Phase transition for block-weighted random planar maps

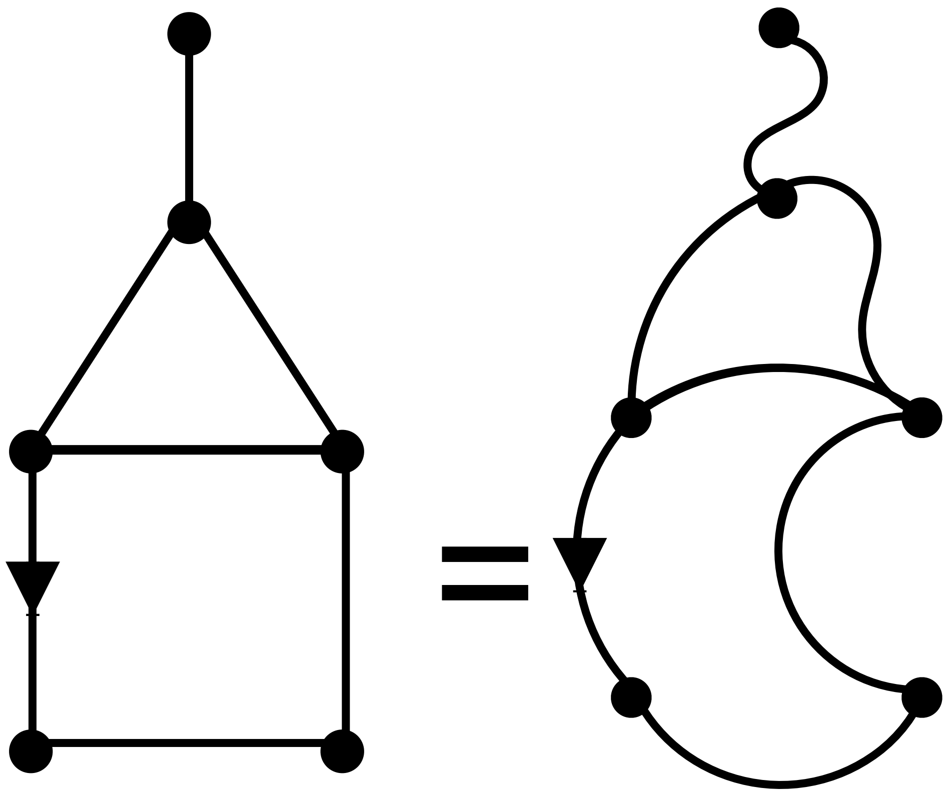
Probability and Geometry in, on and of non-Euclidian spaces

Zéphyr Salvy

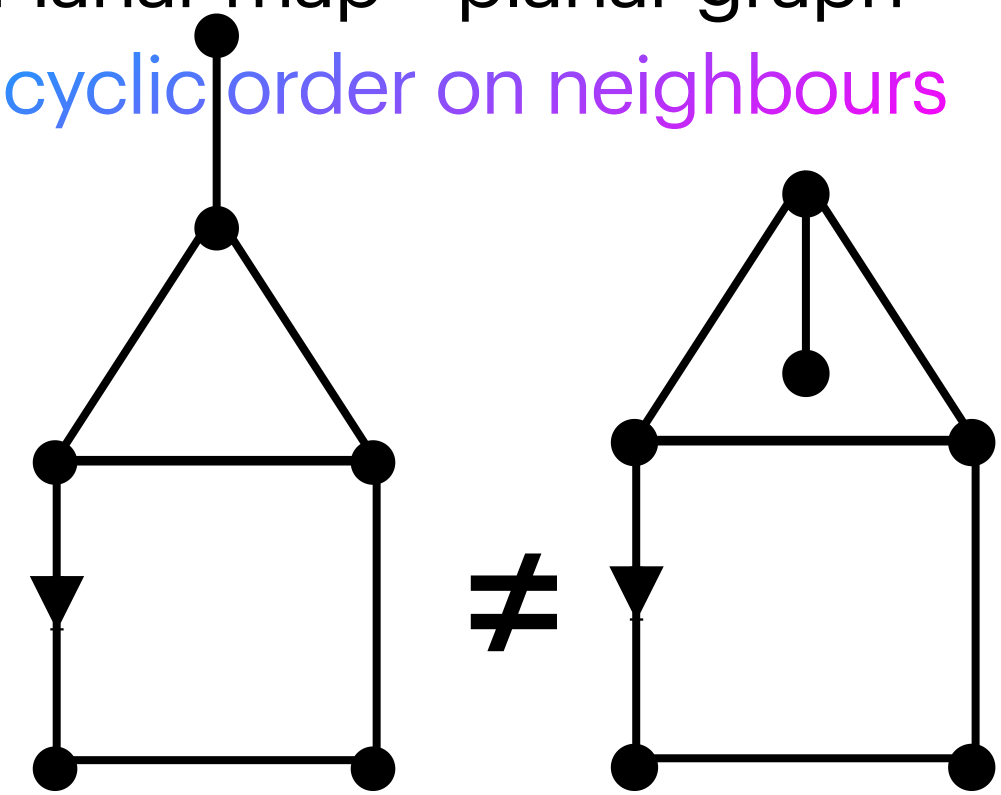
Joint work with William Fleurat

Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms



Planar map = planar graph + cyclic order on neighbours



- Rooted planar map = map endowed with a marked oriented edge (represented by an arrow);
- Size $|\mathfrak{m}|$ = number of edges;
- Corner (does not exist for graphs !) = space between an oriented edge and the next one for the trigonometric order.

Block decomposition

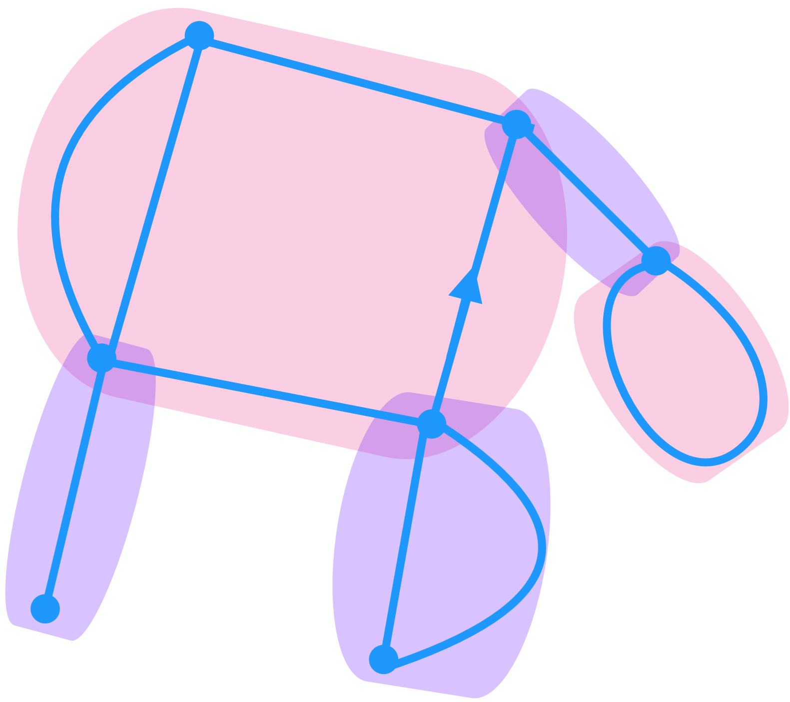
2-connected = two vertices must be removed to disconnect.

Block = maximal (for inclusion) 2-connected submap.

Block decomposition

2-connected = two vertices must be removed to disconnect.

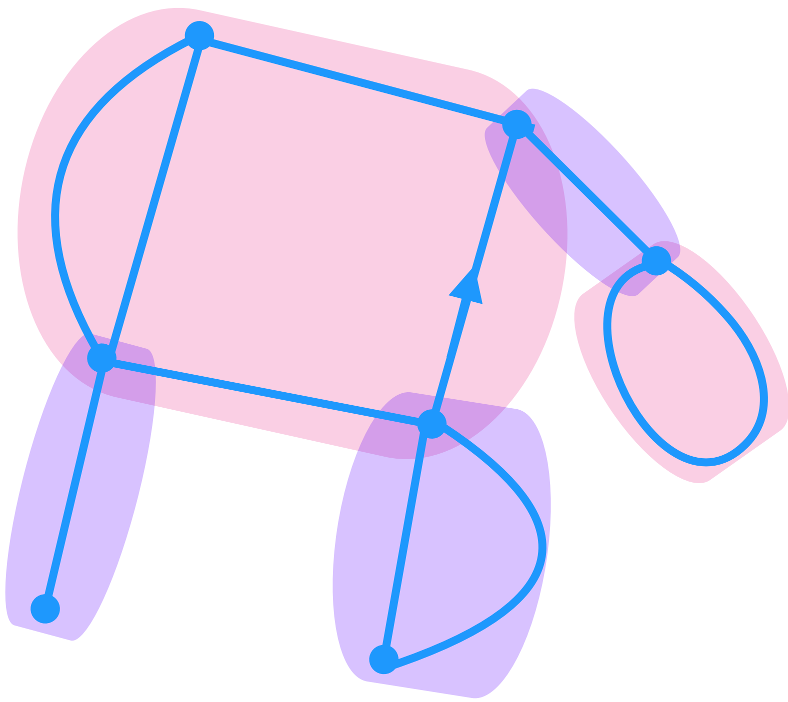
Block = maximal (for inclusion) 2-connected submap.



Block decomposition

2-connected = two vertices must be removed to disconnect.

Block = maximal (for inclusion) 2-connected submap.

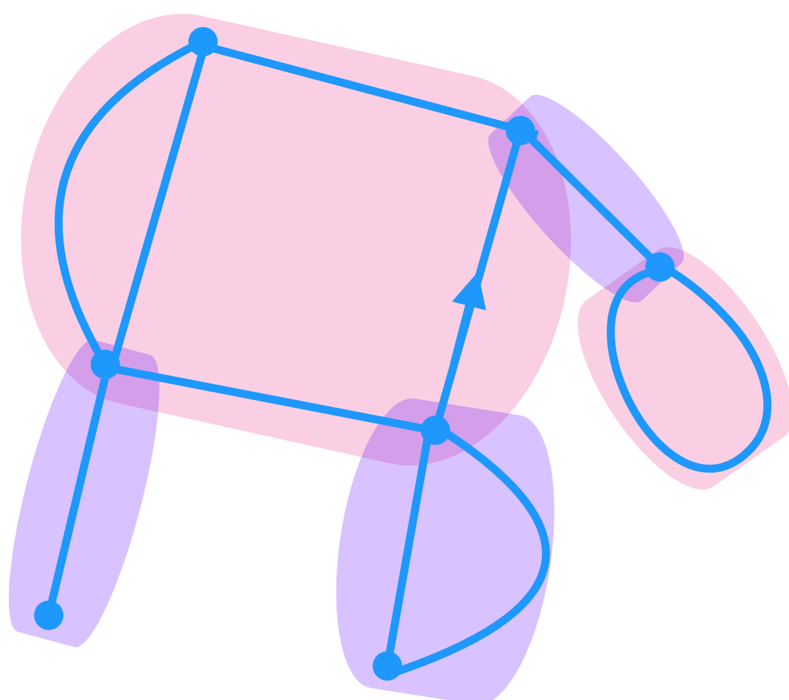


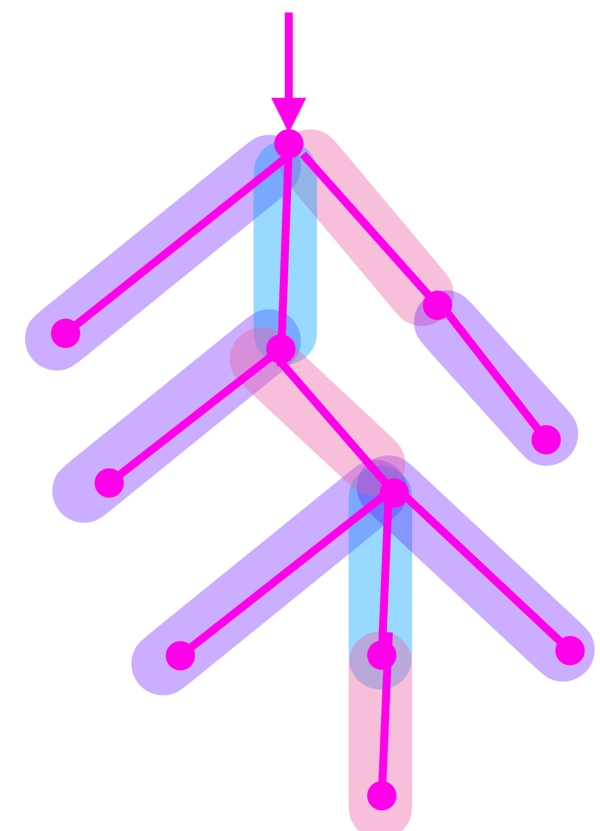
Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

Block decomposition

2-connected = two vertices must be removed to disconnect.

Block = maximal (for inclusion) 2-connected submap.





Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

Only small blocks.

Model

Inspired by [Bonzom 2016].

Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,
$\mathcal{M}_n$ = {maps of size n},
$\mathfrak{m} \in \mathcal{M}_n$,
$Z_{n,u}$ = normalisation.

- $u = 1$: uniform distribution on maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected maps);
- $u \to \infty$: maximising the number of blocks (= trees!).

Given u, asymptotic behaviour when $n \to \infty$?

Results

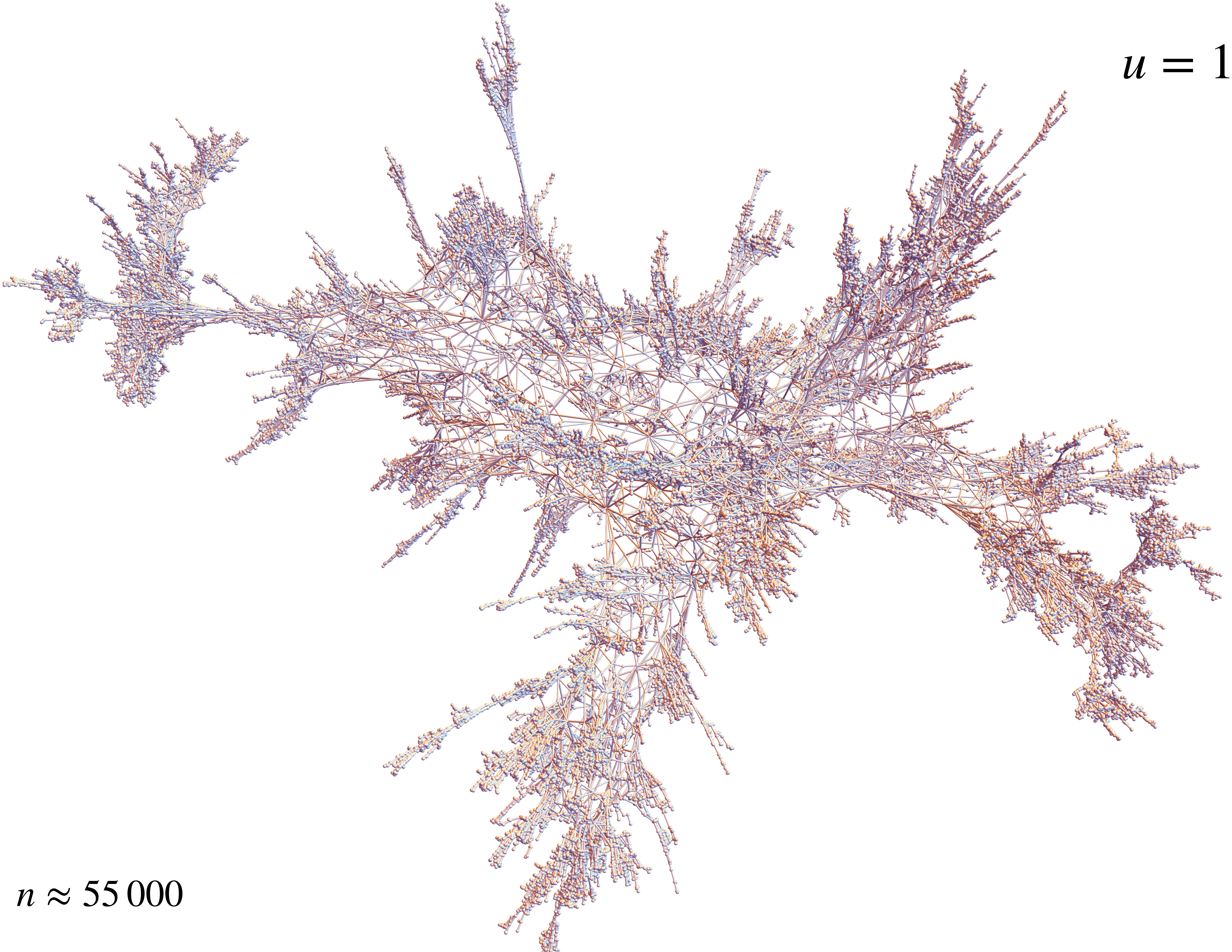
Phase transition

Theorem [Fleurat, S. 23] Model exhibits a phase transition at $u = 9/5$. When $n \to \infty$:

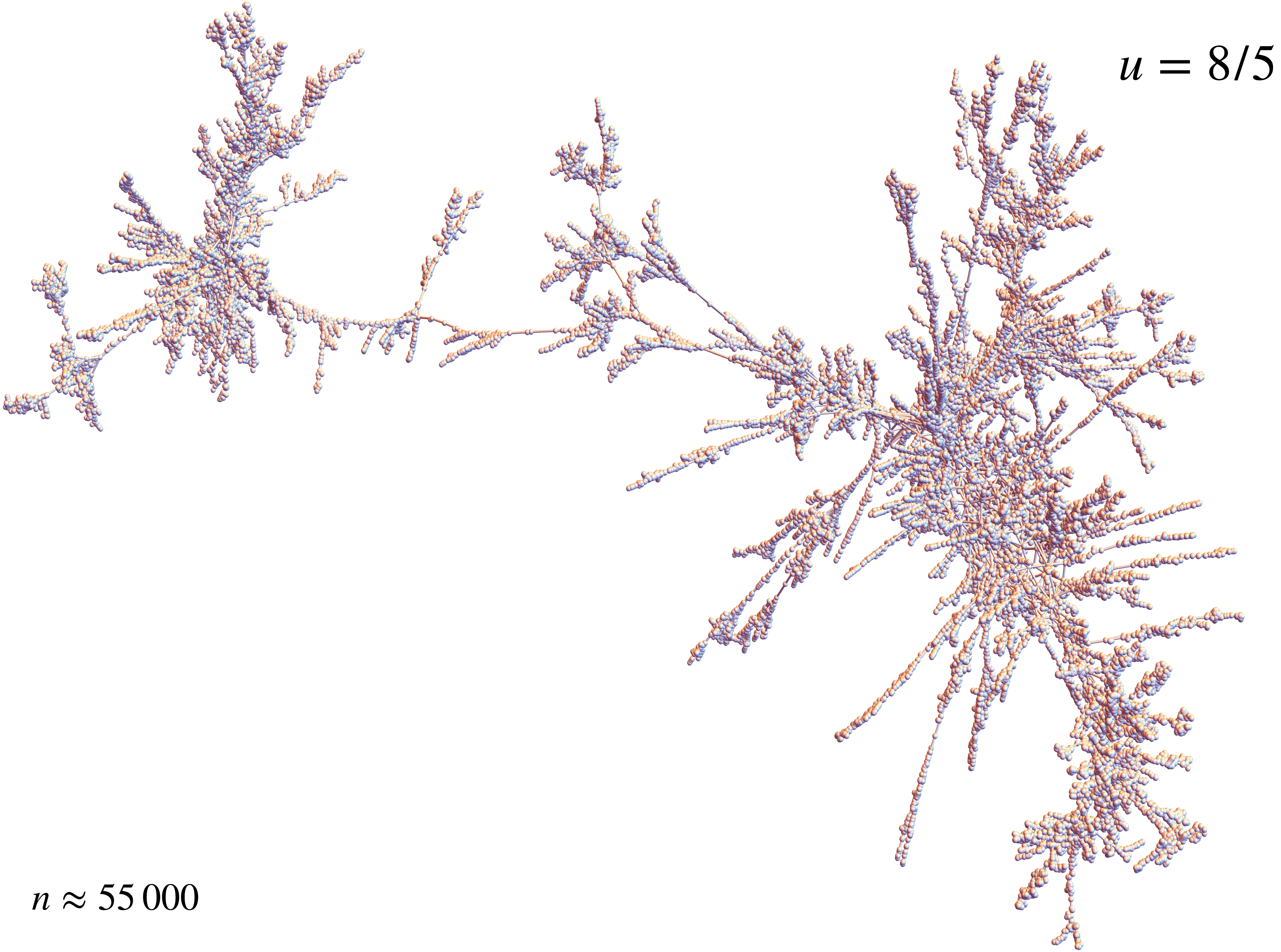
- Subcritical phase $u < 9/5$: "general map phase" one huge block;
- Critical phase $u = 9/5$: a few large blocks;
- Supercritical phase $u > 9/5$: "tree phase" only small blocks.

We obtain explicit results on enumeration, size of blocks and scaling limits in each case.

→ *A phase transition in block-weighted random maps*
W. Fleurat, Z. Salvy (2023)

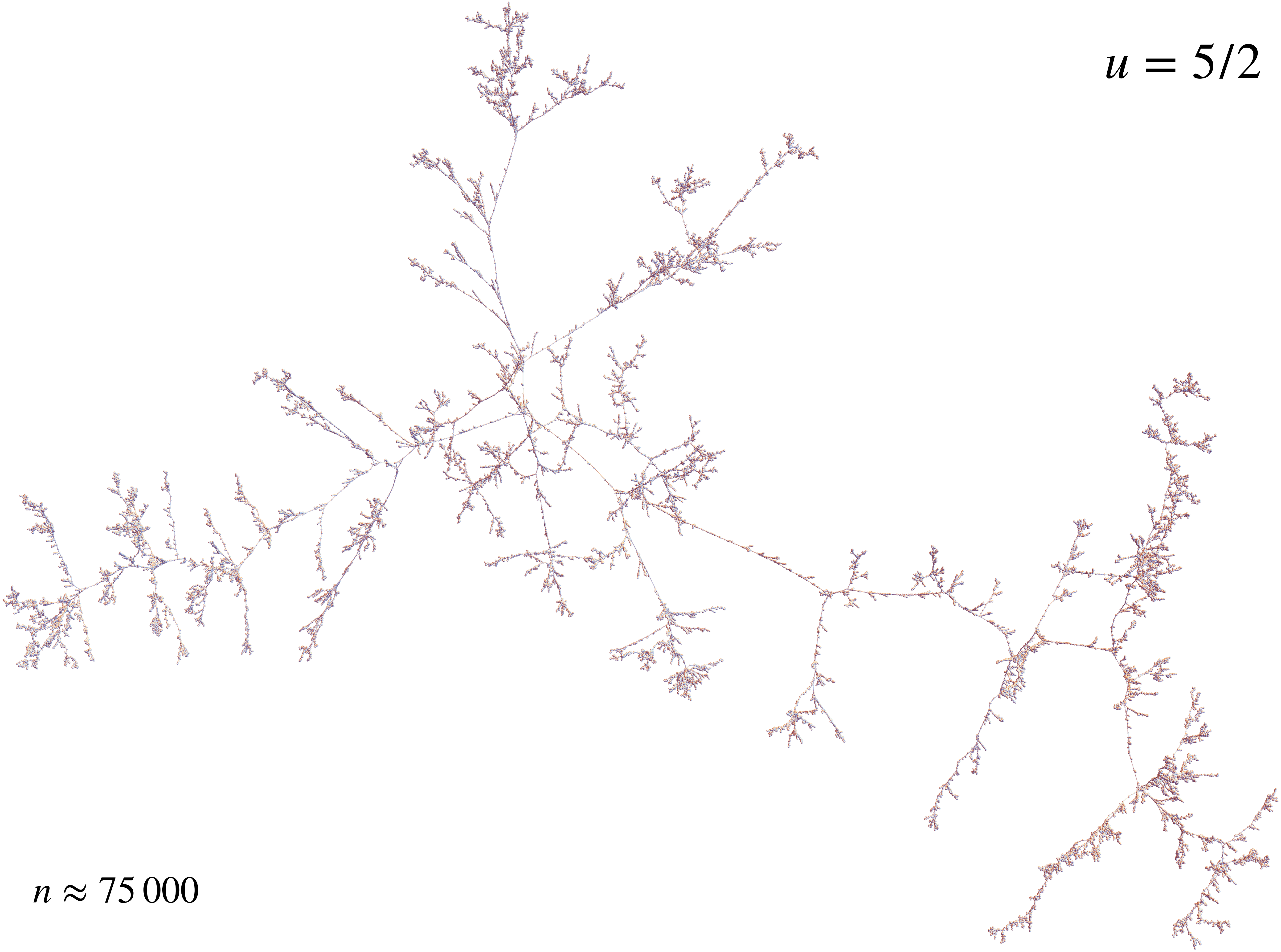


u = 1
n ≈ 55 000



$u = 8/5$
$n \approx 55\,000$

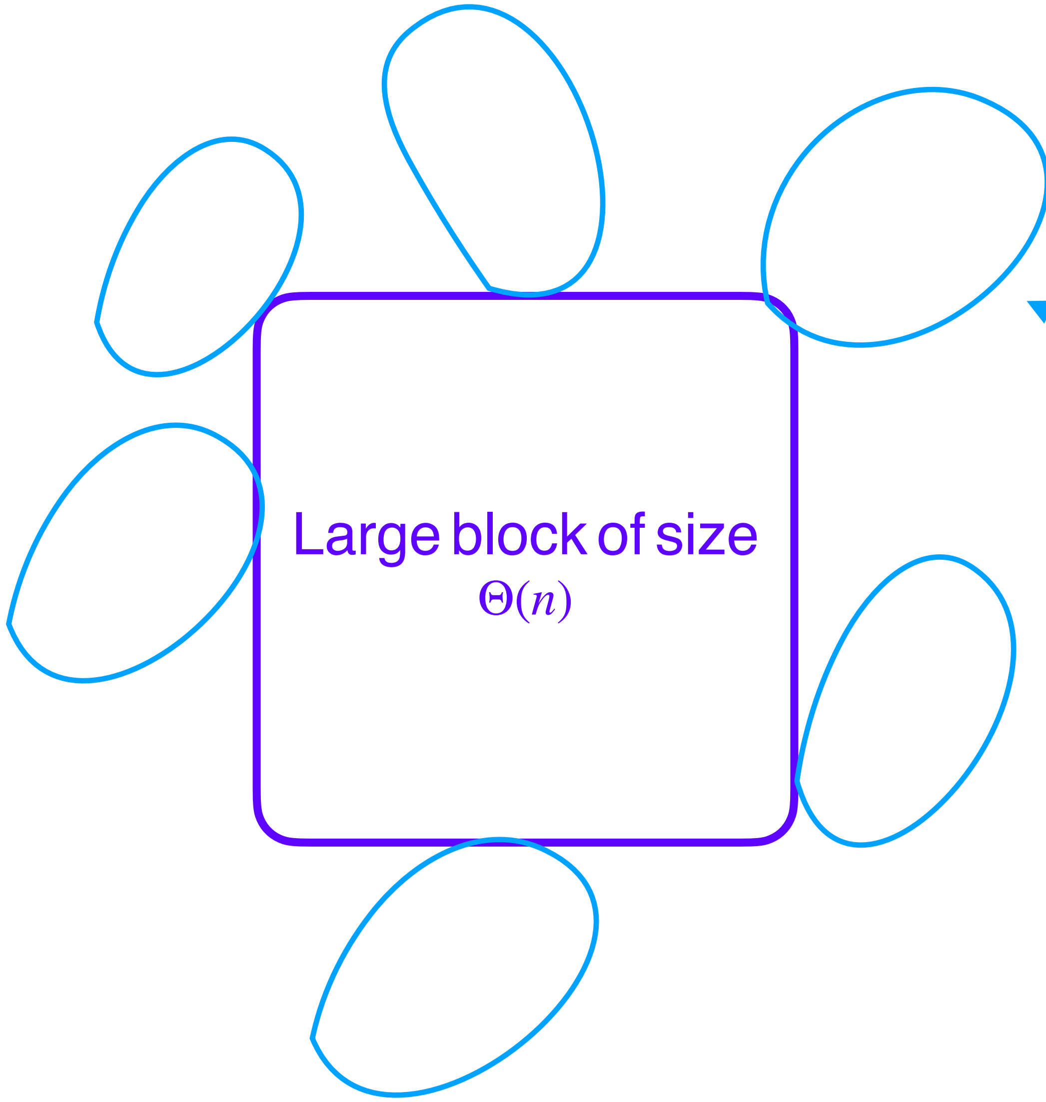
$u = 9/5$
$n \approx 80\,000$


$u = 5/2$
$n \approx 75\,000$

$u = 5$
$n \approx 50\,000$

Focus: scaling limits

Subcritical case = "general map" case



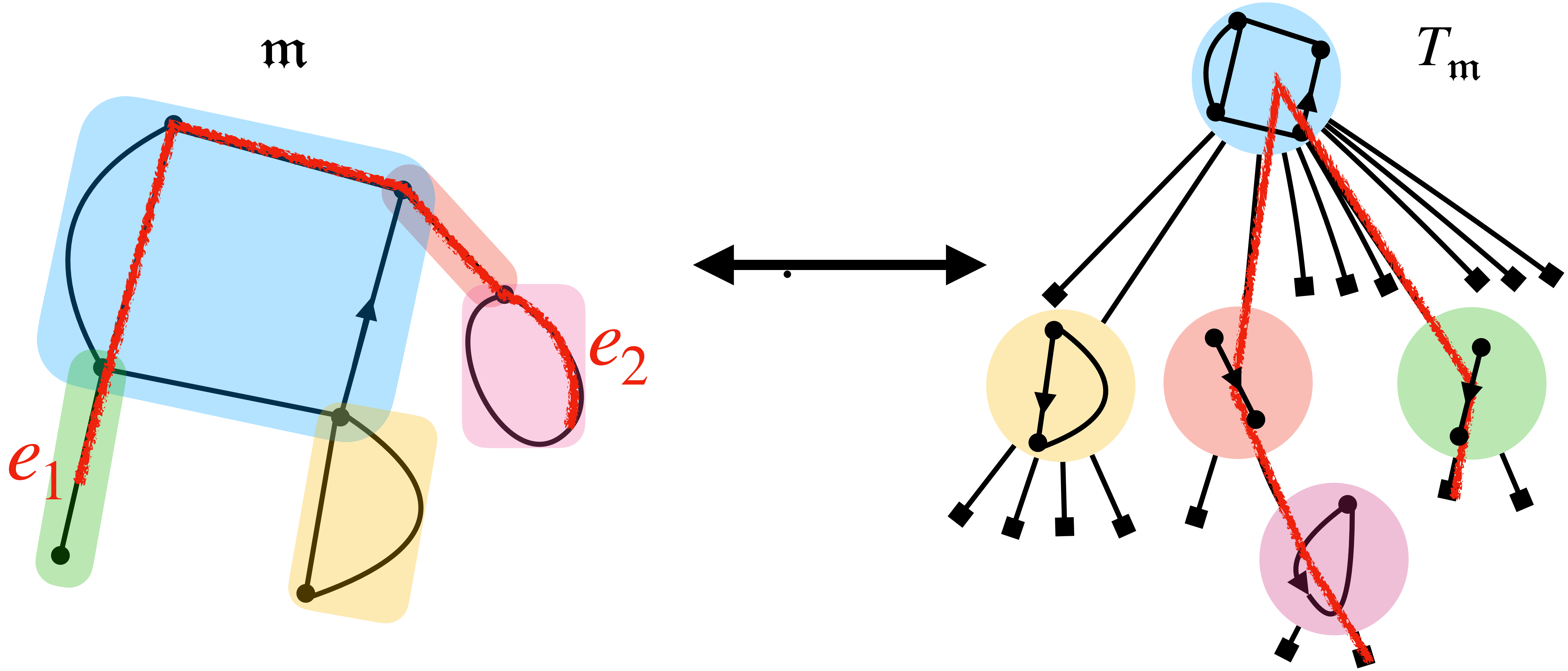
Decorations = groups of smaller blocks

Diameters of decorations = $o(n^{1/4})$.

The scaling limit of M_n (rescaled by $n^{1/4}$) is the scaling limit of the huge block!

uniform, size-concentrated

Critical and supercritical cases



Let $\kappa = \mathbb{E}$("diameter" bipointed block). By a "law of large numbers"-type argument

$$d_{\mathfrak{m}}(e_1, e_2) \simeq \kappa d_{T_{\mathfrak{m}}}(e_1, e_2)\,.$$

So distances in $\mathfrak{m}$ behave like distances in $T_{\mathfrak{m}}$.

Conclusion

Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$
loopless $M_2(z)$	simple $M_3(z)$	$z(1+M)$	–
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1+M)^2$	–
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1+M)$	–
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1+M)$
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1+M)$	–
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1-z(1+M))^2$	$z(1+M)^2$
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1+M)^2$	–
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1+M)$	–
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1+M)^3$	–
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1+M)^2$	–

→ *Unified study of the phase transition for block-weighted random planar maps* Z. Salvy (EUROCOMB'23)

Perspectives

- Extension to decompositions with coreless maps;
- Study of the critical window(s);
- Extension to spanning-tree decorated maps.

Thank you!