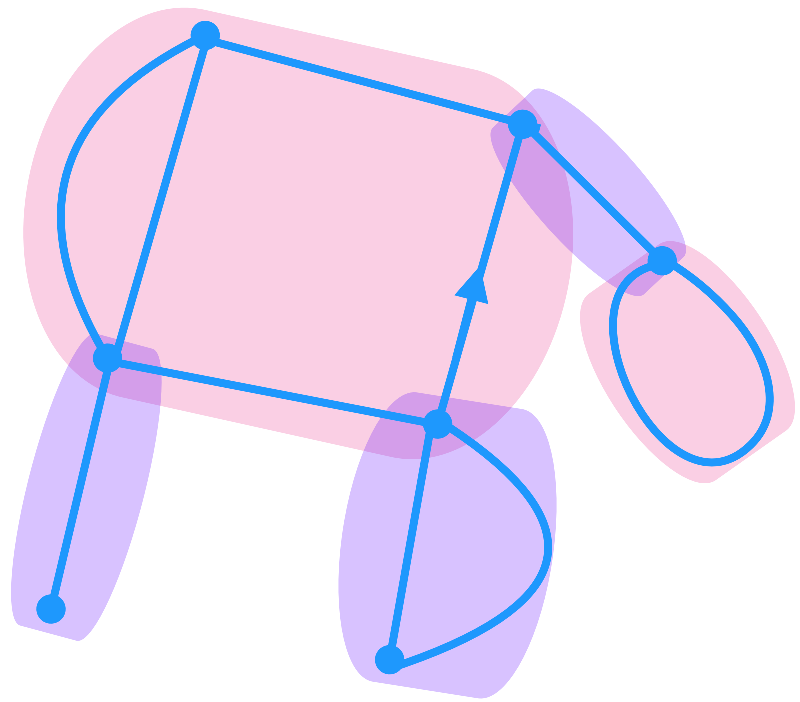

Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

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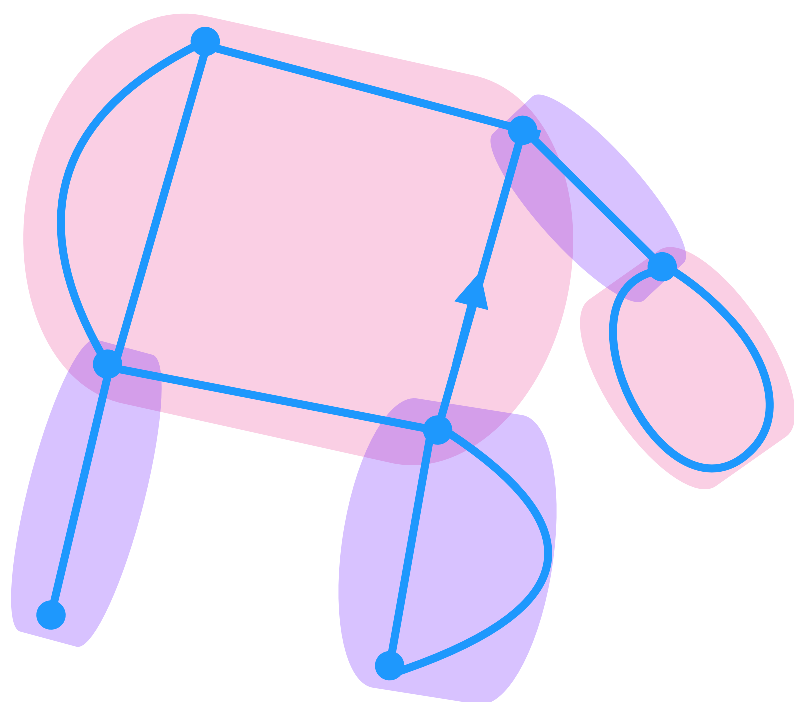

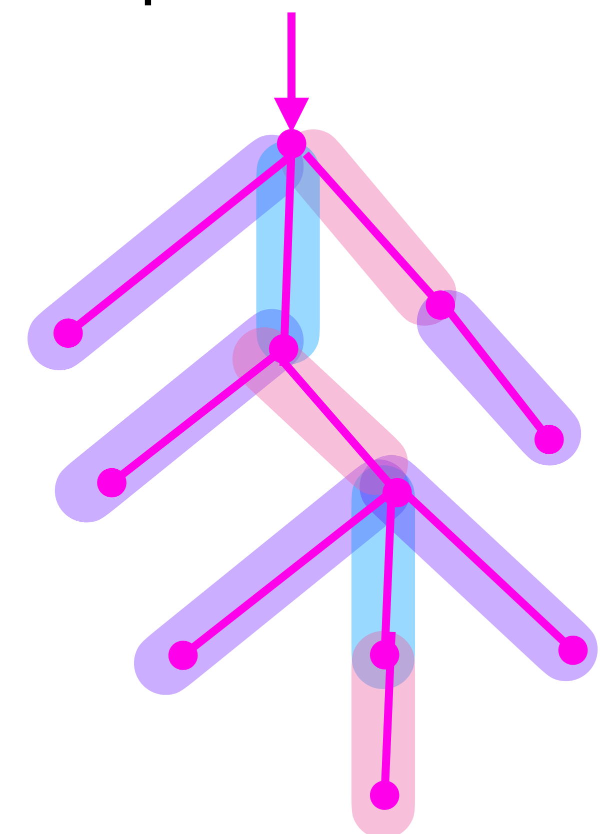

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Only small blocks.

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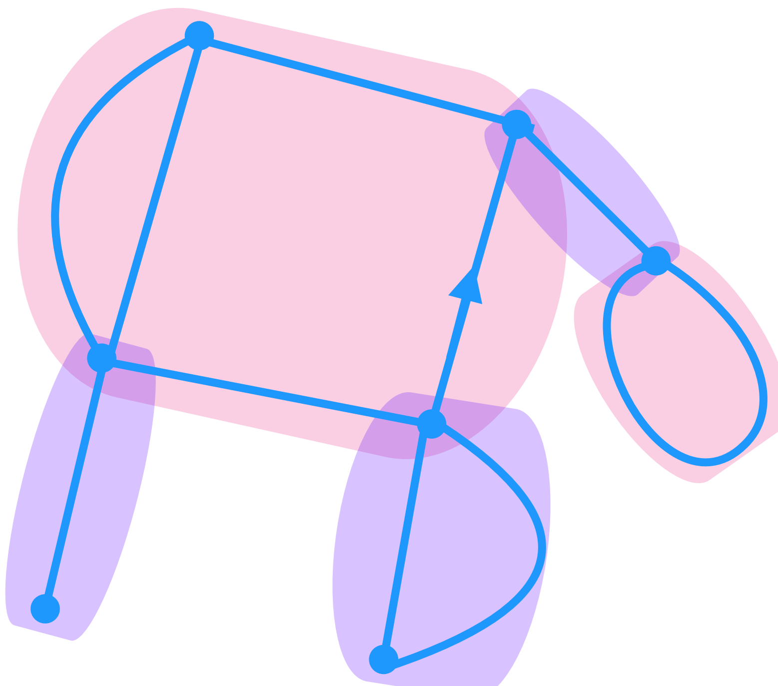

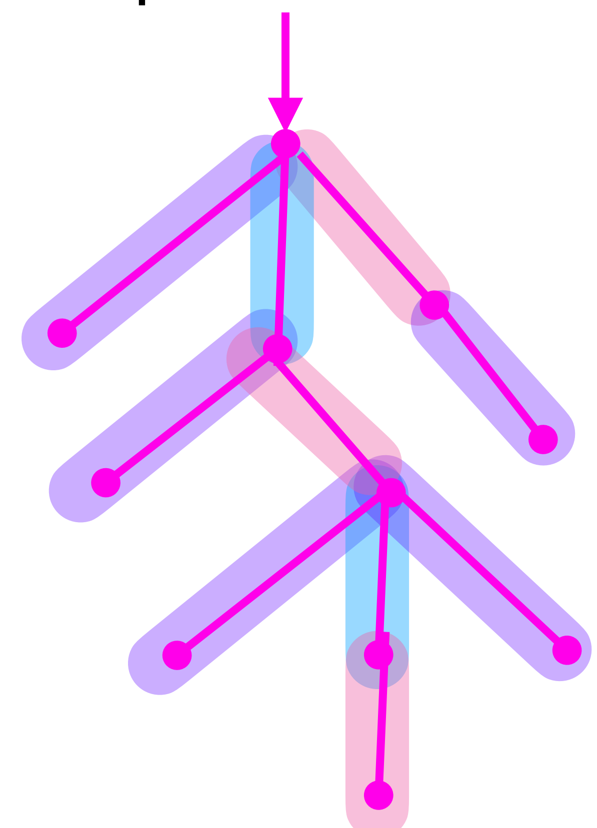

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Interpolating model using blocks!

Outline of the talk

A phase transition in block-weighted random maps

I. Model

II. Block tree of a map and its applications

Interlude. Quadrangulations

III. Scaling limits

IV. Extension to other families of maps

V. Perspectives

I. Model

Model

Inspired by [Bonzom 2016].

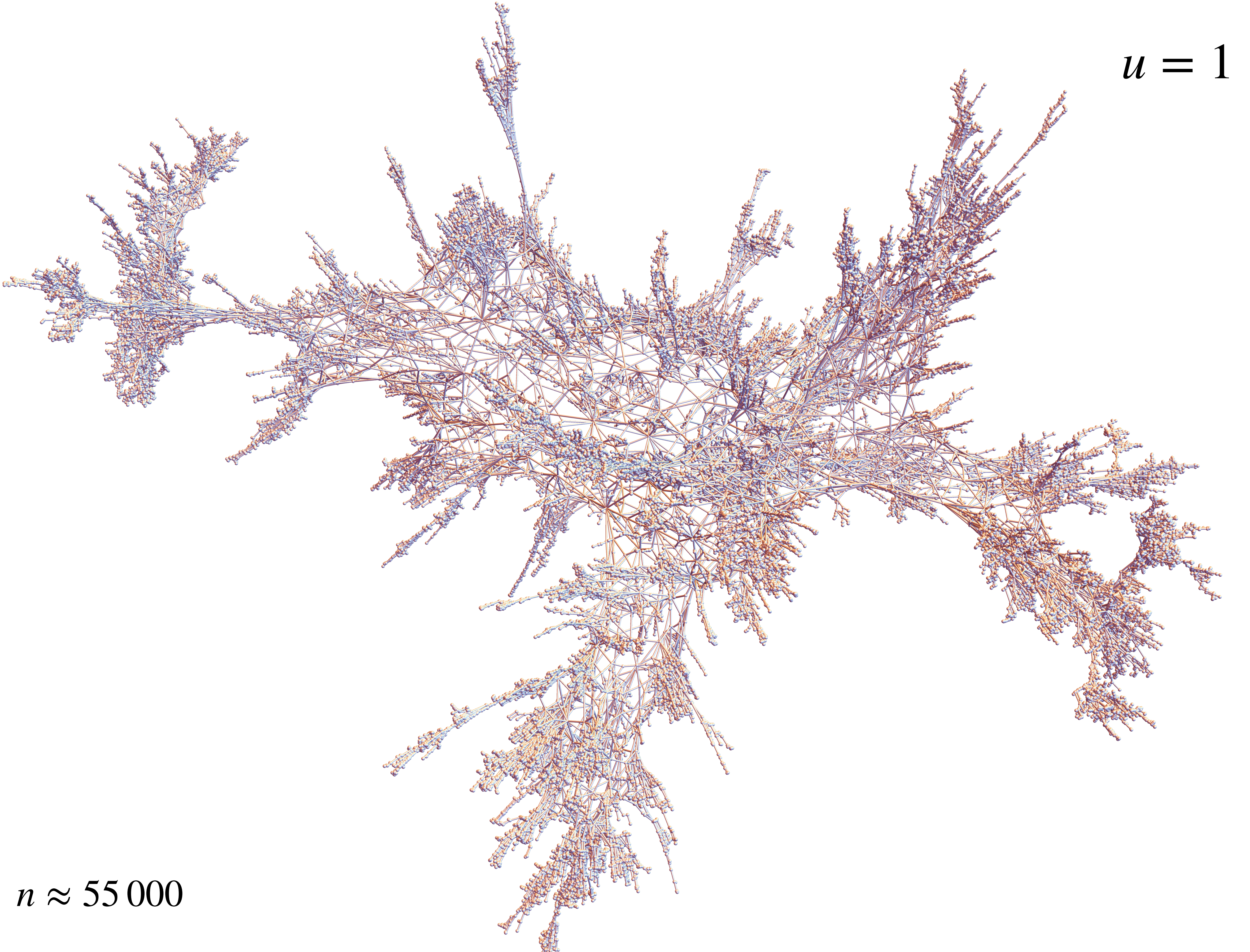
Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

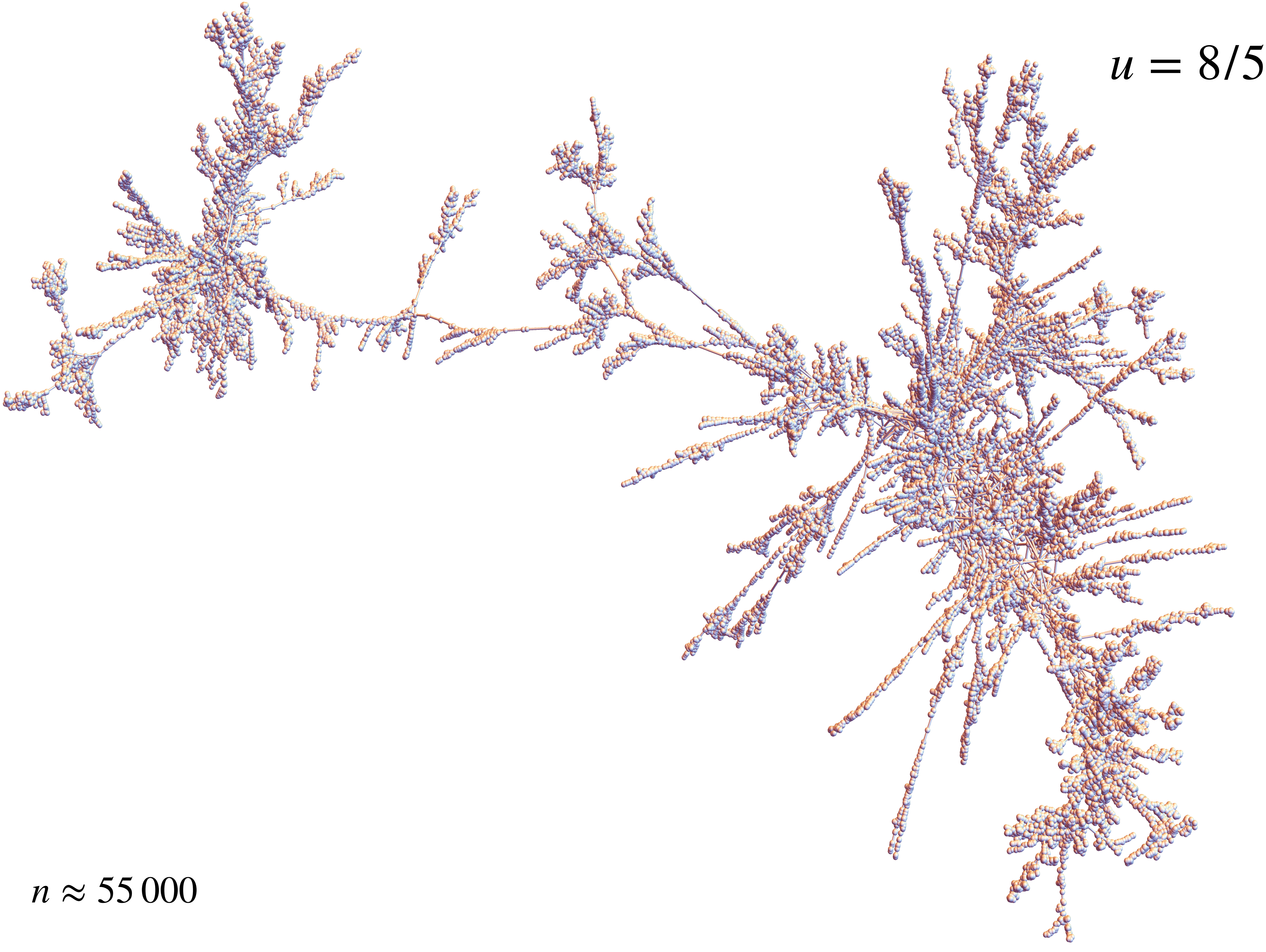
$u > 0$,
$\mathscr{M}_n$ = {maps of size n},
$\mathfrak{m} \in \mathscr{M}_n$,
$Z_{n,u}$ = normalisation.

- $u = 1$: uniform distribution on maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected maps);
- $u \to \infty$: maximising the number of blocks (= trees!).

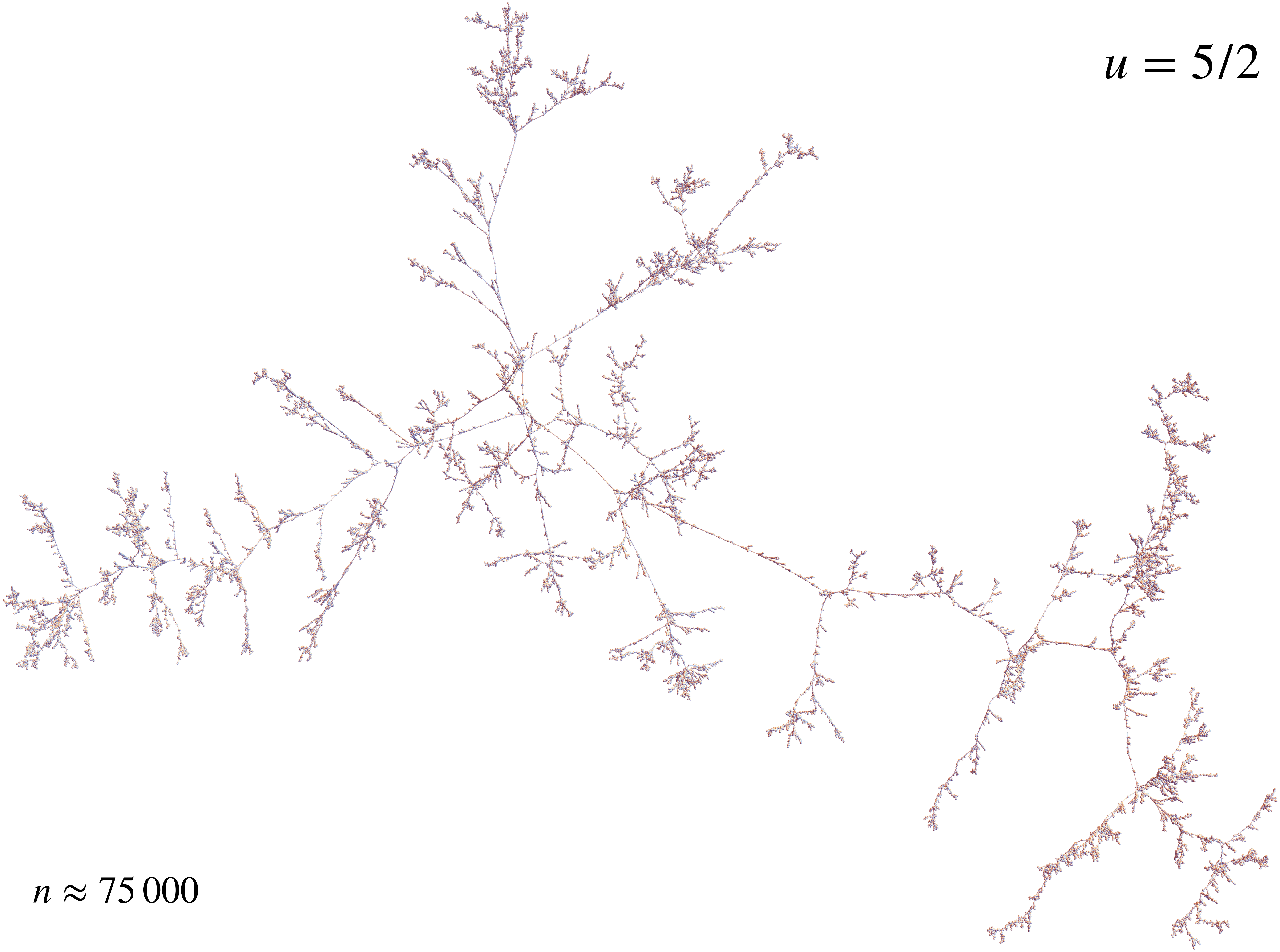
Given u, asymptotic behaviour when $n \to \infty$?



$u = 1$
$n \approx 55\,000$


$u = 8/5$
$n \approx 55\,000$


$u = 9/5$
$n \approx 80\,000$



$u = 5/2$
$n \approx 75\,000$

$u = 5$
$n \approx 50\,000$

Phase transition

Theorem [Fleurat, S. 23] Model exhibits a phase transition at $u = 9/5$. When $n \to \infty$:

- Subcritical phase $u < 9/5$: "general map phase" one huge block;
- Critical phase $u = 9/5$: a few large blocks;
- Supercritical phase $u > 9/5$: "tree phase" only small blocks.

We obtain explicit results on enumeration, size of blocks and scaling limits in each case.

→ *A phase transition in block-weighted random maps*
W. Fleurat, Z. Salvy (2023)

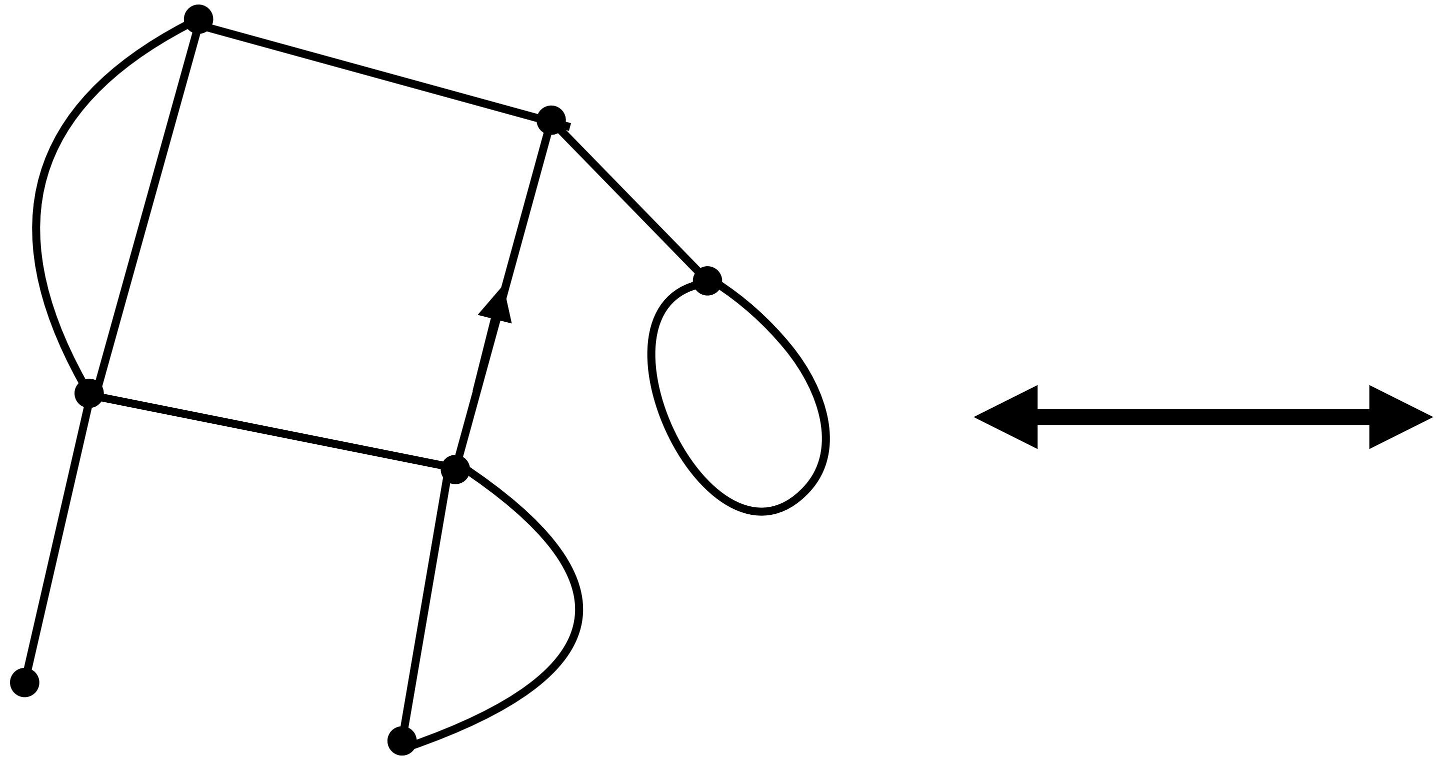
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration			
Size of - the largest block - the second one			
Scaling limit of M_n			

II. Block tree of a map and its applications

Decomposition of a map into blocks

Inspiration from [Tutte 1963]



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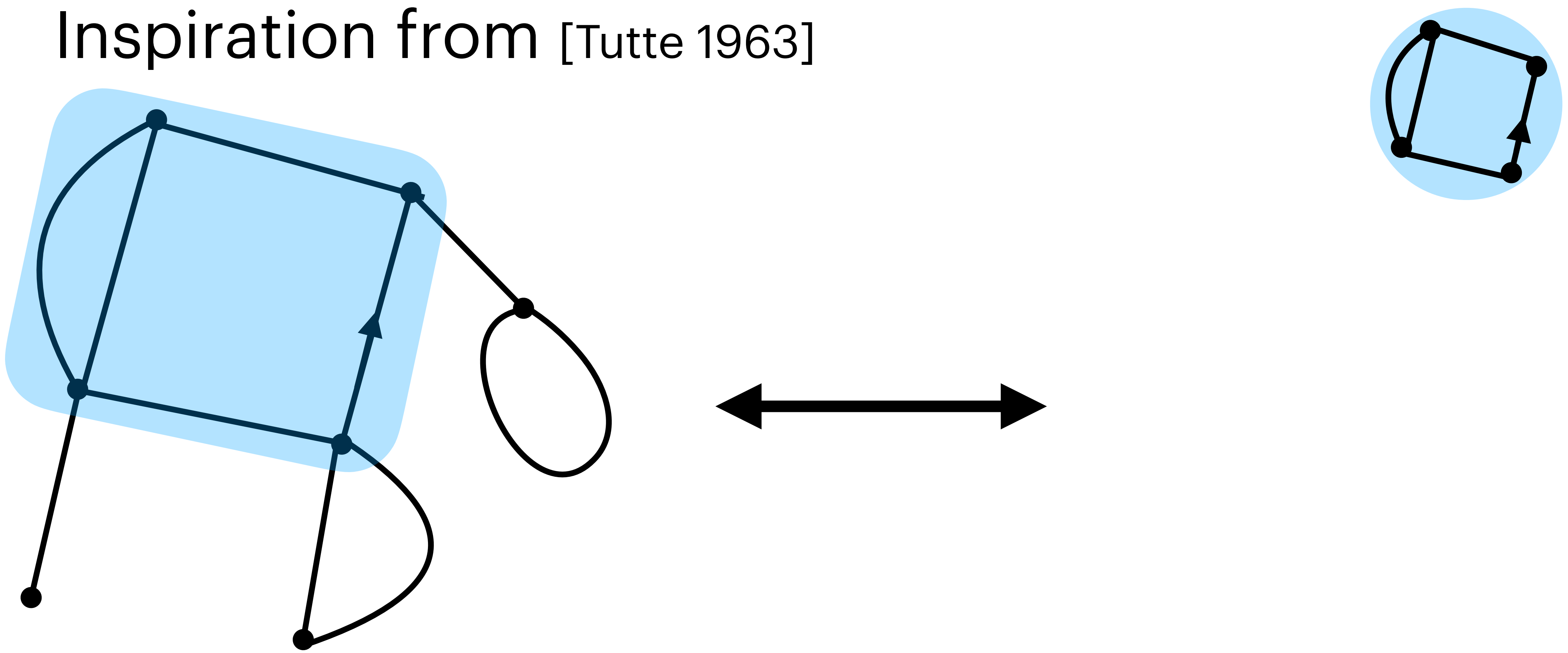

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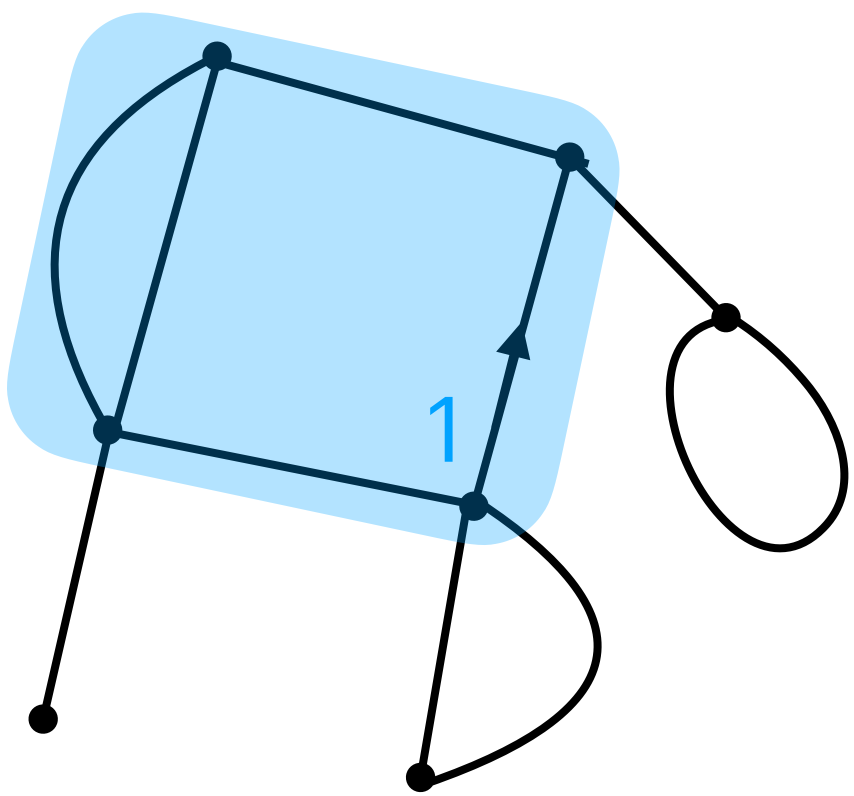

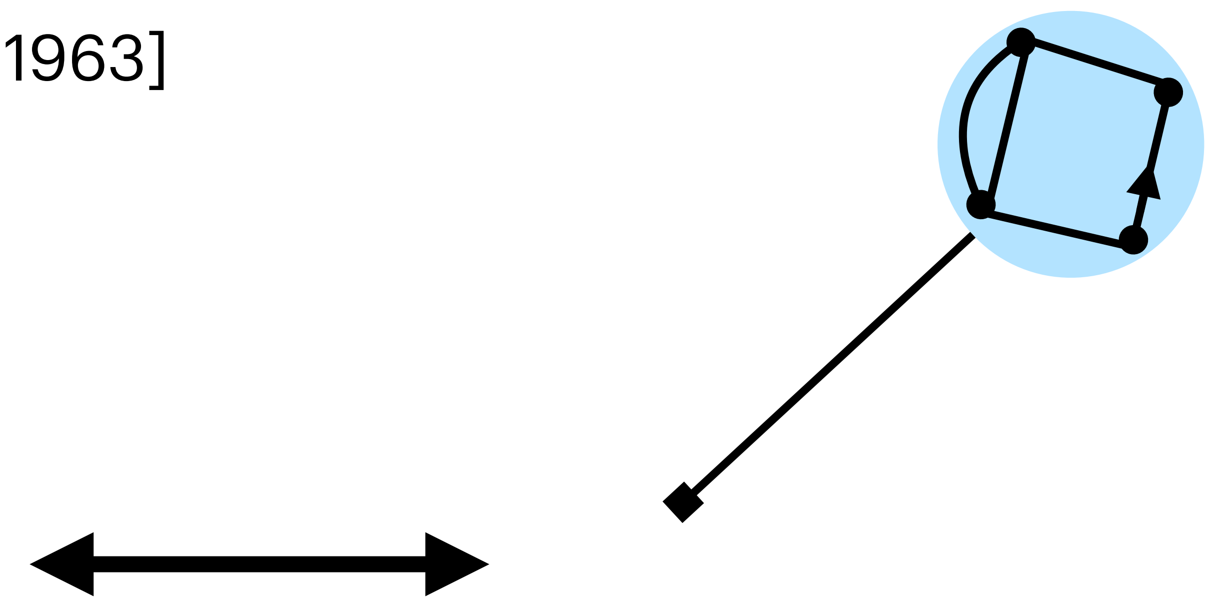

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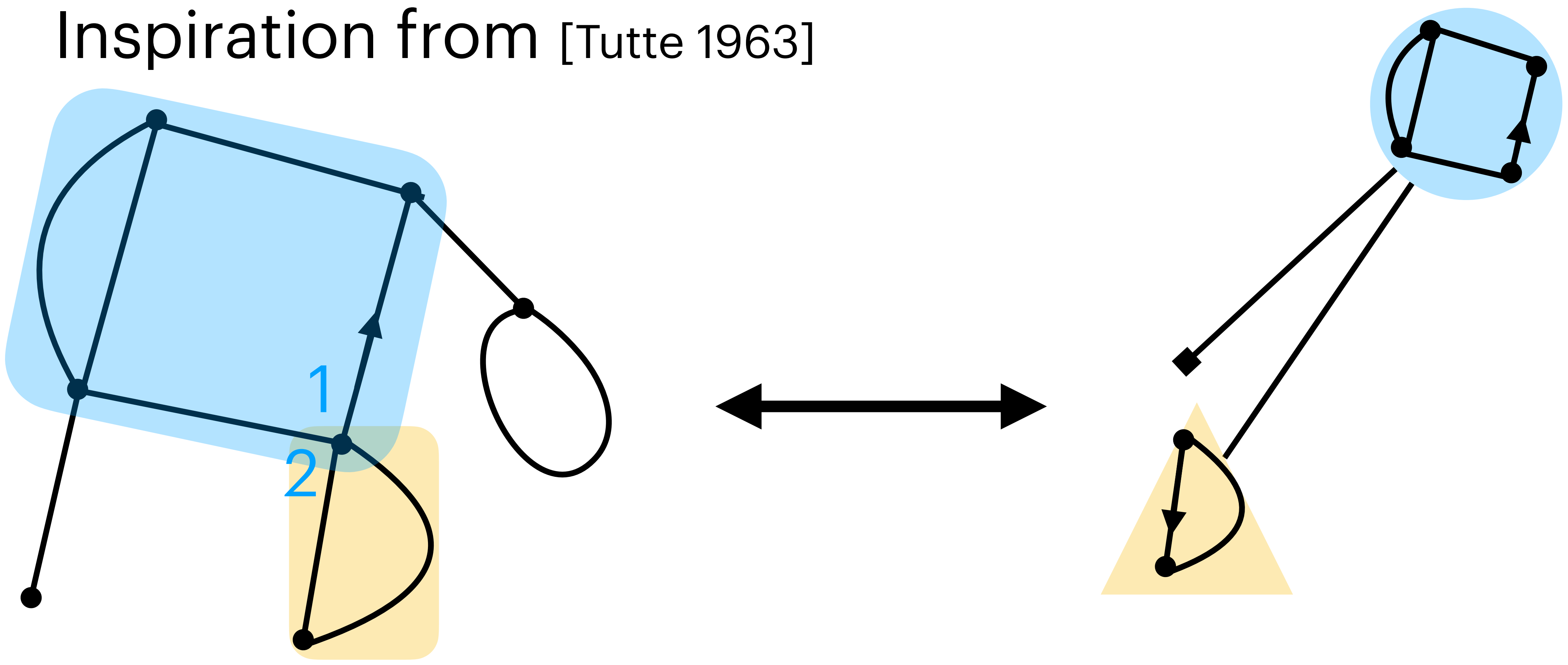

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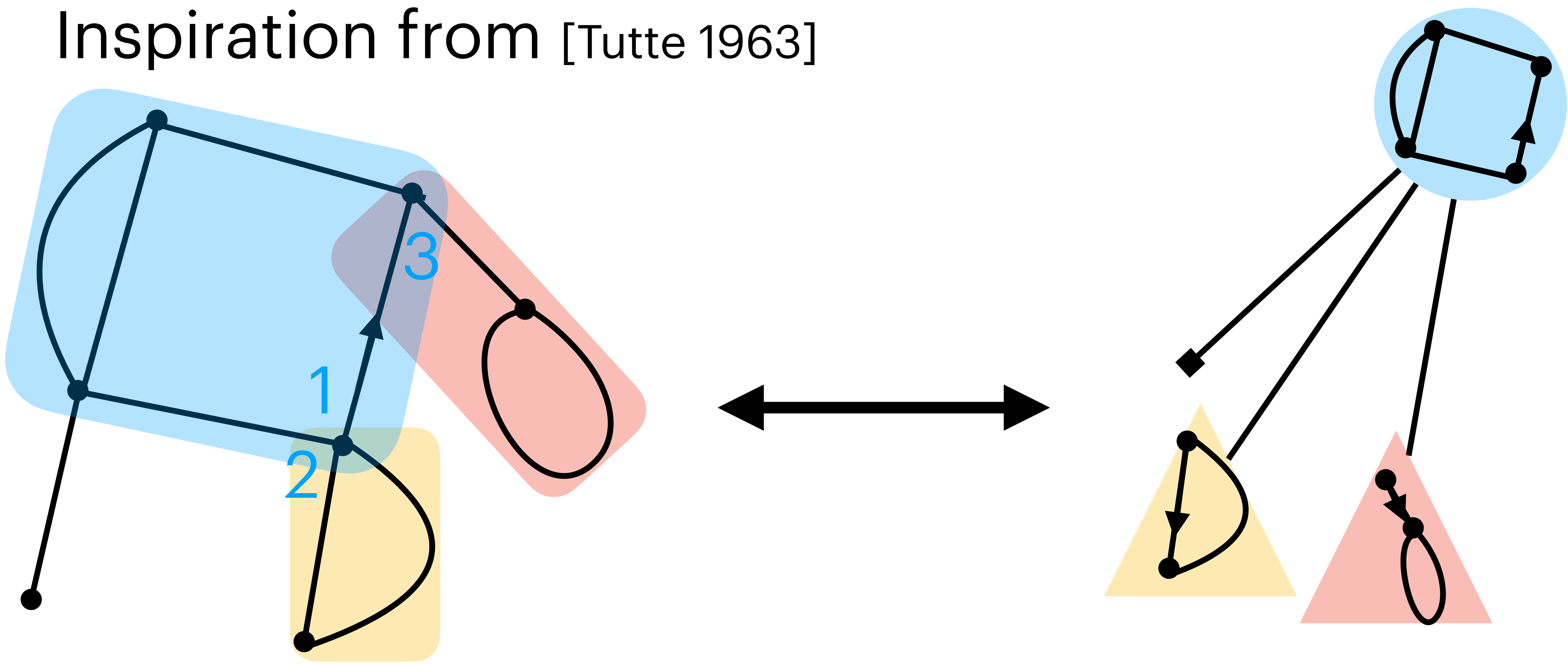

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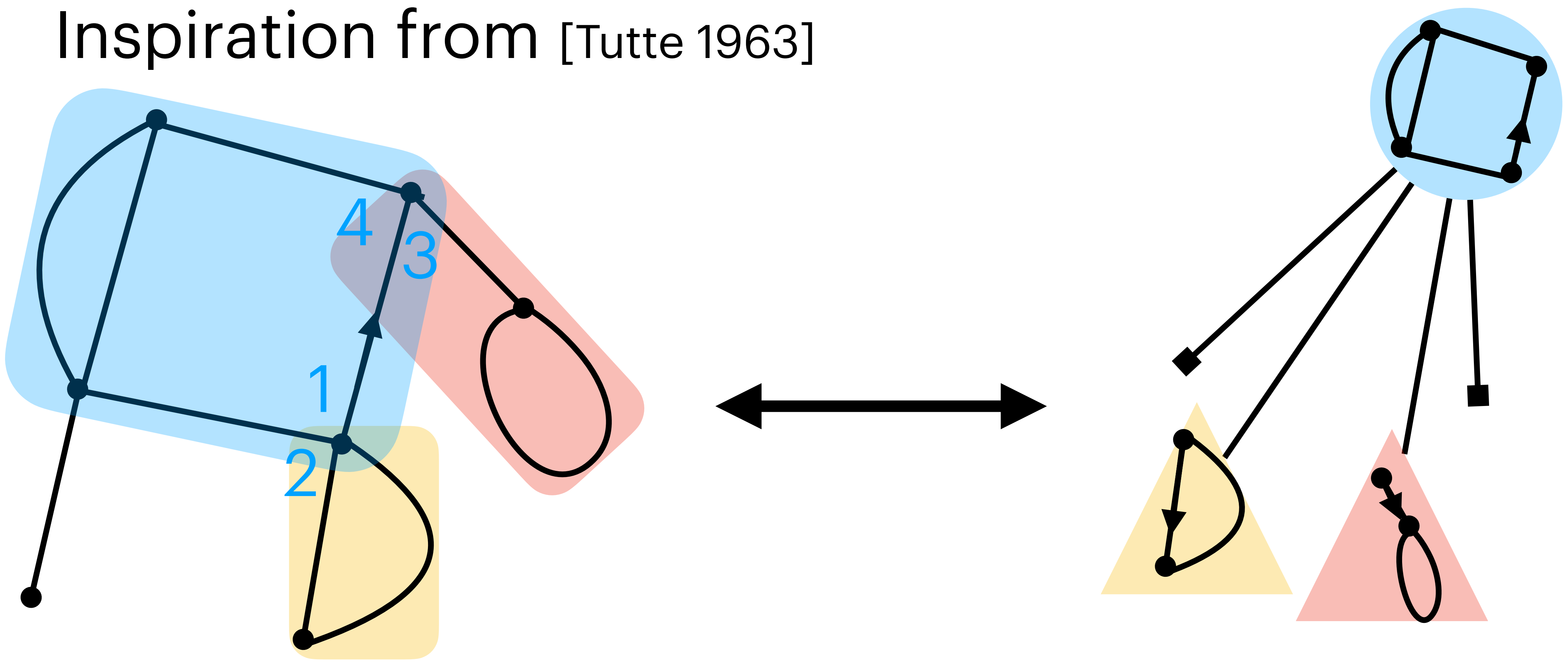

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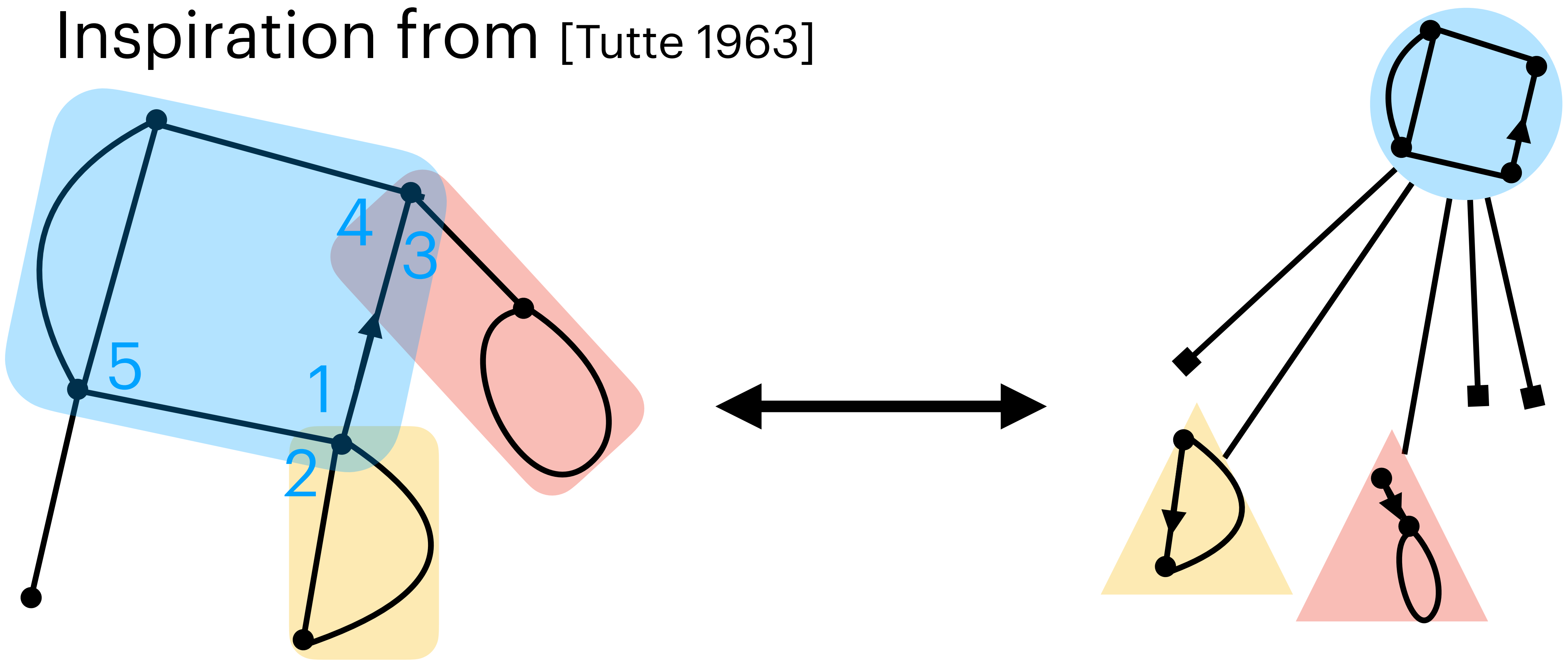

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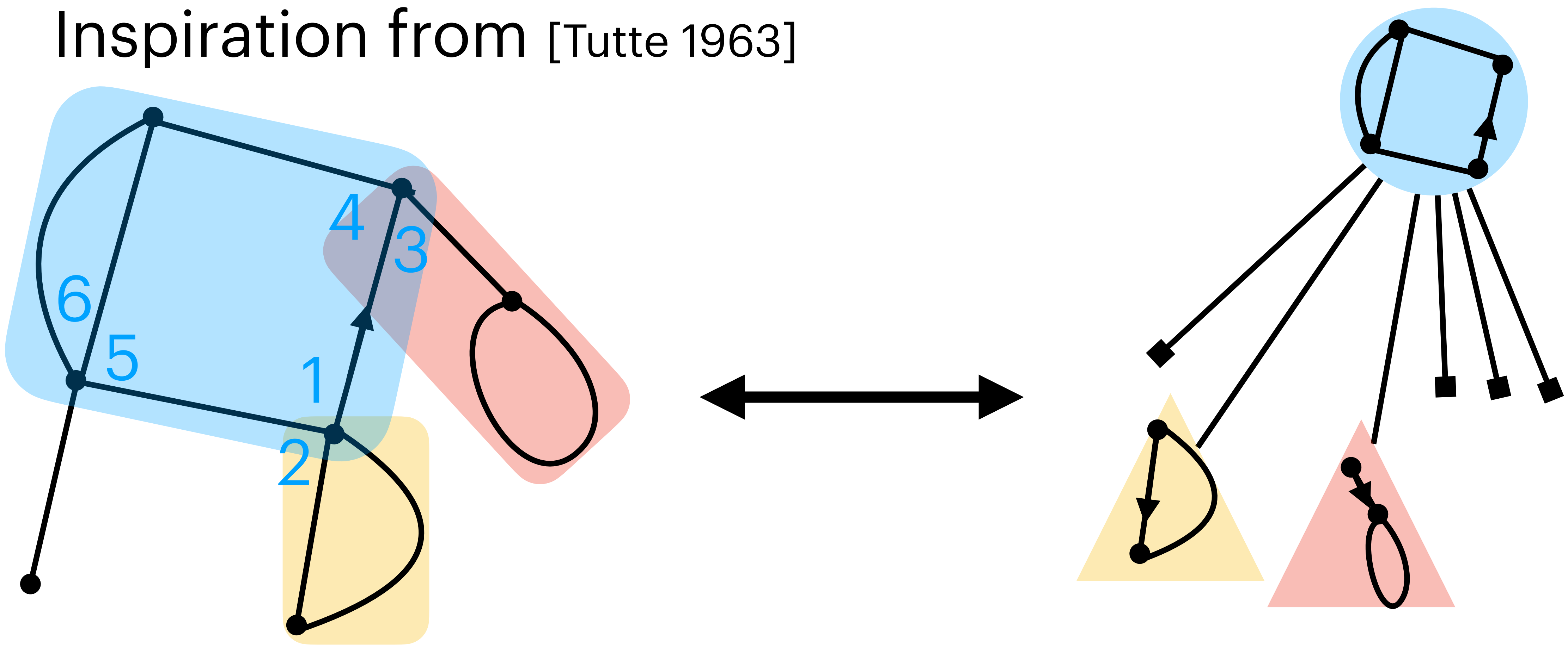

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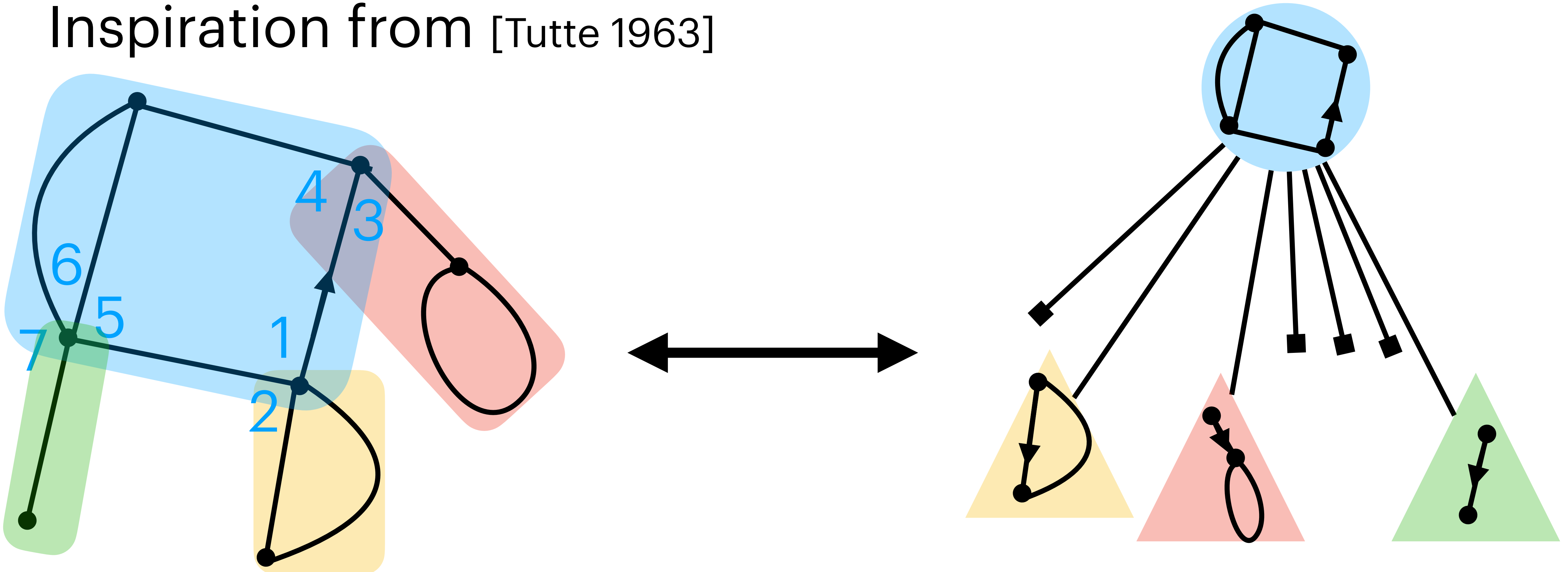

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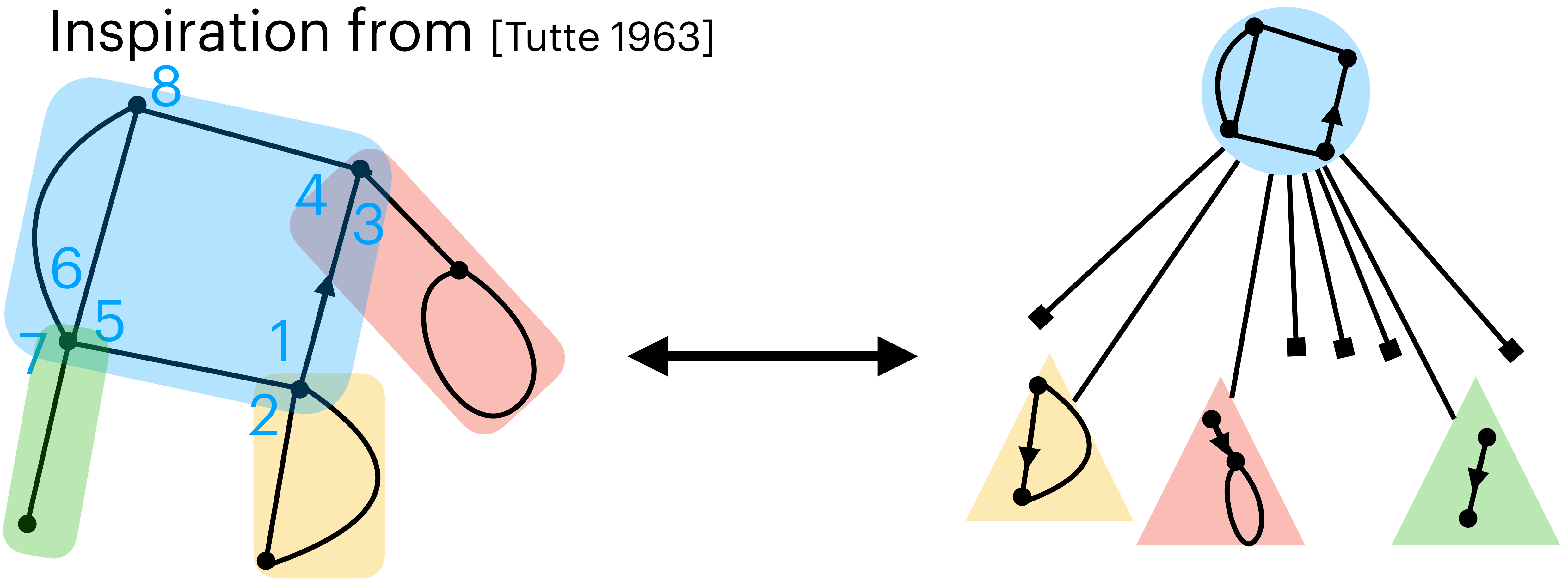

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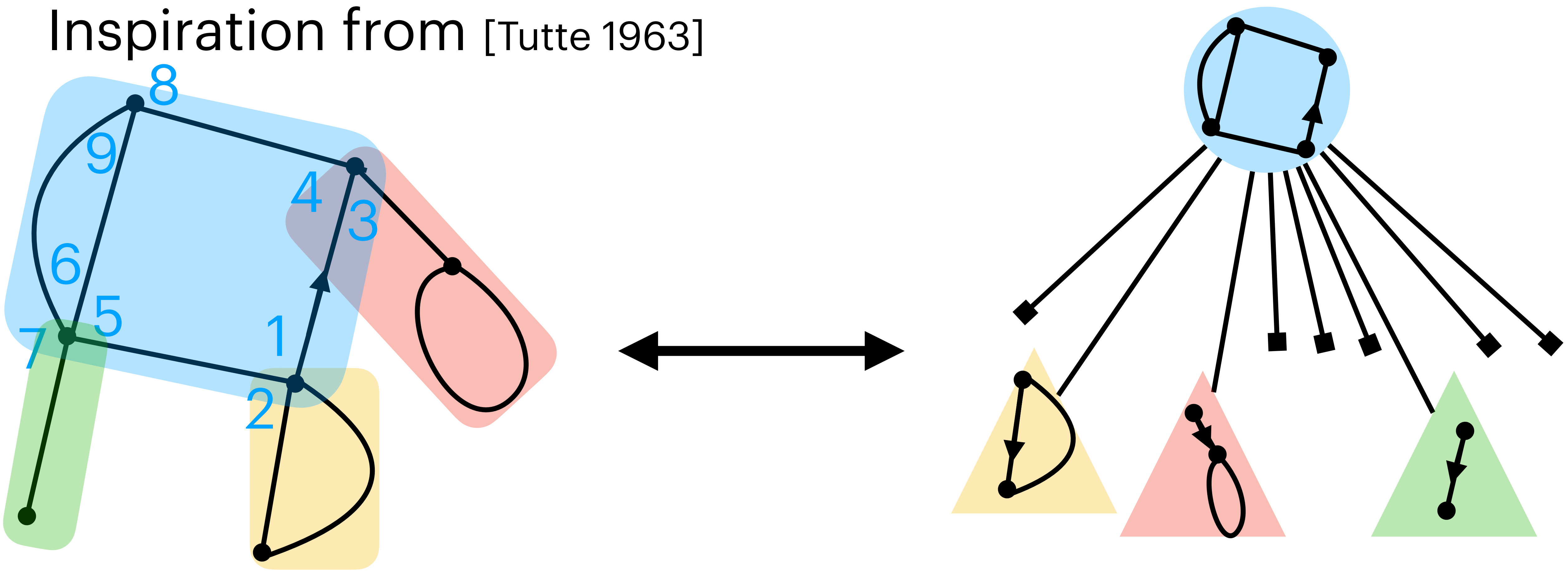

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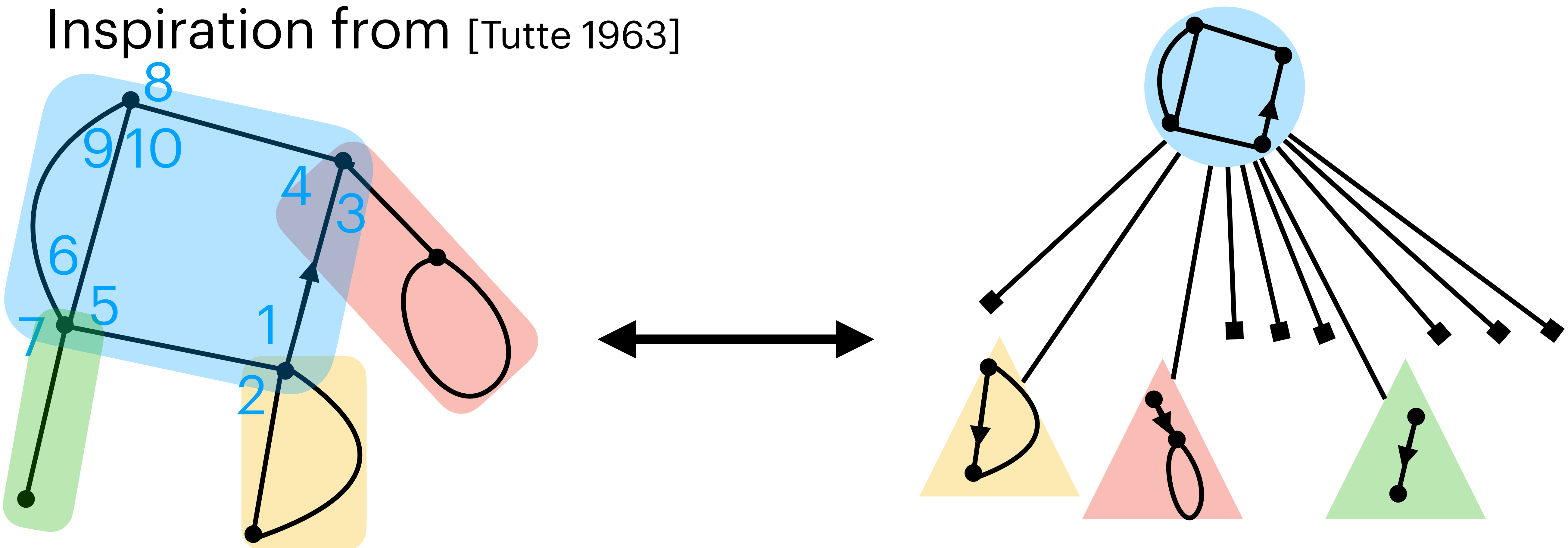

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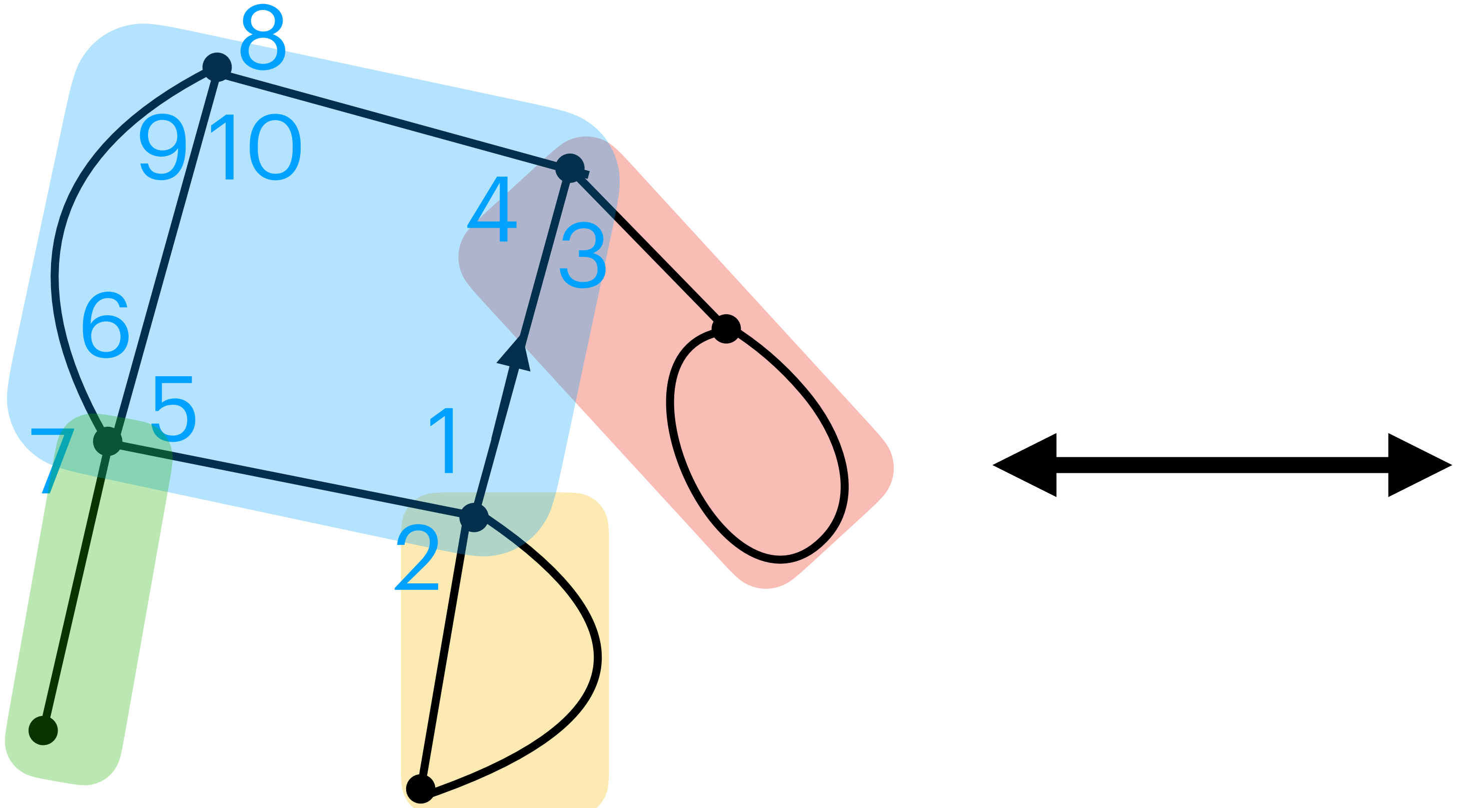

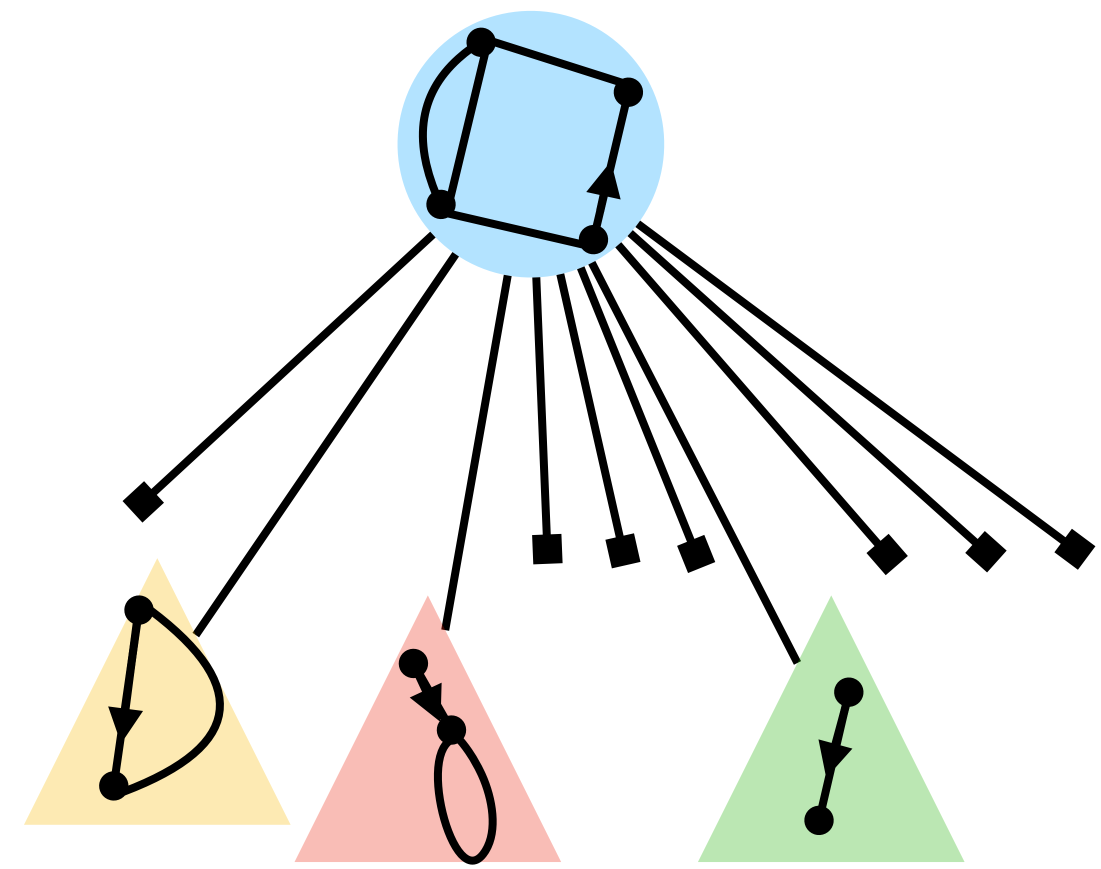

$$M(z) = B(zM^2(z))$$

GS of 2-connected maps

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$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{\mathfrak{m}} u^{\#blocks(\mathfrak{m})}$$

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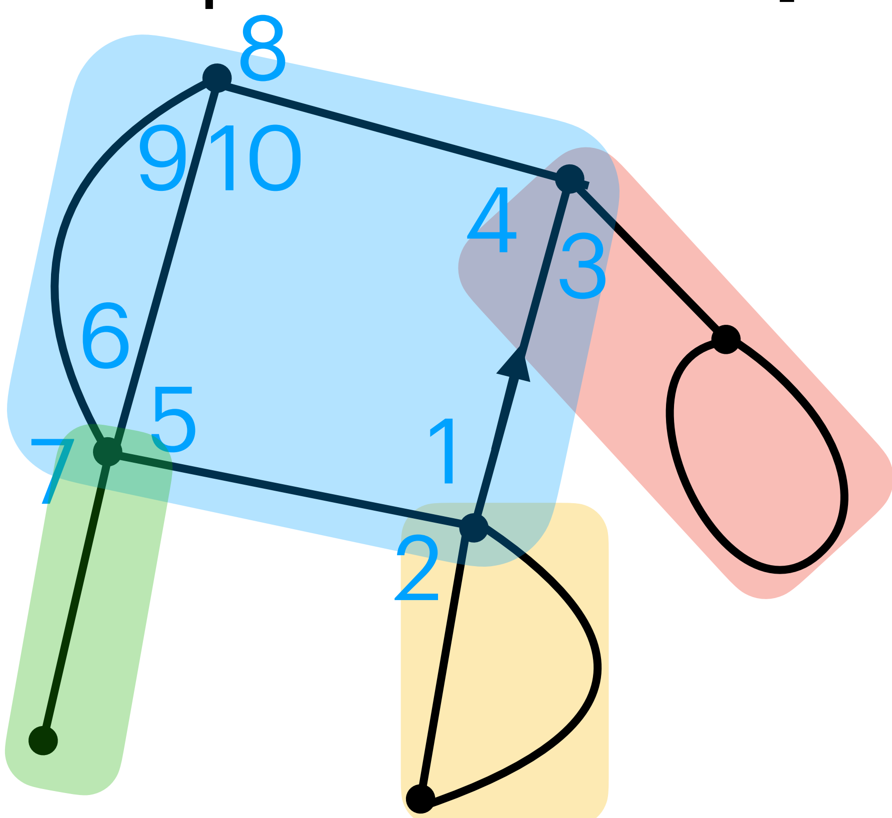

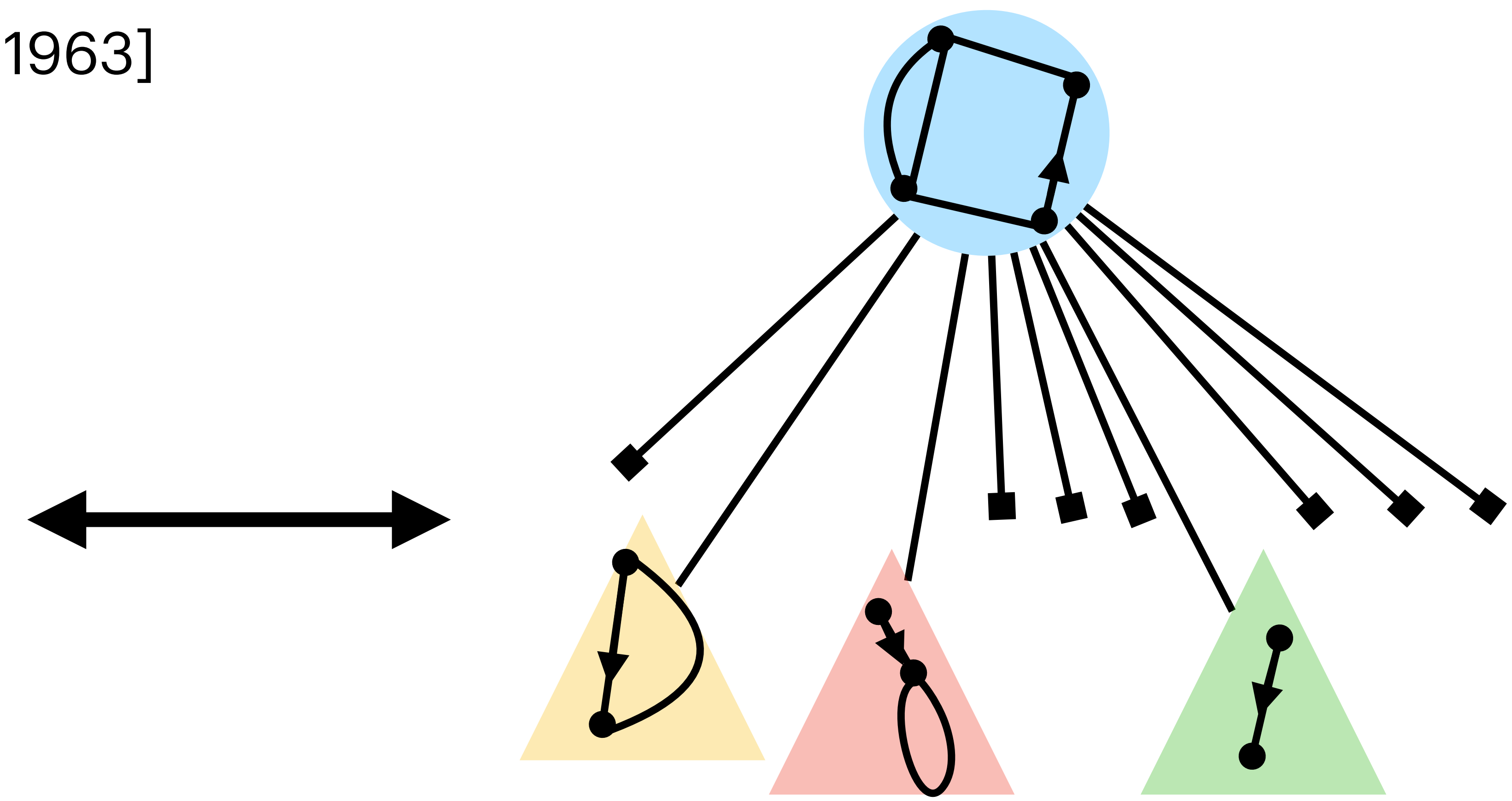

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With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one			
Scaling limit of M_n			

Decomposition of a map into blocks

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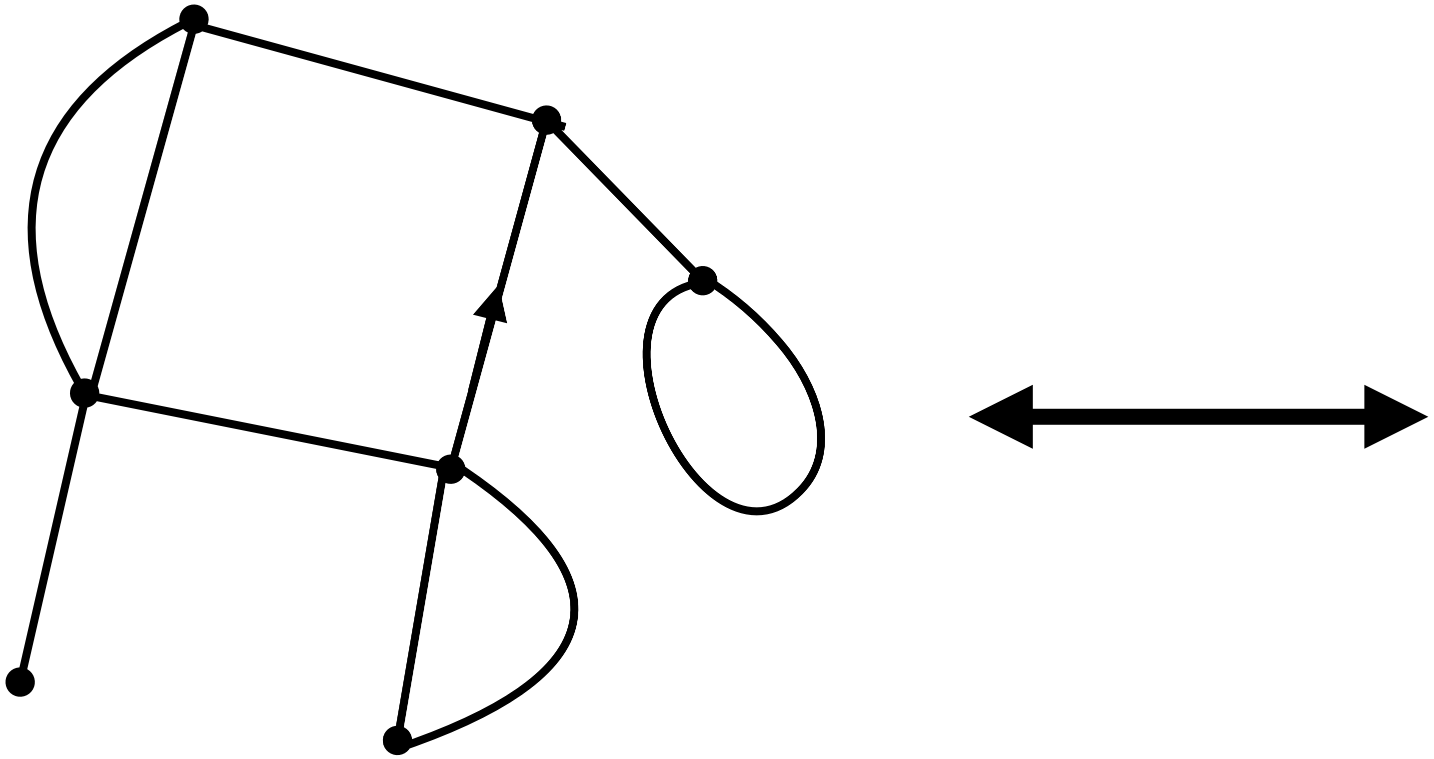

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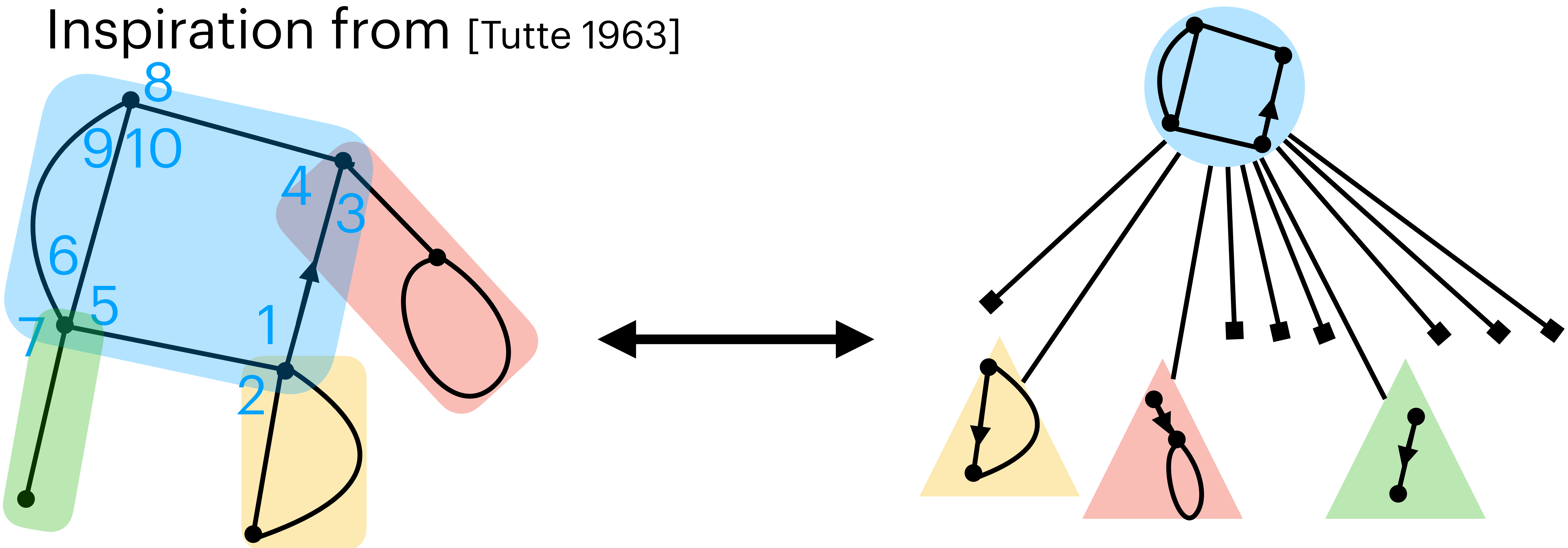

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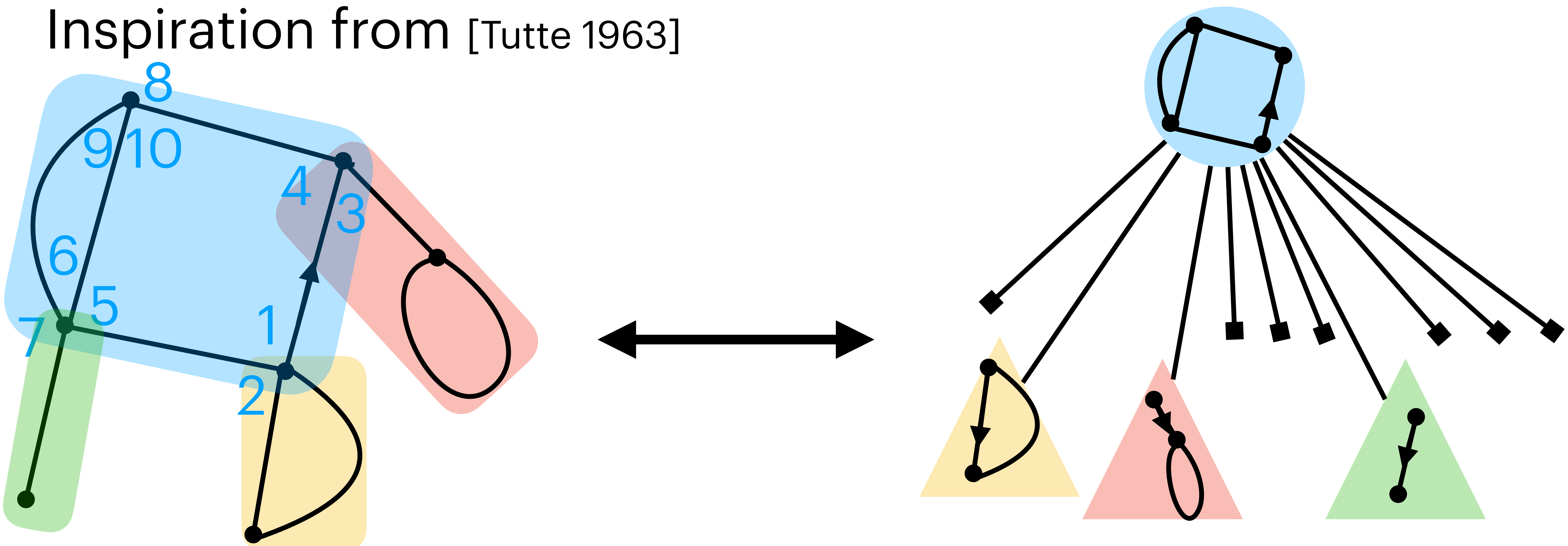

⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

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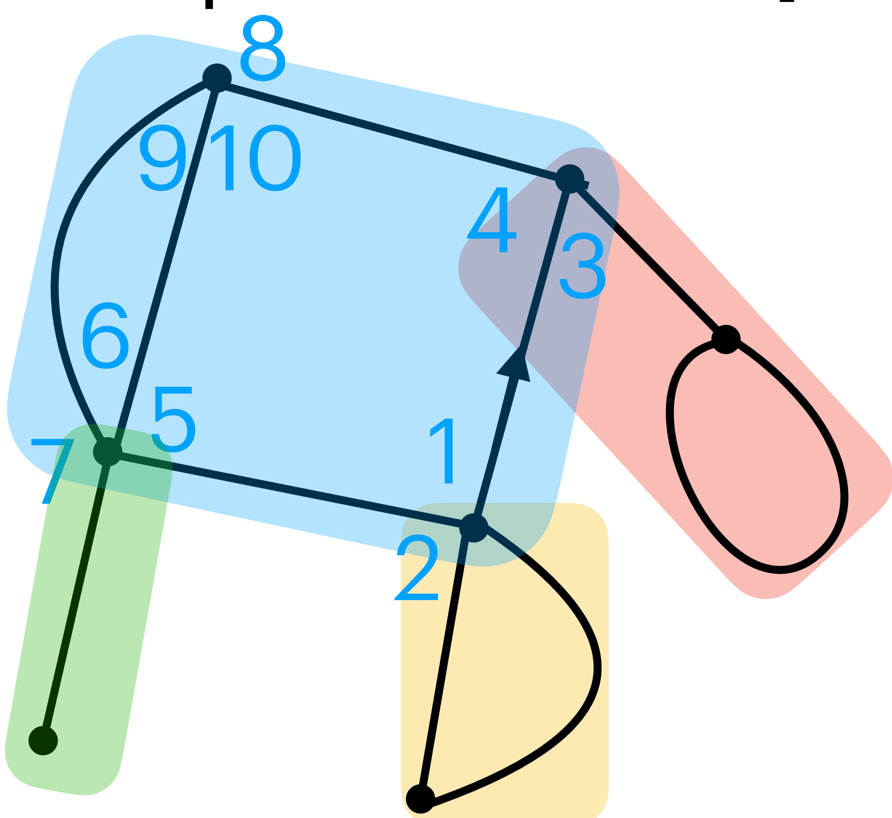

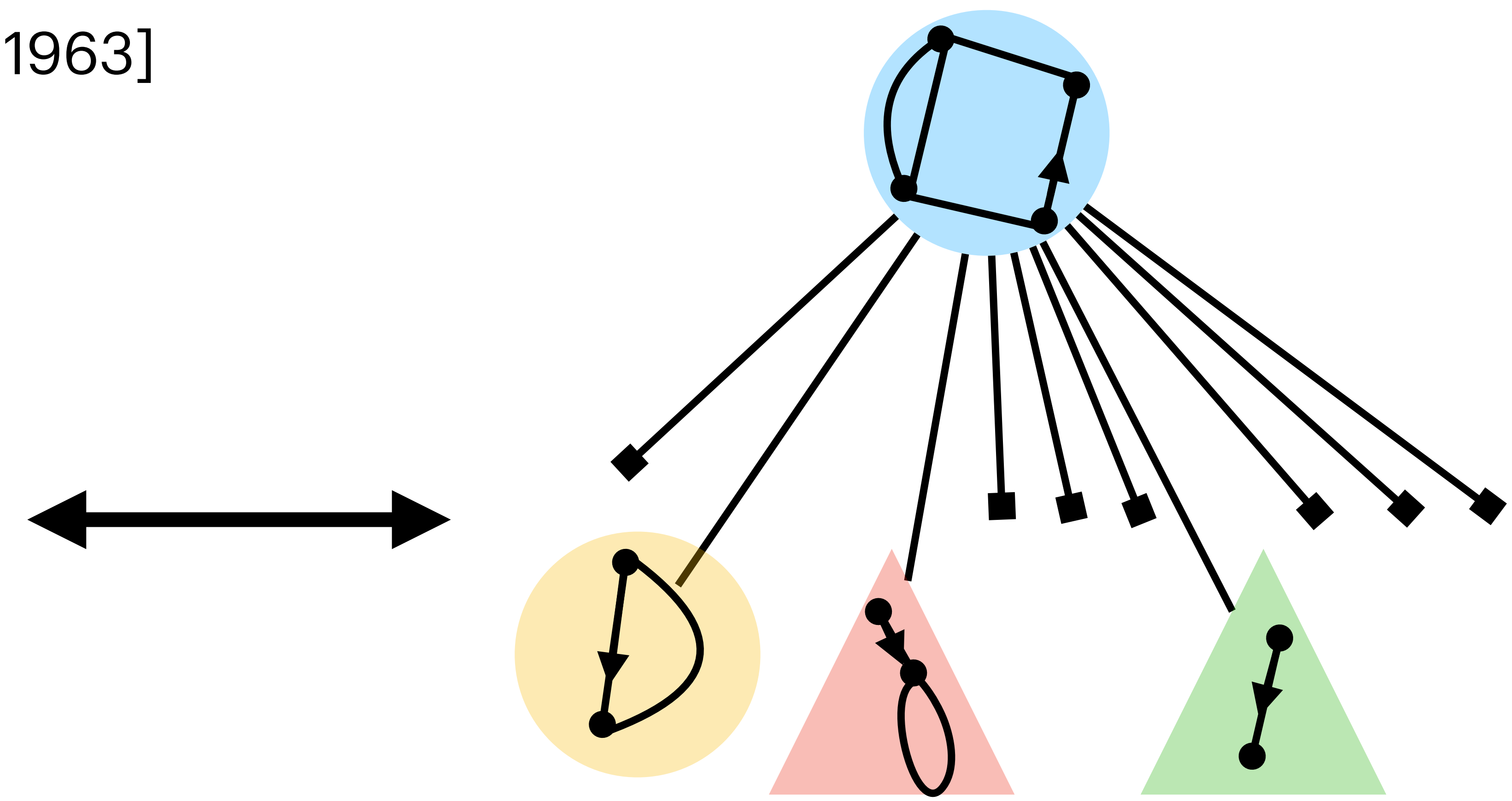

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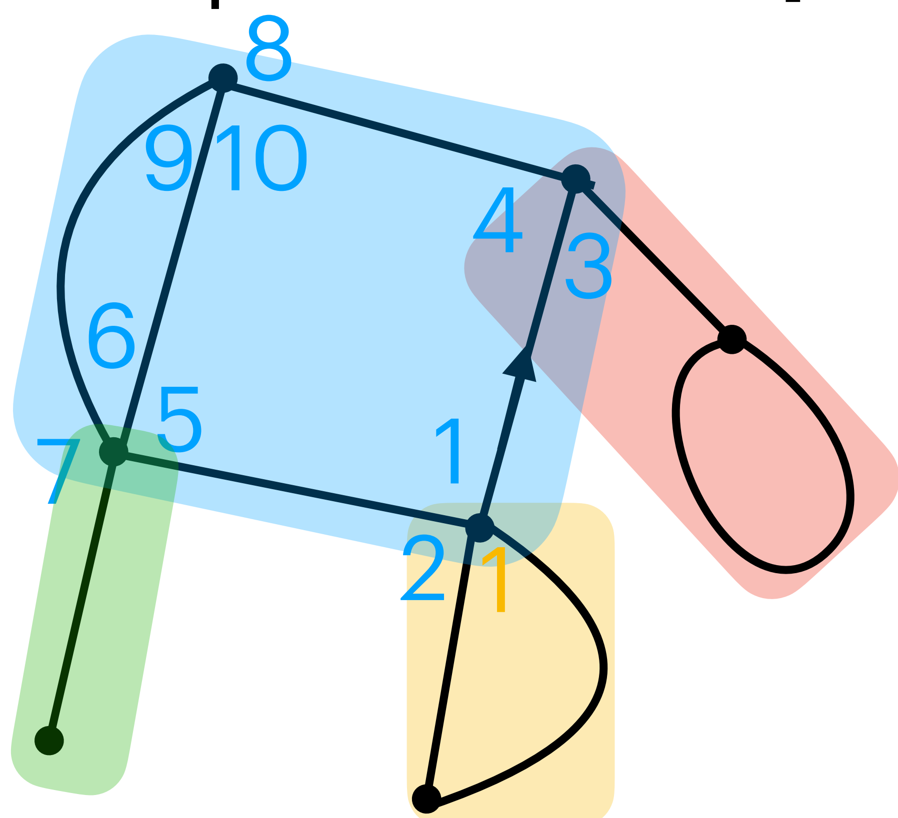

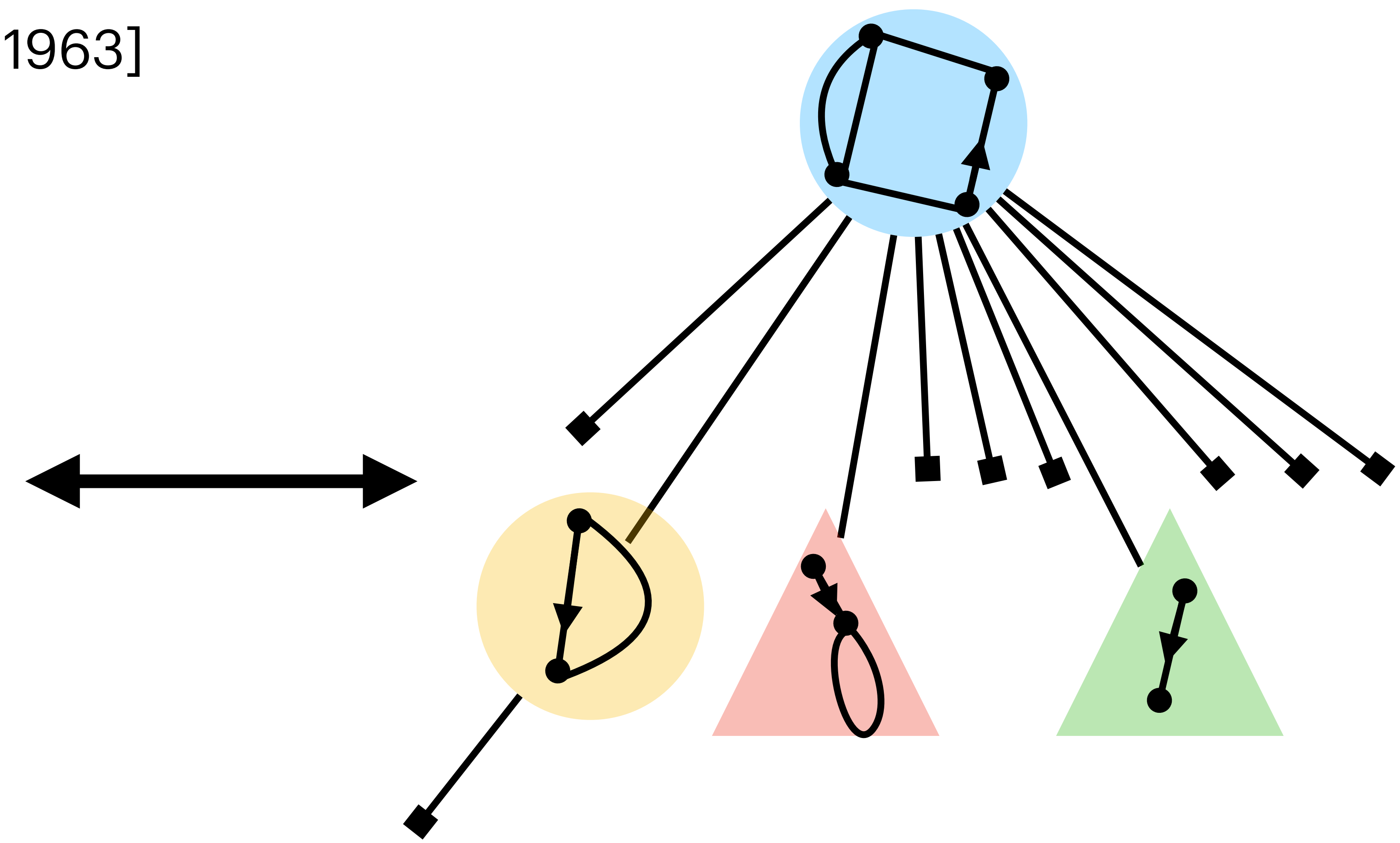

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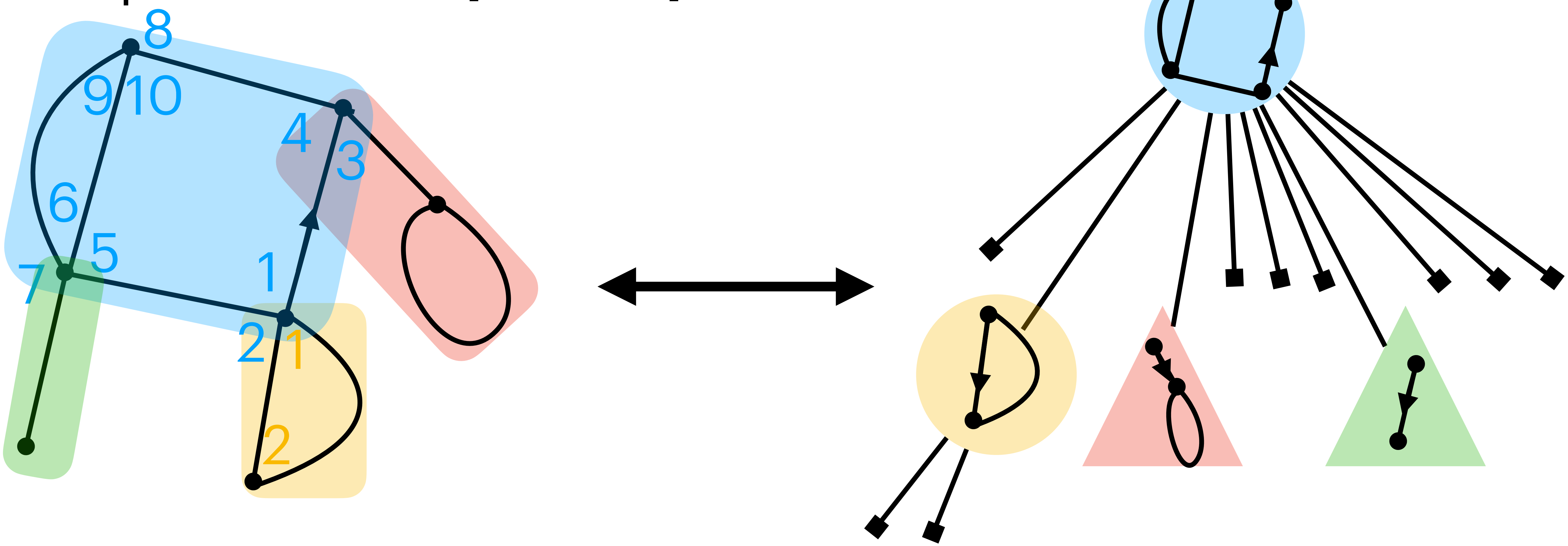

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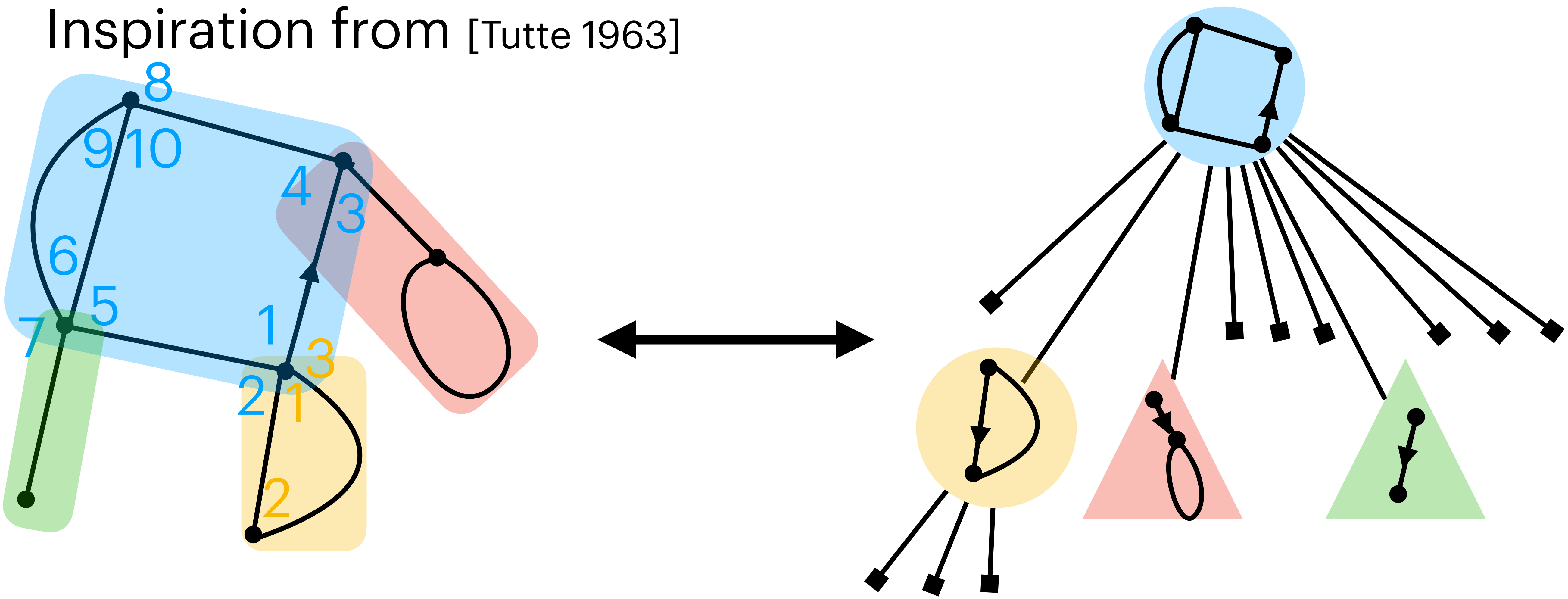

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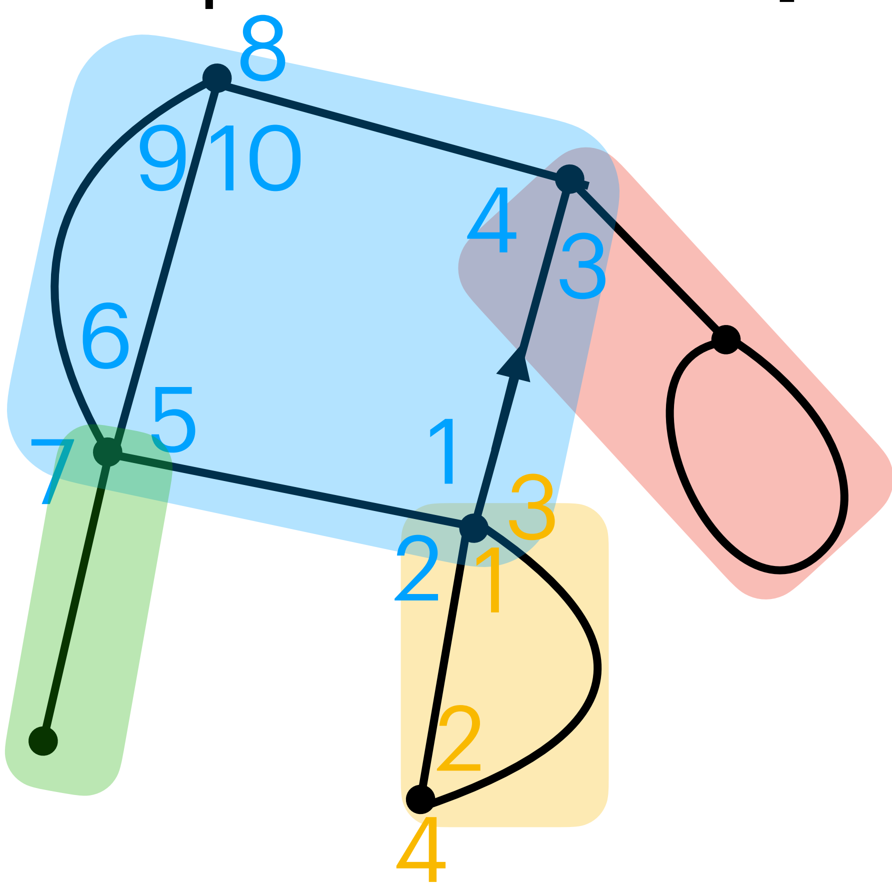

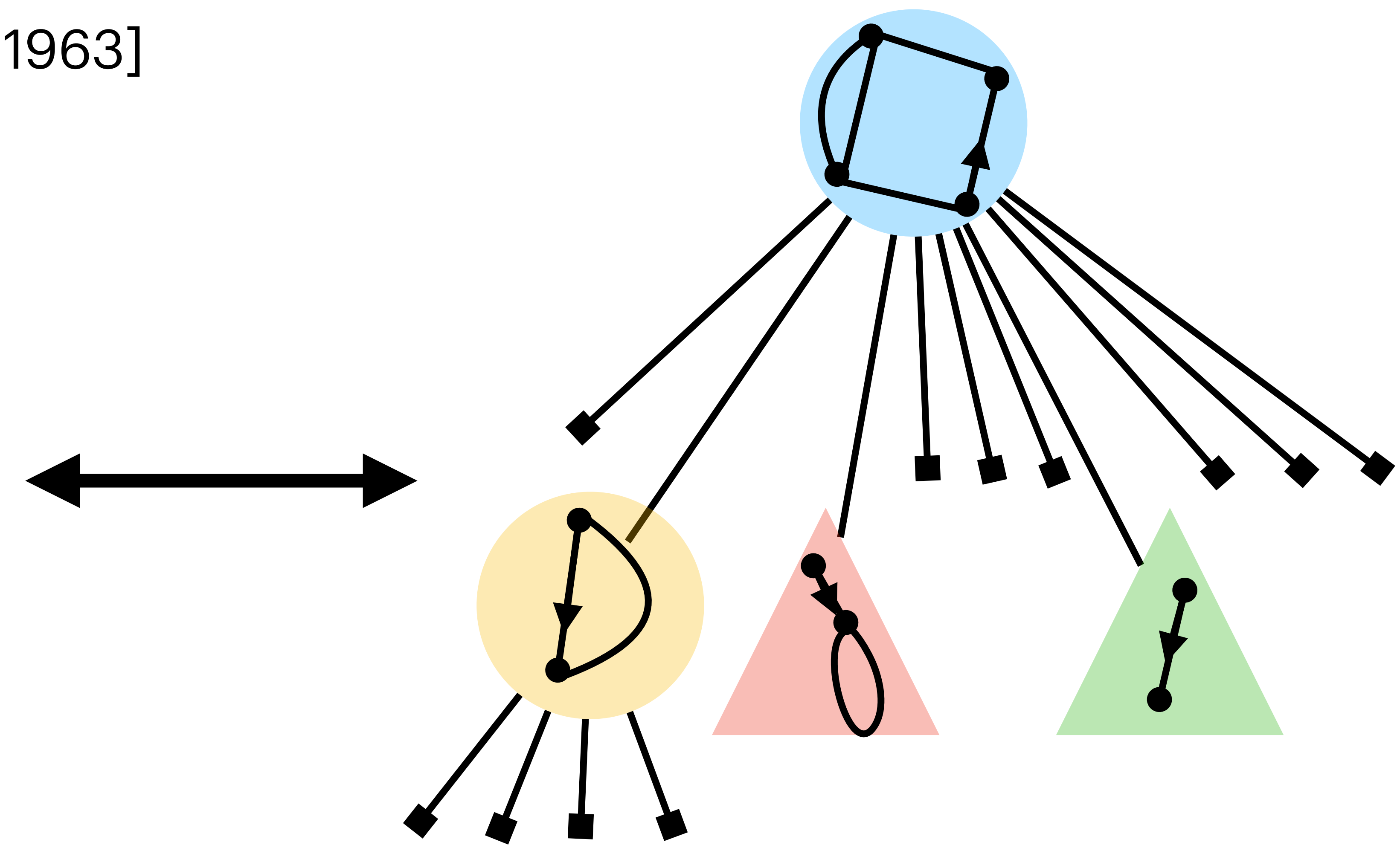

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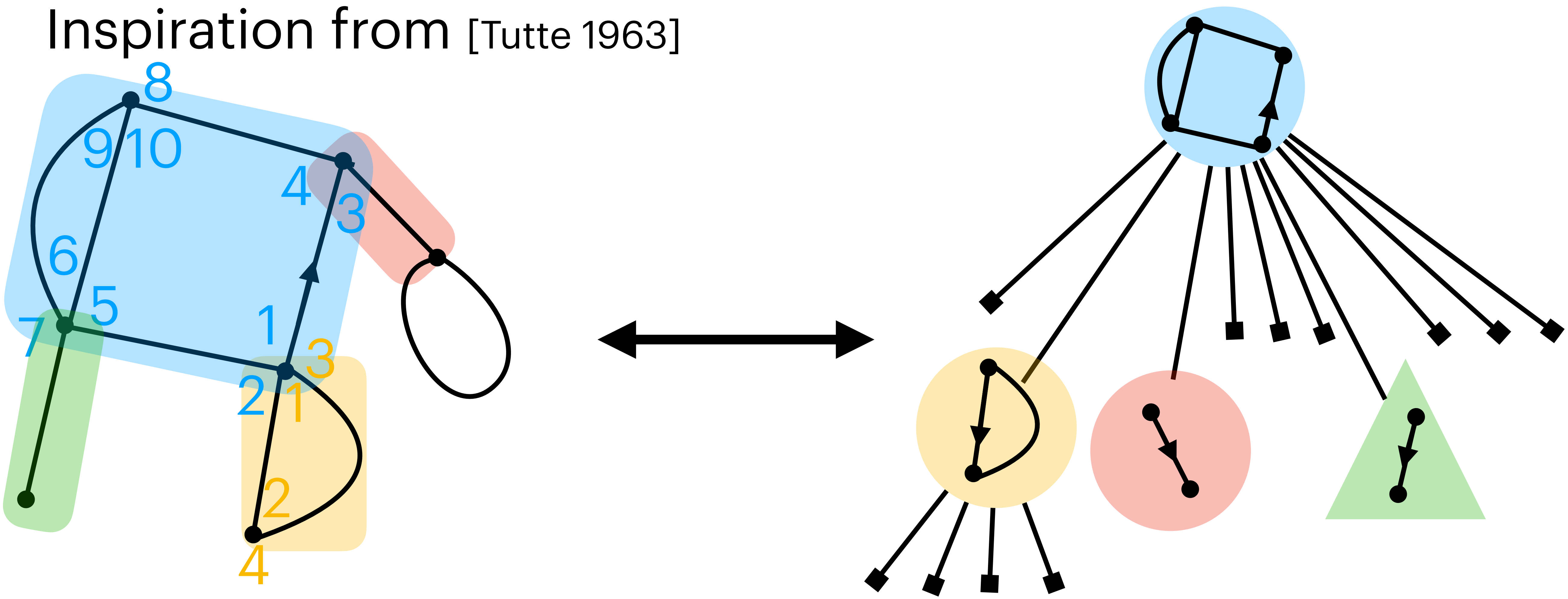

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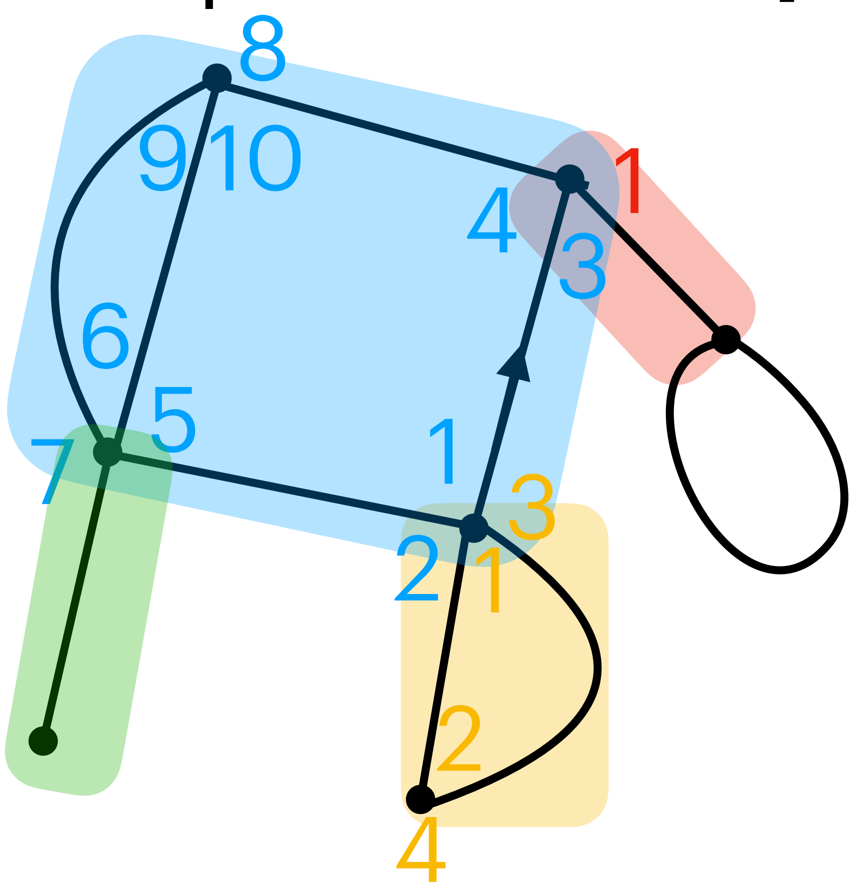
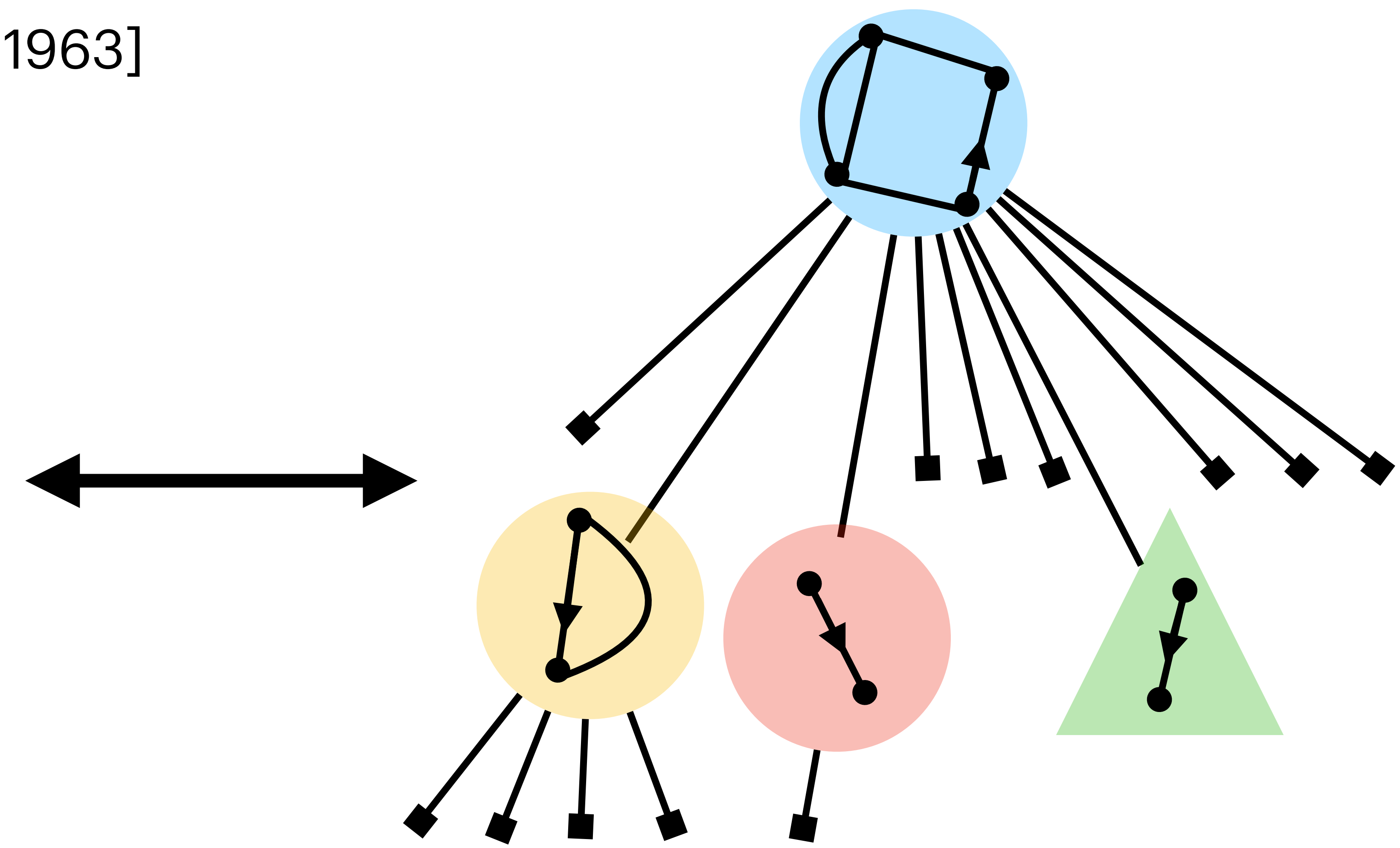




⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

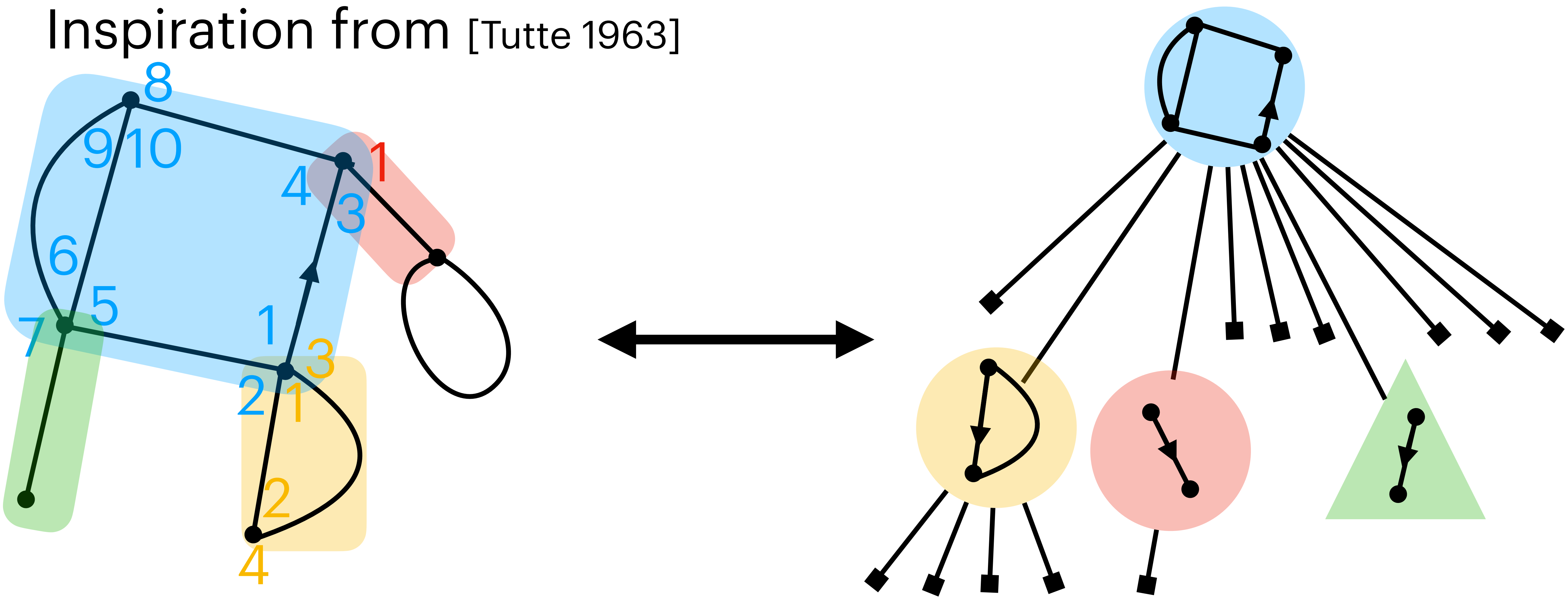
GS of 2-connected maps

With a weight u on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

Decomposition of a map into blocks

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$

Inspiration from [Tutte 1963]



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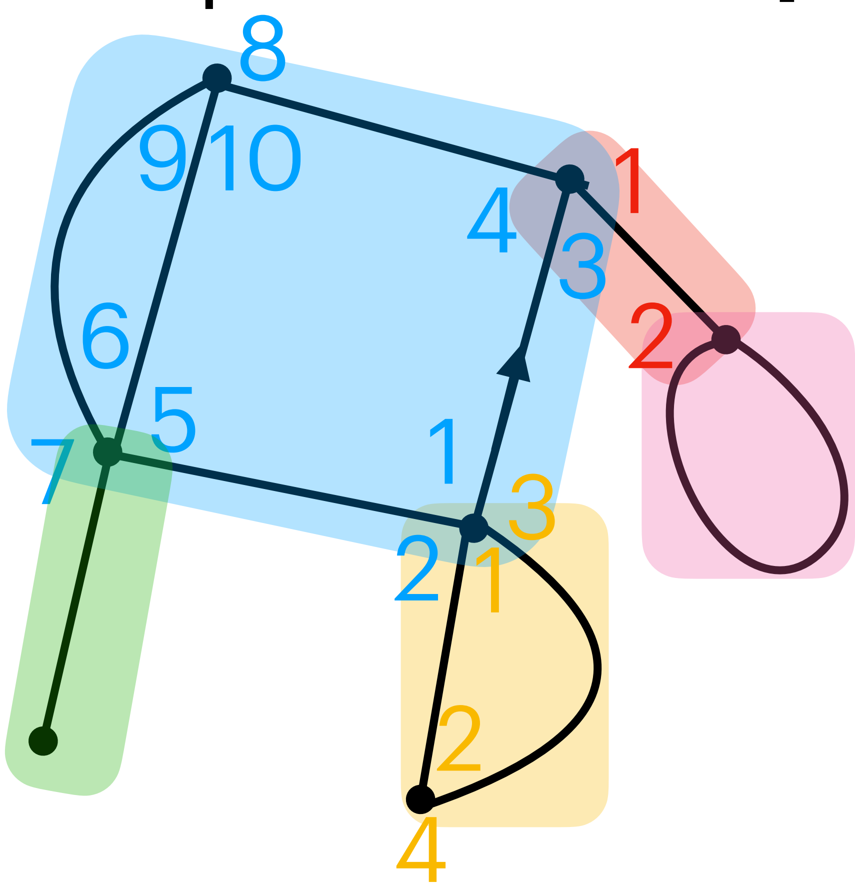

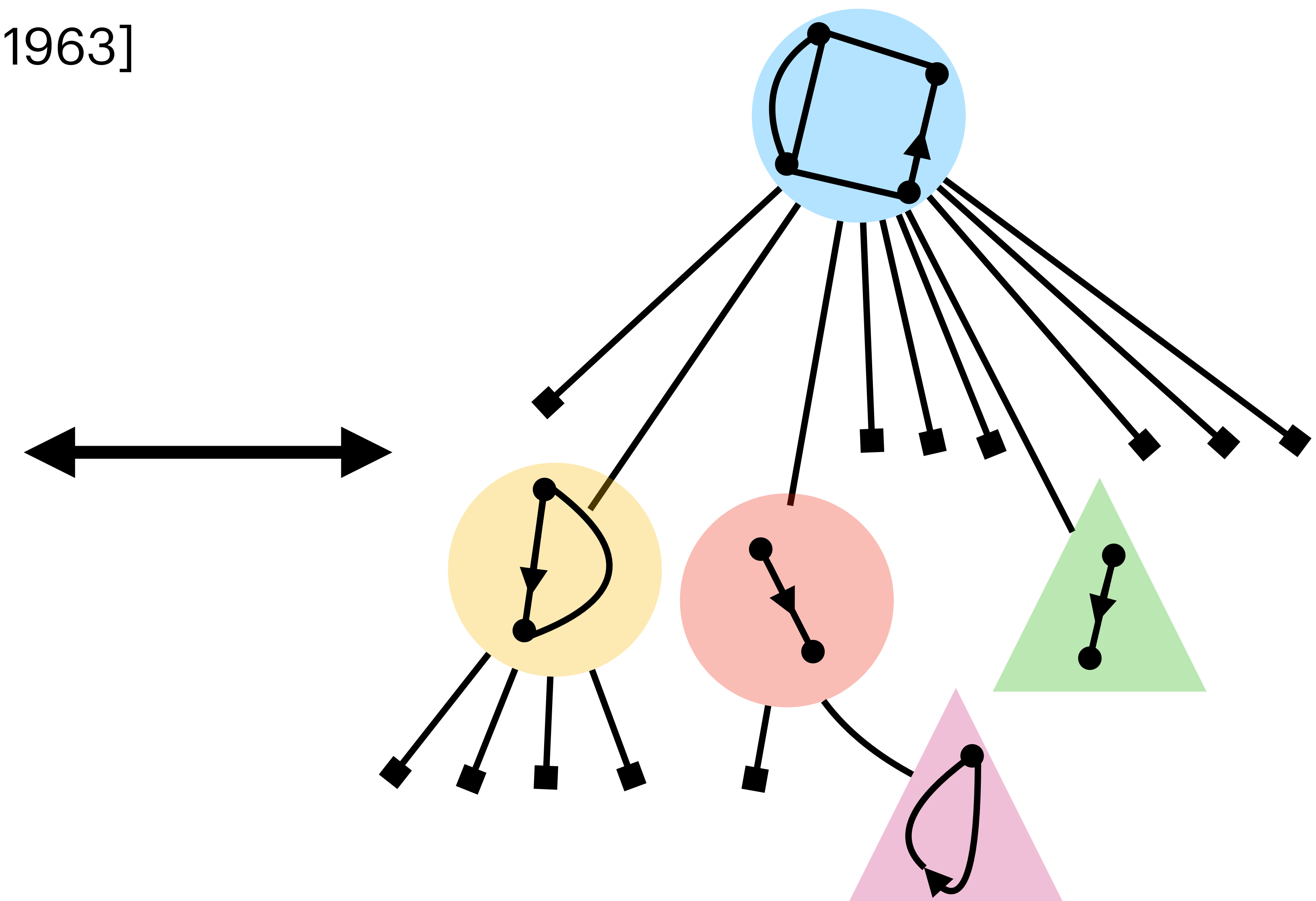

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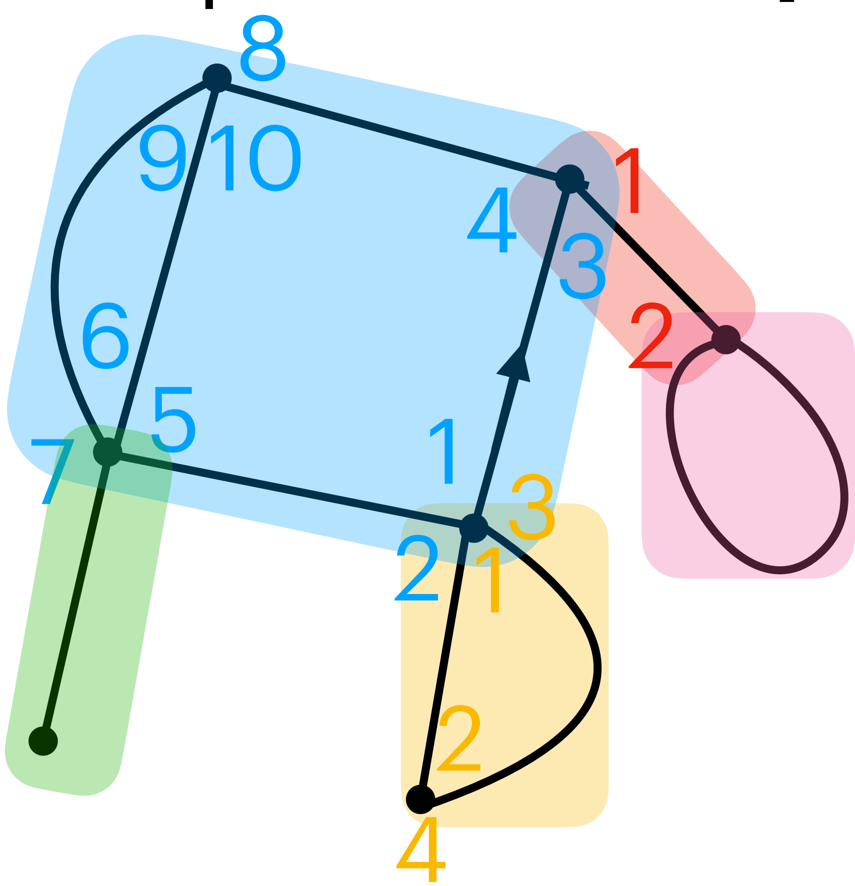

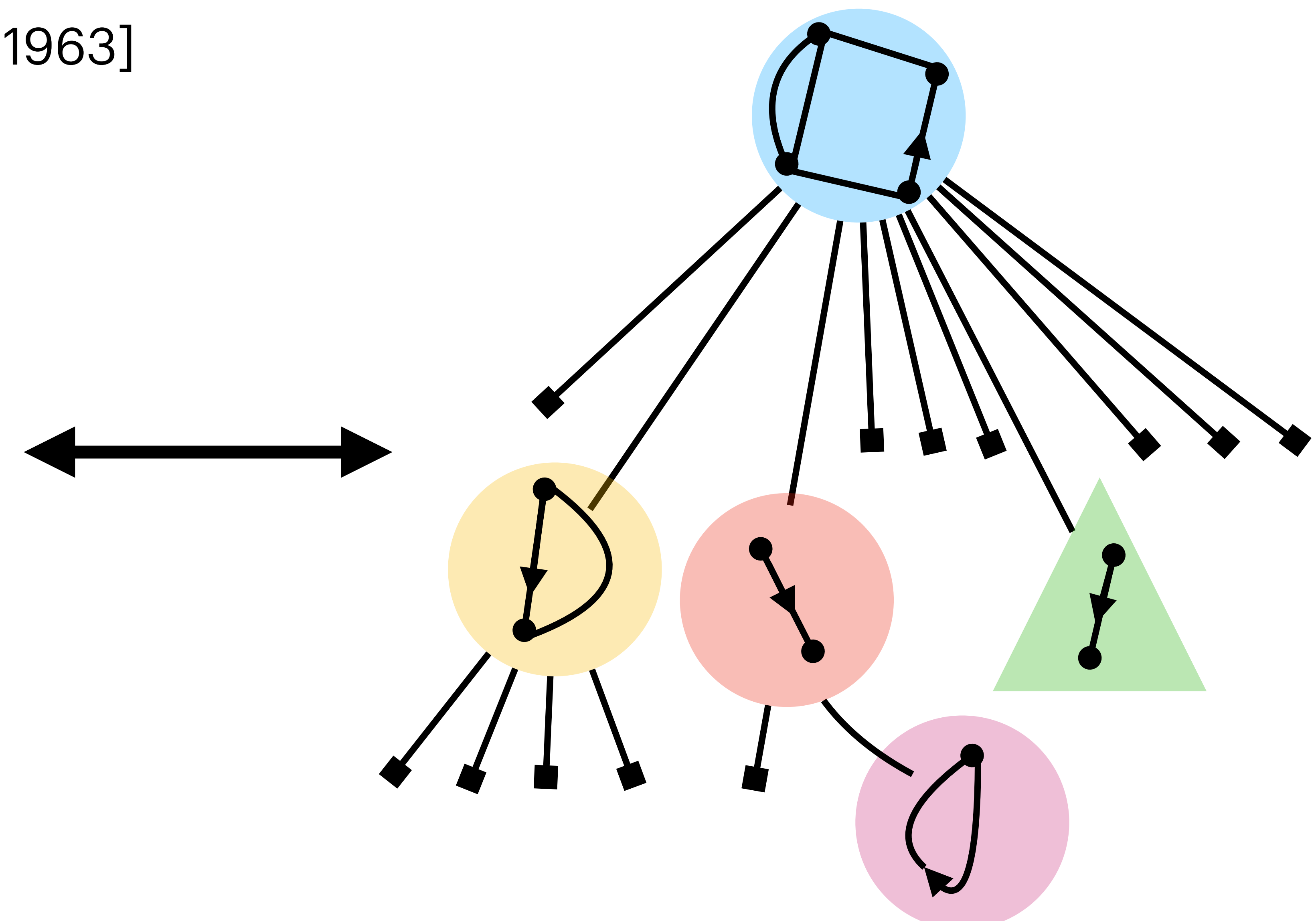

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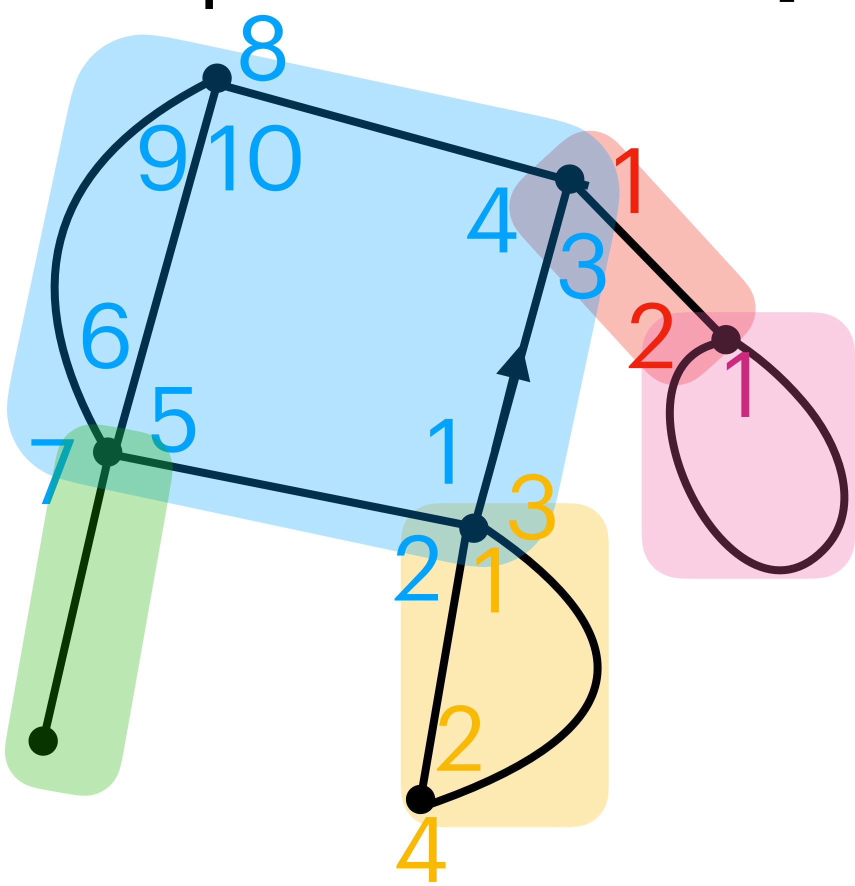

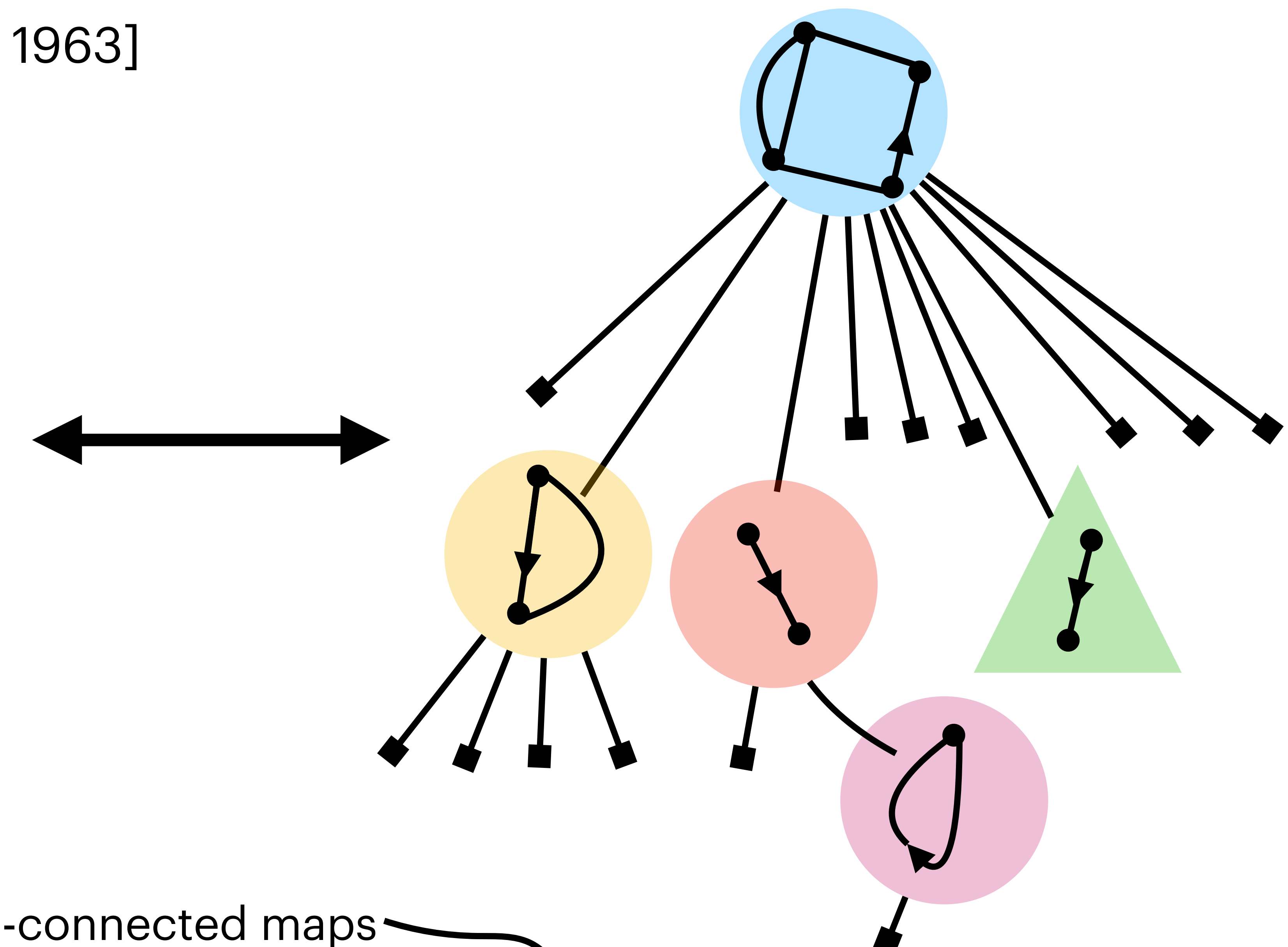

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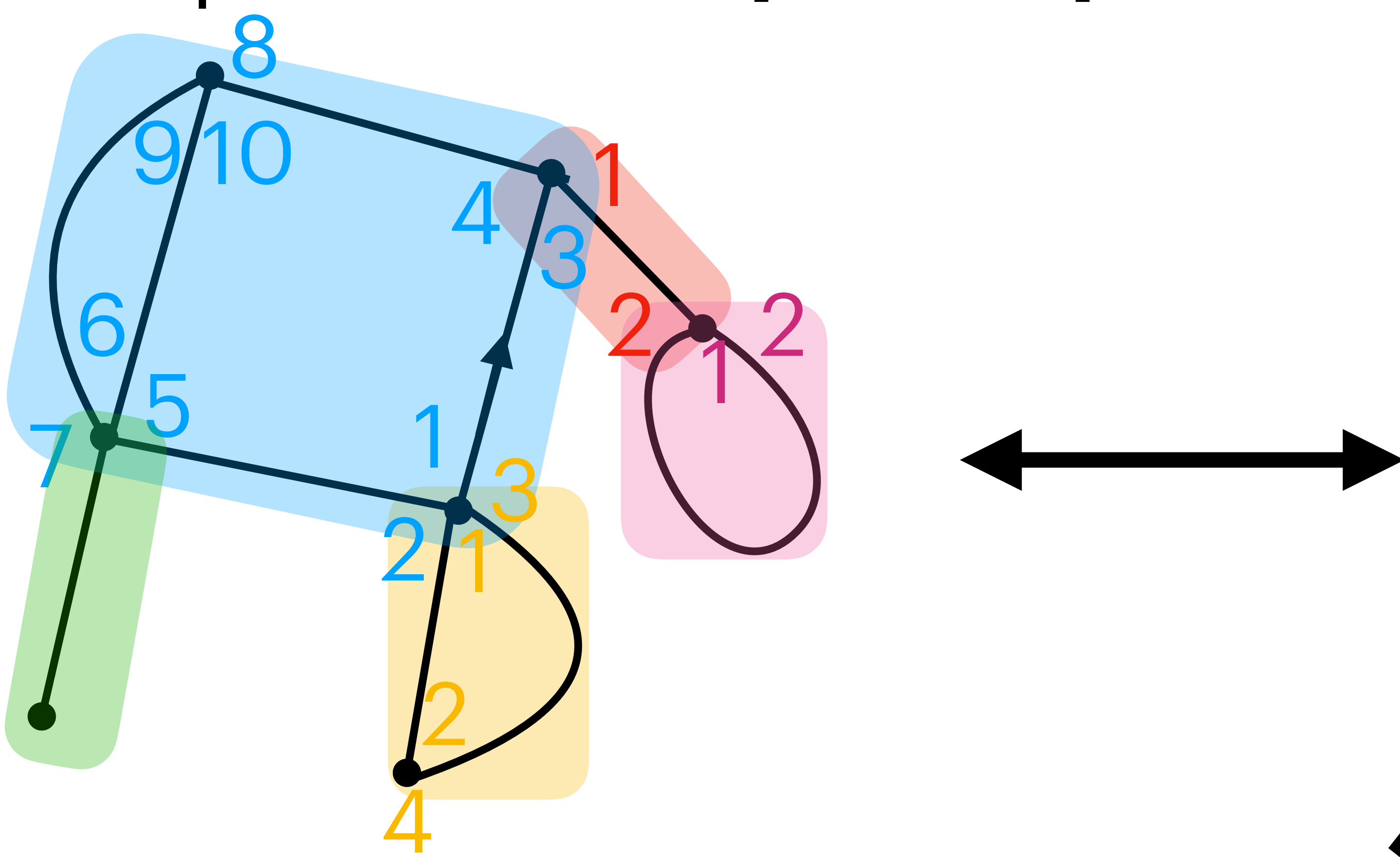

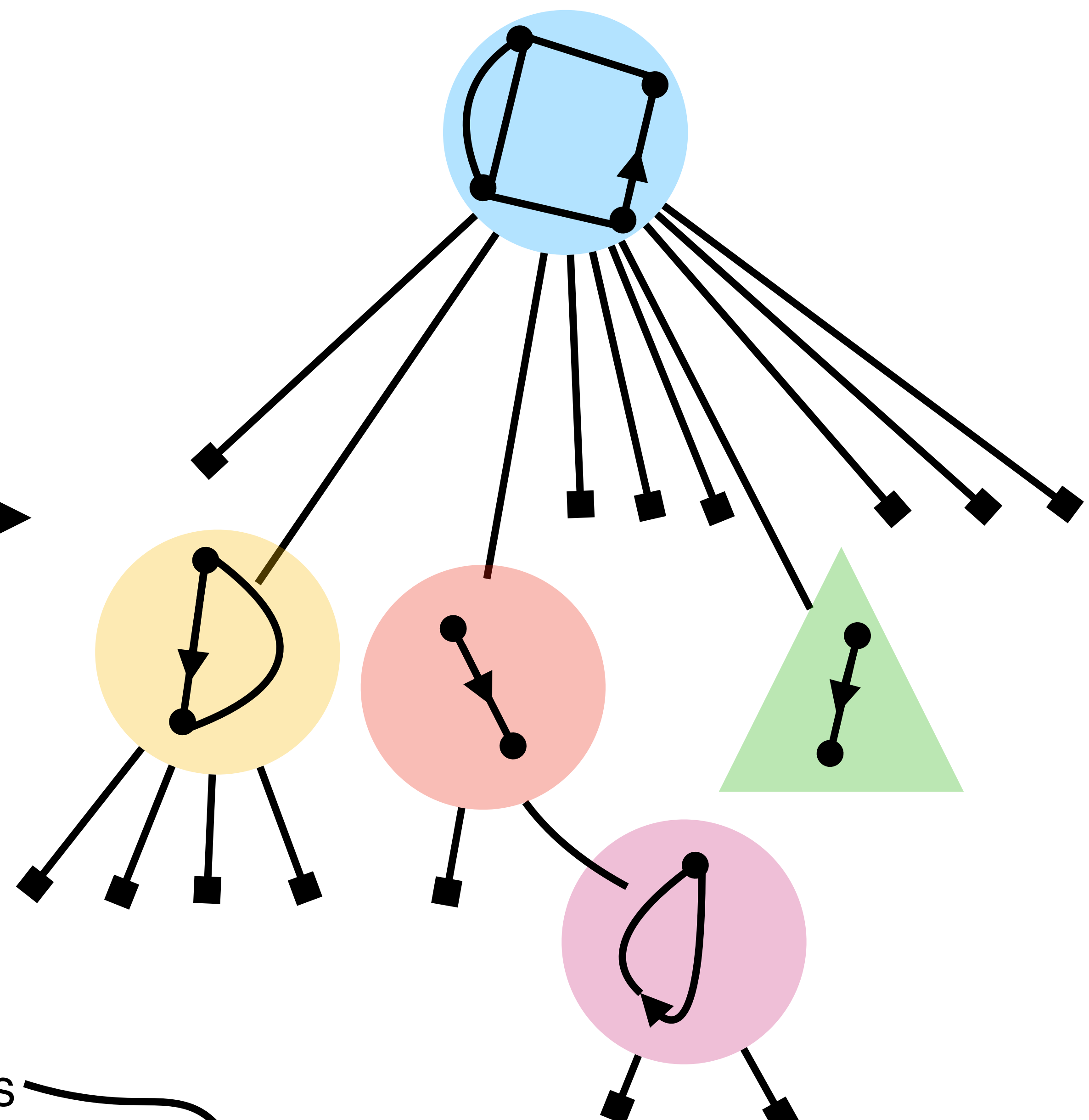

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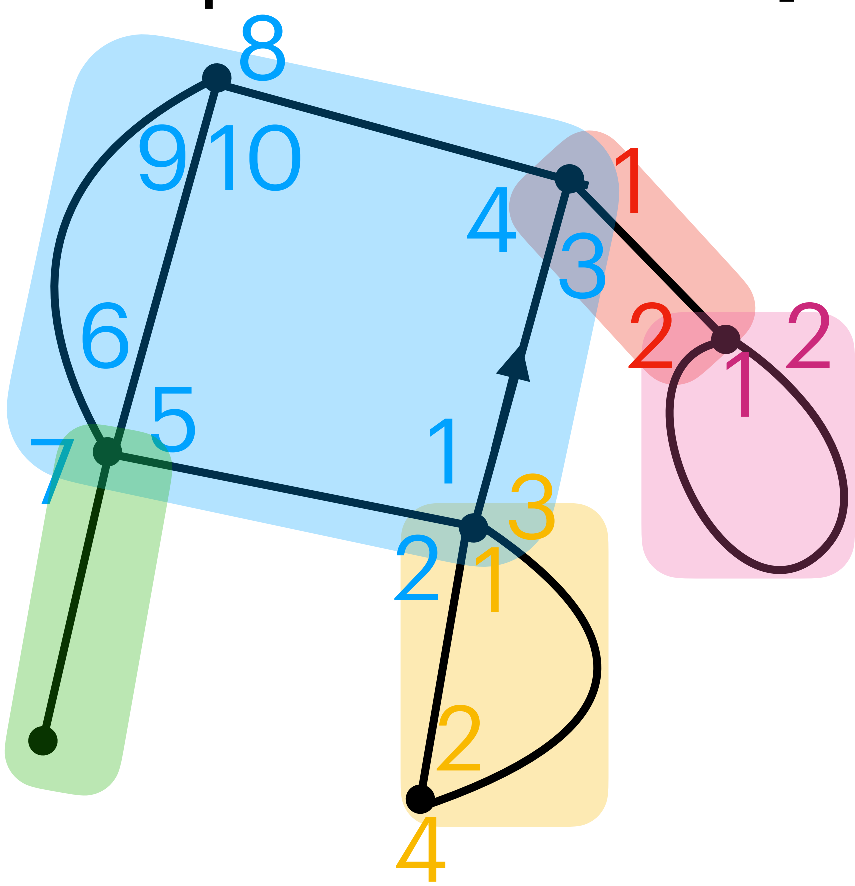

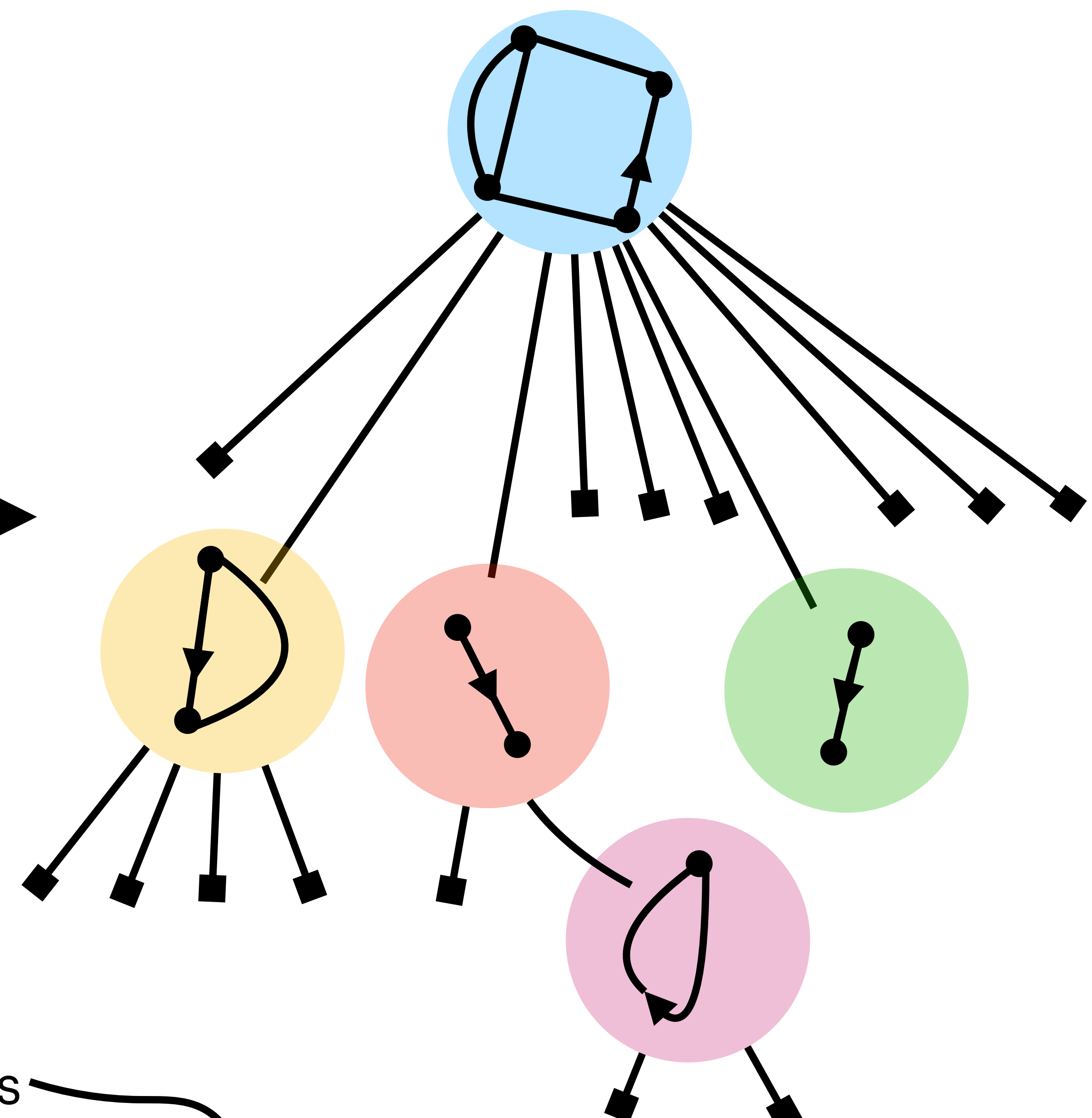

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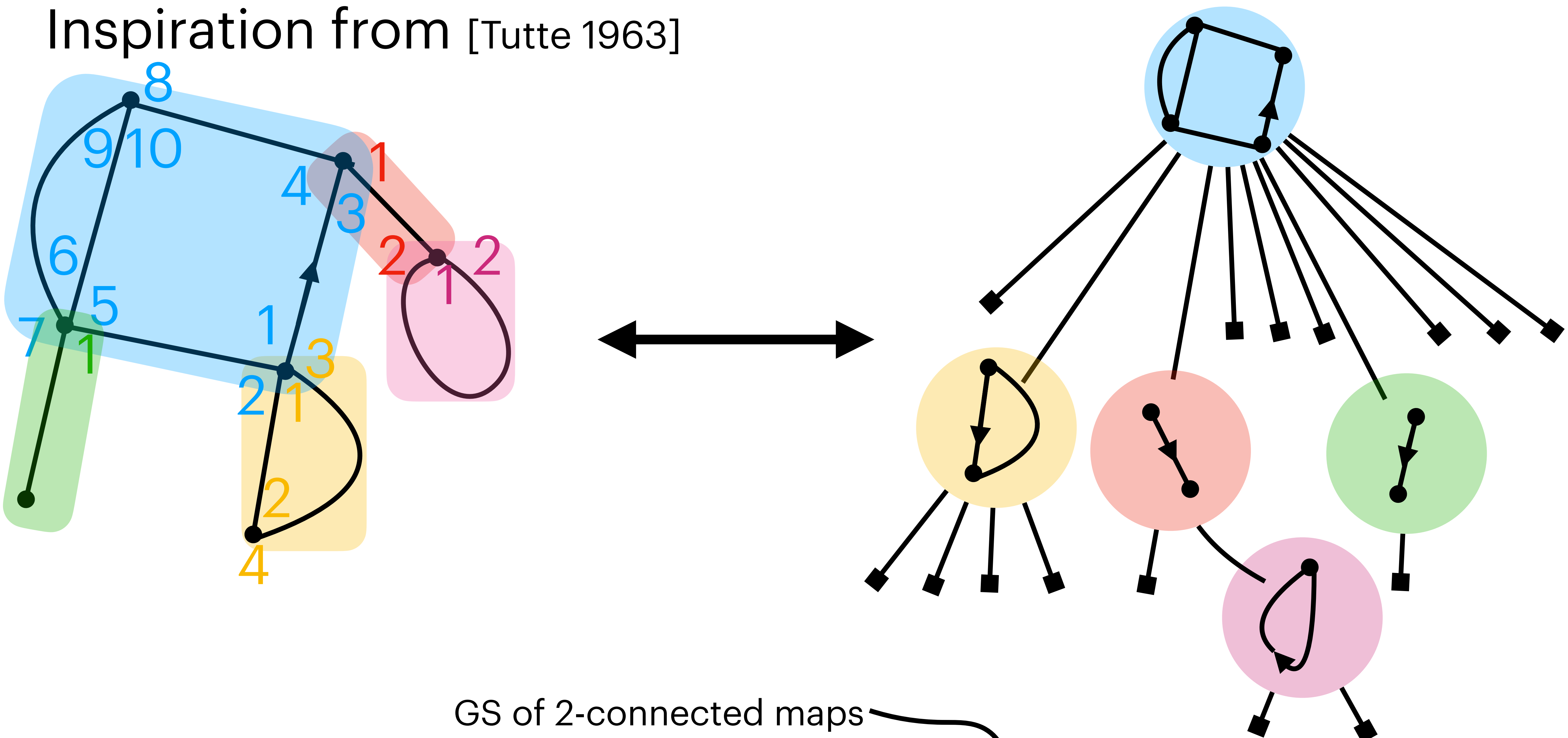

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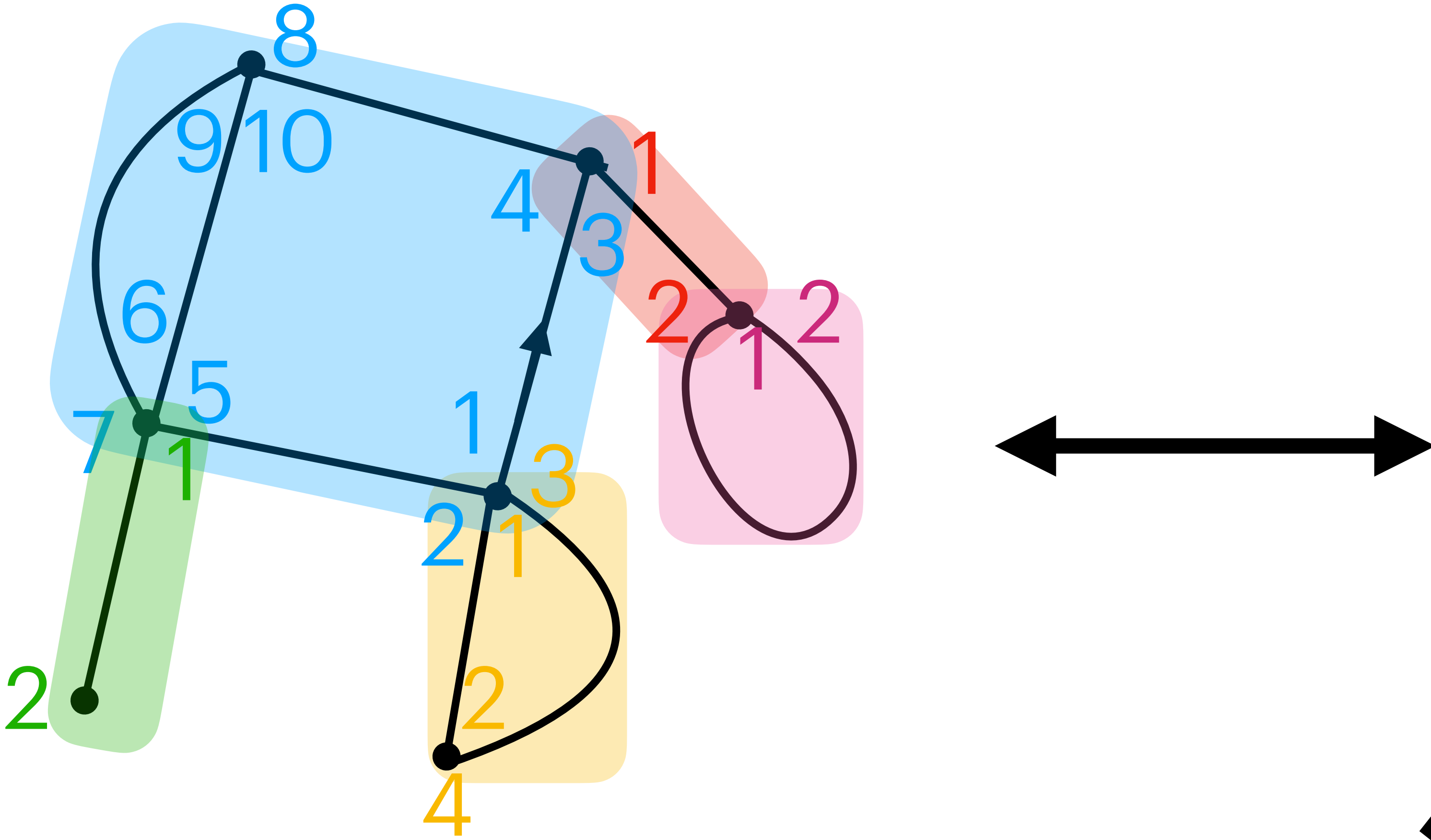
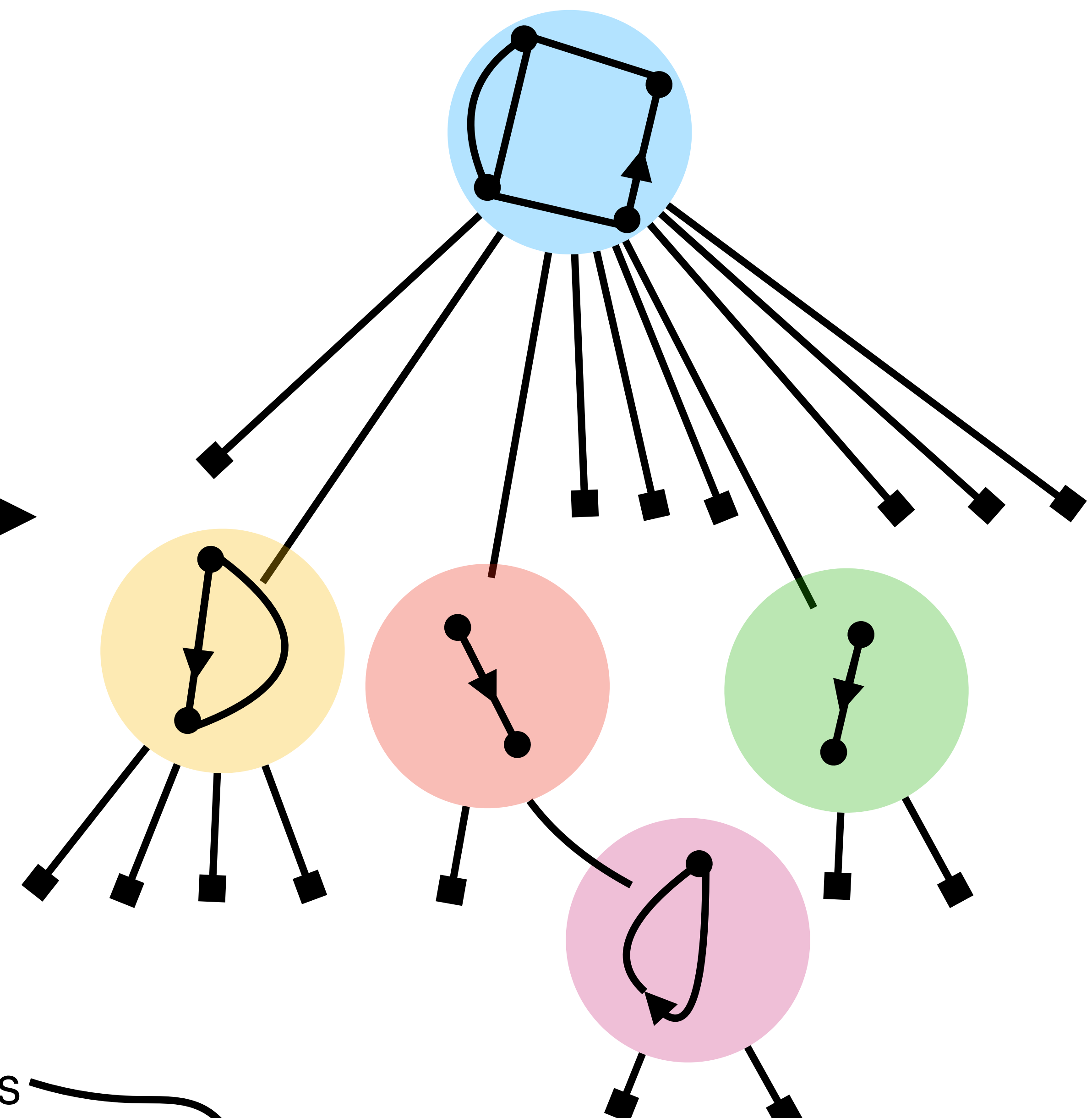
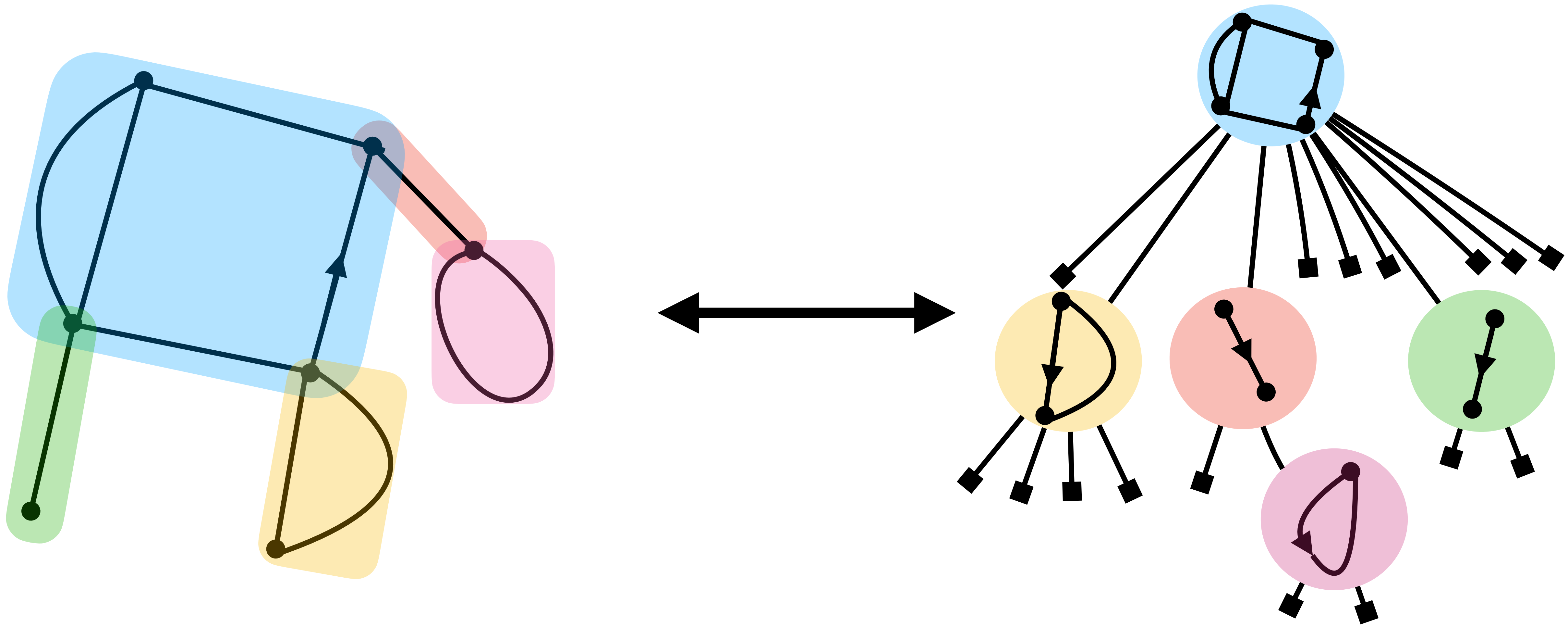




GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks: properties



- Internal node (with k children) of $T_{\mathfrak{m}}$ $\leftrightarrow$ block of $\mathfrak{m}$ of size $k/2$;
- $\mathfrak{m}$ is entirely determined by $T_{\mathfrak{m}}$ and $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ where $\mathfrak{b}_v$ is the block of $\mathfrak{m}$ represented by v in $T_{\mathfrak{m}}$.

T_{M_n} gives the block sizes of a random map M_n.

Galton-Watson trees for map blocks

μ-Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

Galton-Watson trees for map blocks

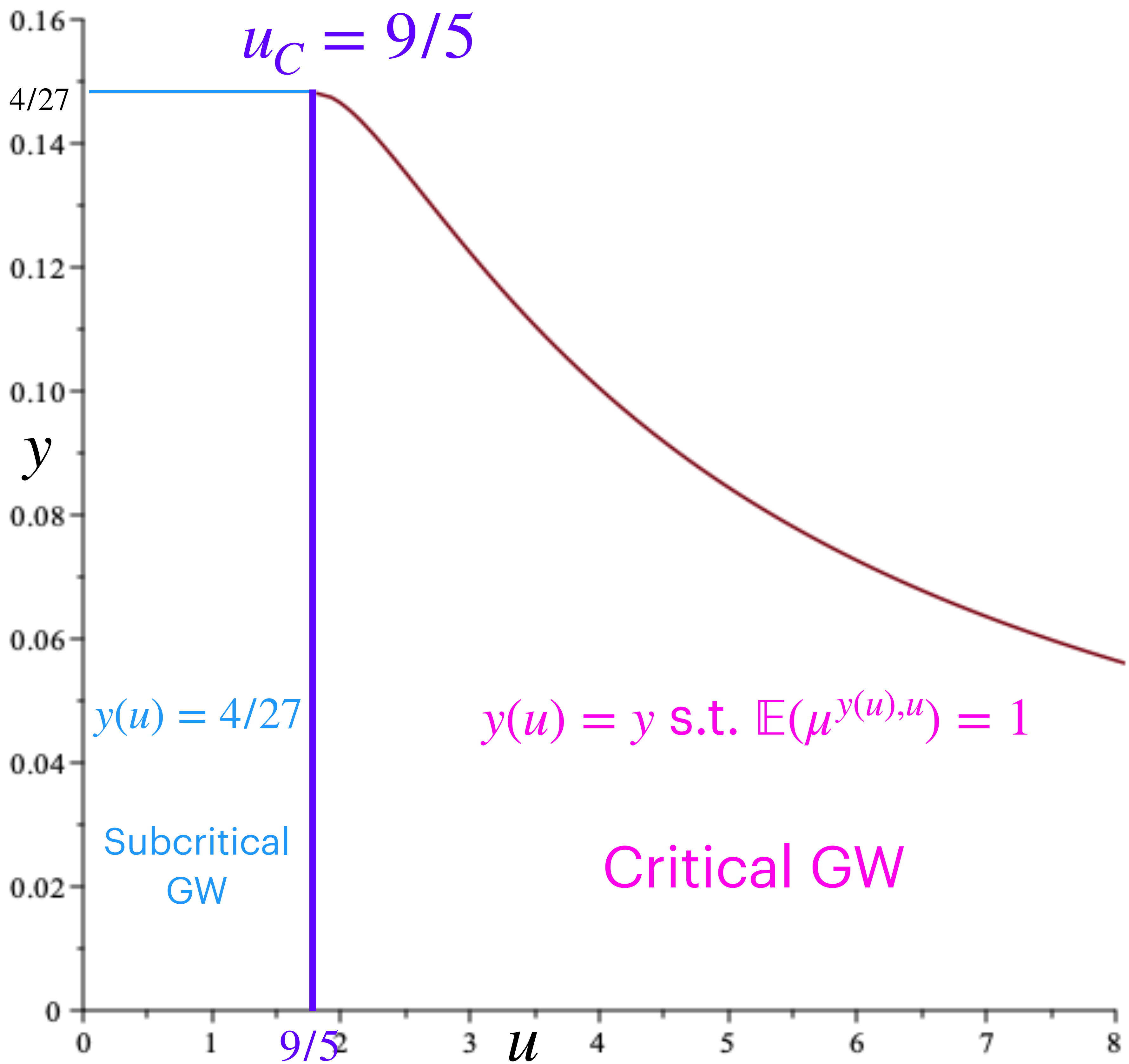
μ-Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

Theorem [Fleurat, S. 23] $u > 0$

If $M_n \hookrightarrow \mathbb{P}_{n,u'}$ then there exists an (explicit) $y = y(u)$ s.t. T_{M_n} has the law of a Galton-Watson tree of reproduction law $\mu^{y,u}$ conditioned to be of size $2n$, with

$$\mu^{y,u}(\{2k\}) = \frac{B_k y^k u^{\mathbf{1}_{k\neq 0}}}{uB(y) + 1 - u}.$$

Phase transition



"Map regime" "Tree regime"

Largest blocks?

- Degrees of T_{M_n} give the block sizes of the map M_n;
- Largest degrees of a Galton-Waston tree are well-known [Janson 2012].

Rough intuition

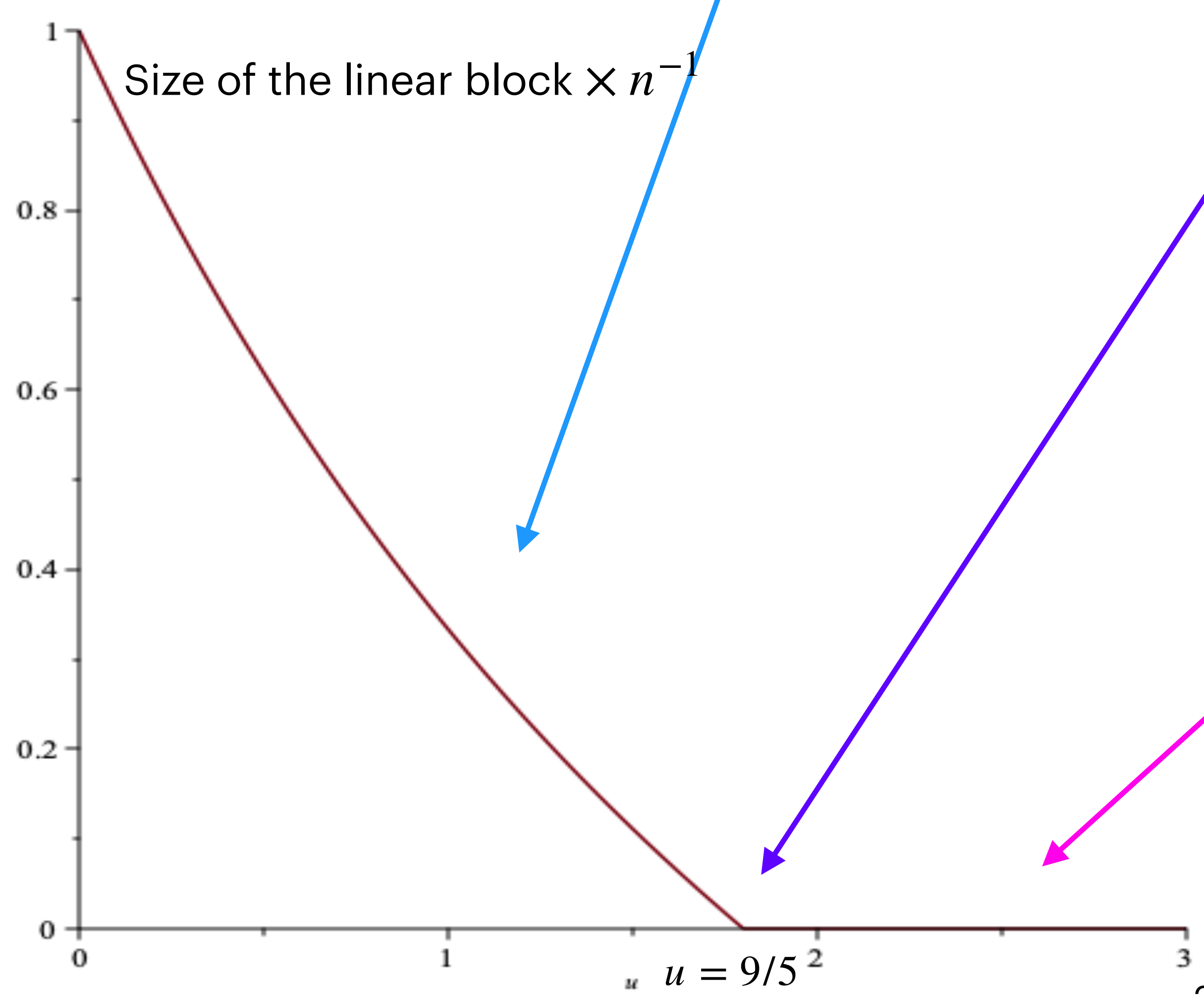
	$u < 9/5$	$u = 9/5$	$u > 9/5$
$\mu^{y(u),u}(\{2k\})$	$\sim c_u k^{-5/2}$		$\sim c_u \pi_u^k k^{-5/2}$
Galton-Watson tree	subcritical	critical	

Dichotomy between situations:

- Subcritical: condensation, cf [Jonsson Stefánsson 2011];
- Supercritical: behaves as maximum of independent variables.

Size $L_{n,k}$ of the k-th largest block

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
$L_{n,1}$	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
$L_{n,2}$	$\Theta(n^{2/3})$ [Stufler 2020]		



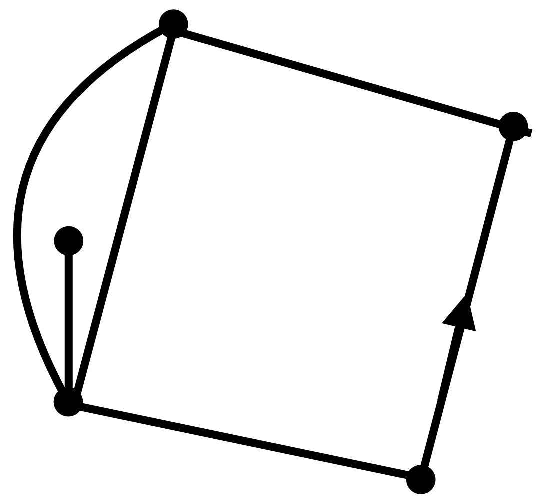
Results

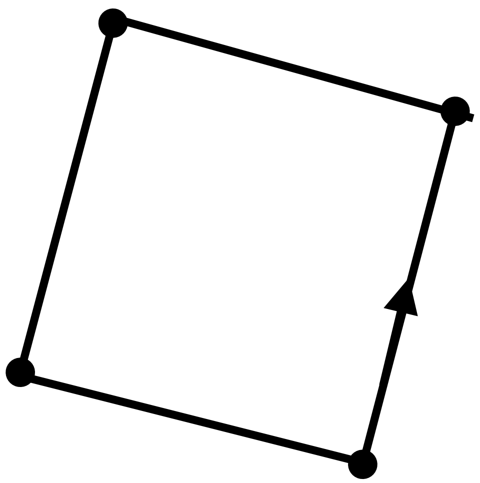
For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

Interlude: quadrangulations

Quadrangulations

Def: map with all faces of degree 4.



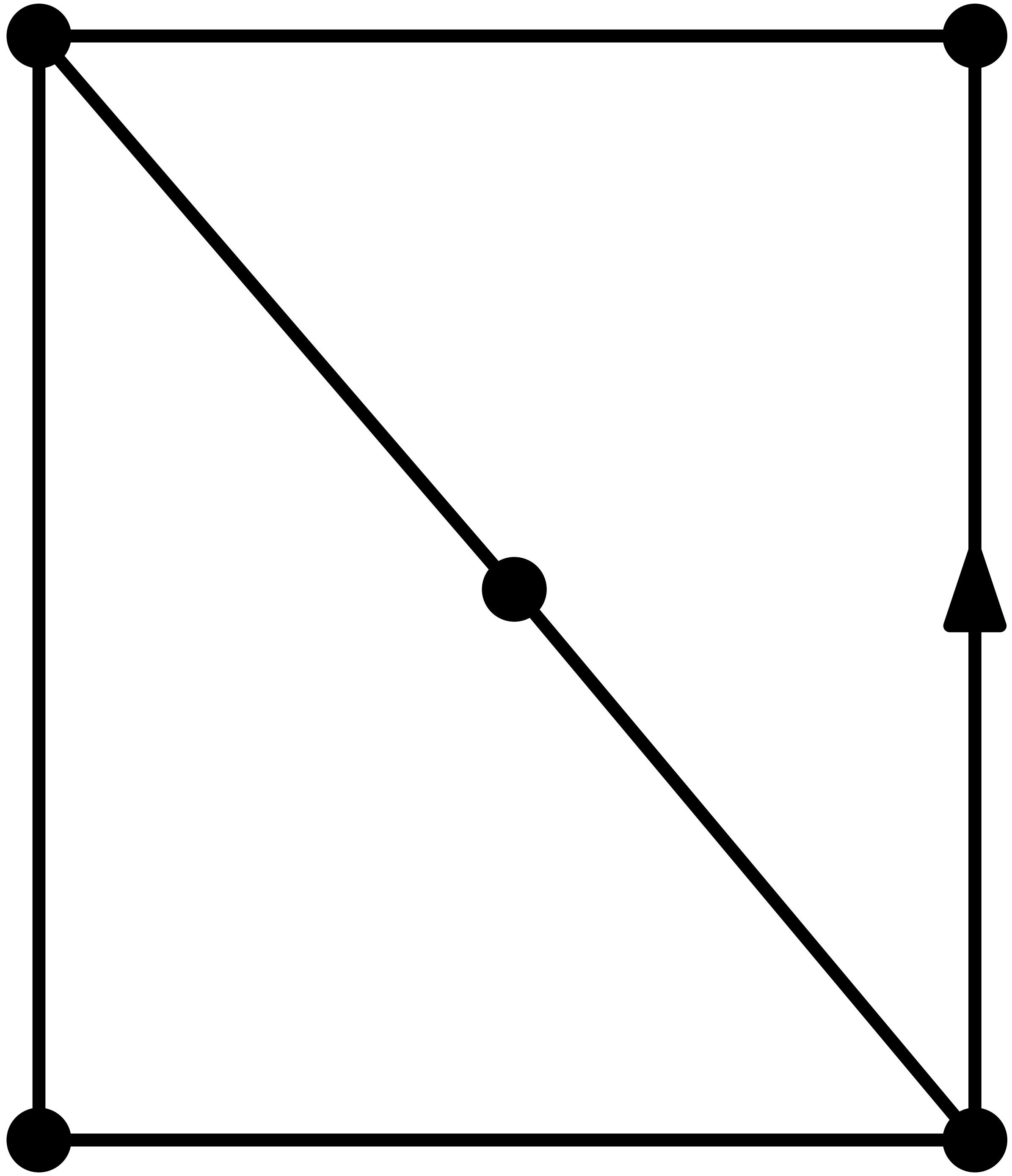


Simple quadrangulation = no multiple edges.

Size $|\mathbf{q}|$ = number of *faces*.

$$V(\mathbf{q}) = |\mathbf{q}| + 2, \quad E(\mathbf{q}) = 2|\mathbf{q}|.$$

Construction of a quadrangulation from a simple core



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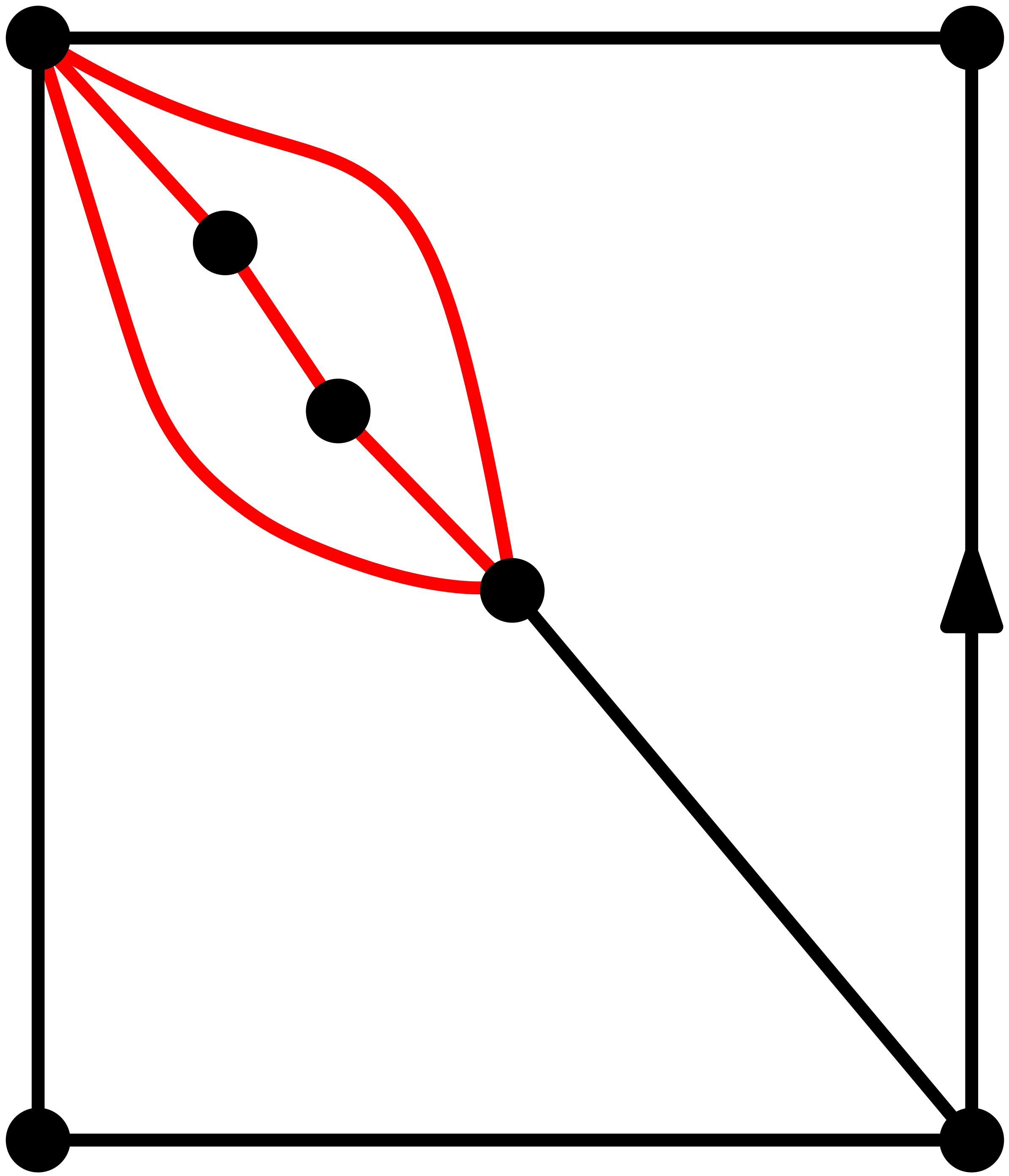

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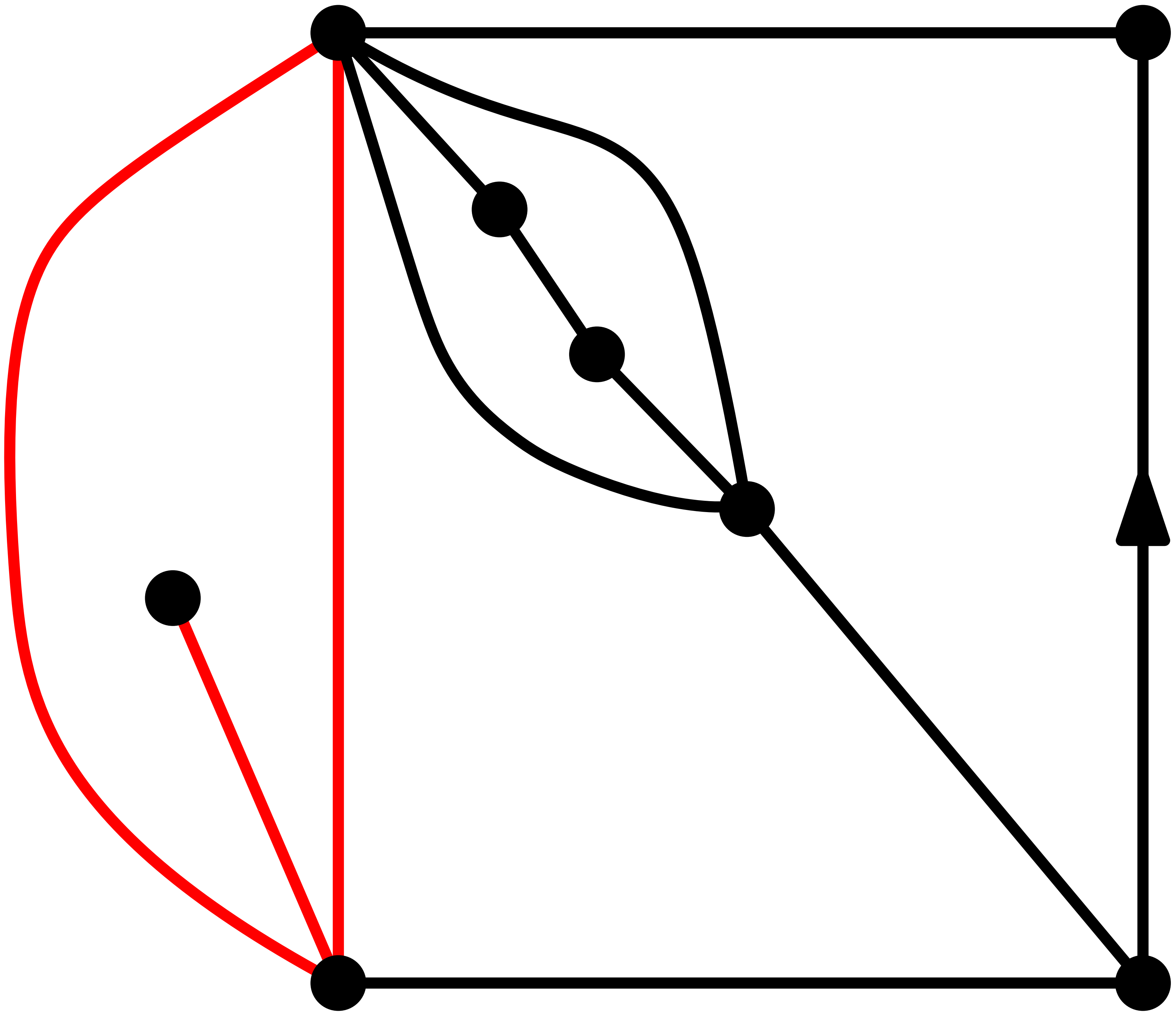
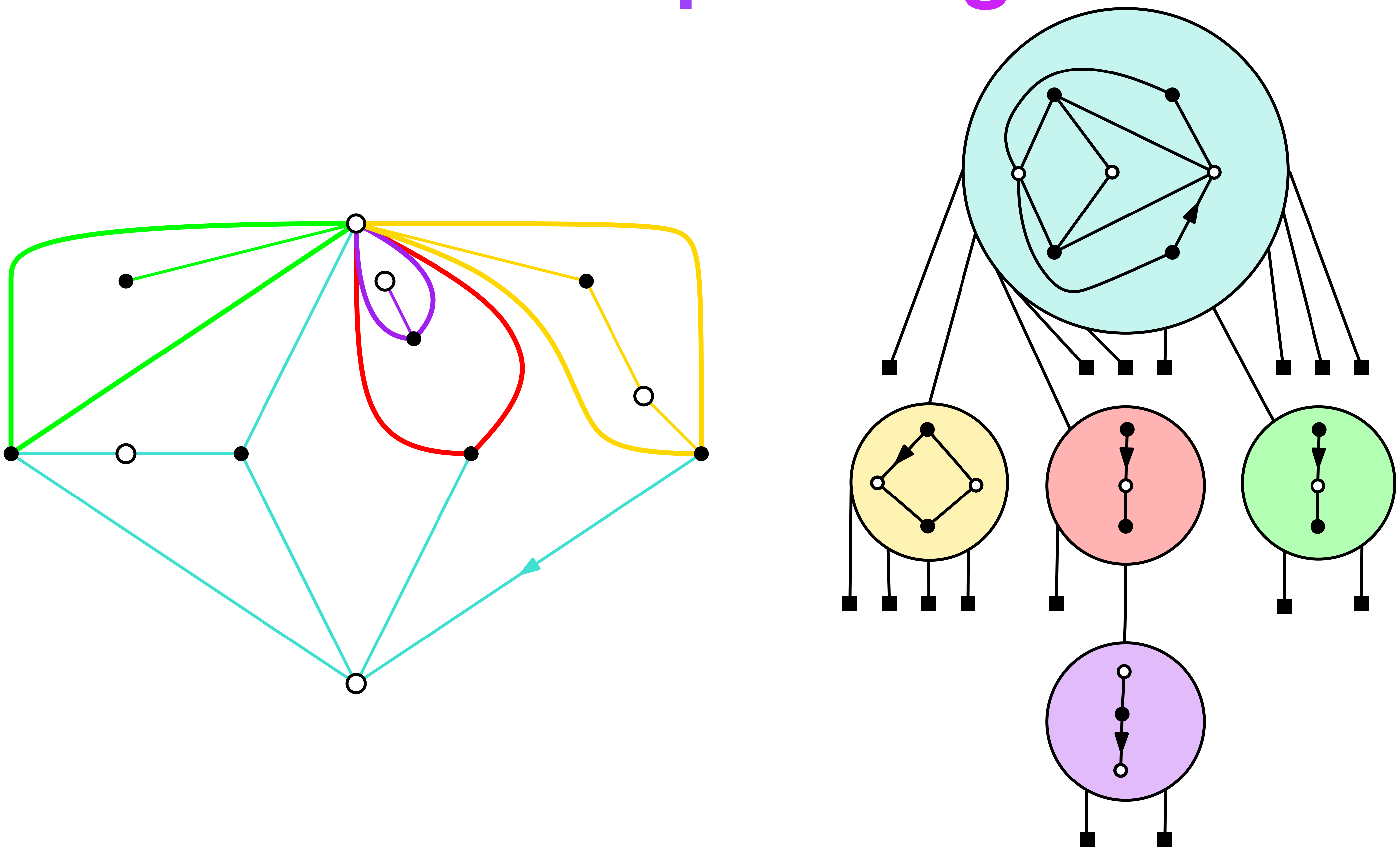


Block tree for a quadrangulation

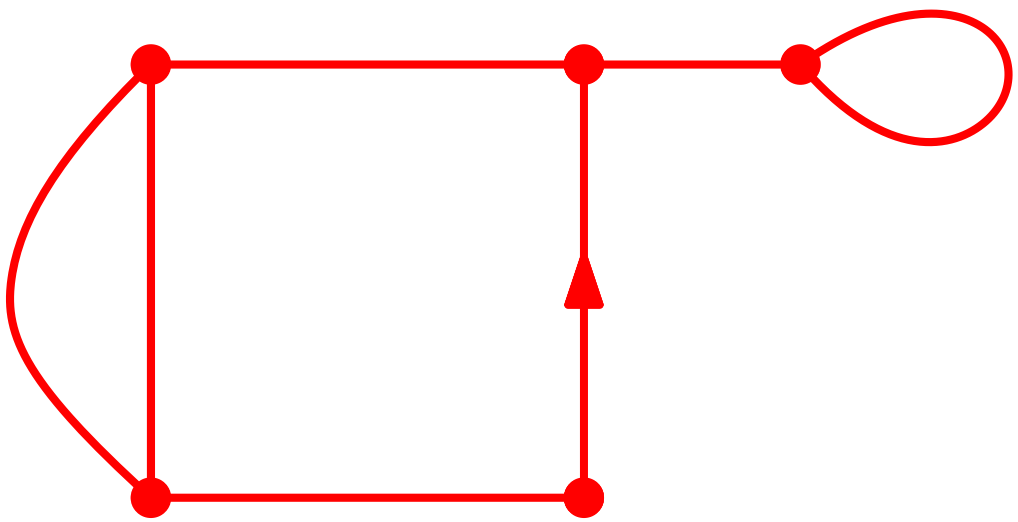


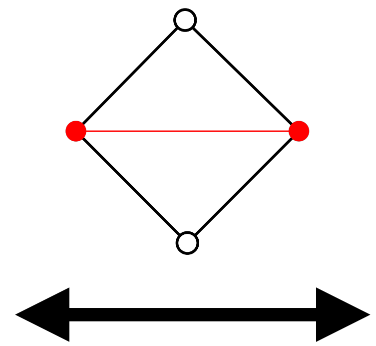
With a weight u on blocks: $Q(z,u) = uS(zQ^2(z,u)) + 1 - u$

Remember: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

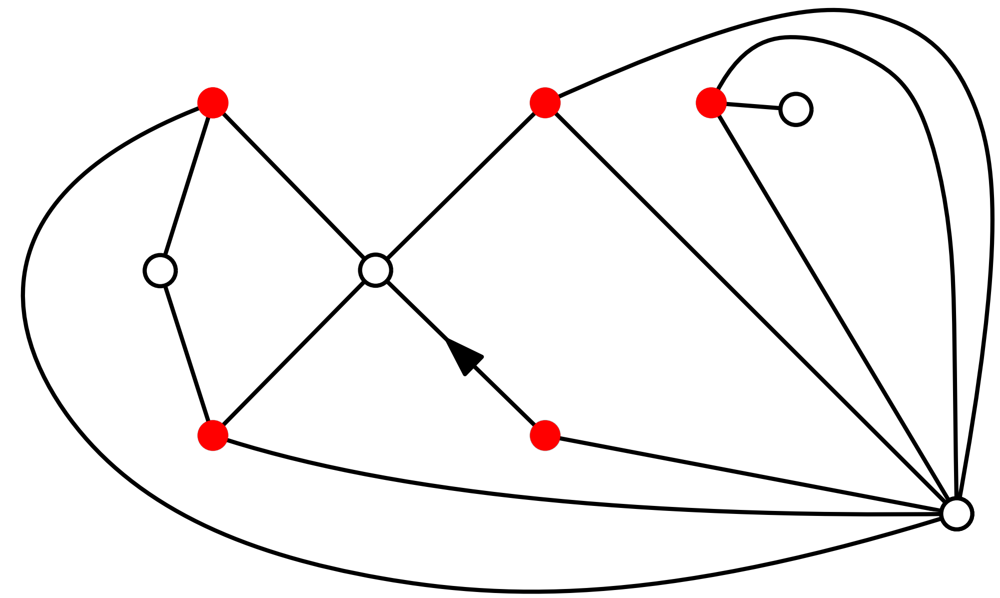
Tutte's bijection

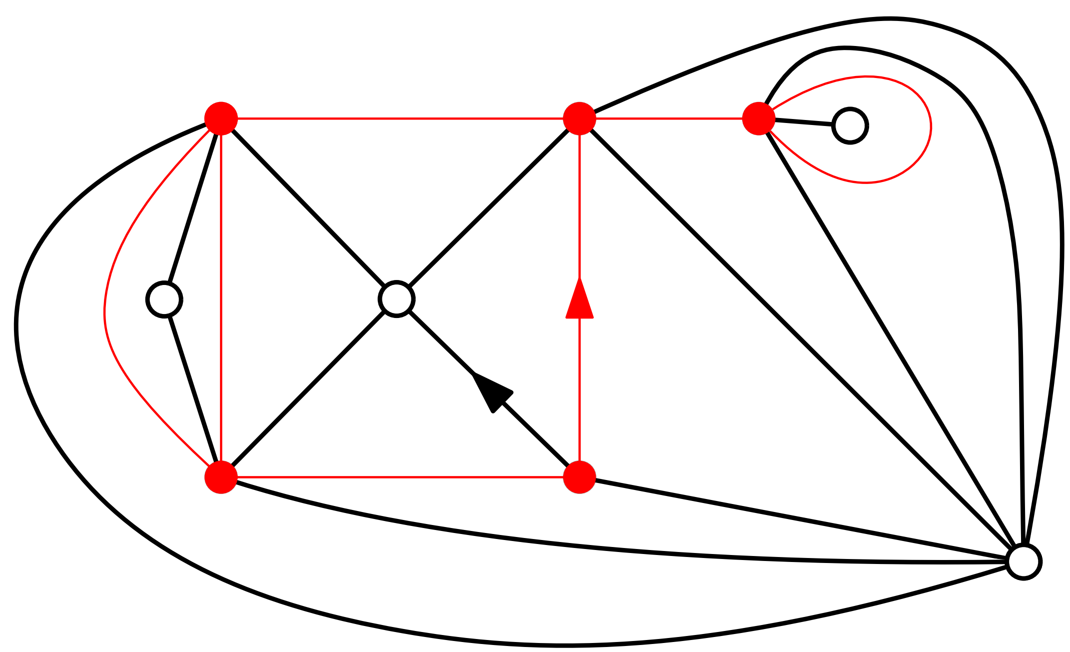
Map





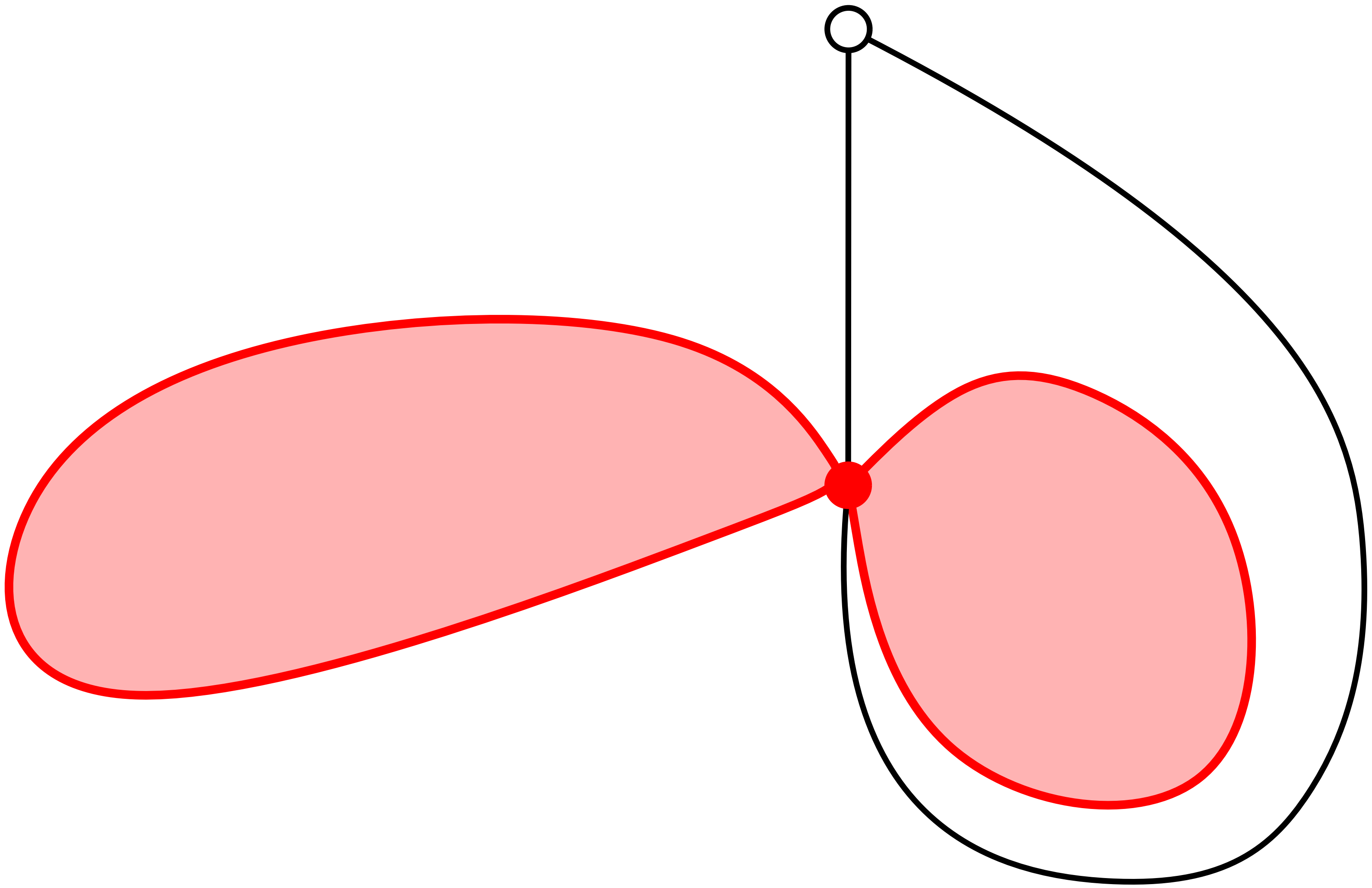
Quadrangulation





[Tutte 1963]

Tutte's bijection for 2-connected maps

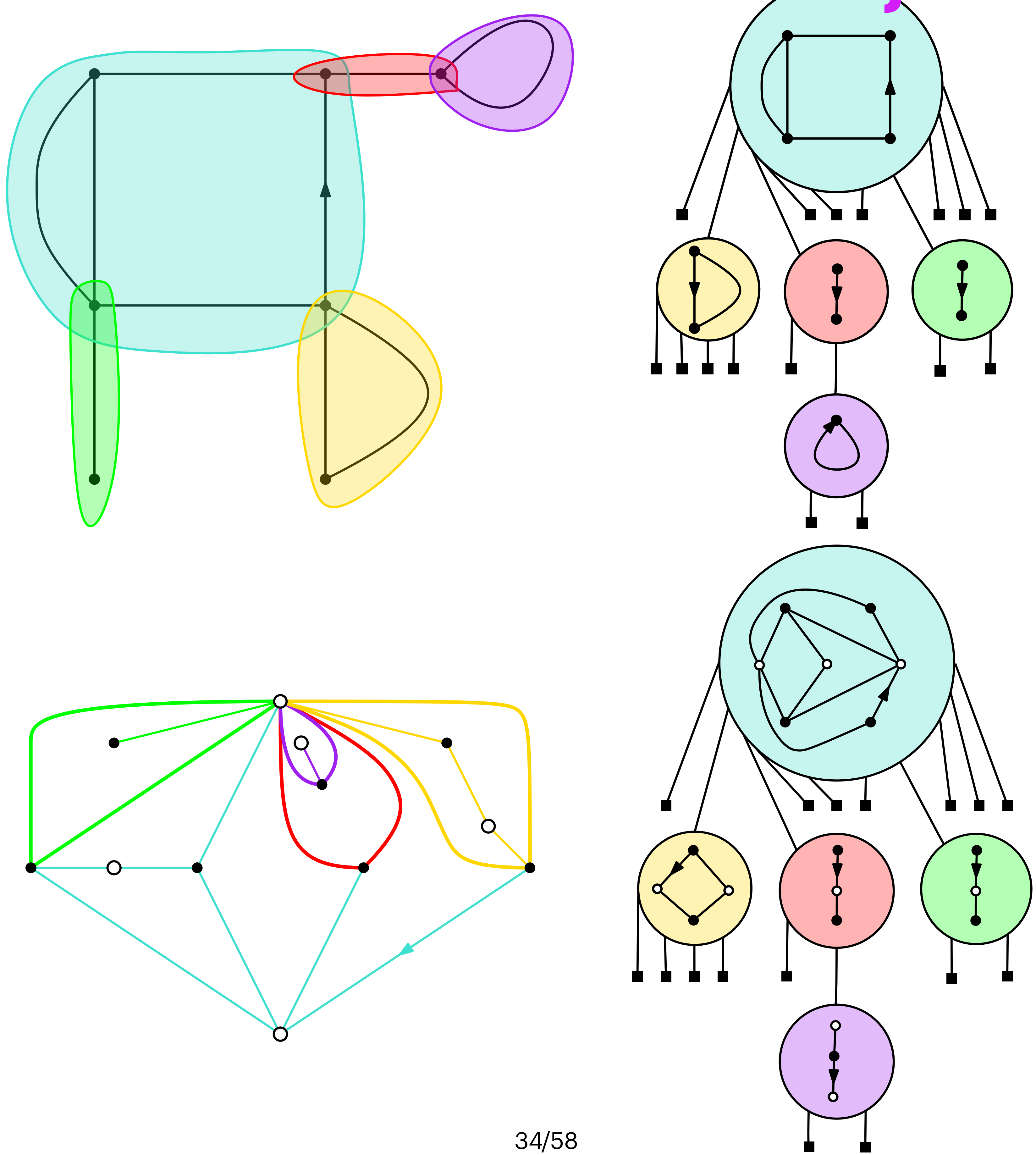


Cut vertex => multiple edge

2-connected maps <=> simple quadrangulations

[Brown 1965]

Block trees under Tutte's bijection



Implications on results

We choose: $\mathbb{P}_{n,u}(\mathfrak{q}) = \dfrac{u^{\#blocks(\mathfrak{q})}}{Z_{n,u}}$ where

$u > 0$,

$\mathcal{Q}_n$ = {quadrangulations of size n},

$\mathfrak{q} \in \mathcal{Q}_n$,

$Z_{n,u}$ = normalisation.

Results on the size of (2-connected) blocks can be transferred immediately for quadrangulations and their simple blocks.

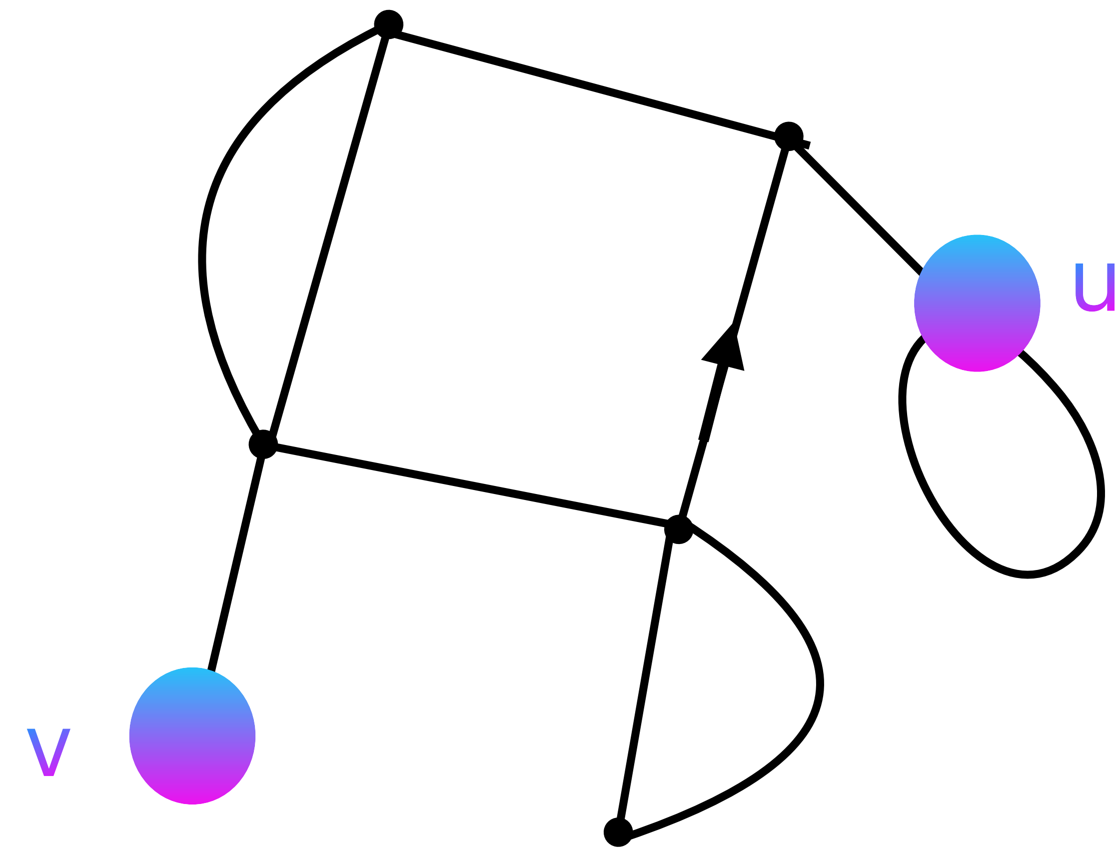
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016] for 2-c case	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

III. Scaling limits

Scaling limits

Convergence of the whole object considered as a metric space (with the graph distance), after renormalisation.



$d(u, v) = 4$

$M_n \hookrightarrow \mathbb{P}_{n,u}$ (carte ou quadrangulation)

What is the limit of the sequence of metric spaces $((M_n, d/n^?))_{n\in\mathbb{N}}$?

(Convergence for Gromov-Hausdorff topology)

Scaling limit of supercritical and critical maps

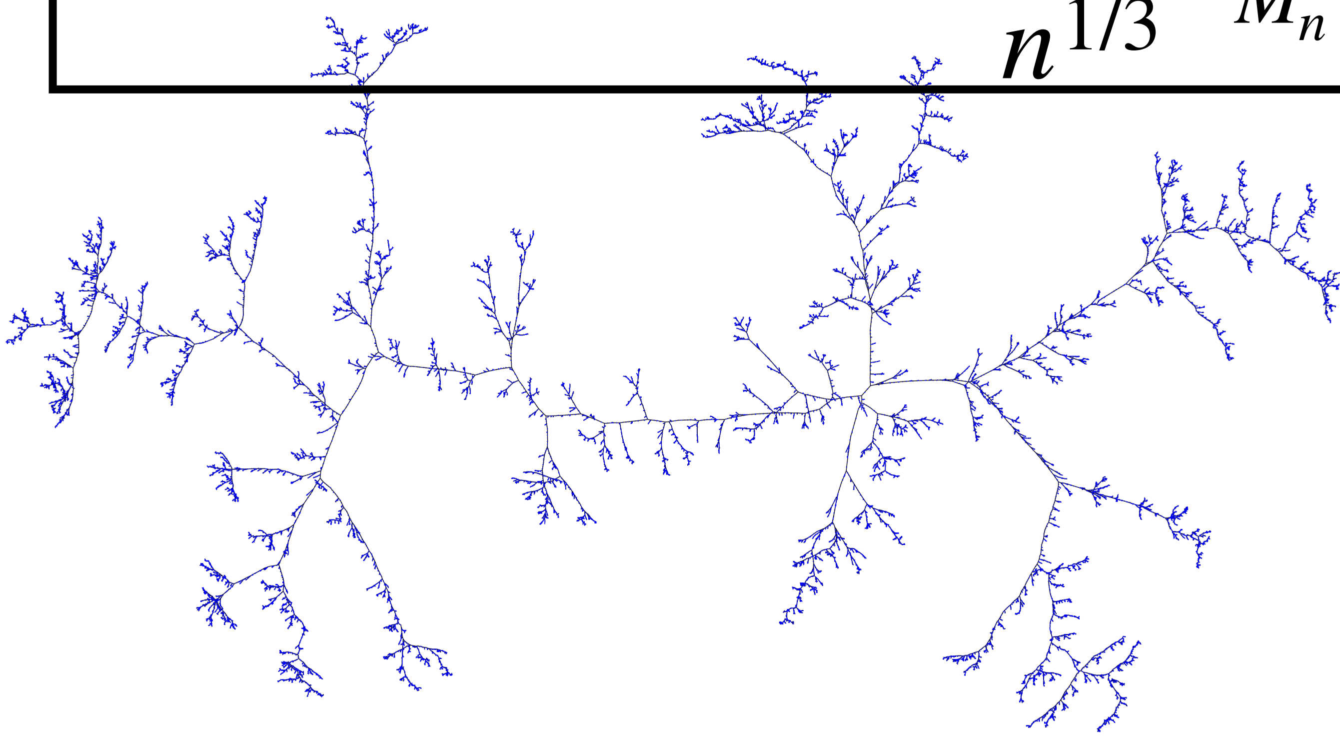
<u>Lemma</u> For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

- If $u > 9/5$,

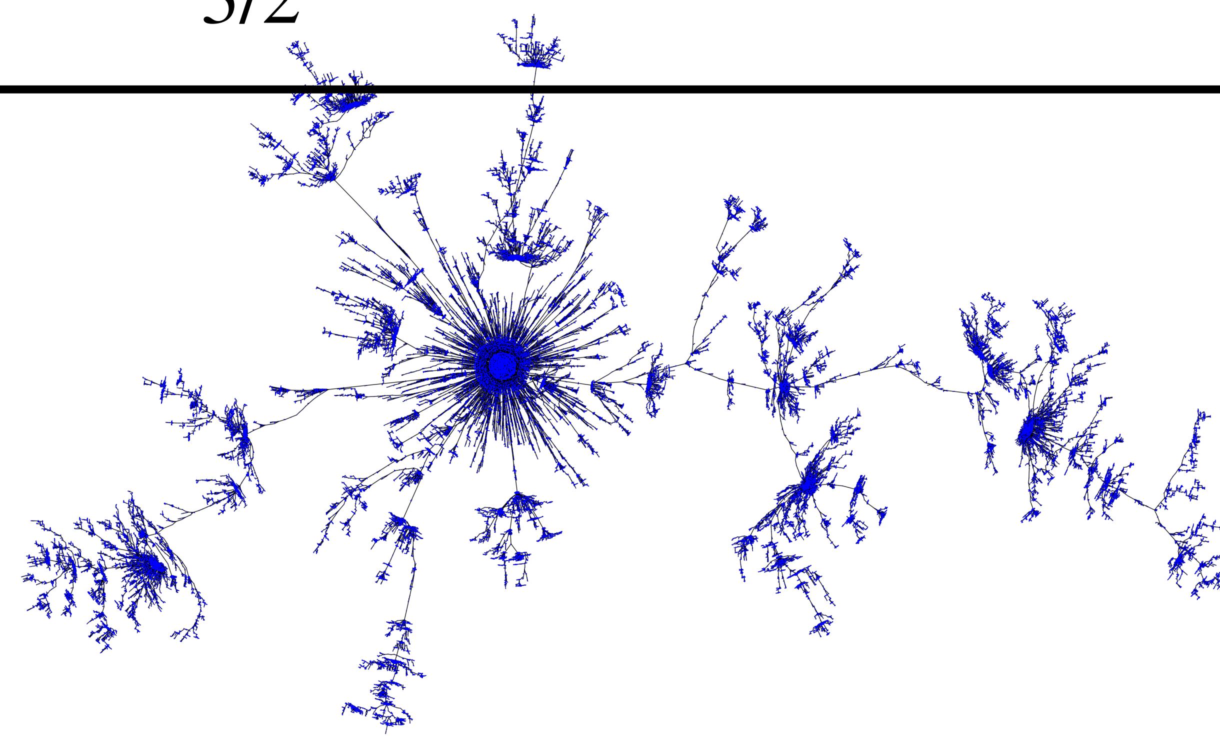
$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e.$$

- If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}.$$



Brownian Tree $\mathcal{T}_e$



Stable Tree $\mathcal{T}_{3/2}$

Scaling limit of supercritical and critical maps

Lemma For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

- If $u > 9/5$,

$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e.$$

- If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}.$$

Proof Known scaling limits of critical Galton-Watson trees

- with finite variance [Aldous 1993, Le Gall 2006];
- infinite variance and polynomial tails [Duquesne 2003]

Scaling limit of supercritical and critical maps

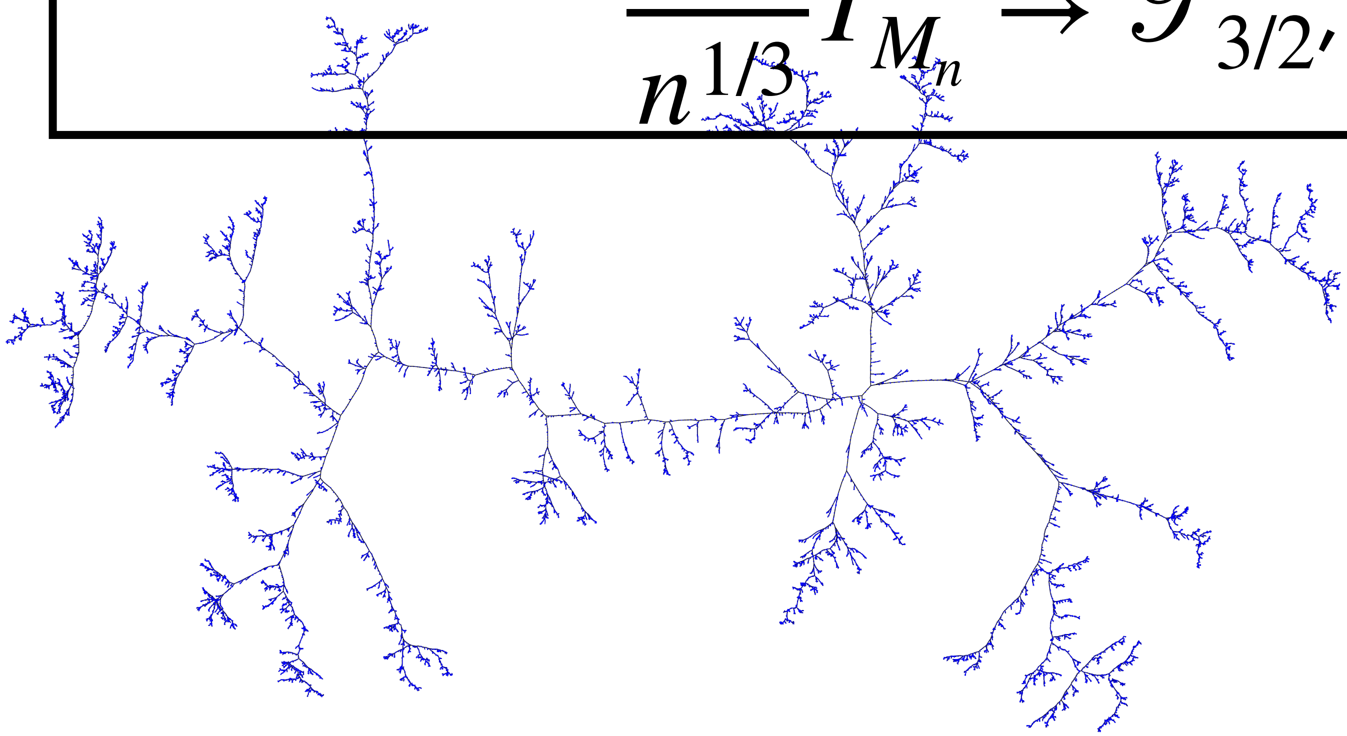
Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

- [Stufler 2020] If $u > 9/5$,

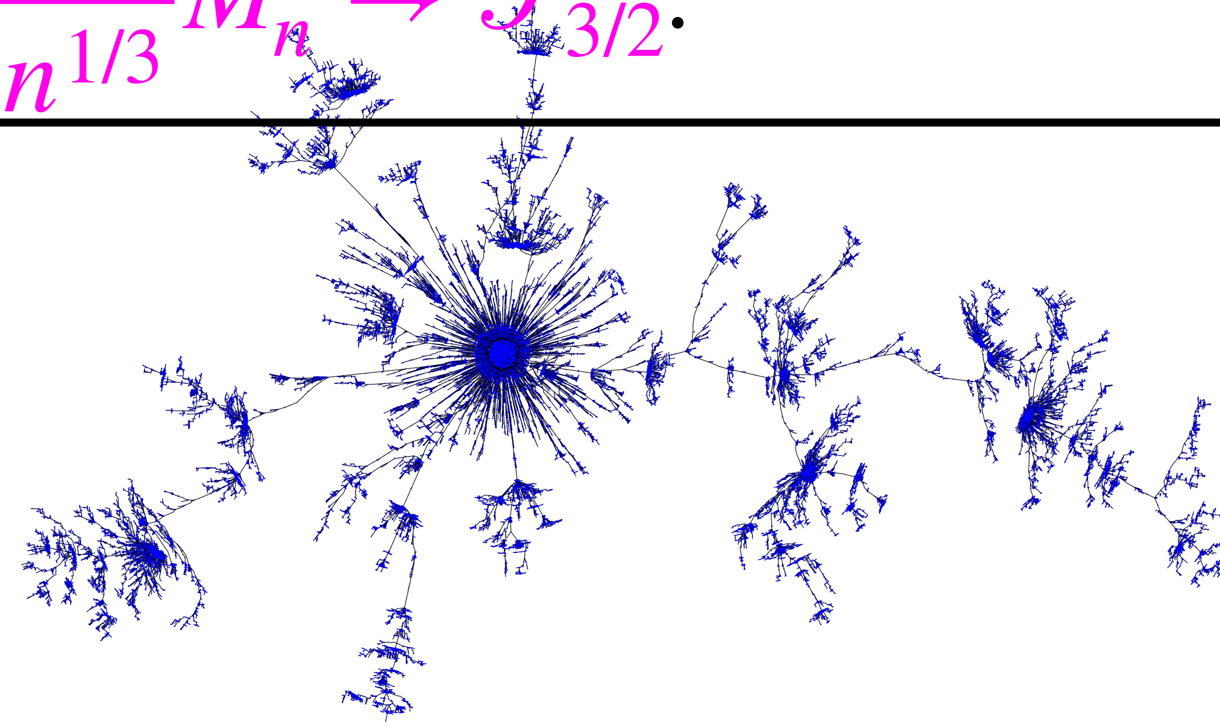
$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e, \quad \frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e.$$

- [Fleurat, S. 23] If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}, \quad \frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}.$$



Brownian Tree $\mathcal{T}_e$



Stable Tree $\mathcal{T}_{3/2}$

Scaling limit of supercritical and critical maps

Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

- [Stufler 2020] If $u > 9/5$,

$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e, \quad \frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e.$$

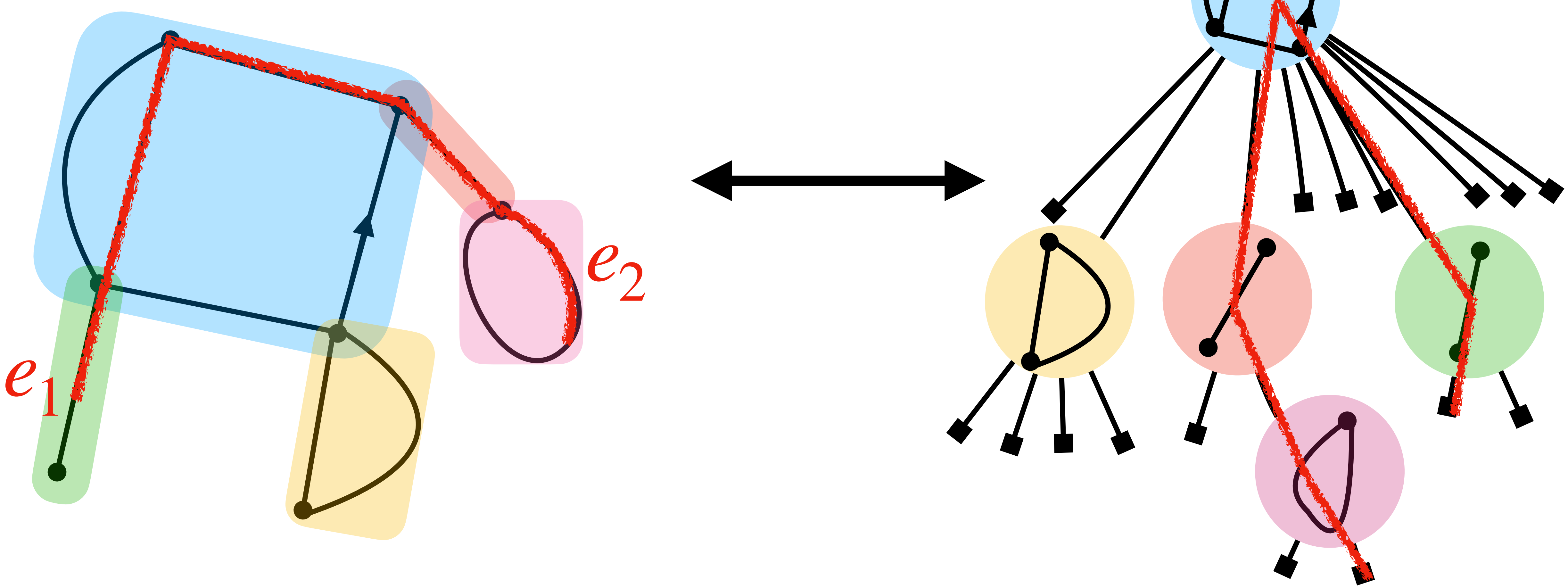
- [Fleurat, S. 23] If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}, \quad \frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}.$$

Proof Distances in M_n behave like distances in T_{M_n}!

Supercritical and critical cases

Difficult part = show that distances in M_n behave like distances in T_{M_n}.

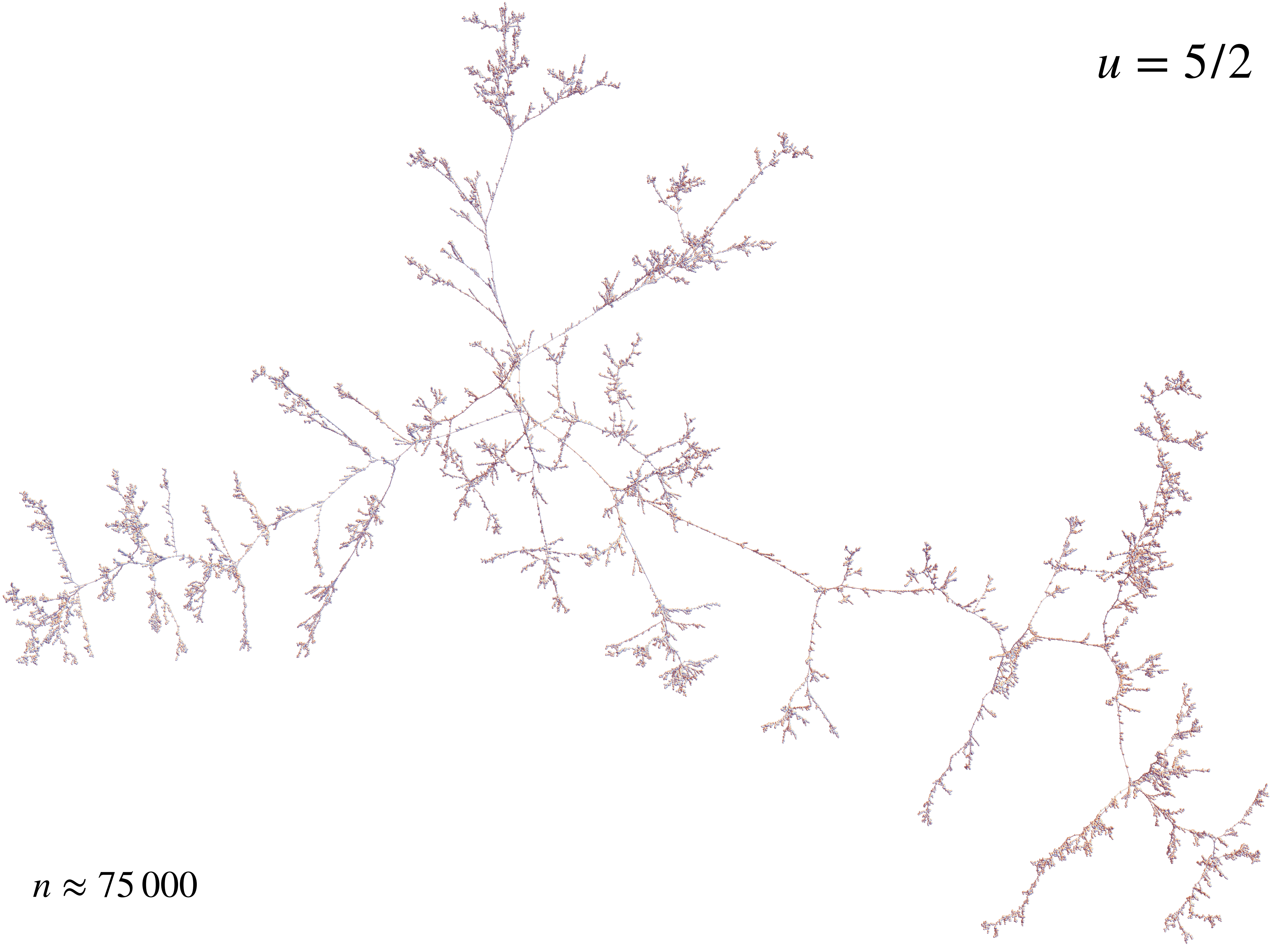


Let $\kappa = \mathbb{E}$("diameter" bipointed block). By a “law of large numbers”-type argument

$$d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2)\,.$$

Difficult for the critical case => large deviation estimates


$u = 9/5$
$n \approx 80\,000$


$u = 5/2$
$n \approx 75\,000$


$u = 5$
$n \approx 50\,000$

Scaling limits of subcritical maps

Theorem [Fleurat, S. 23] If $u < 9/5$, for $M_n \hookrightarrow \mathbb{P}_{n,u}$ and B_n a uniform block of size n:

$$d_{GH}\left(\frac{C_1(u)}{n^{1/4}}M_n, \frac{1}{n^{1/4}}B_n\right) \to 0.$$

So, if $cn^{-1/4}B_n \to \mathcal{S}_e$, then

$$\frac{C_1(u)}{cn^{1/4}}M_n \to \mathcal{S}_e.$$

Brownian Sphere $\mathcal{S}_e$

See [Addario-Berry, Wen 2019] for a similar result and method.

Scaling limits of subcritical maps

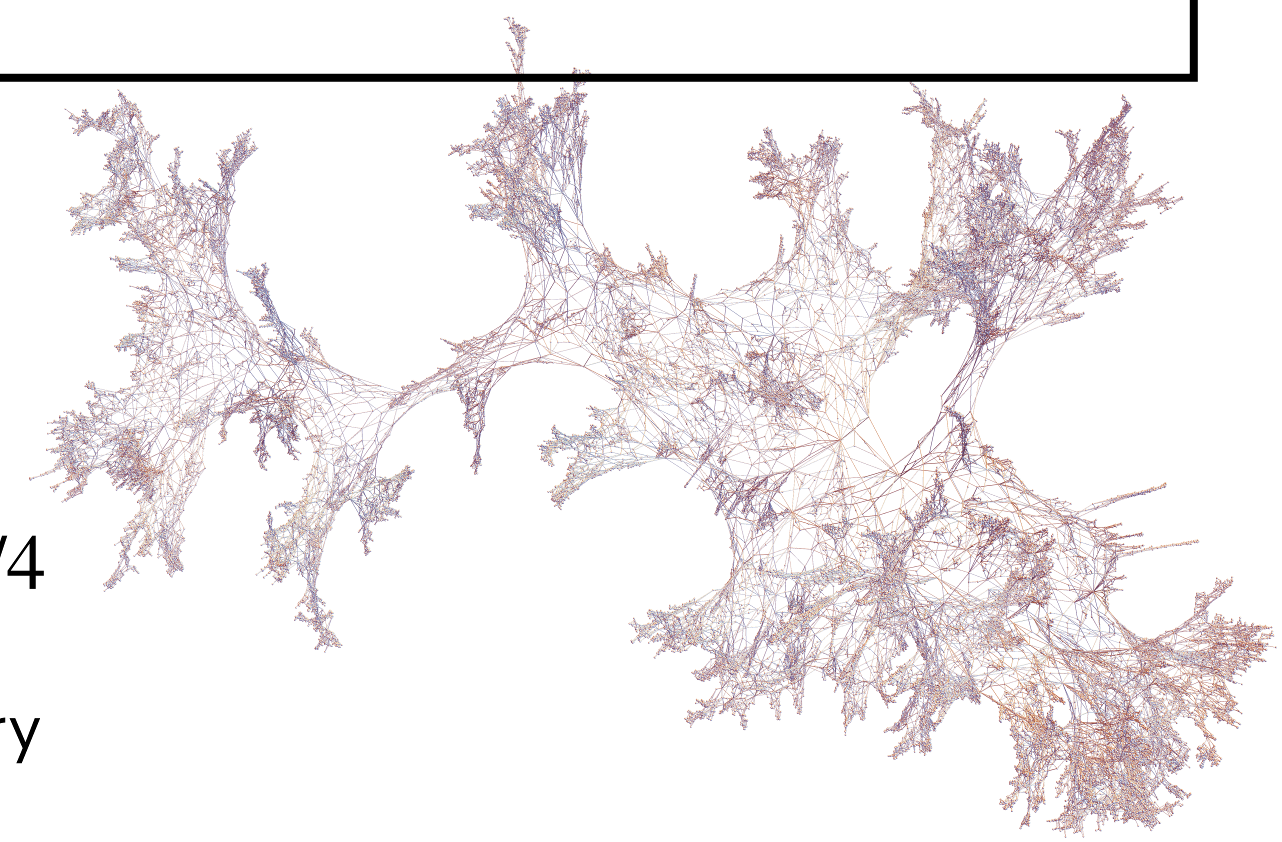
Theorem [Fleurat, S. 23] If $u < 9/5$, for $Q_n \hookrightarrow \mathbb{P}_{n,u}$ a quadrangulation:

$$\frac{C_1(u)}{n^{1/4}} Q_n \to \mathcal{S}_e.$$

Moreover, Q_n and its simple core converge jointly to the same Brownian sphere.

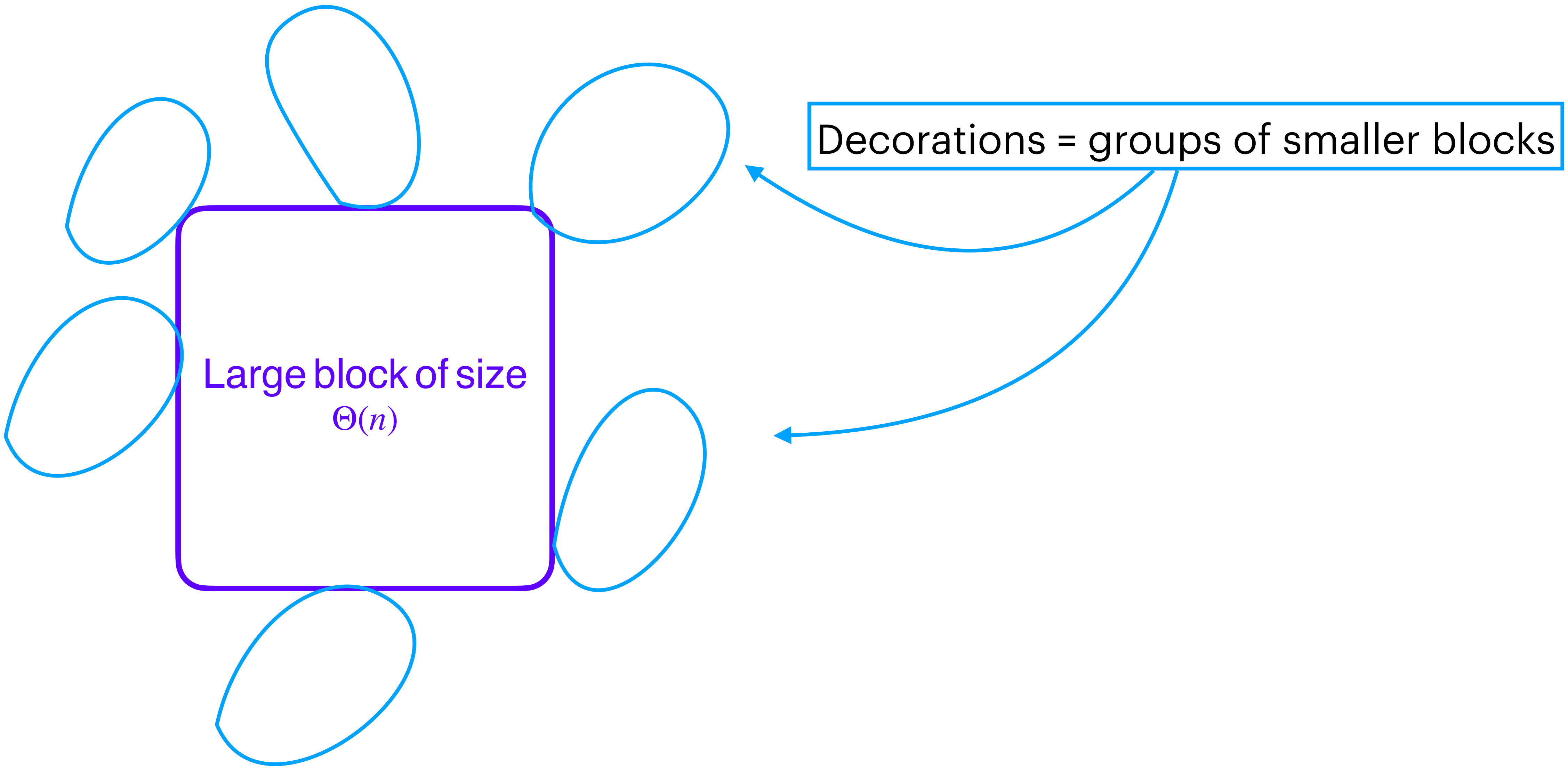
Proof

- Previous theorem
- Scaling limit of uniform simple quad. rescaled by $n^{1/4}$ = Brownian sphere [Addario-Berry Albenque 2017].



Brownian Sphere $\mathcal{S}_e$

Subcritical case



Diameter of a decoration ≤ blocks to cross × max diameter of blocks

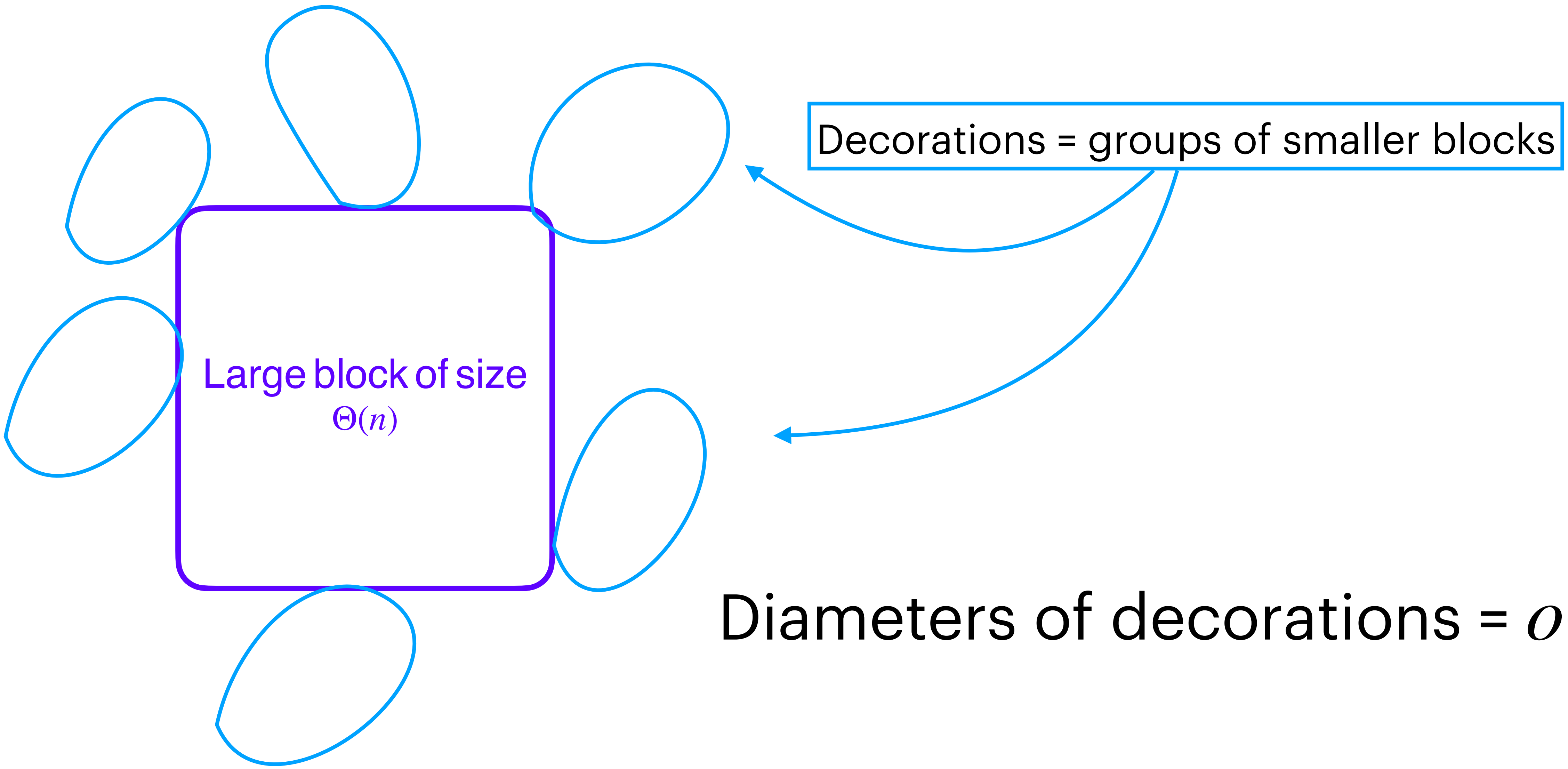
$$\leq diam(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = diam(T_{M_n}) \times O(n^{1/6+\delta})$$

$$= O(n^{1/6+2\delta}) = o(n^{1/4}) .$$

T_{M_n} is a subcritical Galton-Watson tree

[Chapuy Fusy Giménez Noy 2015]

Subcritical case



Diameters of decorations = $o(n^{1/4})$.

Diameter of a decoration ≤ blocks to cross × max diameter of blocks

$$\leq diam(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = diam(T_{M_n}) \times O(n^{1/6+\delta})$$

$$= O(n^{1/6+2\delta}) = o(n^{1/4}) .$$

T_{M_n} is a subcritical Galton-Watson tree

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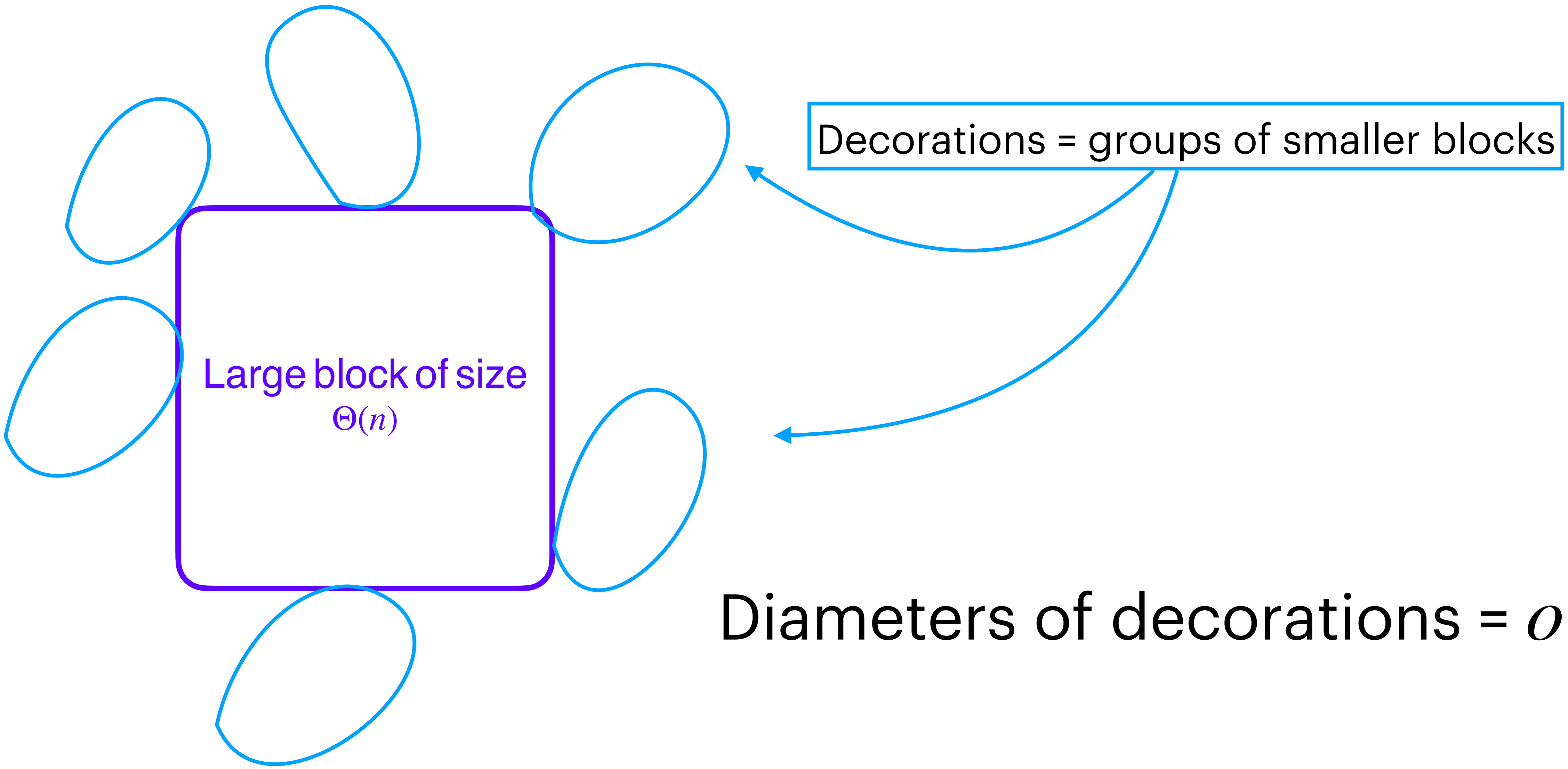
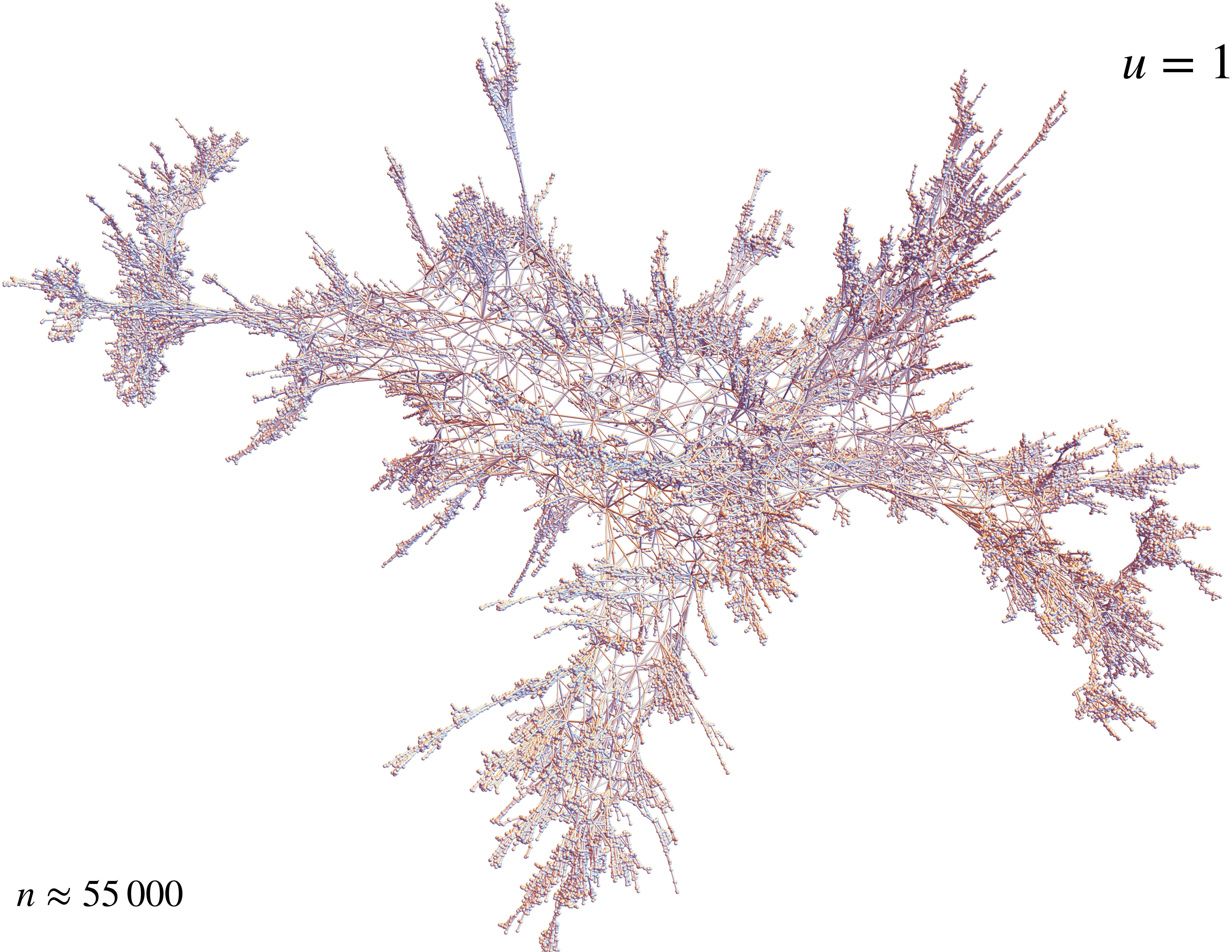
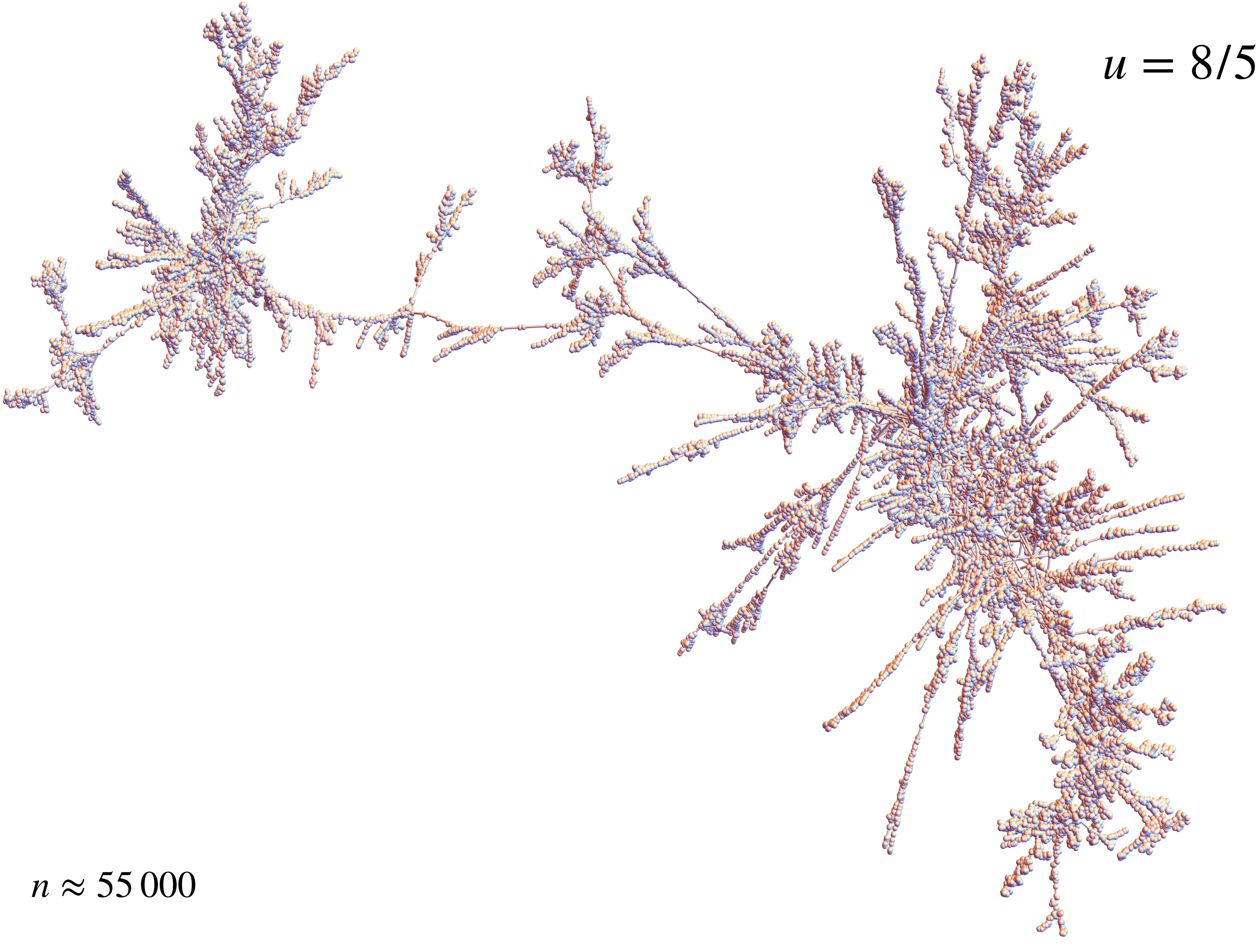
Subcritical case



Diameters of decorations = $o(n^{1/4})$.

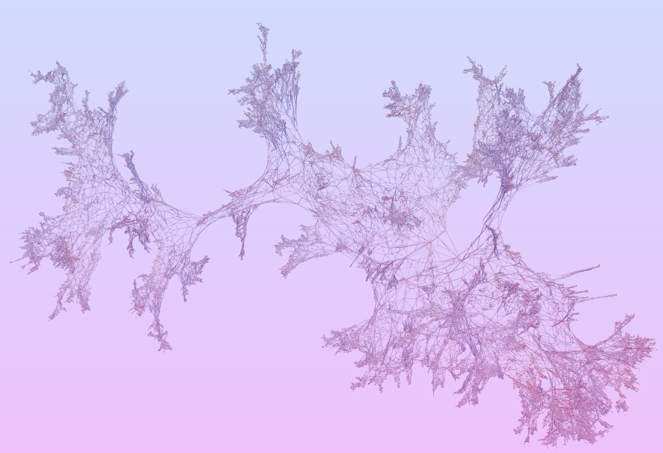
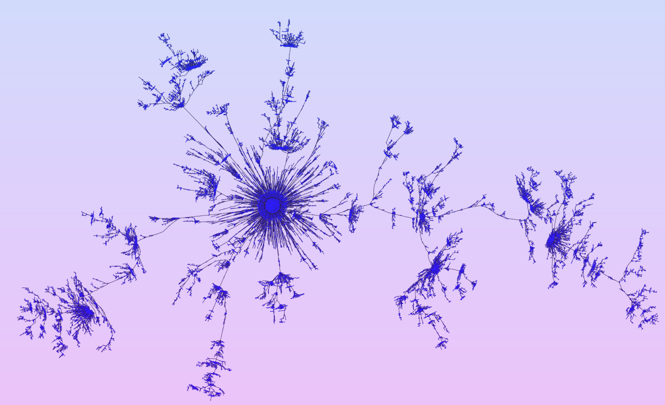
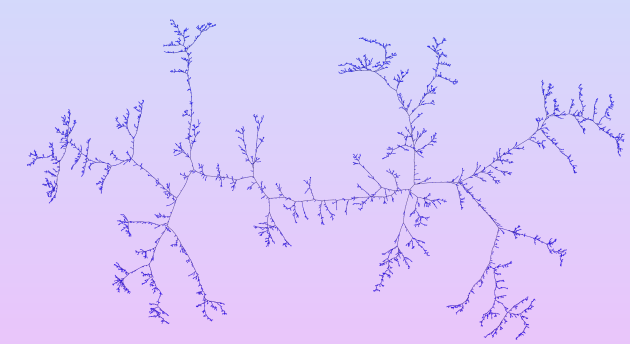
The scaling limit of M_n (rescaled by $n^{1/4}$) is the scaling limit of uniform blocks!


$u = 1$
$n \approx 55\,000$



$u = 8/5$
$n \approx 55\,000$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{y,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n	$\frac{C_1(u)}{n^{1/4}} M_n \to \mathcal{S}_e$  Assuming the convergence of 2-connected maps towards the Brownian sphere	$\frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$ 	$\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e$ [Stufler 2020] 

IV. Extension to other families of maps

Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1+\mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$
loopless $M_2(z)$	simple $M_3(z)$	$z(1+M)$	–
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1+M)^2$	–
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1+M)$	–
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1+M)$
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1+M)$	–
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1+M)^2$	–
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1+M)$	–
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1+M)^3$	–
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1+M)^2$	–

Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$	u_C
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$	
loopless $M_2(z)$	simple $M_3(z)$	$z(1+M)$	–	81/17
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1+M)^2$	–	9/5
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1+M)$	–	135/7
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1+M)$	
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1+M)$	–	36/11
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$	
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1+M)^2$	–	52/27
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1+M)$	–	68/3
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1+M)^3$	–	16/7
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1+M)^2$	–	64/37

→ *Unified study of the phase transition for block-weighted random planar maps* Z. Salvy (EUROCOMB'23)

Statement of the results

Theorem [S. 23] Model of the preceding table without coreless maps exhibits a phase transition at some explicit u_C. When $n \to \infty$:

- Subcritical phase $u < u_C$: "general map phase" one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: "tree phase" only small blocks.

We obtain explicit results on enumeration and size of blocks in each case.

V. Perspectives

Extension to decompositions with coreless maps

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1 - z(1 + M))^2$	$z(1 + M)^2$
loopless $M_2(z)$	simple $M_3(z)$	$z(1 + M)$	–
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1 + M)^2$	–
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1 + M)$	–
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1 + M)$
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1 + M)$	–
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1 - z(1 + M))^2$	$z(1 + M)^2$
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1 + M)^2$	–
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1 + M)$	–
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1 + M)^3$	–
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1 + M)^2$	–

Critical window?

Phase transition very sharp => what if $u = 9/5 \pm \varepsilon(n)$?

- Block size results still hold if $u_n = 9/5 - \varepsilon(n)$, $\varepsilon^3 n \to \infty$;
- For $u_n = 9/5 + \varepsilon(n)$, this is the case as well: when $\varepsilon^3 n \to \infty$

$$L_{n,1} \sim 2.7648\, \varepsilon^{-2} \ln(\varepsilon^3 n)$$

(analogous to [Bollobás 1984]'s result for Erdős-Rényi graphs!);

- Results exist for scaling limits in ER graphs [Addario-Berry, Broutin, Goldschmidt 2010], open question in our case.

Is there a critical window? If so, what is its width?

Extension to tree-rooted maps

$$u_C = \frac{9\pi(4-\pi)}{420\pi - 81\pi^2 - 512} \simeq 3.02$$

Thank you!