Phase transition for block-weighted random planar maps

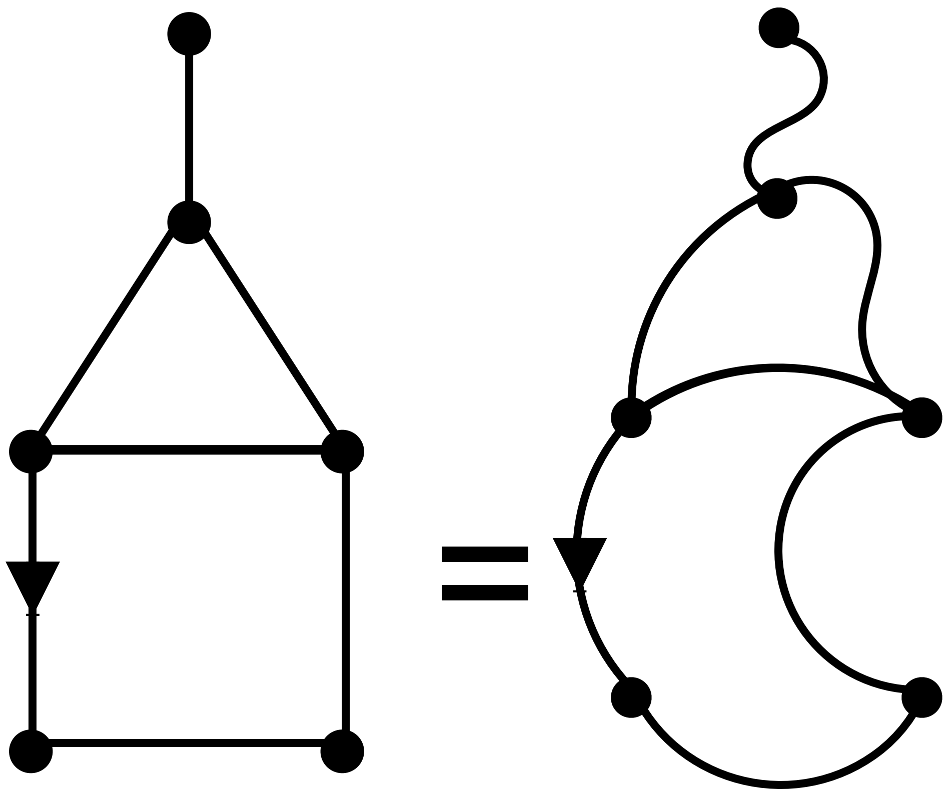
EUROCOMB'23

Zéphyr Salvy

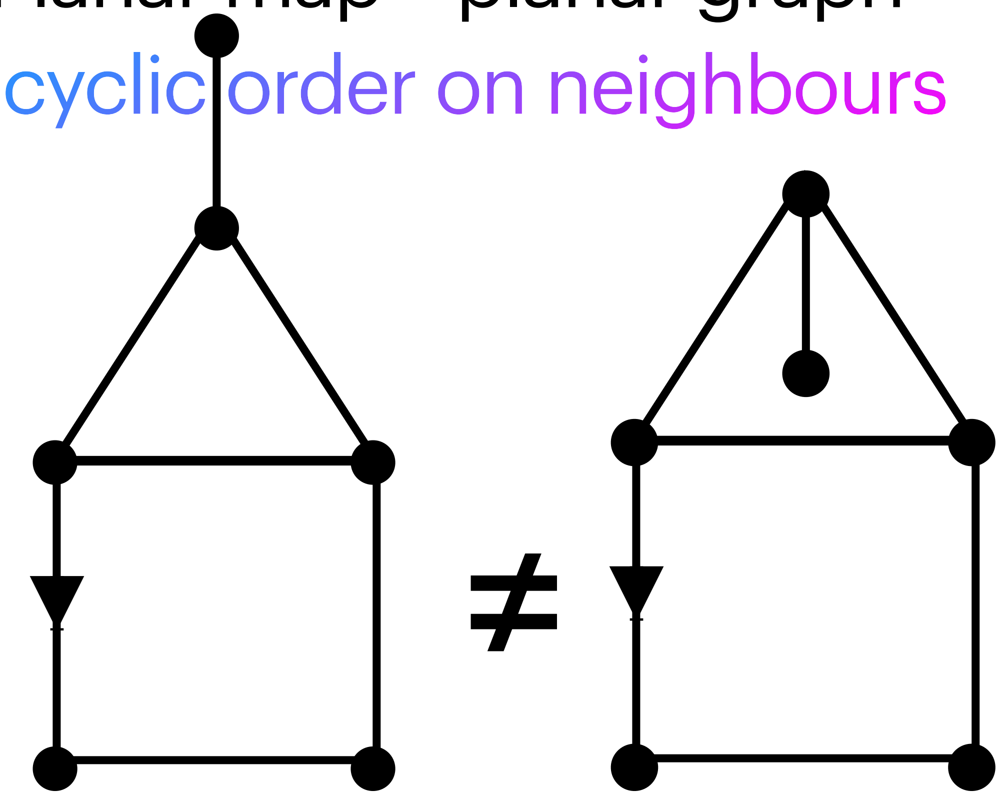
LIGM, Université Gustave Eiffel

Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms

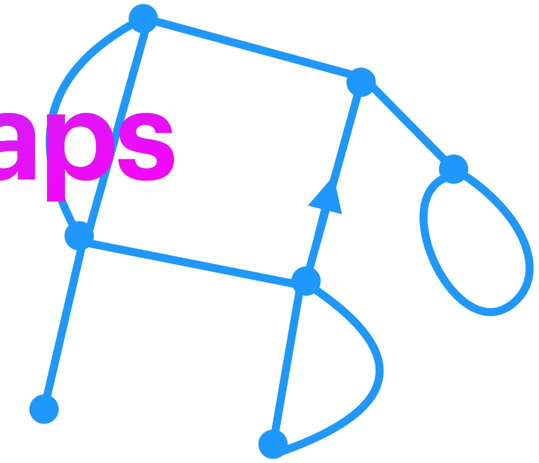


Planar map = planar graph + cyclic order on neighbours



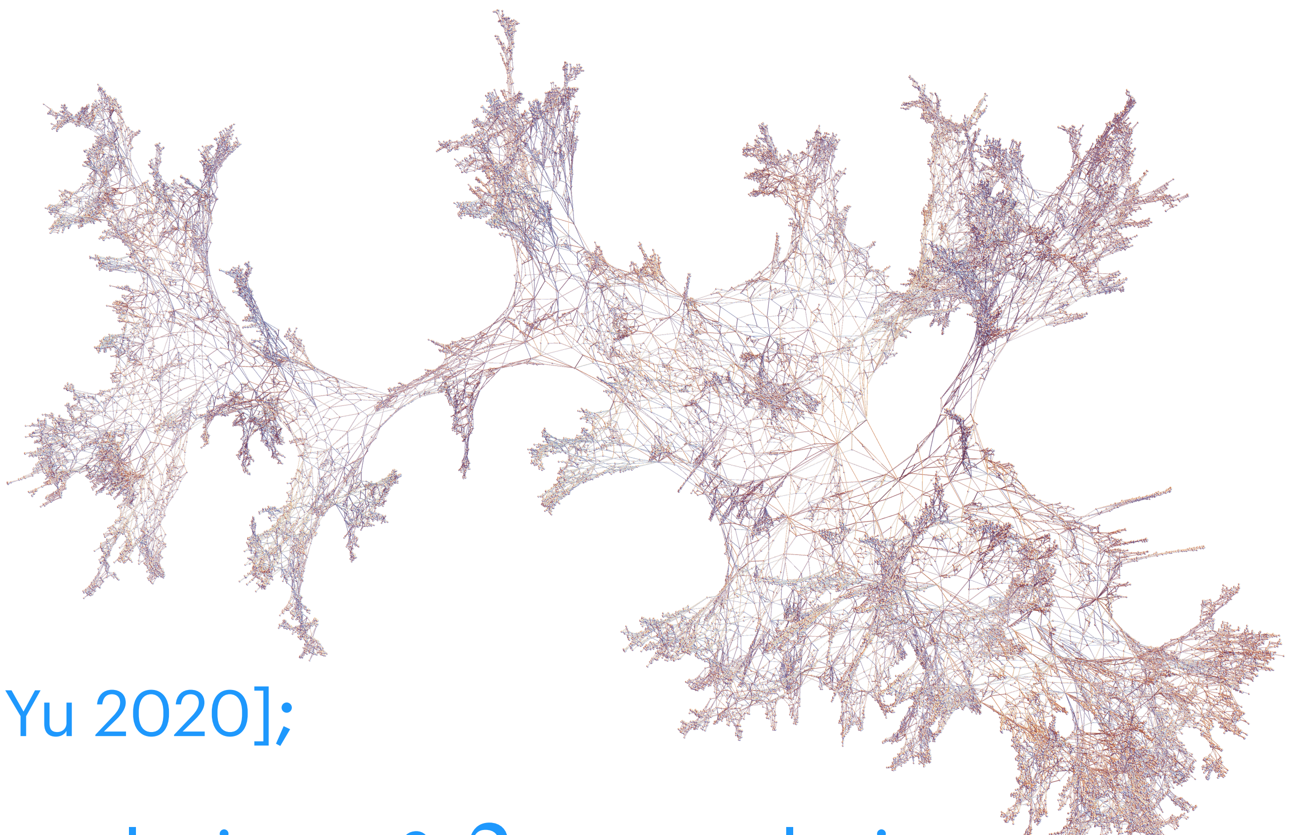
- Rooted planar map = map endowed with a marked oriented edge (represented by an arrow);
- Size $|\mathfrak{m}|$ = number of edges;
- Corner (does not exist for graphs !) = space between an oriented edge and the next one for the trigonometric order.

Universality results for planar maps



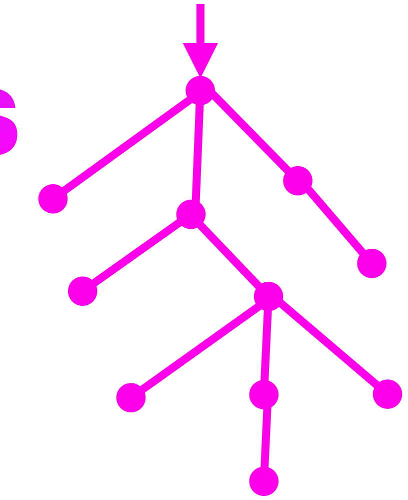
- Enumeration: $\kappa\rho^{-n}n^{-5/2}$ [Tutte 1963];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for quadrangulations [Le Gall 2013, Miermont 2013] and uniform maps [Bettinelli, Jacob, Miermont 2014];

Brownian Sphere $\mathcal{S}_e$

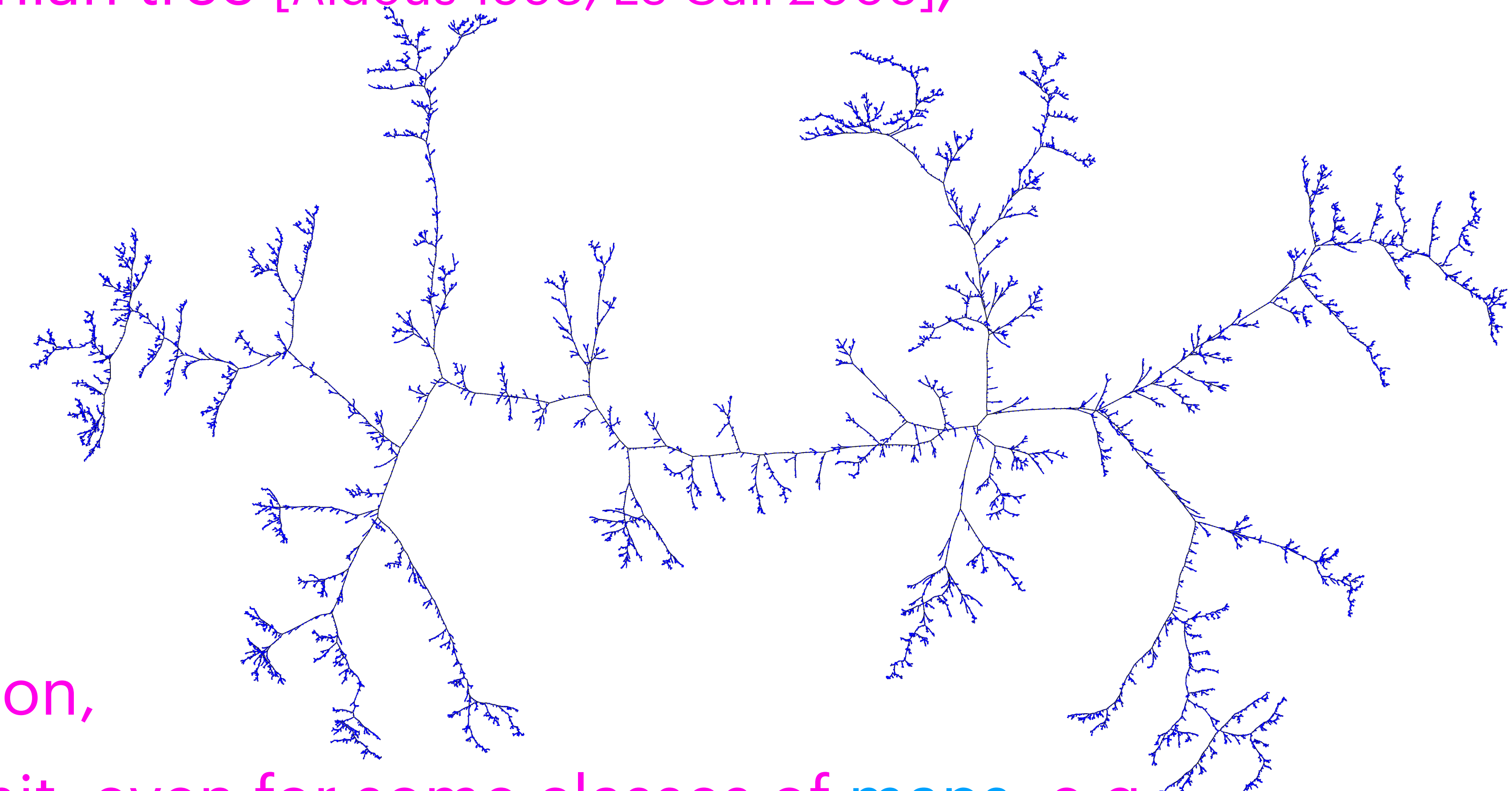


- Universality:
  - Same enumeration [Drmota, Noy, Yu 2020];
  - Same scaling limit, *e.g.* for triangulations & $2q$-angulations [Le Gall 2013], simple quadrangulations [Addario-Berry, Albenque 2017].

Universality results for plane trees



- Enumeration: $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];



- Universality:
  - Same enumeration,
  - Same scaling limit, even for some classes of maps; e.g. outerplanar maps [Caraceni 2016], maps with a boundary of size >> $n^{1/2}$ [Bettinelli 2015].

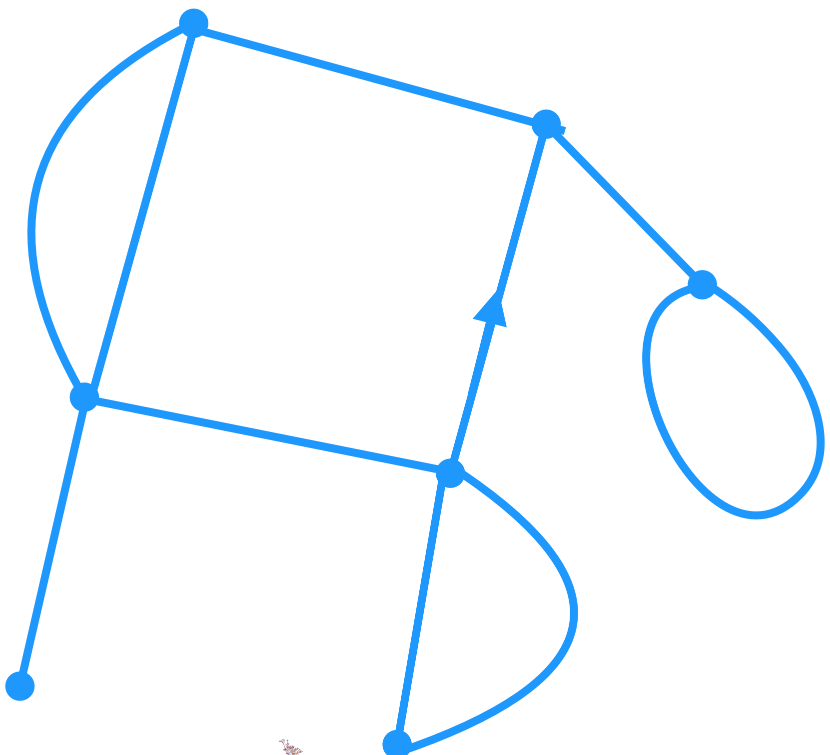
Models with (very) constrained boundaries

Motivation

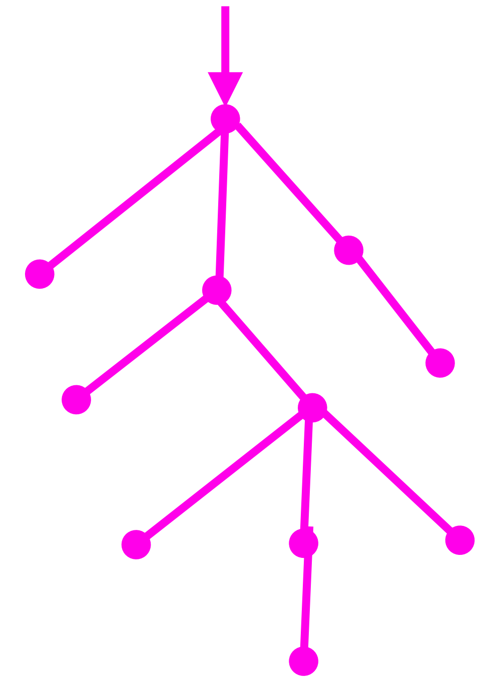
Inspired by [Bonzom 2016].

Two rich situations with universality results:

Planar maps



Plane trees



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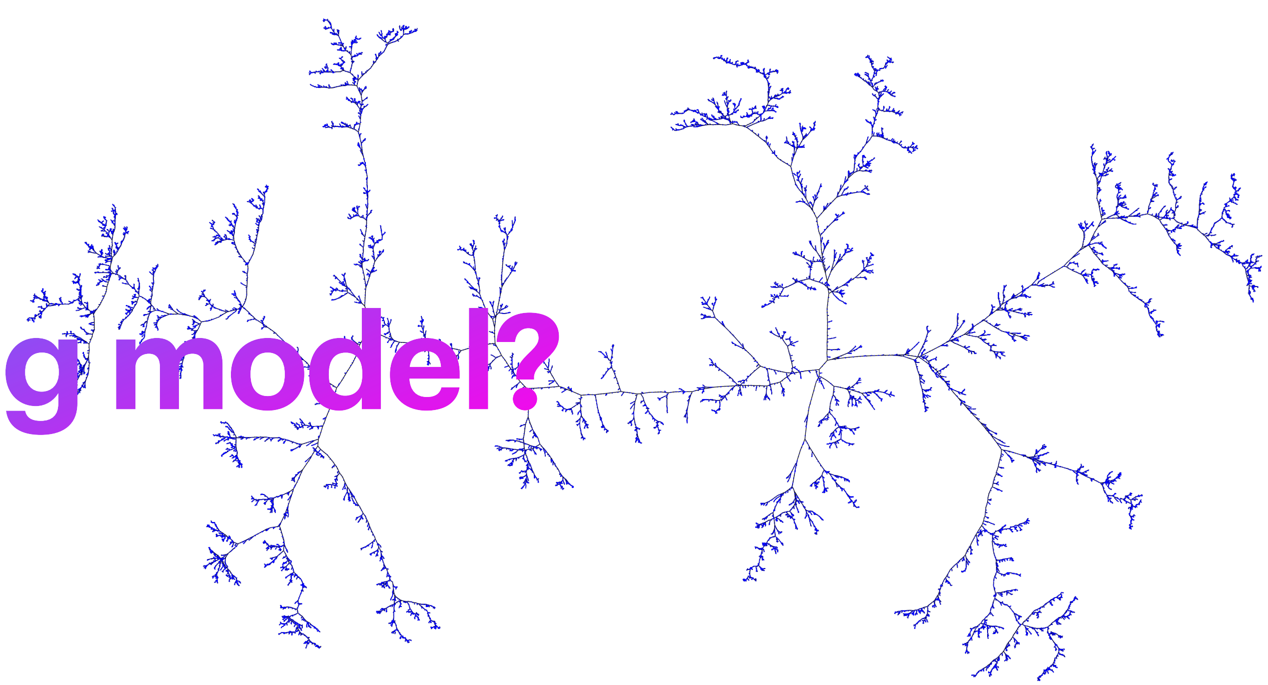

Brownian Tree $\mathscr{T}_e$

Interpolating model?

Model definition

2-connected = two vertices must be removed to disconnect.

Block = maximal (for inclusion) 2-connected submap.

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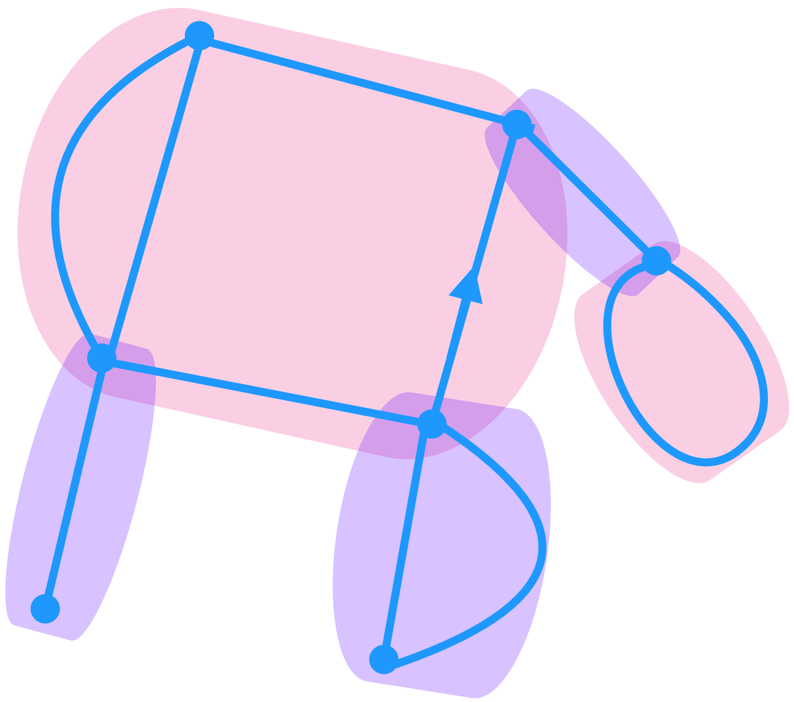

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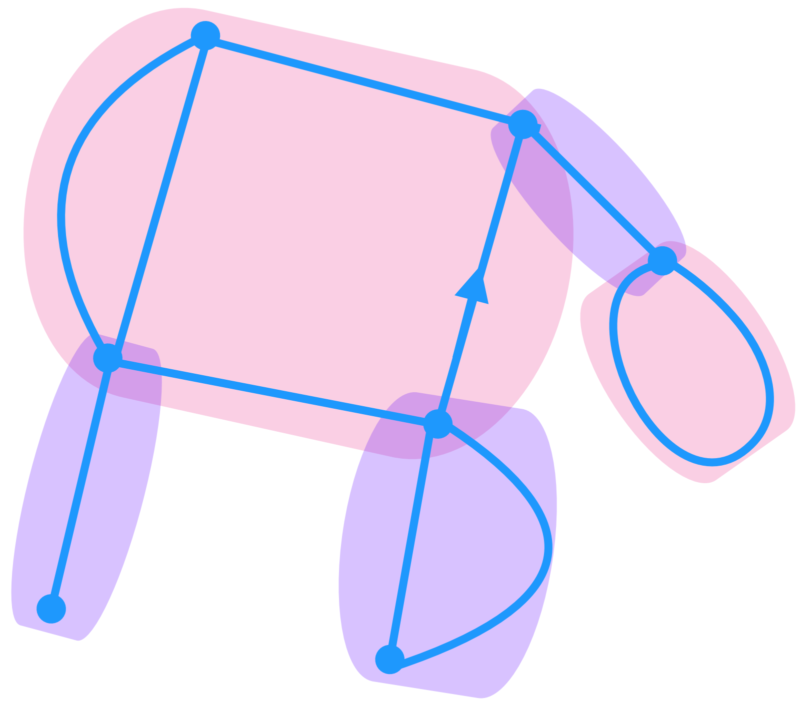

Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

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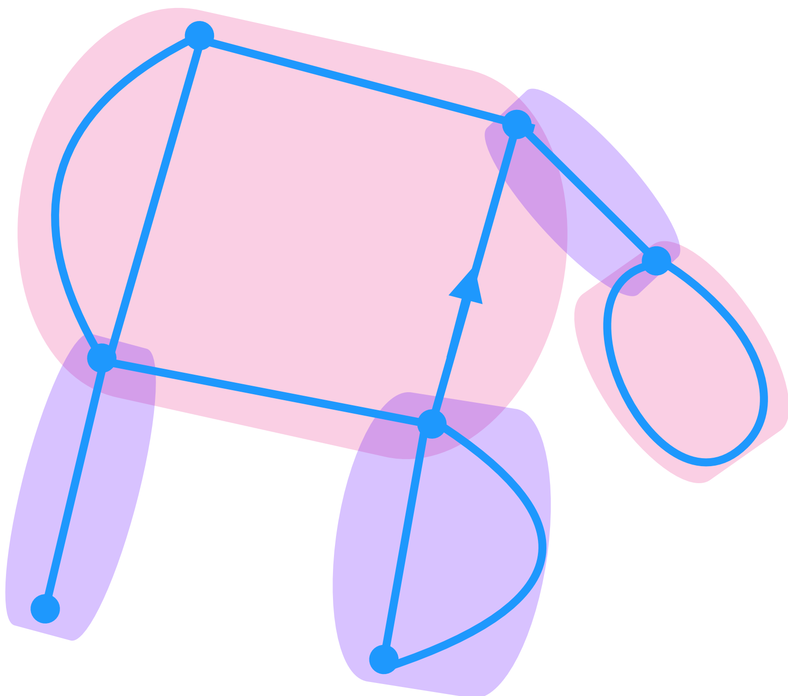

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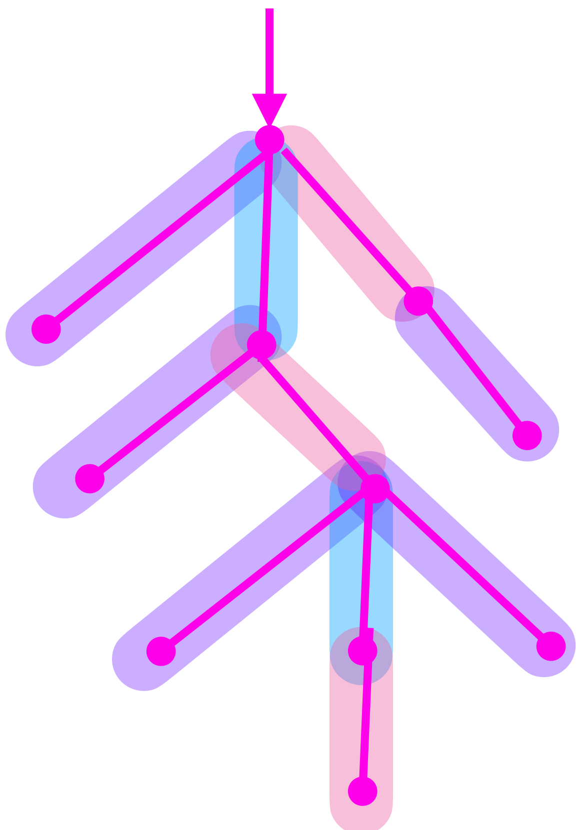

Only small blocks.

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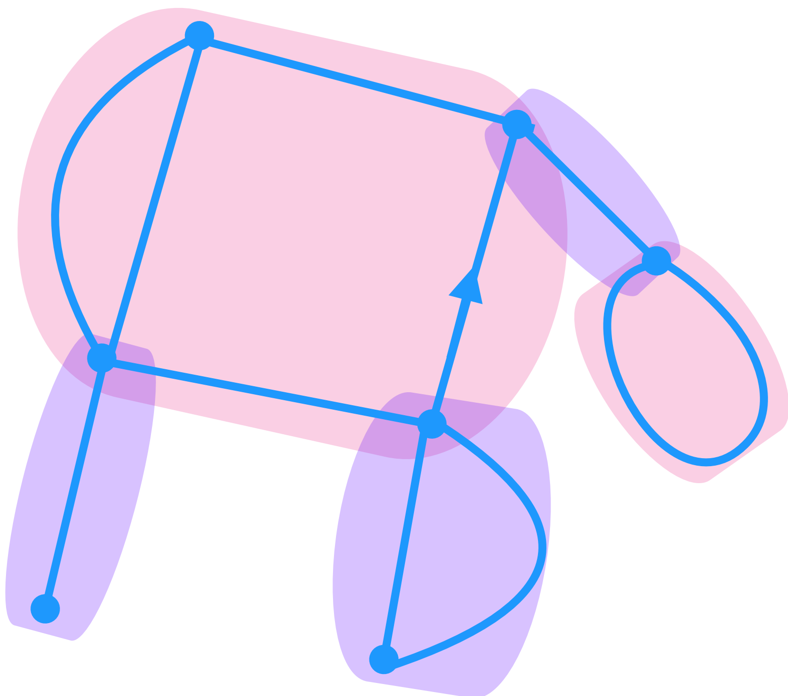

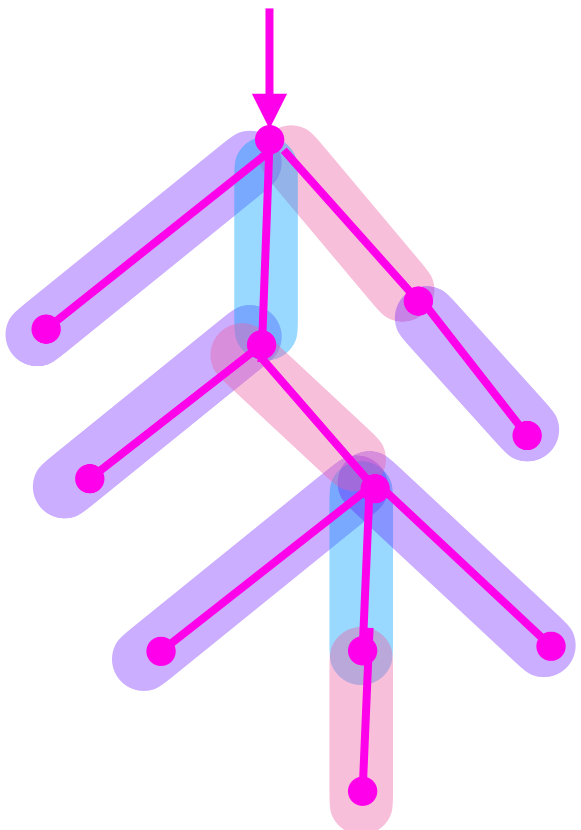

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Interpolating model using blocks!

Model

Inspired by [Bonzom 2016].

Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,

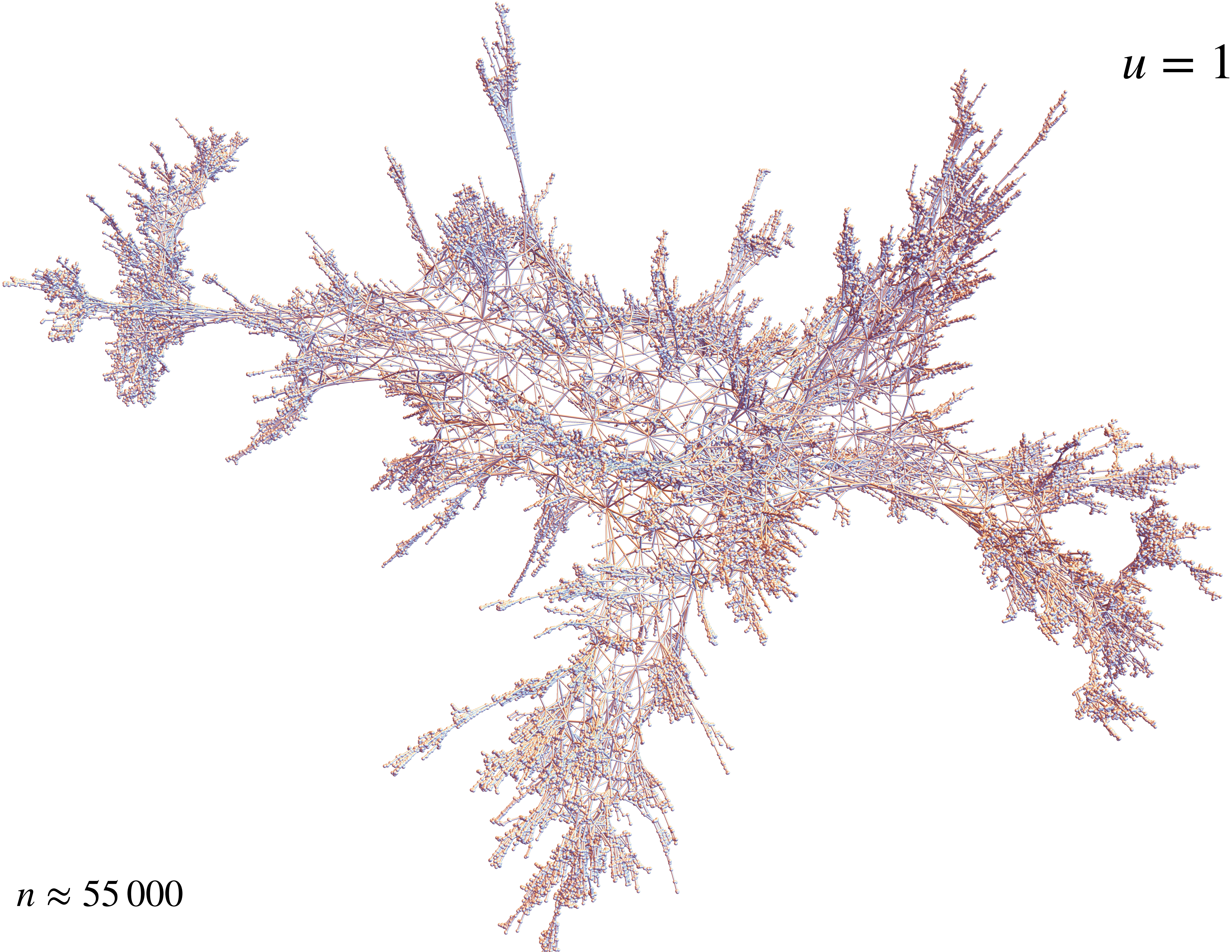
$\mathcal{M}_n$ = {maps of size n},

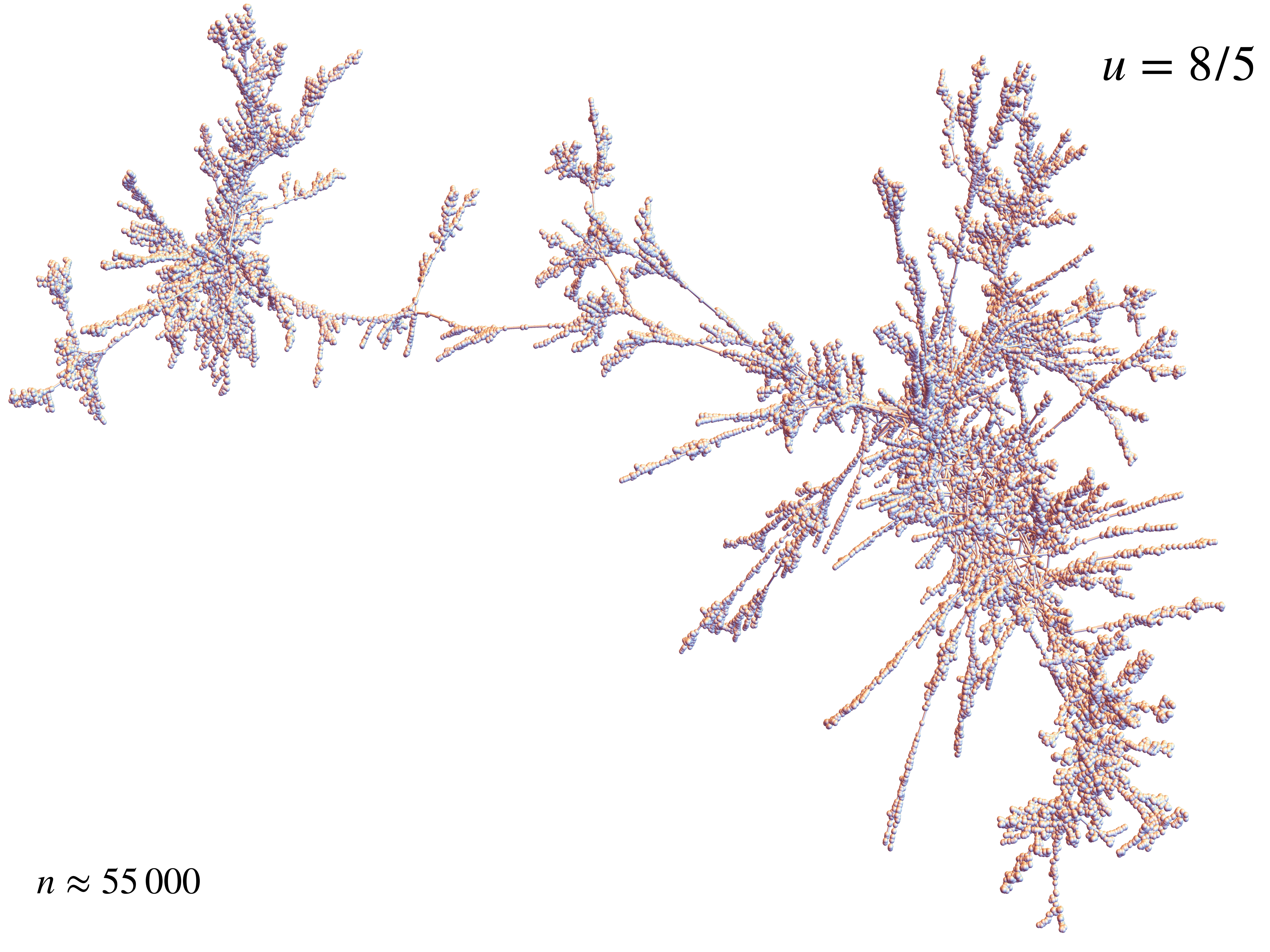
$\mathfrak{m} \in \mathcal{M}_n$,

$Z_{n,u}$ = normalisation.

- $u = 1$: uniform distribution on maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected maps);
- $u \to \infty$: maximising the number of blocks (= trees!).

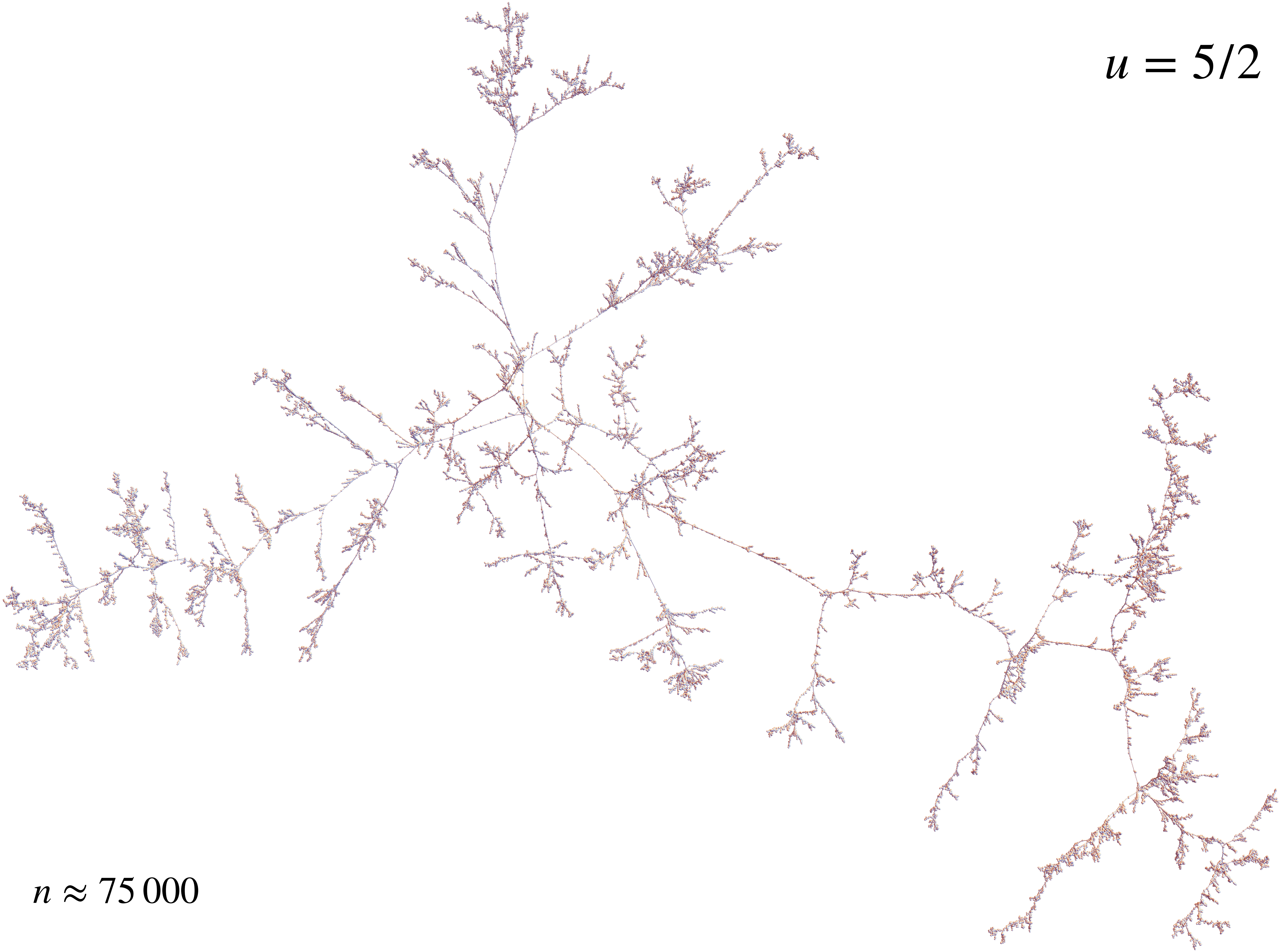
Given u, asymptotic behaviour when $n \to \infty$?

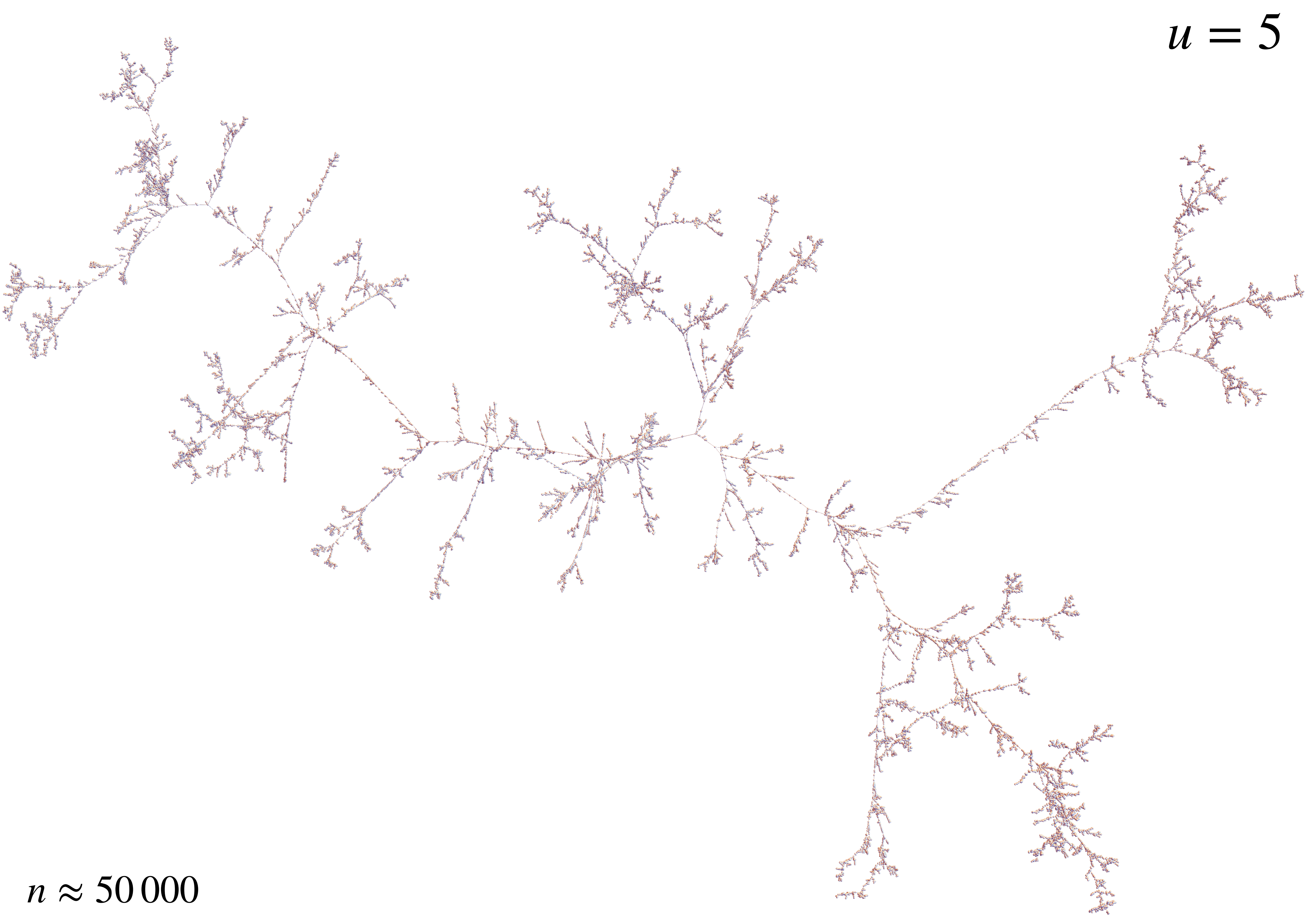

$u = 1$
$n \approx 55\,000$



$u = 8/5$
$n \approx 55\,000$


$u = 9/5$
$n \approx 80\,000$


$u = 5/2$
$n \approx 75\,000$


$u = 5$
$n \approx 50\,000$

Phase transition

Theorem [Fleurat, S. 23] Model exhibits a phase transition at $u = 9/5$. When $n \to \infty$:

- Subcritical phase $u < 9/5$: "general map phase" one huge block;
- Critical phase $u = 9/5$: a few large blocks;
- Supercritical phase $u > 9/5$: "tree phase" only small blocks.

We obtain explicit results on enumeration, size of blocks and scaling limits in each case.

→ *A phase transition in block-weighted random maps*
W. Fleurat, Z. Salvy (2023)

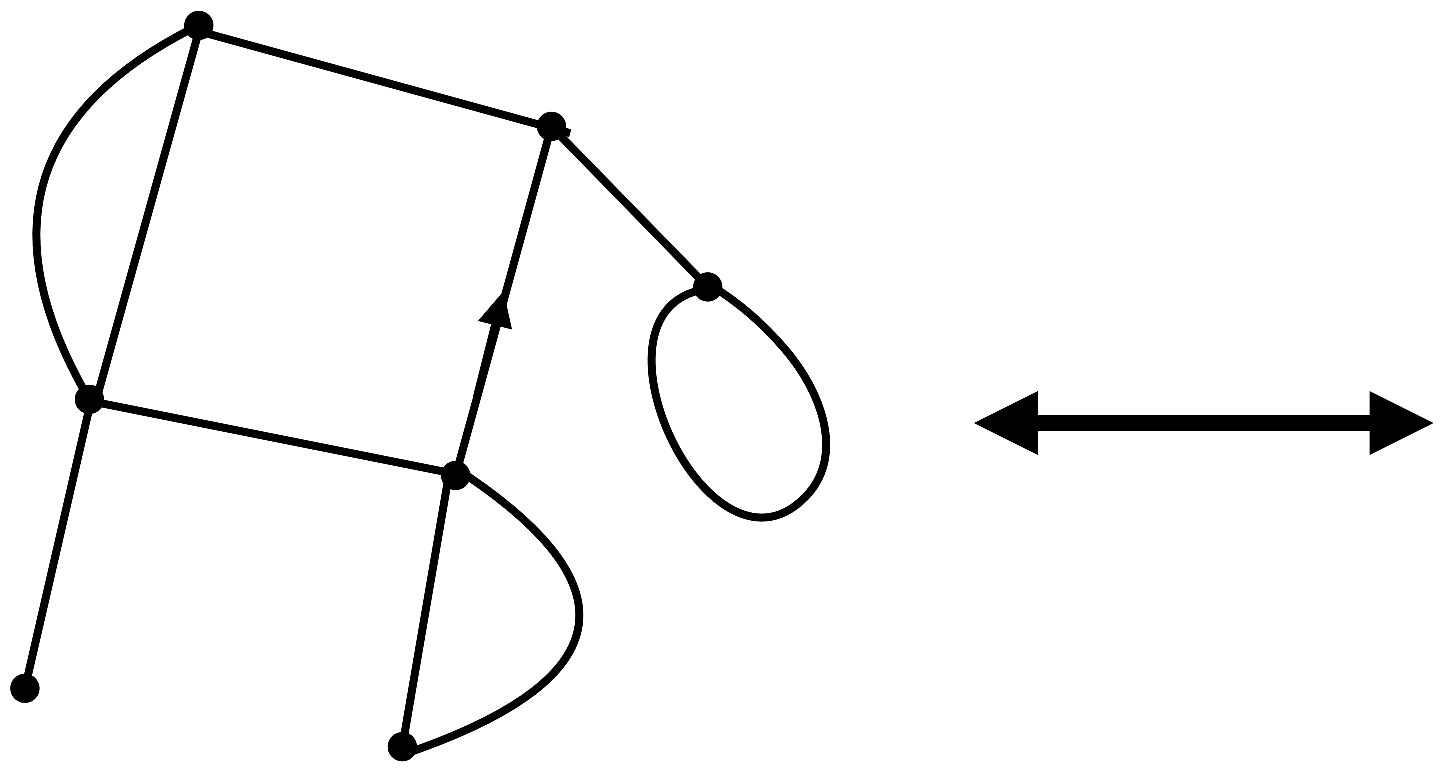
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration			
Size of - the largest block - the second one			
Scaling limit of M_n			

I. Block tree of a map and its applications

Decomposition of a map into blocks

Inspiration from [Tutte 1963]



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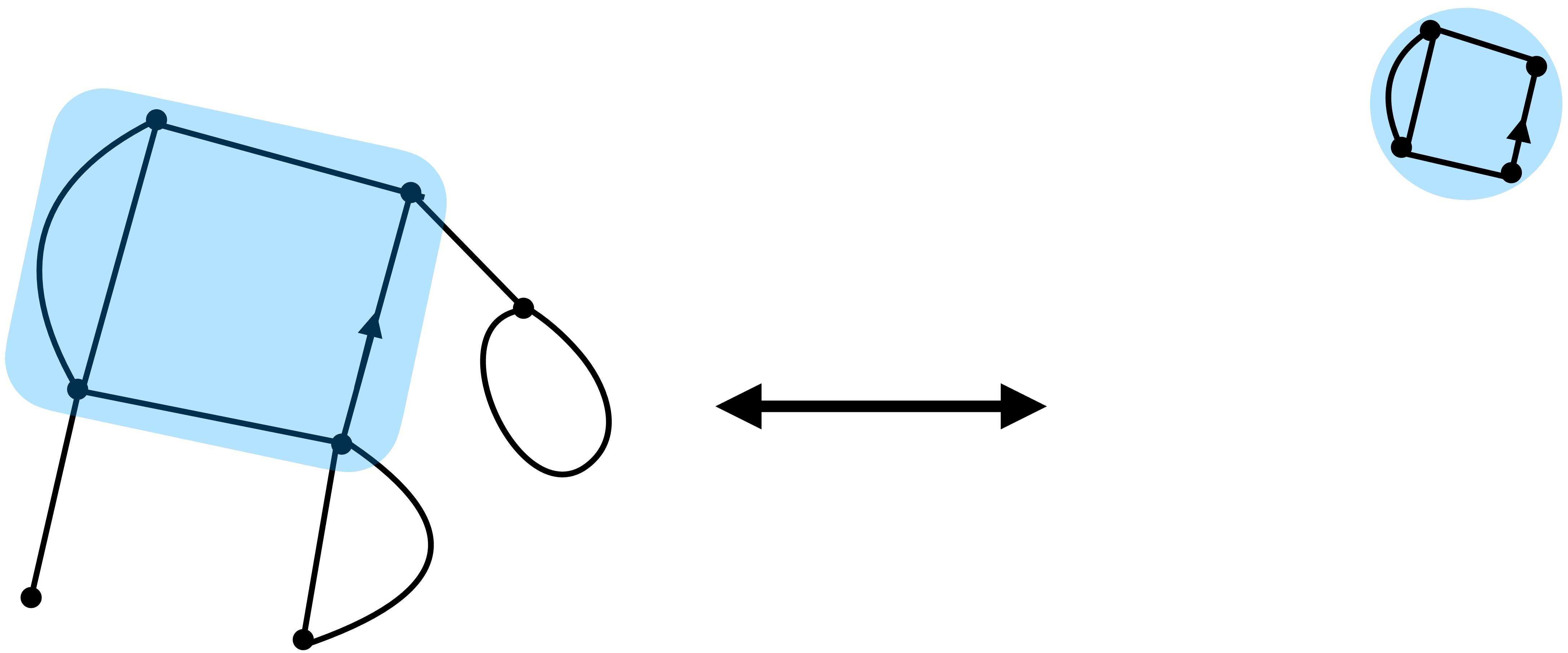

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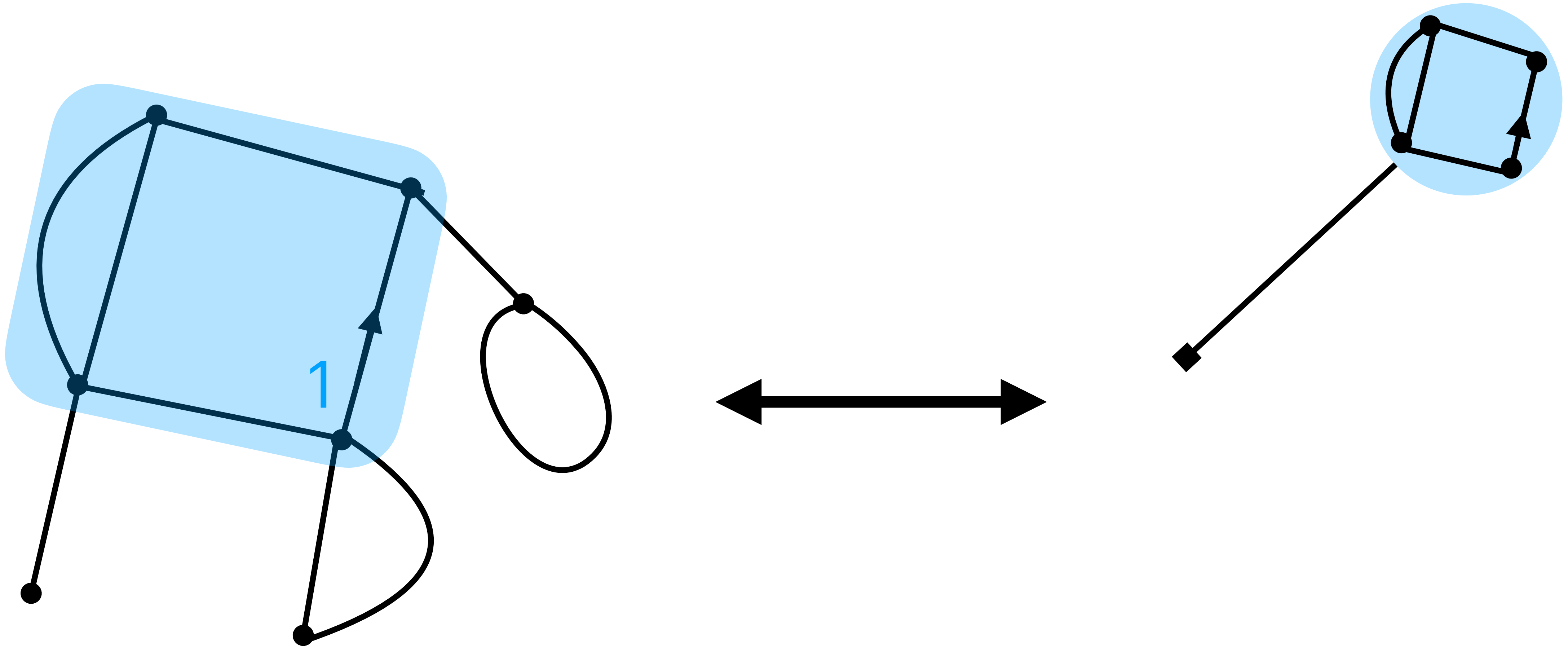

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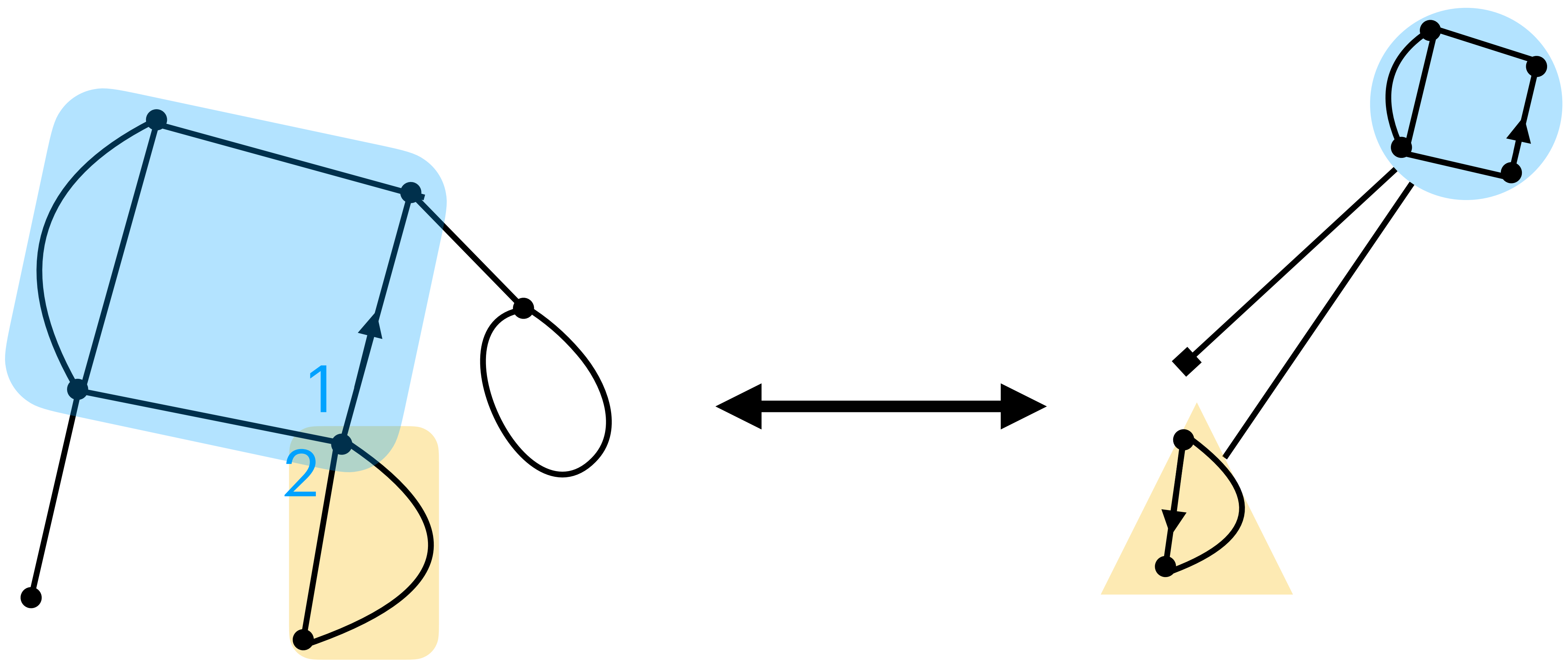

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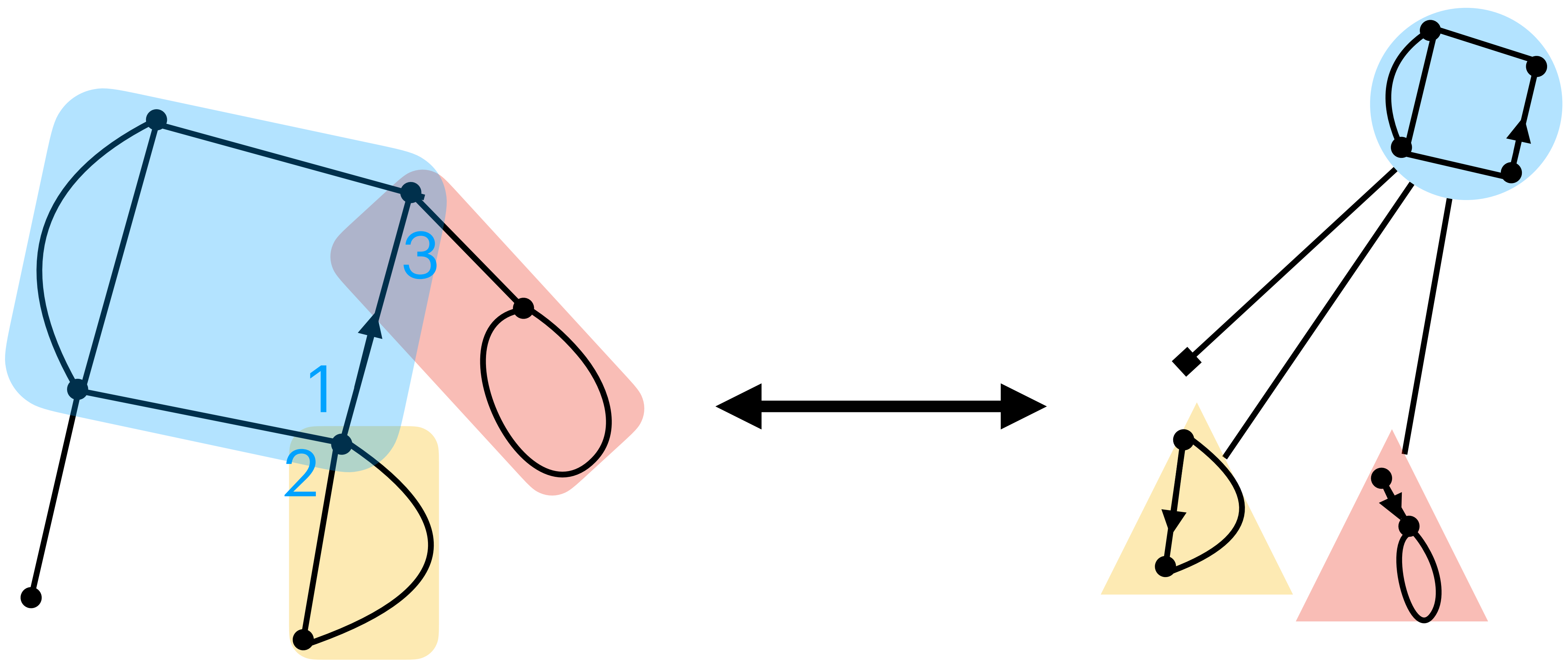

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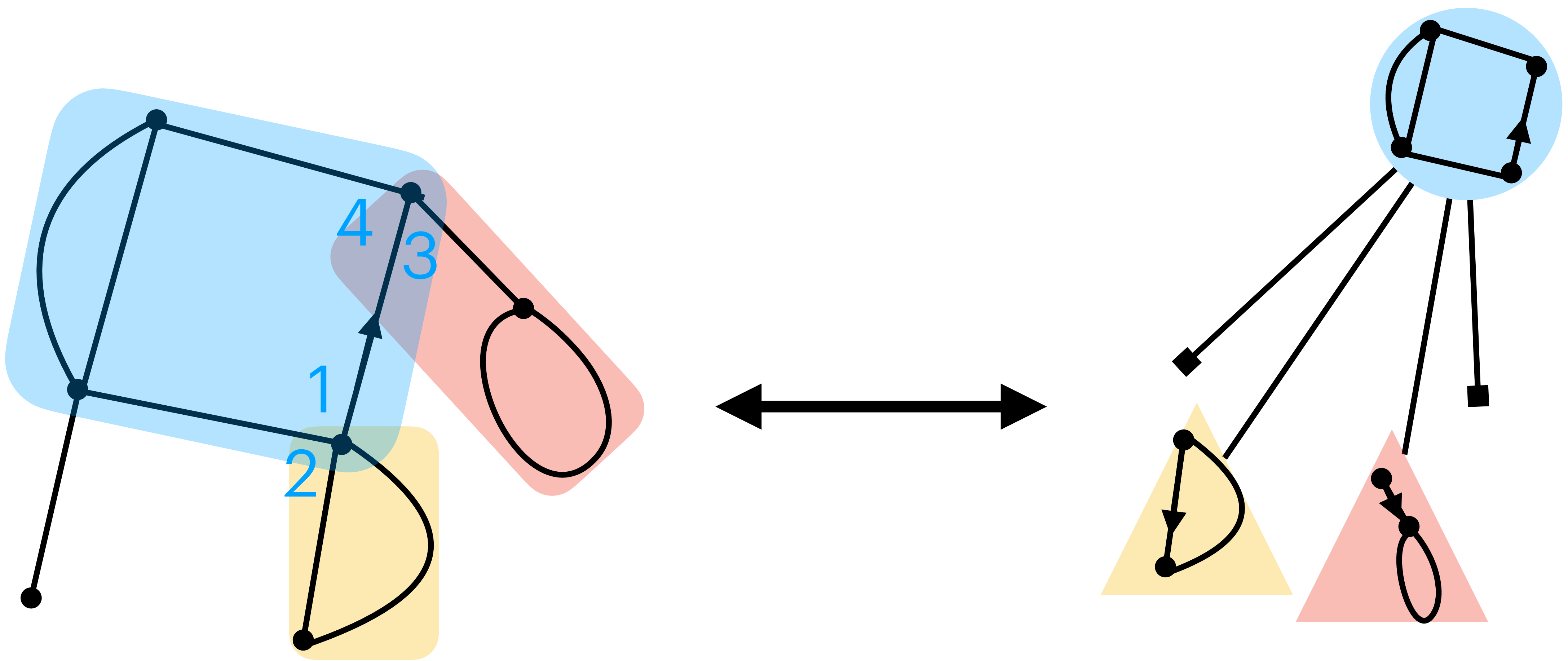

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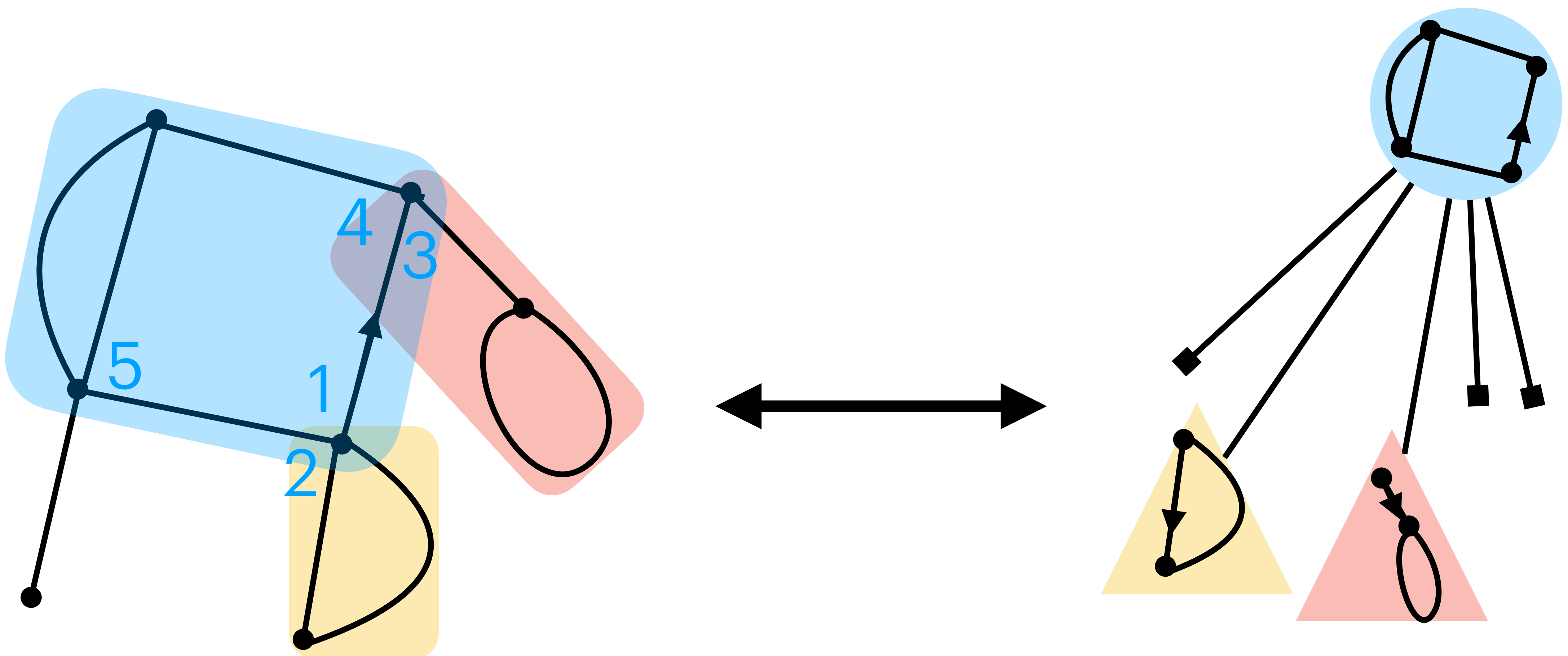

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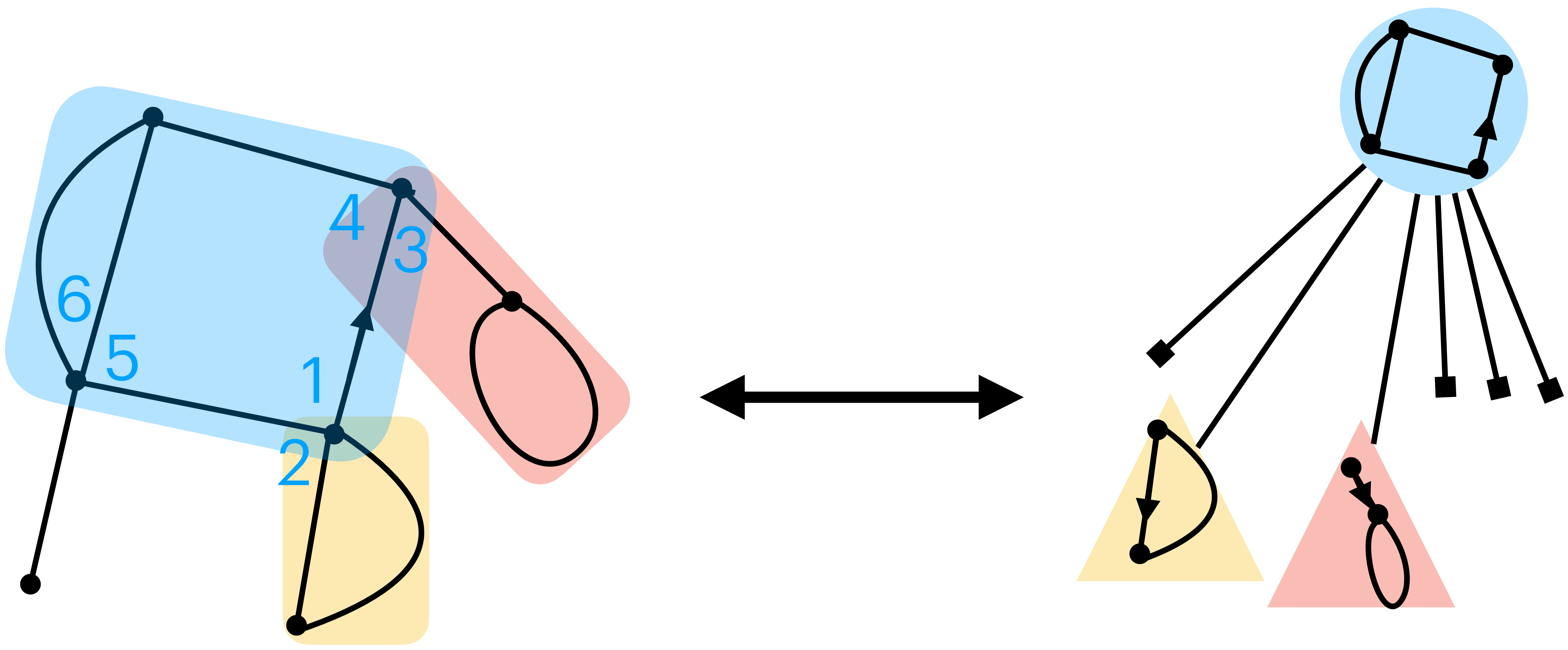

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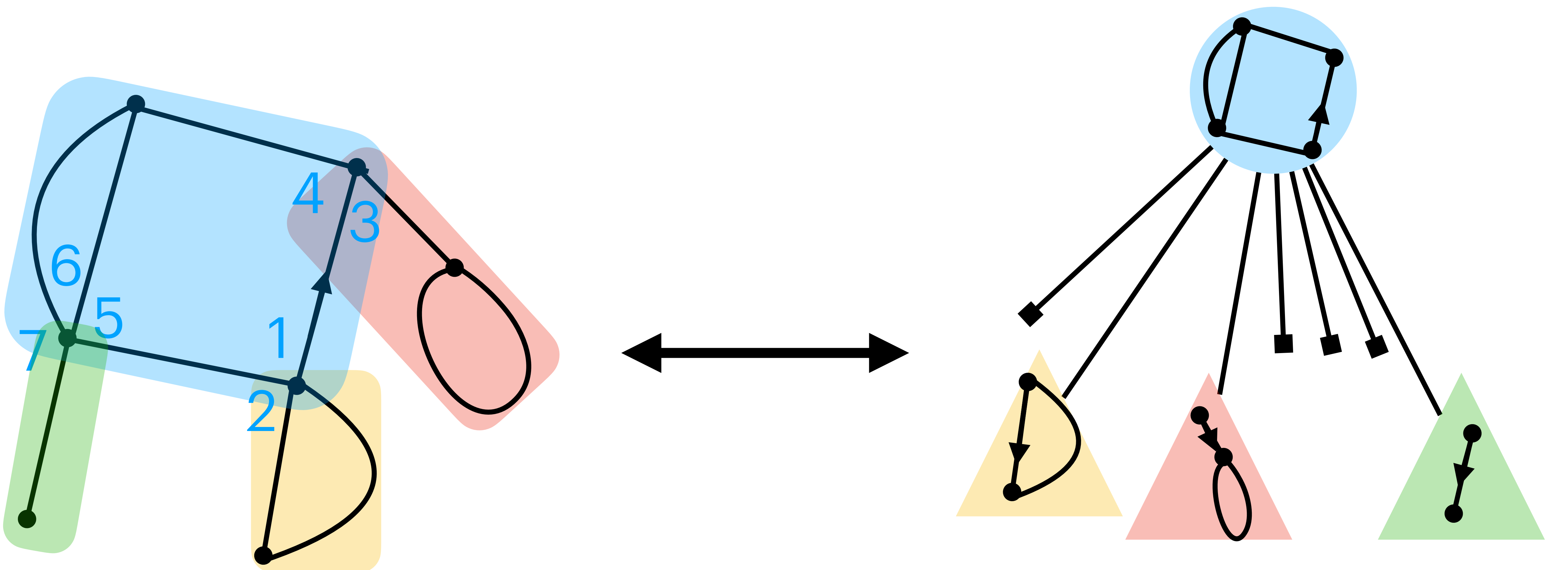

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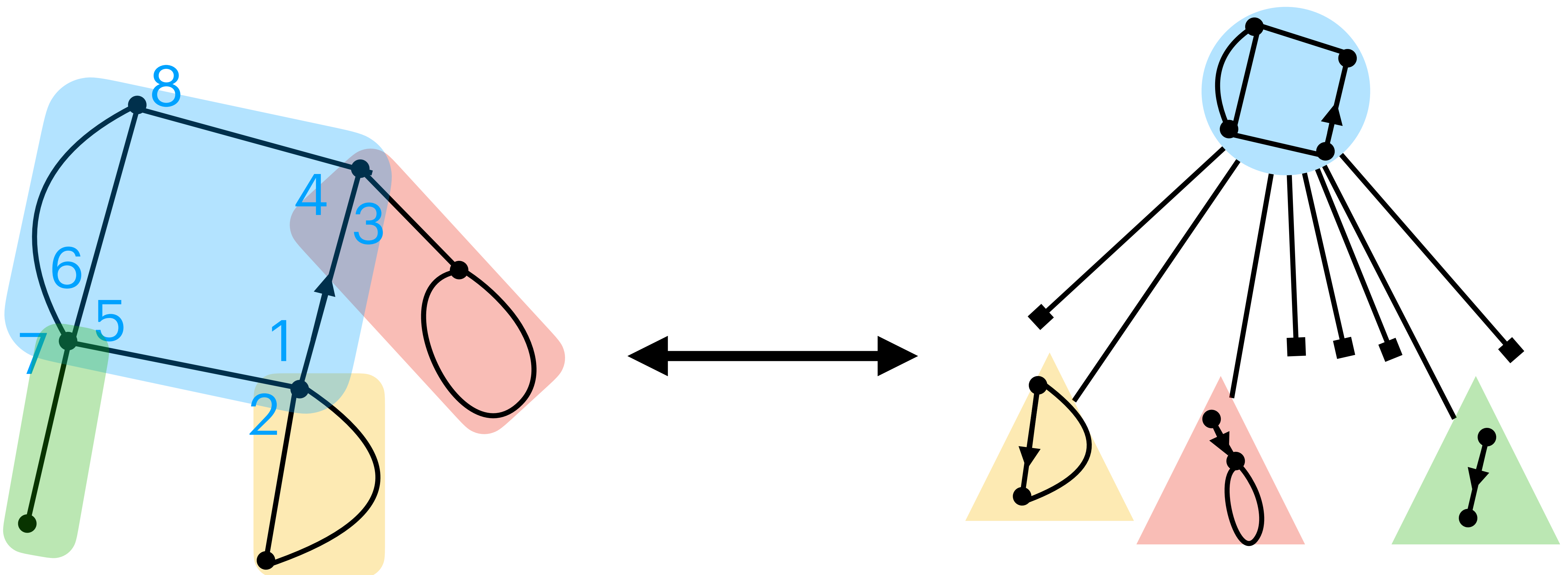

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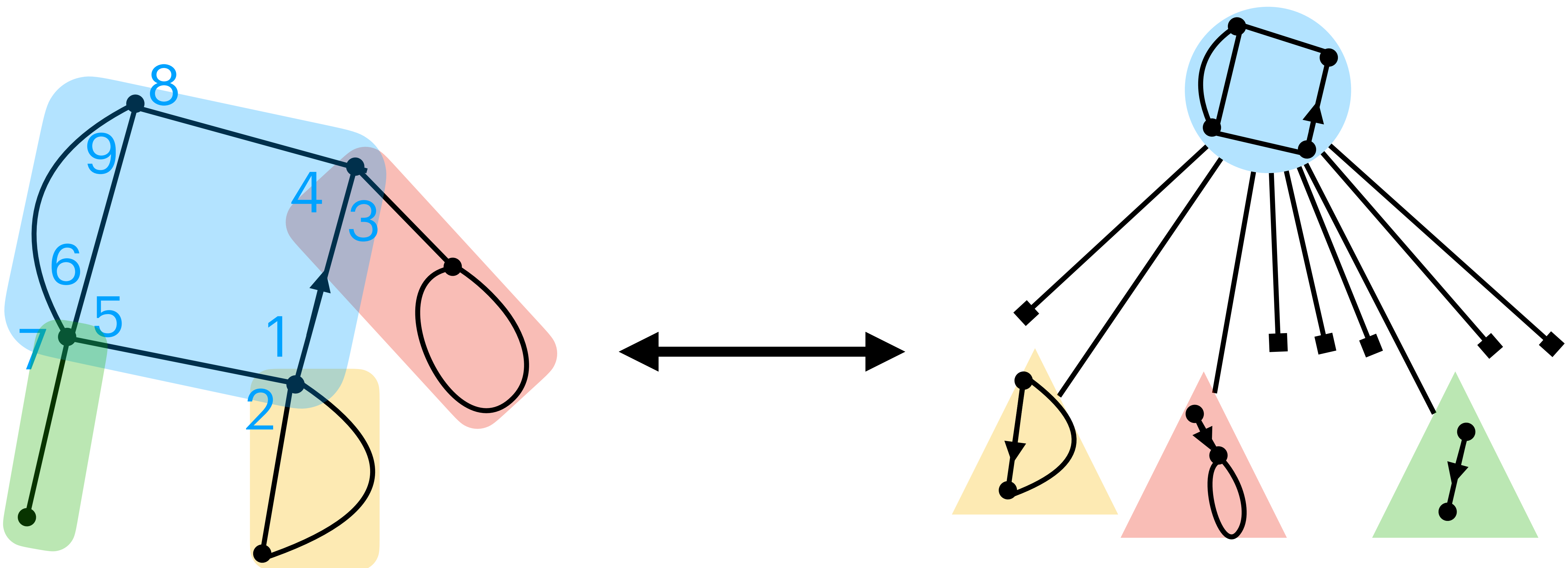

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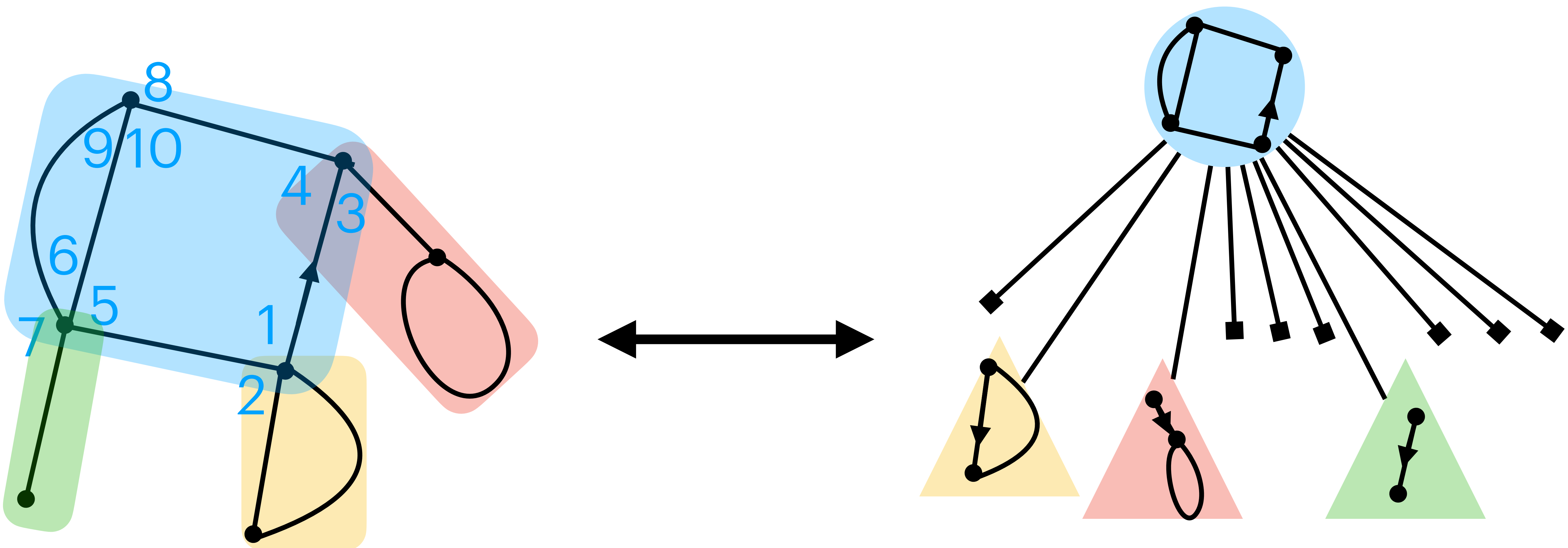

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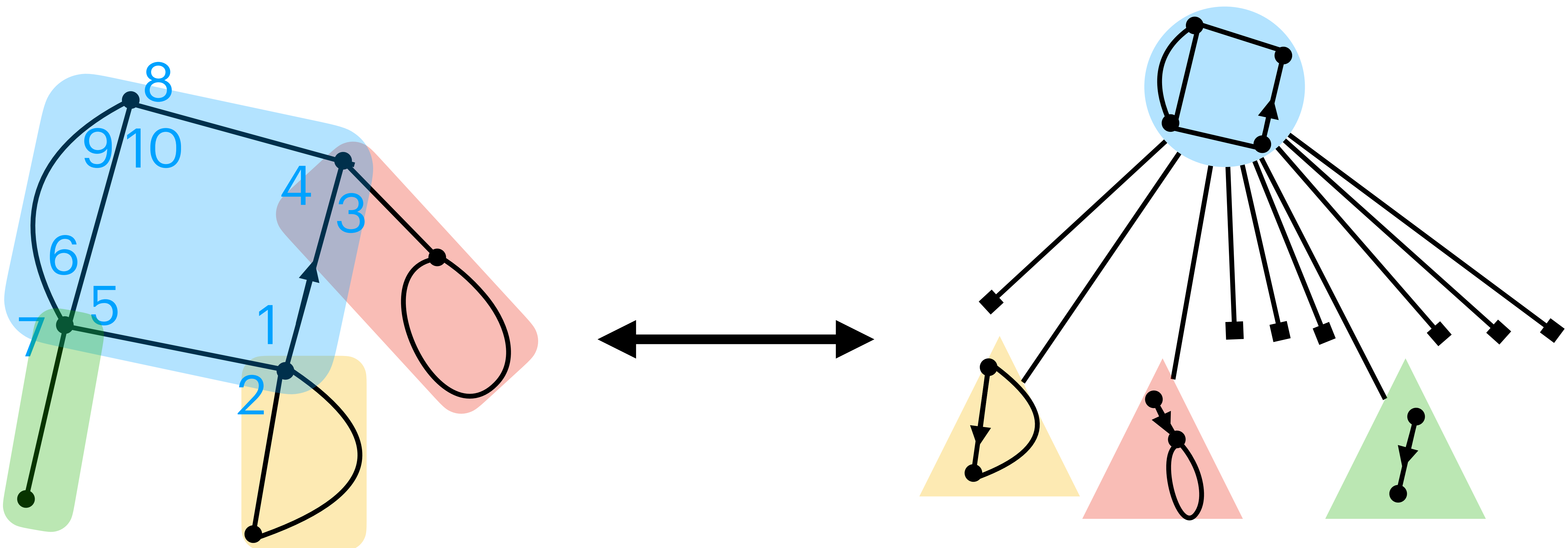

GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one			
Scaling limit of M_n			

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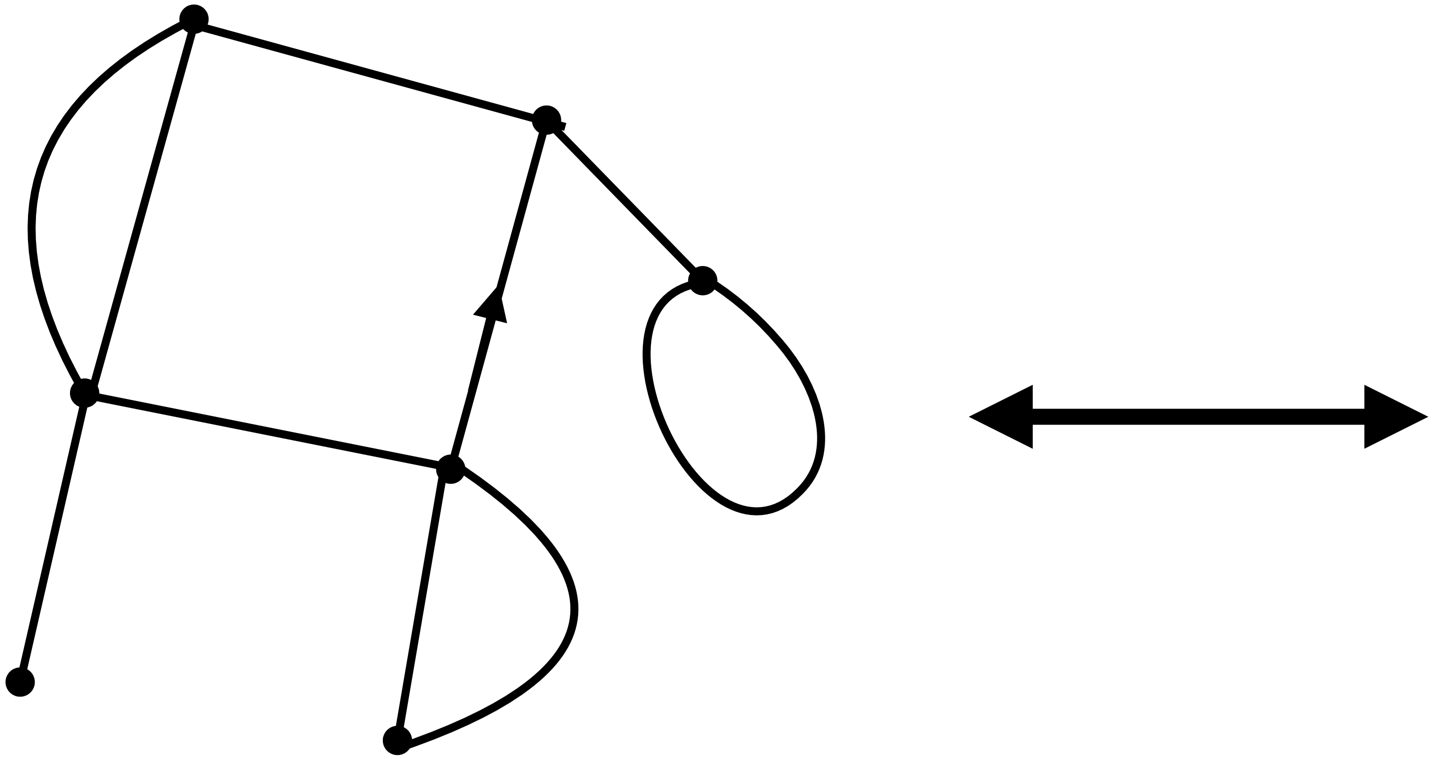

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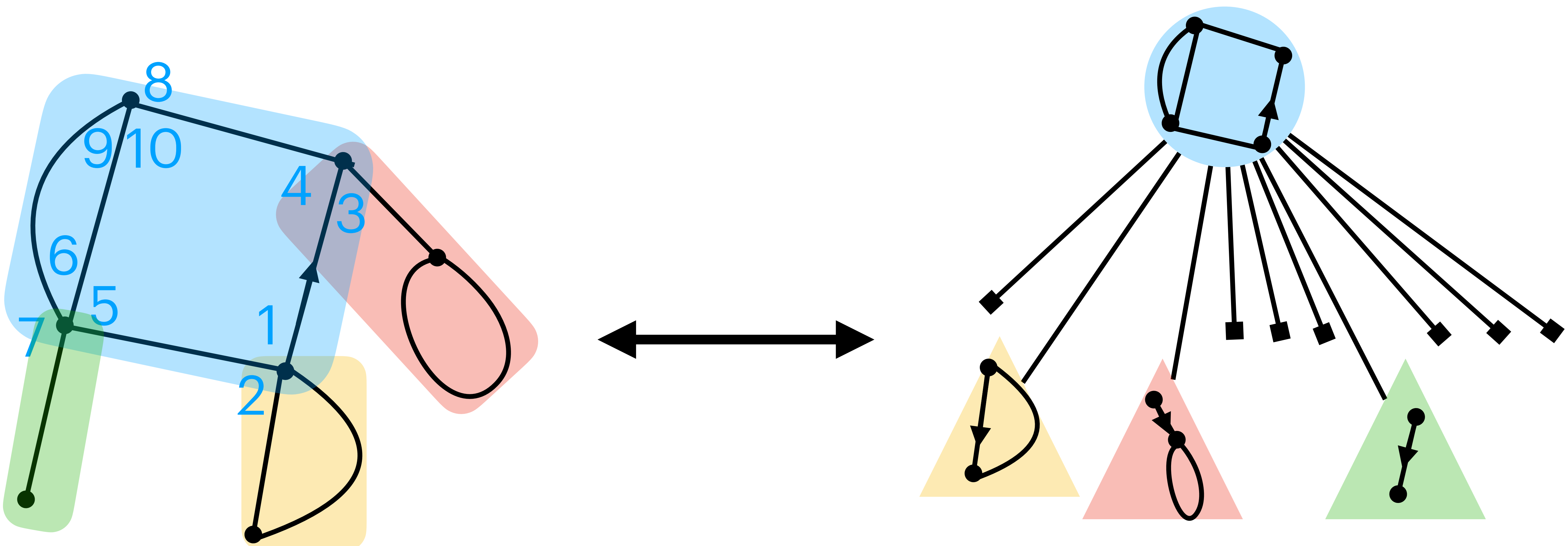

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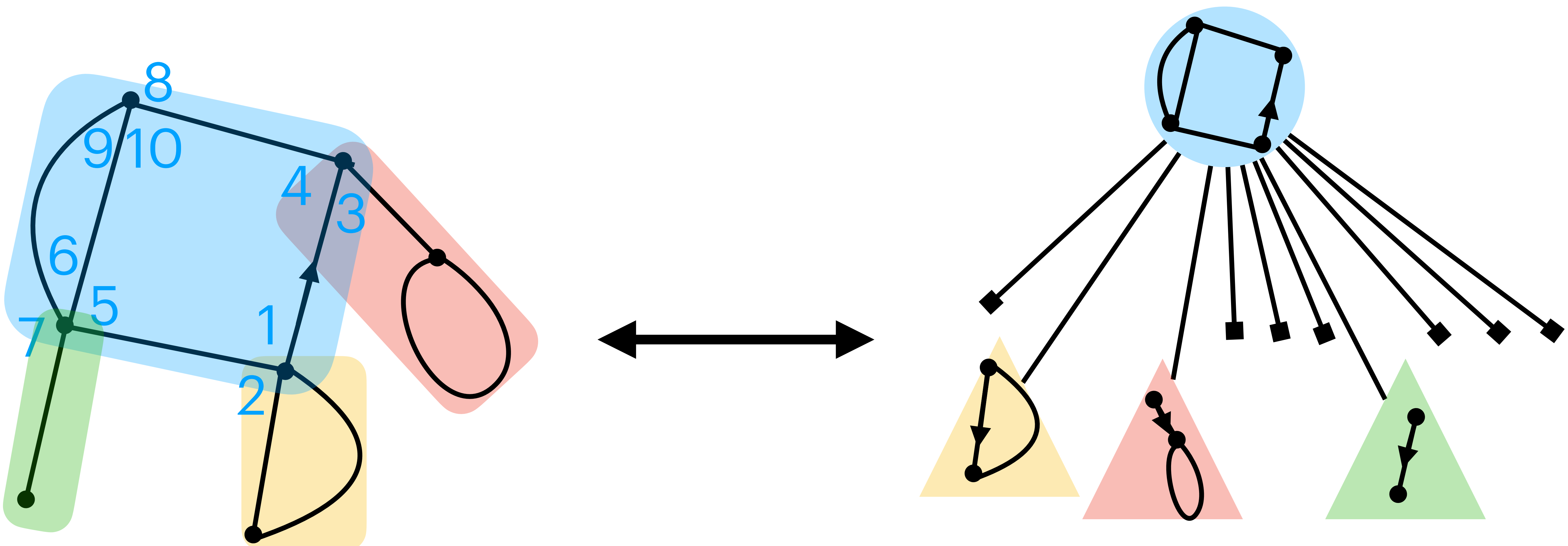

⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

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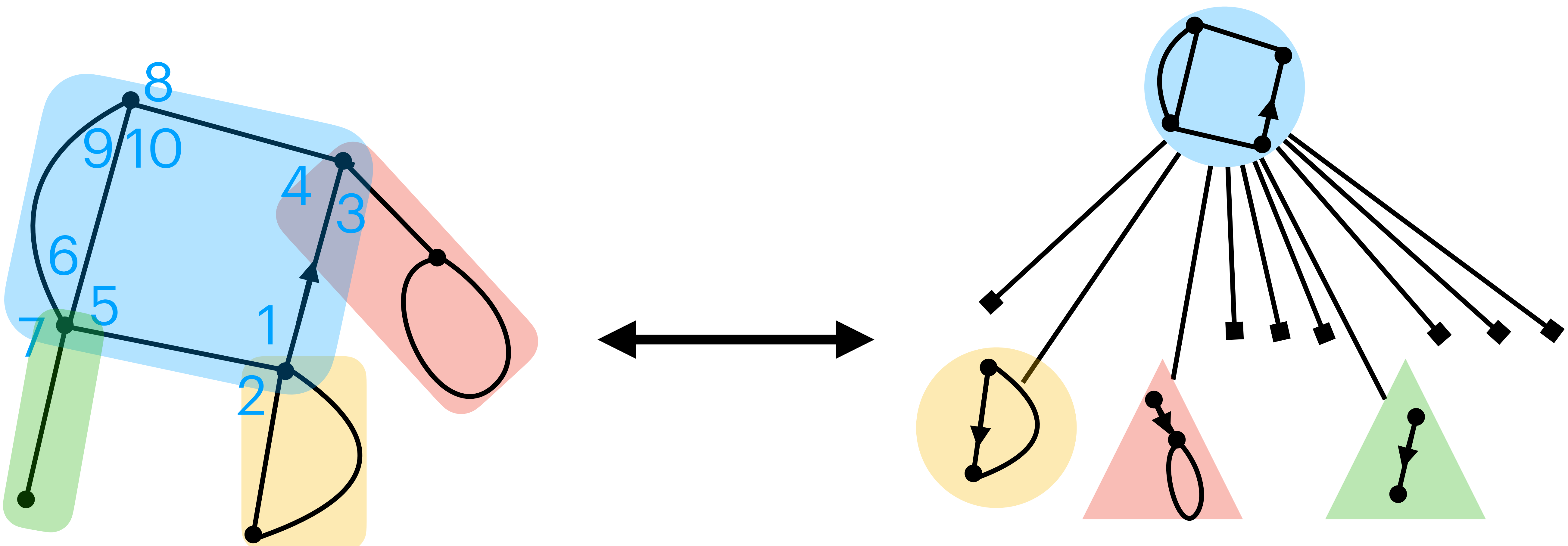

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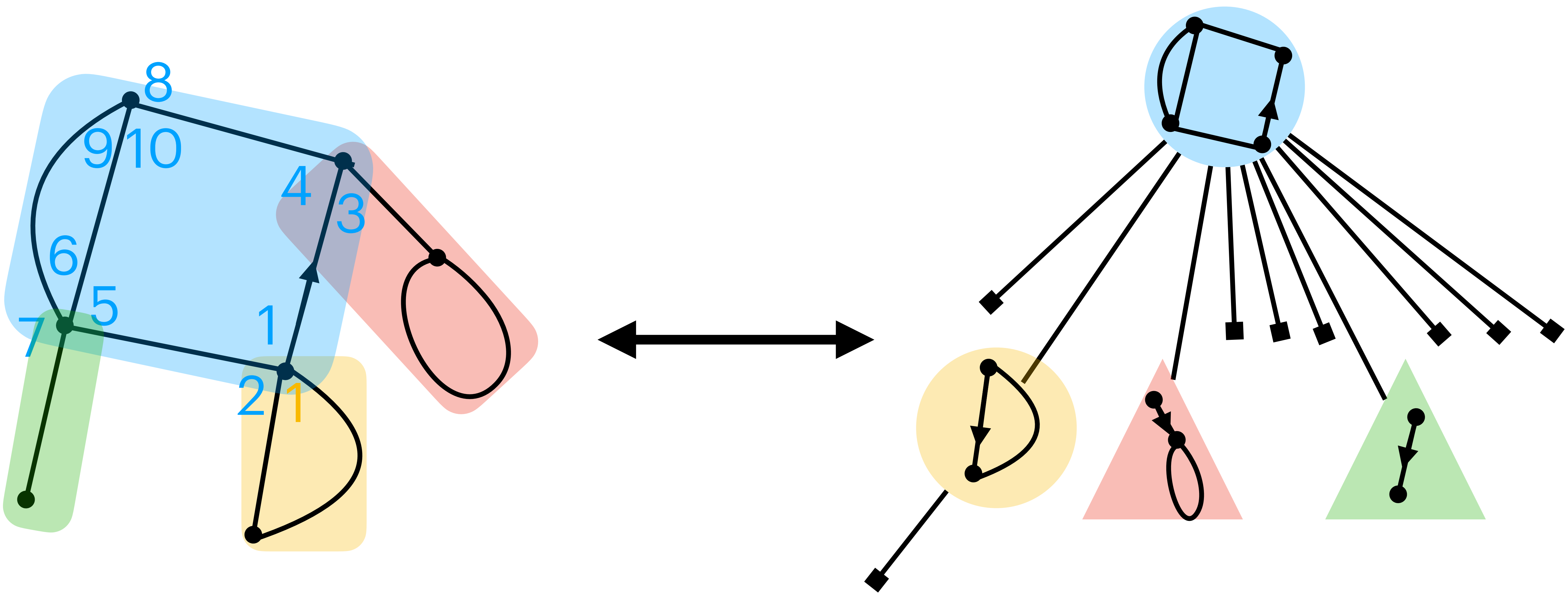

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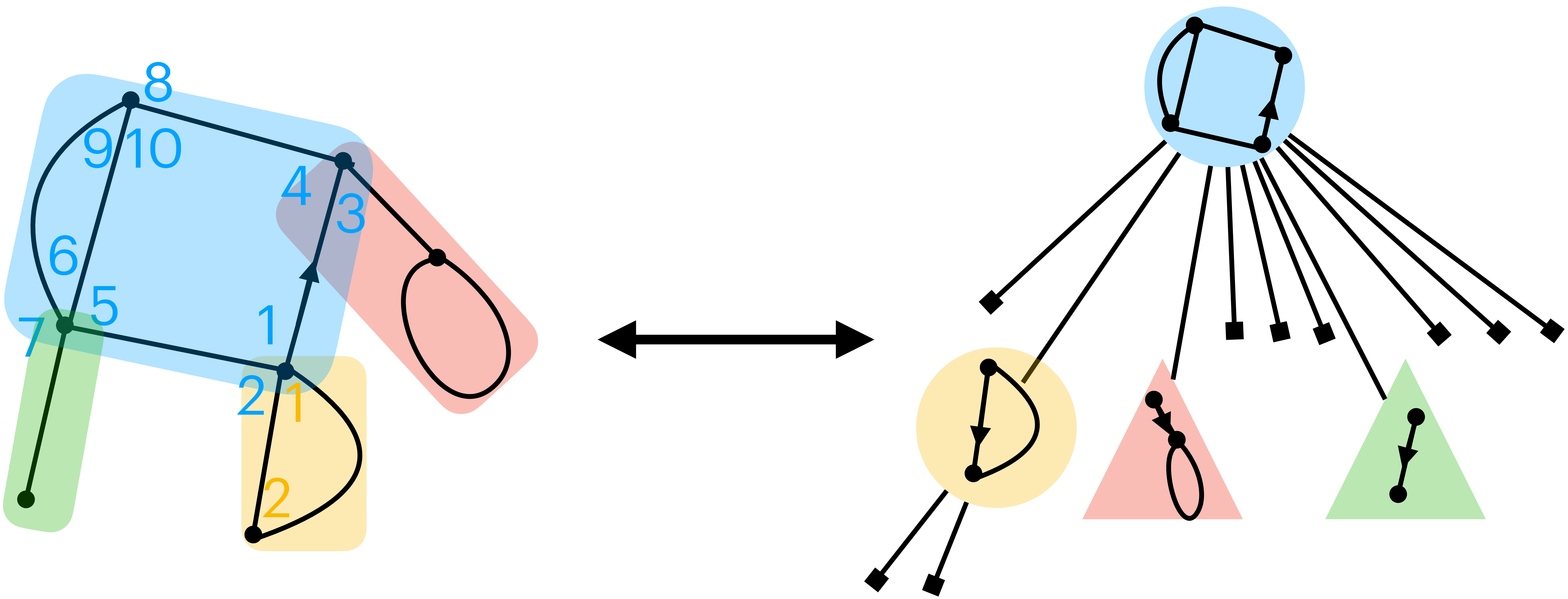

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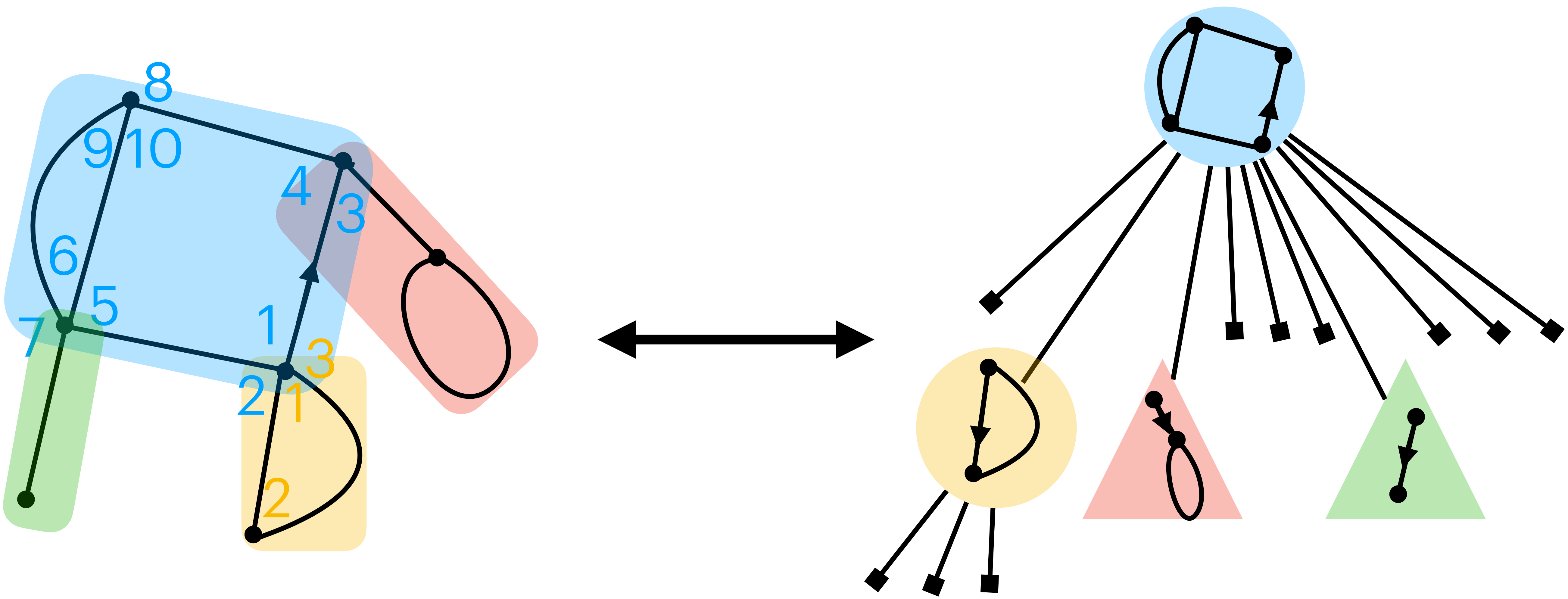

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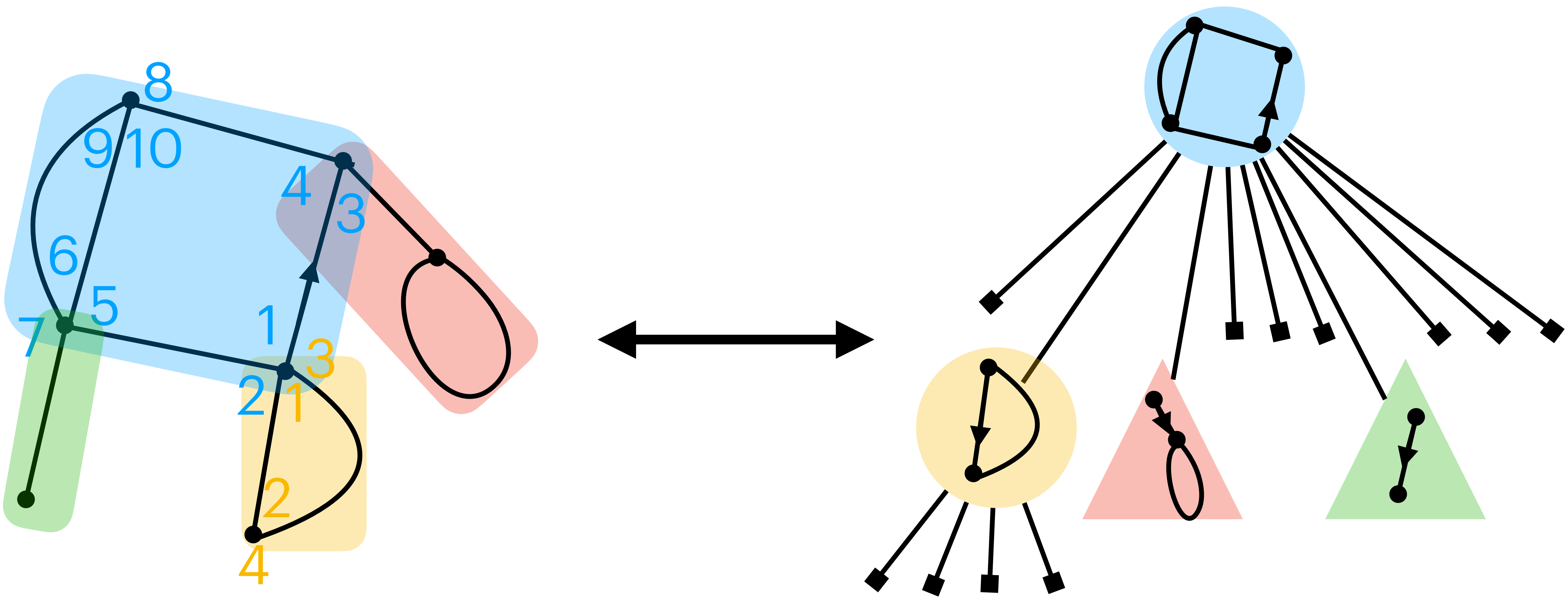

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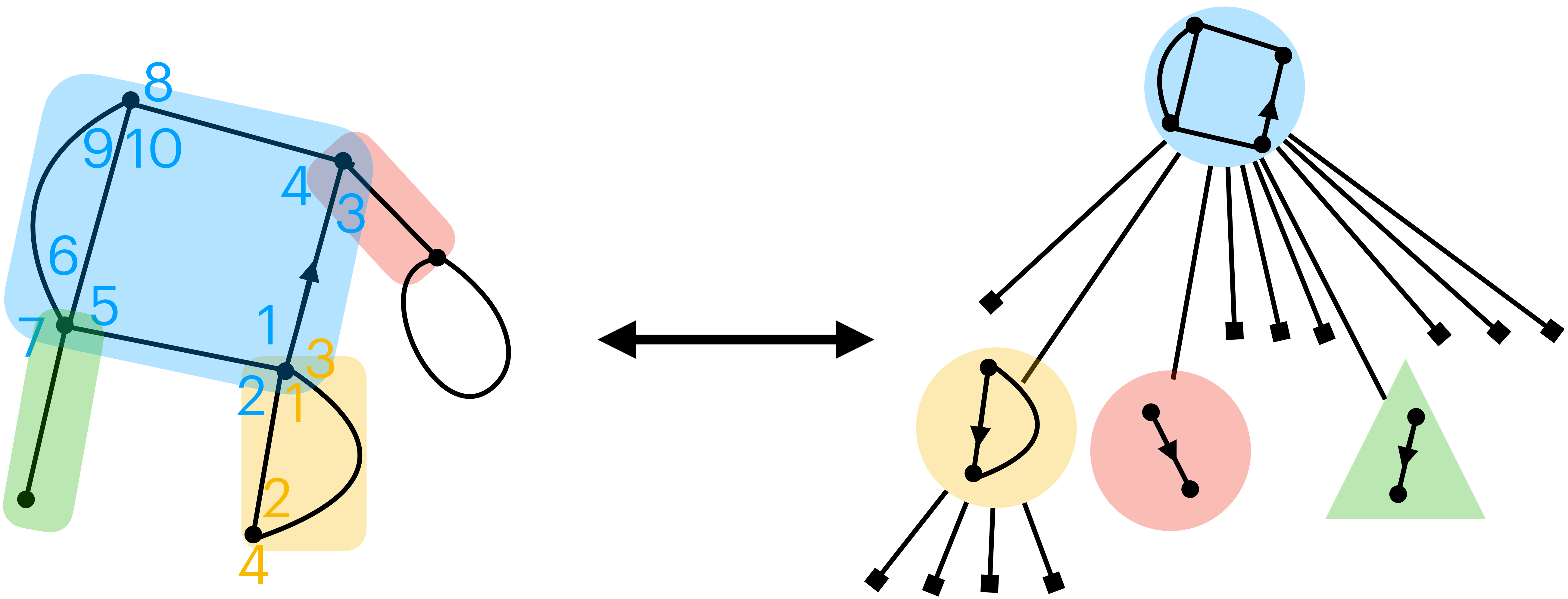

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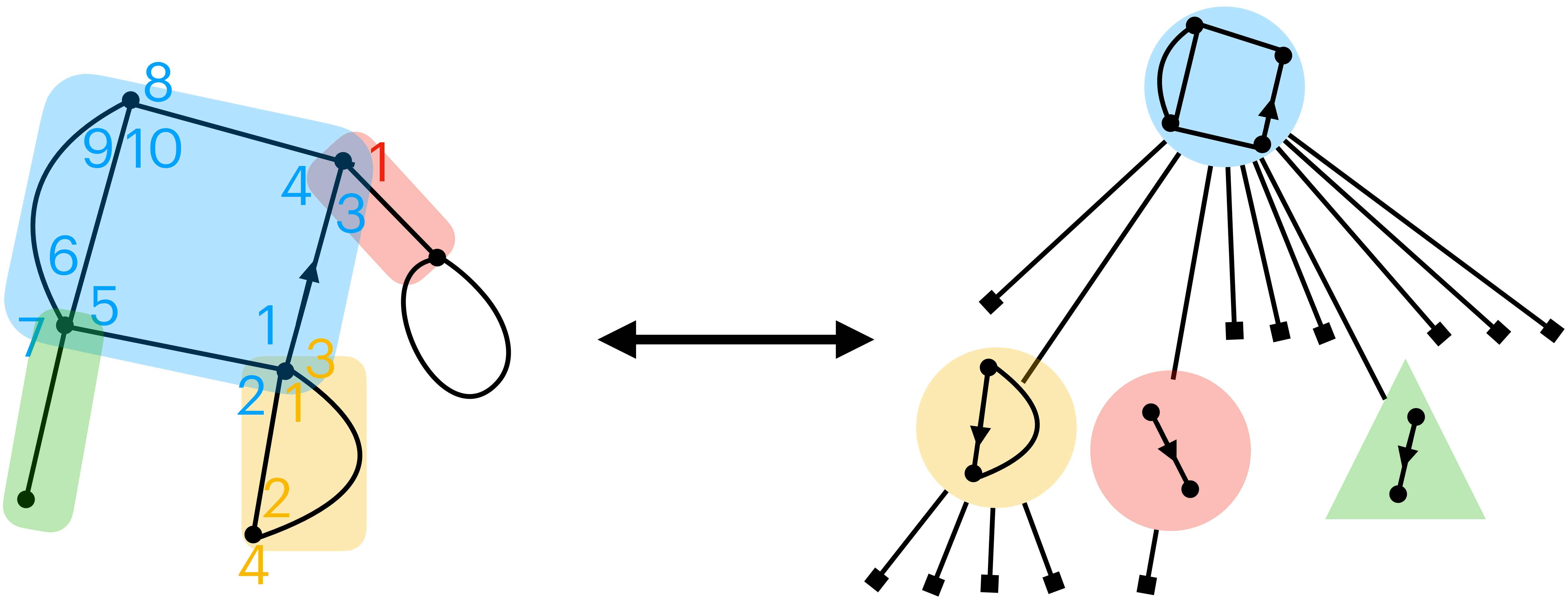

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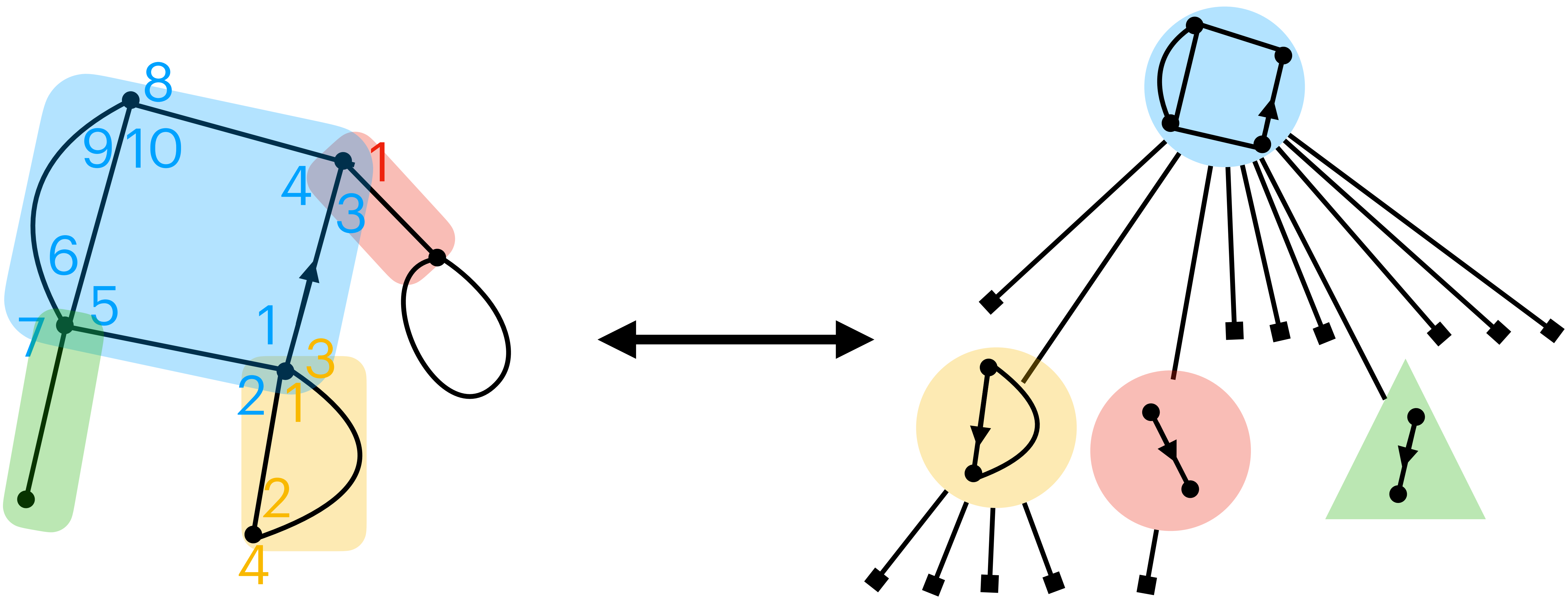

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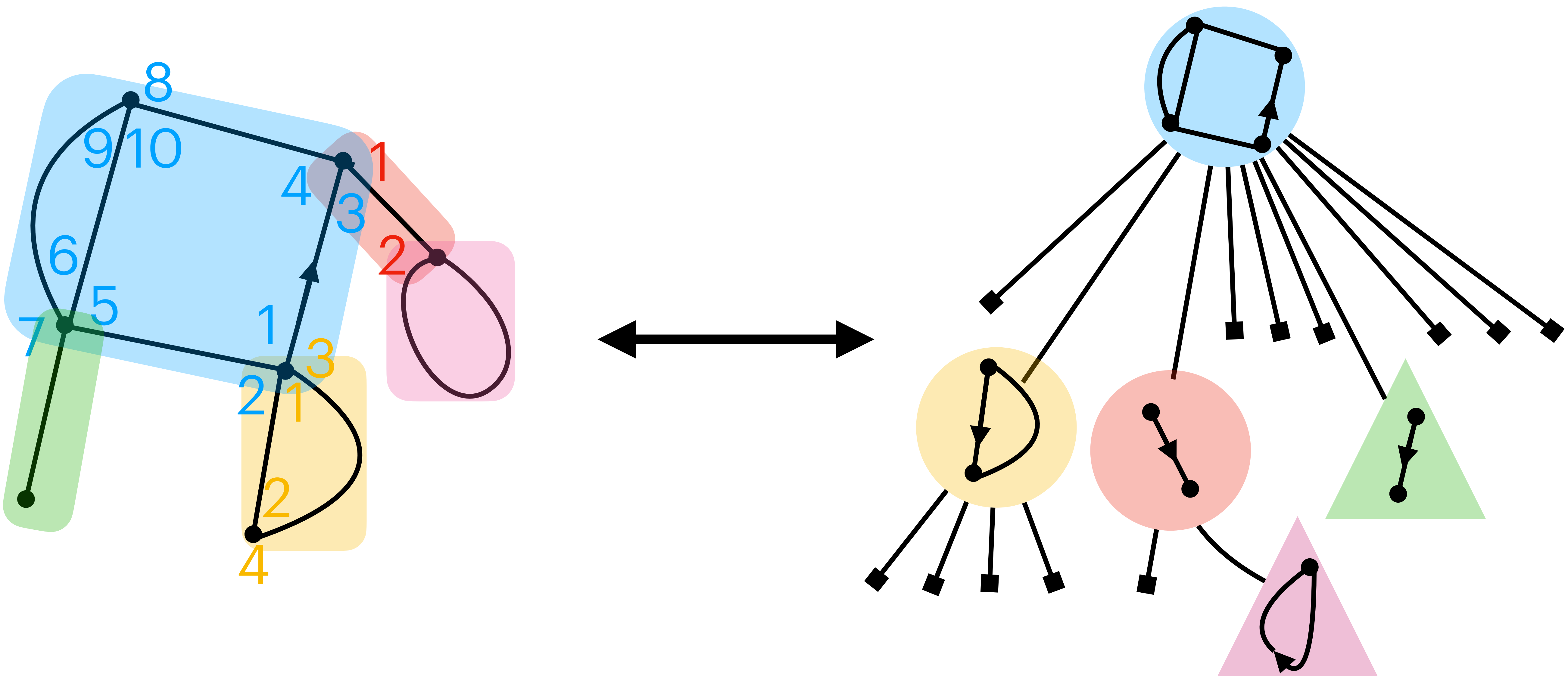

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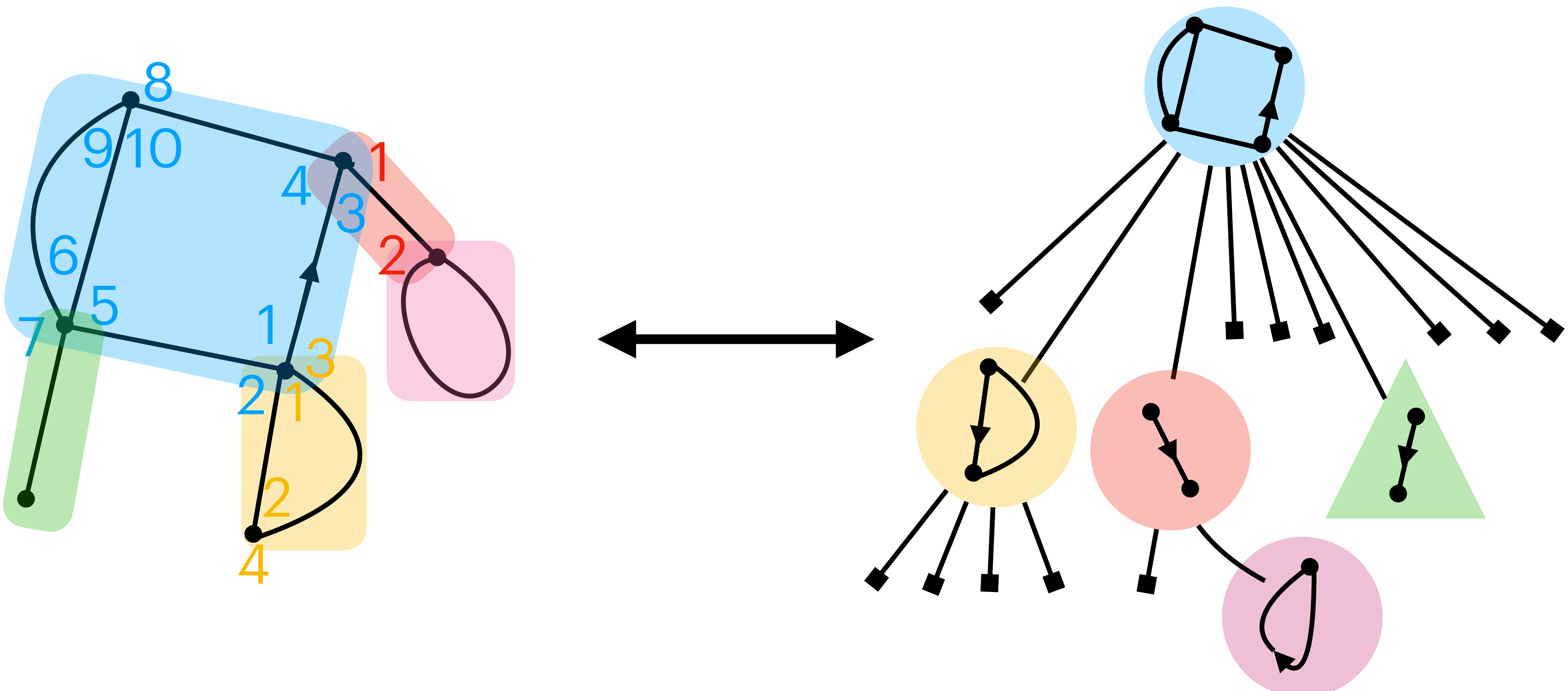

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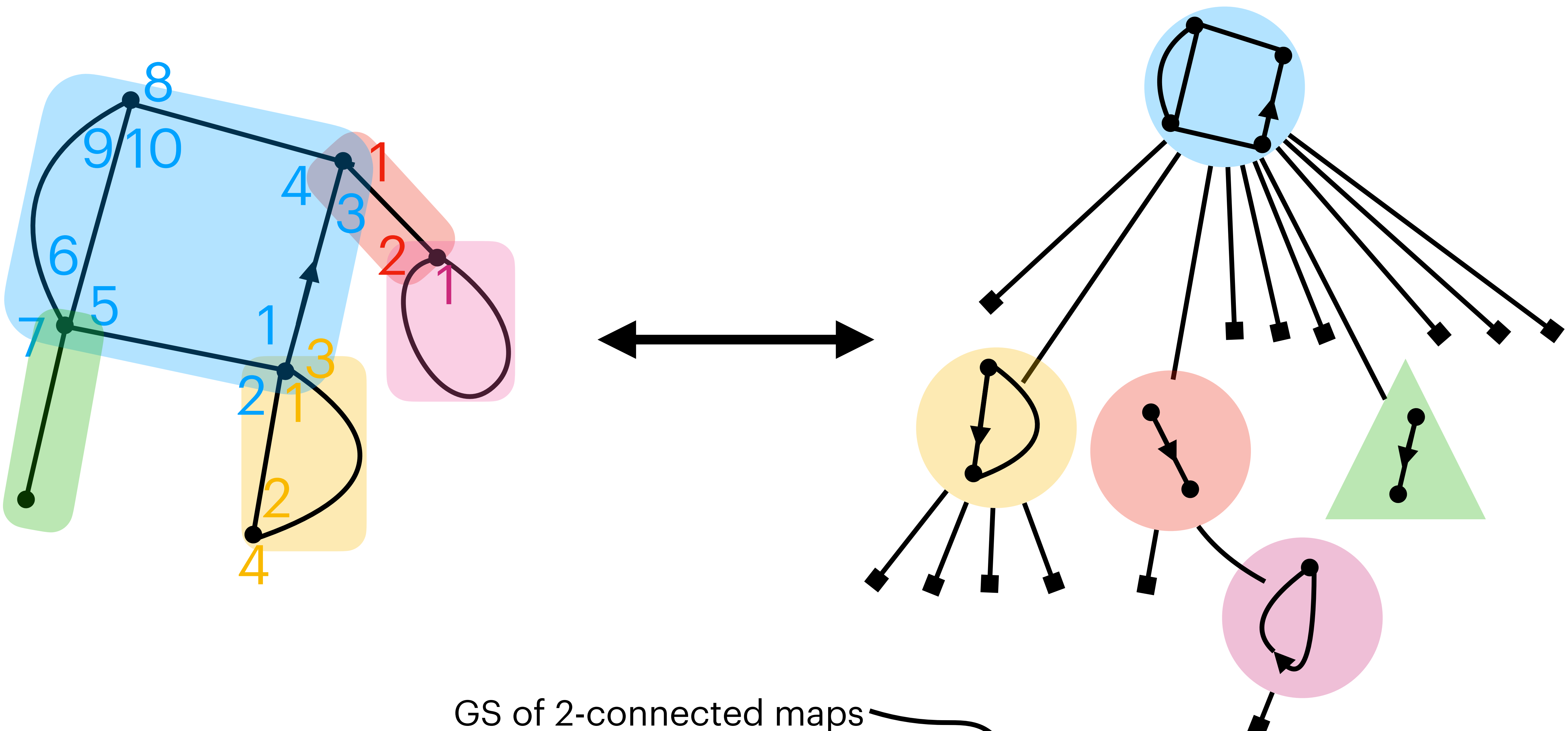

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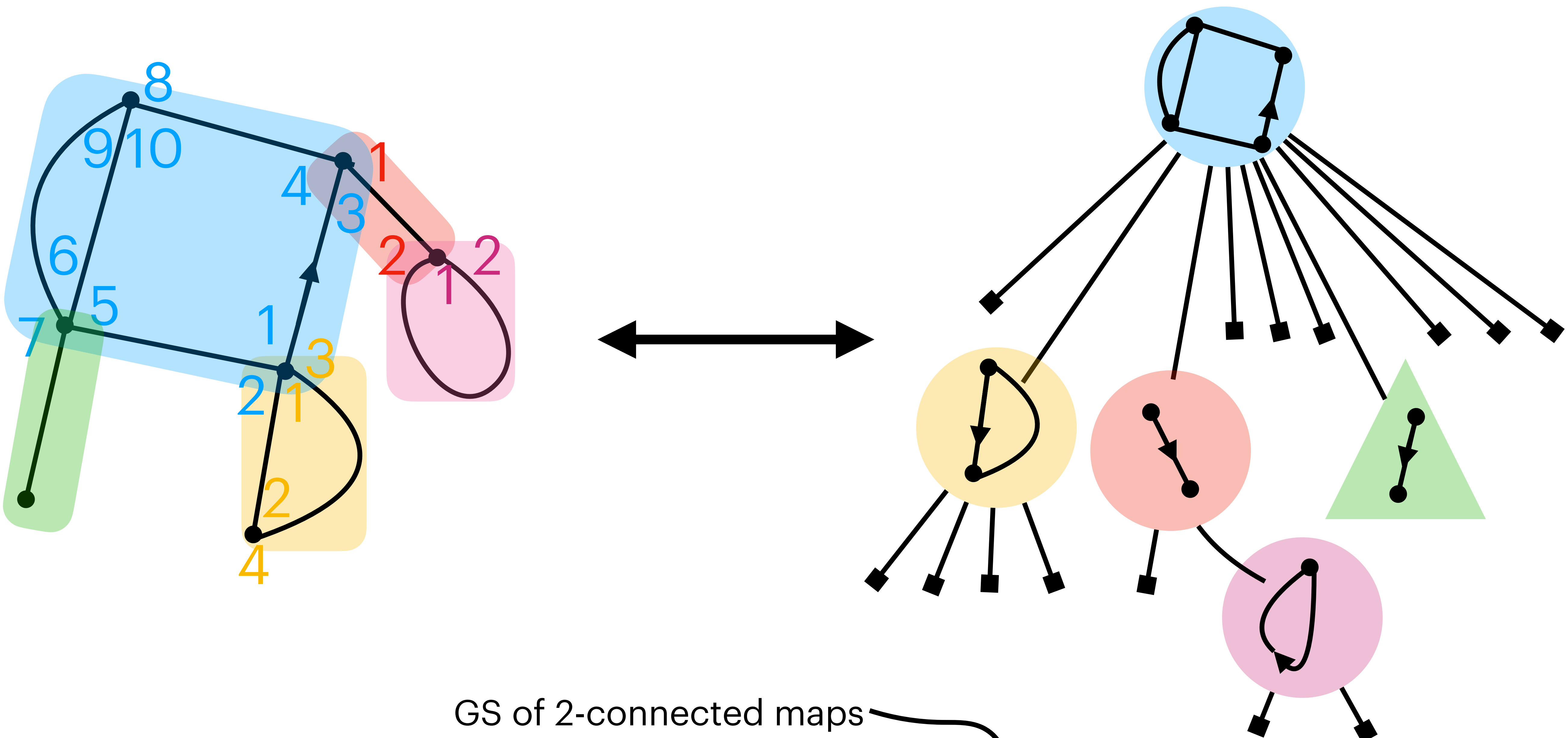
Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



GS of 2-connected maps

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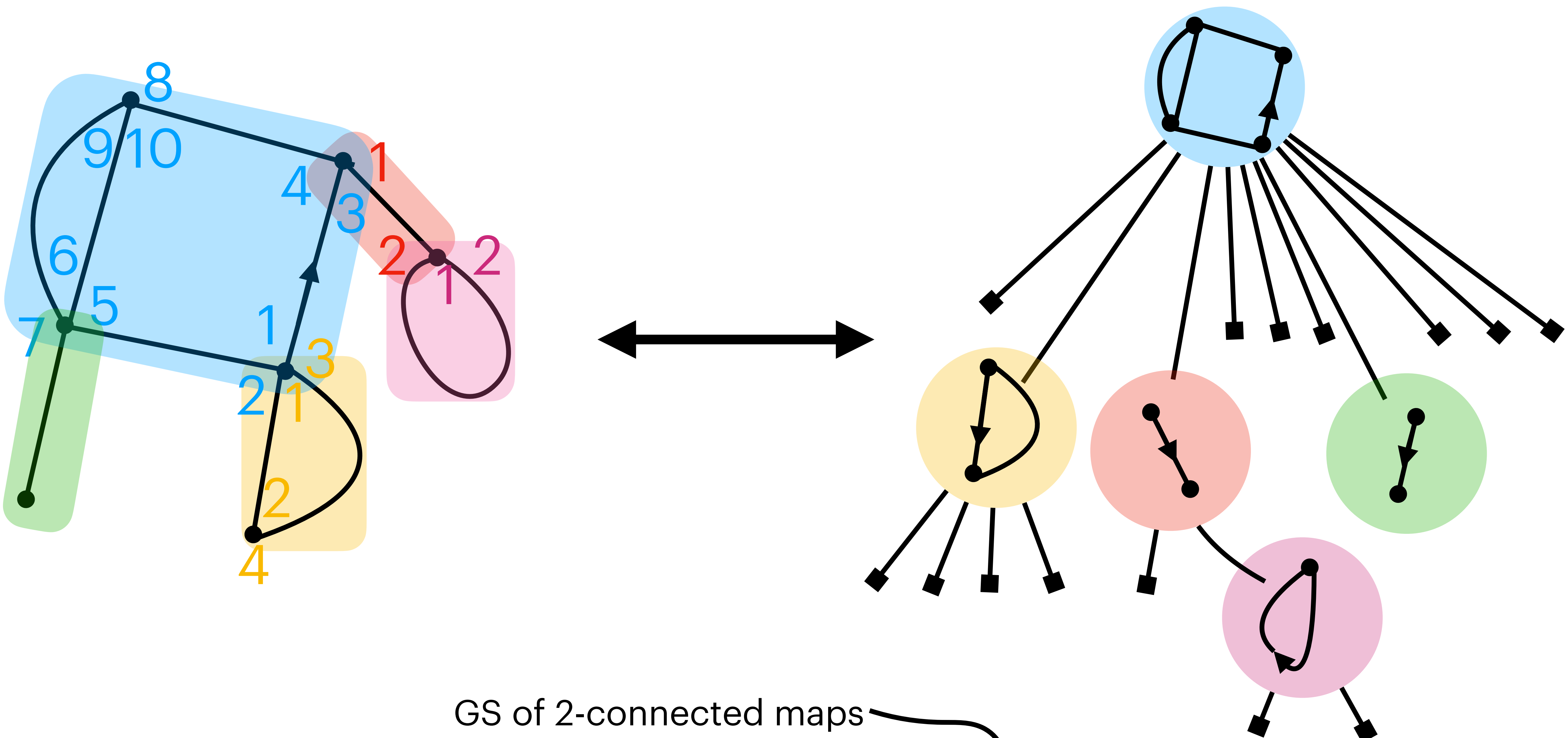

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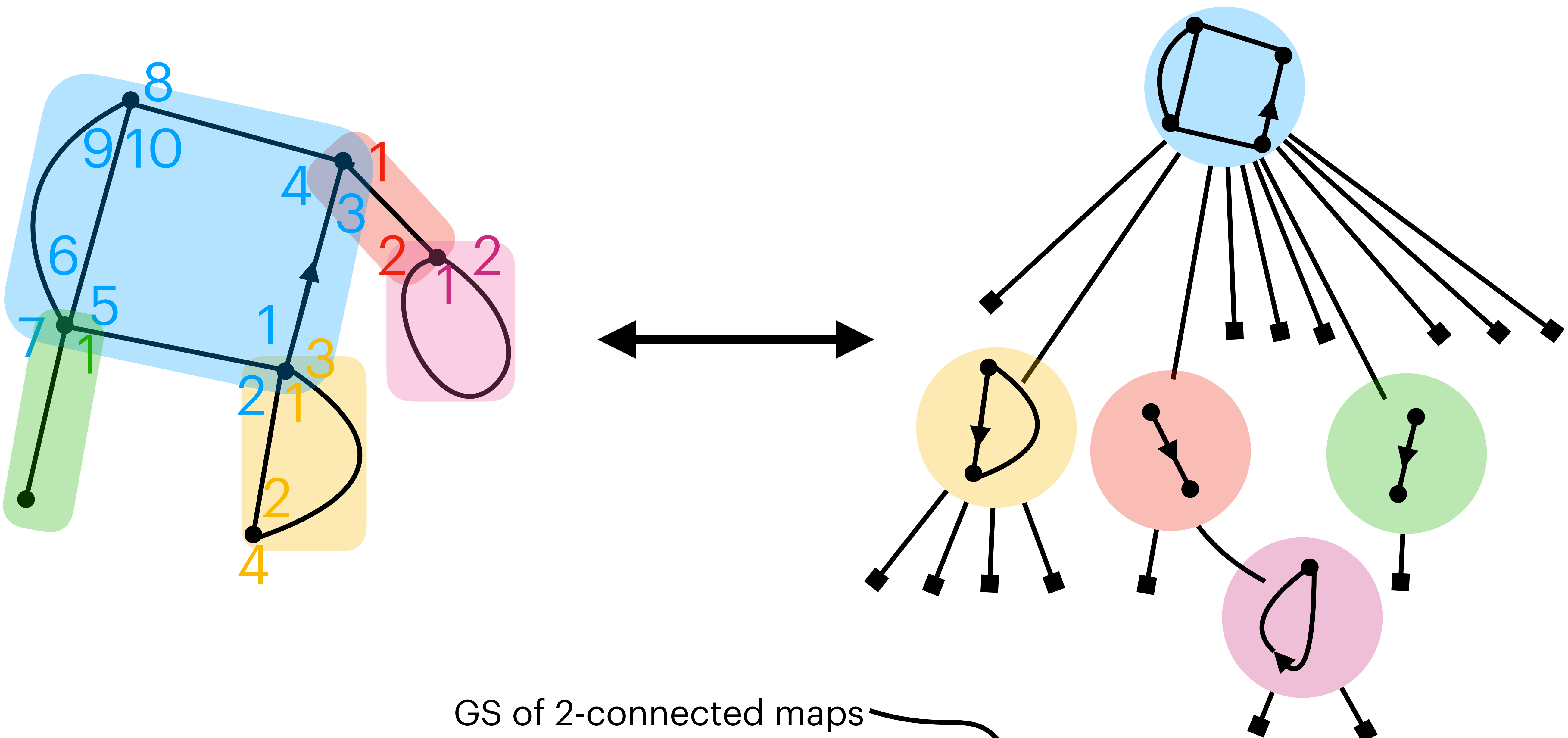

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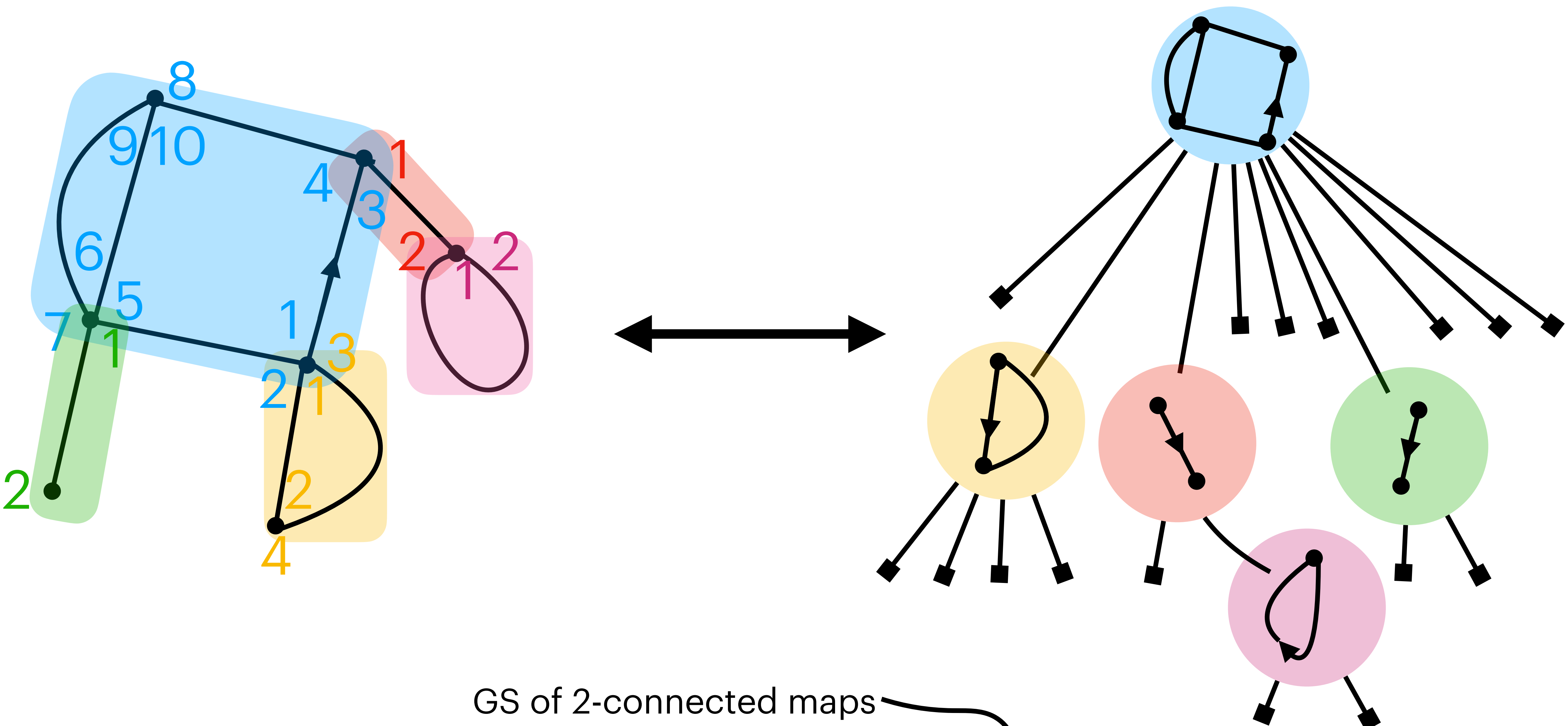
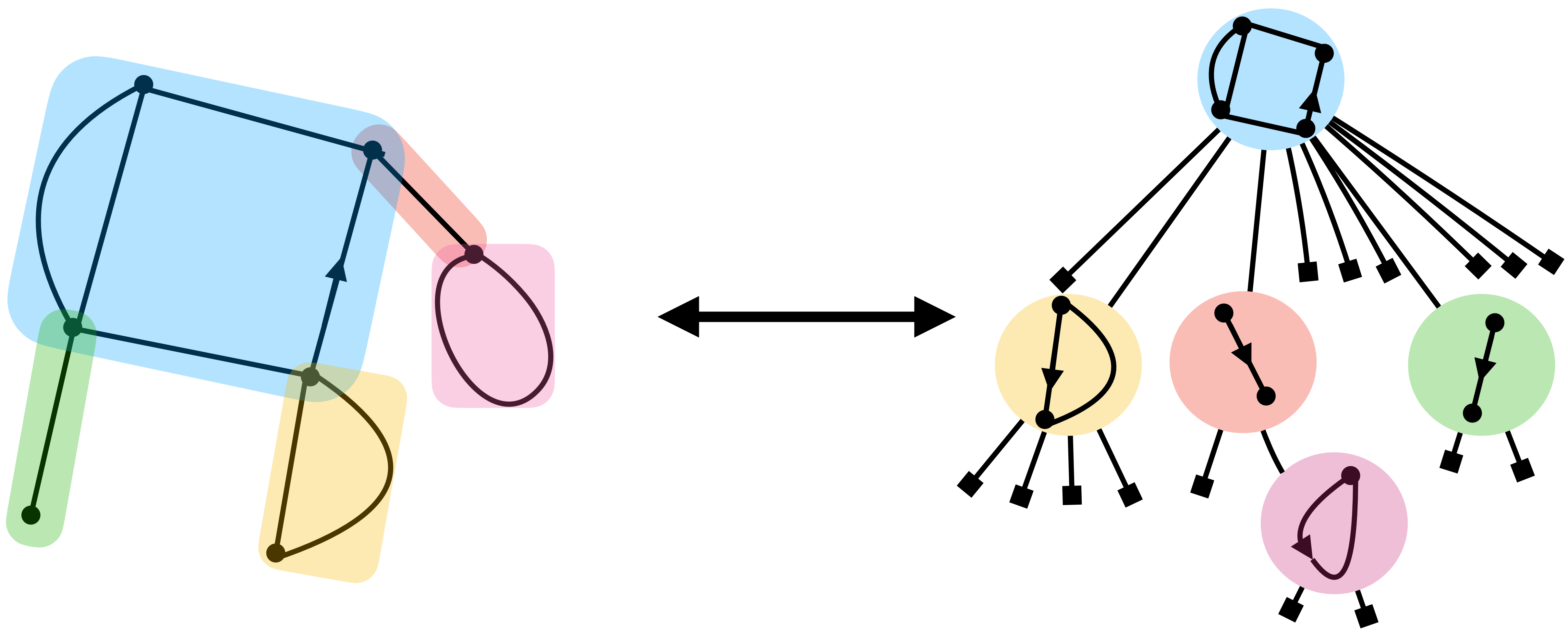


GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks: properties



- Internal node (with k children) of $T_{\mathfrak{m}}$ $\leftrightarrow$ block of $\mathfrak{m}$ of size $k/2$;
- $\mathfrak{m}$ is entirely determined by $T_{\mathfrak{m}}$ and $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ where $\mathfrak{b}_v$ is the block of $\mathfrak{m}$ represented by v in $T_{\mathfrak{m}}$.

T_{M_n} gives the block sizes of a random map M_n.

Galton-Watson trees for map blocks

μ-Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

Galton-Watson trees for map blocks

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Theorem [Fleurat, S. 23]

$u > 0$

If $M_n \hookrightarrow \mathbb{P}_{n,u}$, then there exists an (explicit) $y = y(u)$ s.t. T_{M_n} has the law of a Galton-Watson tree of reproduction law $\mu^{y,u}$ conditioned to be of size $2n$, with

$$\mu^{y,u}(\{2k\}) = \frac{B_k y^k u^{\mathbf{1}_{k \neq 0}}}{uB(y) + 1 - u}.$$

Largest blocks?

- Degrees of T_{M_n} give the block sizes of the map M_n;
- Largest degrees of a Galton-Waston tree are well-known [Janson 2012].

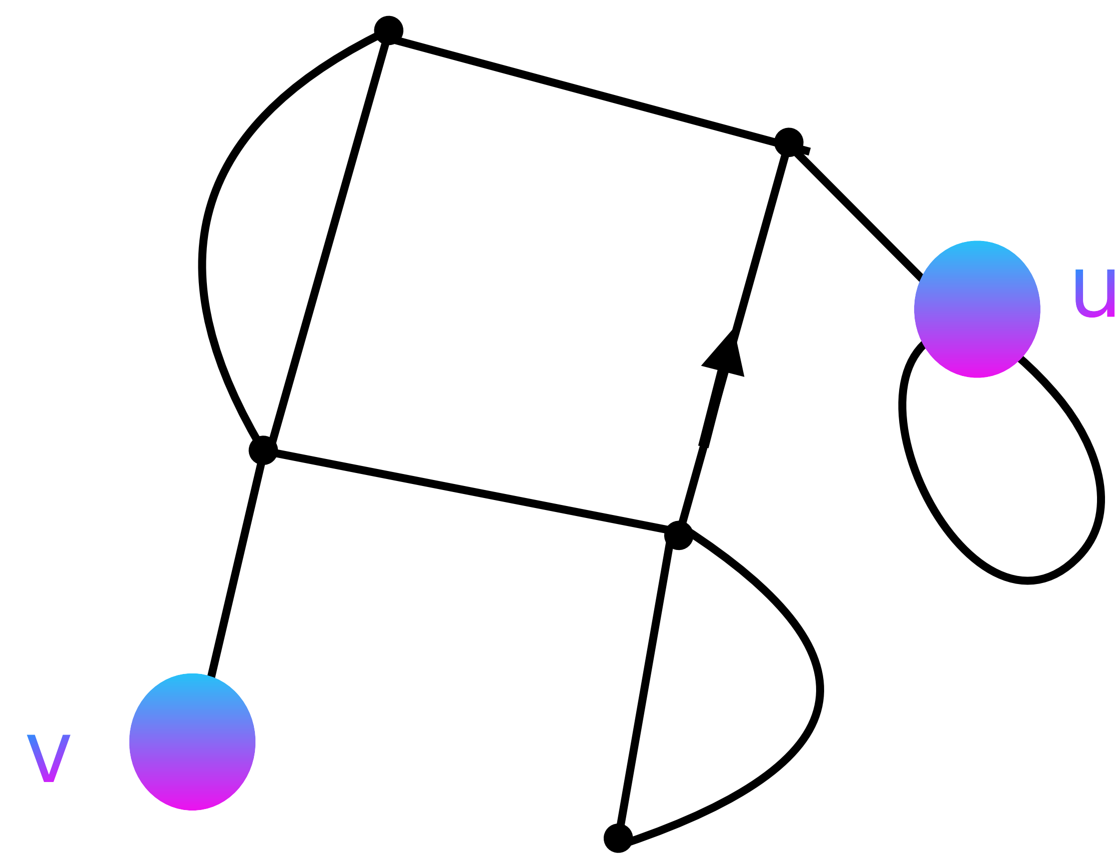
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{y,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

II. Scaling limits

Scaling limits

Convergence of the whole object considered as a metric space (with the graph distance), after renormalisation.



$$d(u, v) = 4$$

$M_n \hookrightarrow \mathbb{P}_{n,u}$

What is the limit of the sequence of metric spaces $((M_n, d/n^?))_{n\in\mathbb{N}}$?

(Convergence for Gromov-Hausdorff metric)

Scaling limit of supercritical and critical maps

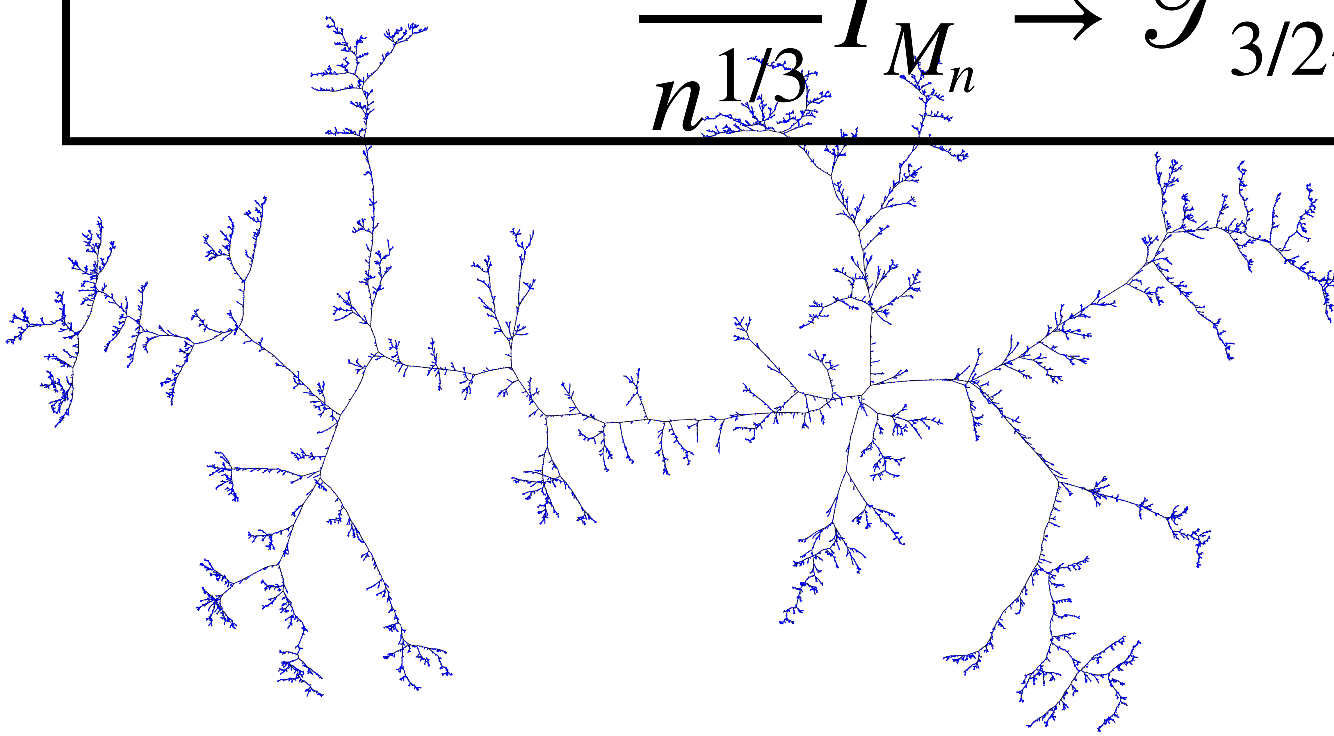
Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

- [Stufler 2020] If $u > 9/5$,

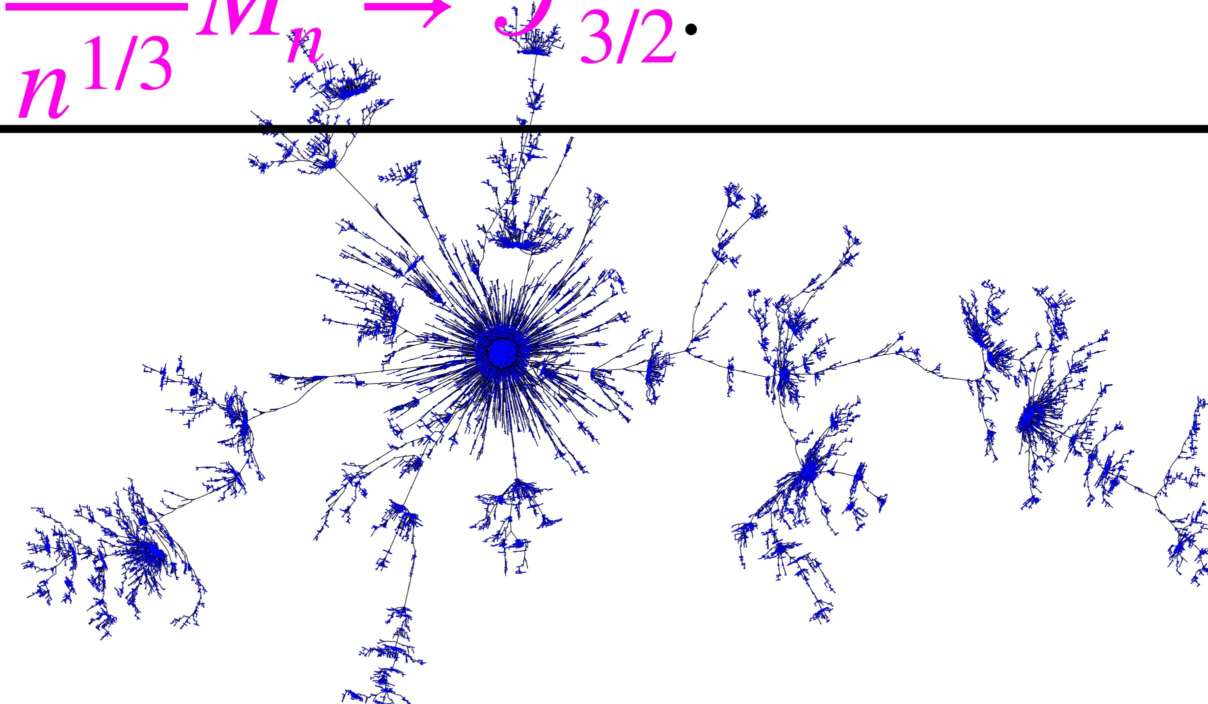
$$\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e, \quad \frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e.$$

- [Fleurat, S. 23] If $u = 9/5$,

$$\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}, \quad \frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}.$$

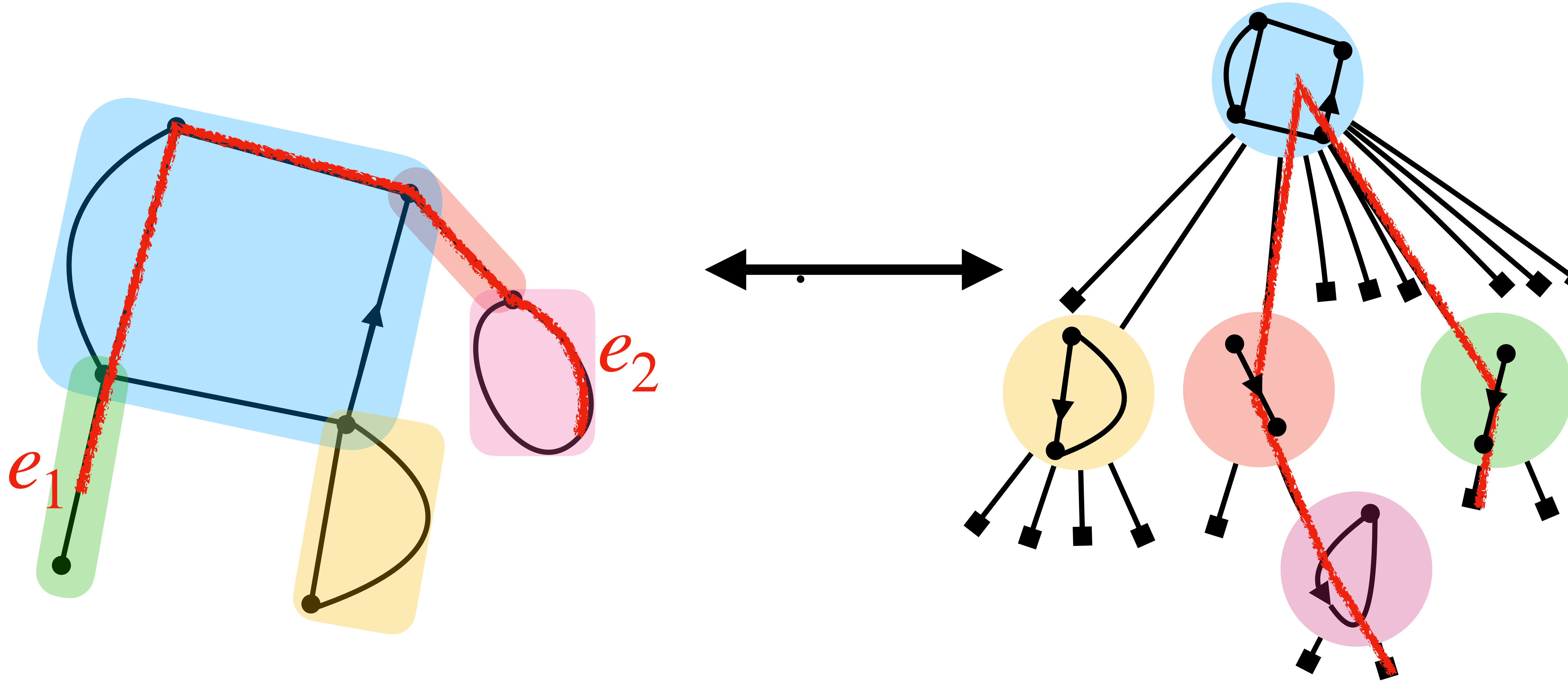


Brownian Tree $\mathcal{T}_e$



Stable Tree $\mathcal{T}_{3/2}$

Critical and supercritical and cases

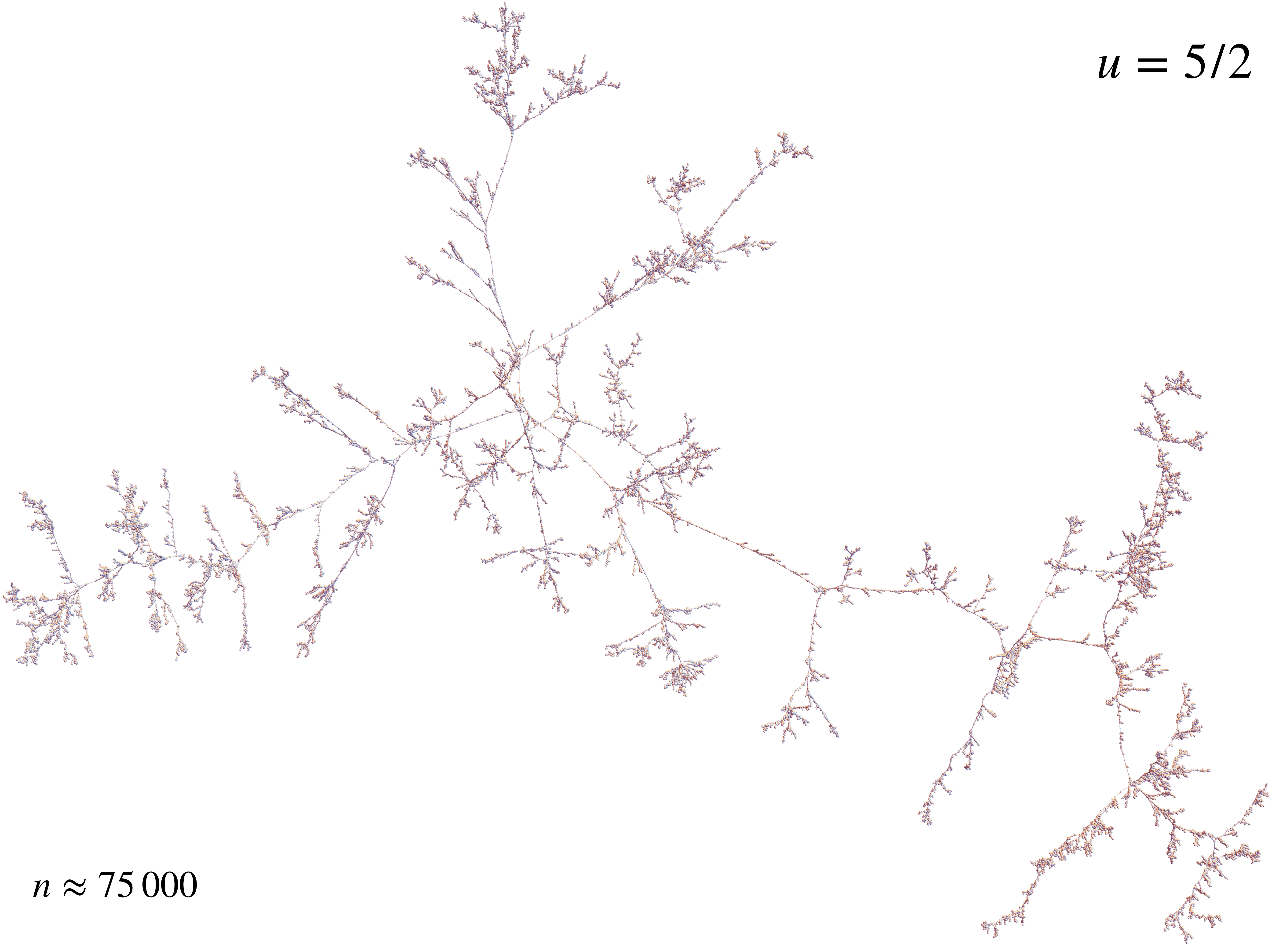


Let $\kappa = \mathbb{E}($"diameter" bipointed block$)$. By a “law of large numbers”-type argument

$$d_{\mathfrak{m}}(e_1, e_2) \simeq \kappa d_{T_{\mathfrak{m}}}(e_1, e_2)\,.$$

So distances in $\mathfrak{m}$ behave like distances in $T_{\mathfrak{m}}$.


$u = 9/5$
$n \approx 80\,000$


$u = 5/2$
$n \approx 75\,000$


$u = 5$
$n \approx 50\,000$

Scaling limits of subcritical maps

<u>Theorem</u> [Fleurat, S. 23] If $u < 9/5$, for $M_n \hookrightarrow \mathbb{P}_{n,u}$ and B_n a uniform 2-connected map of size n:

$$d_{GH}\left(\frac{C_1(u)}{n^{1/4}}M_n, \frac{1}{n^{1/4}}B_n\right) \to 0.$$

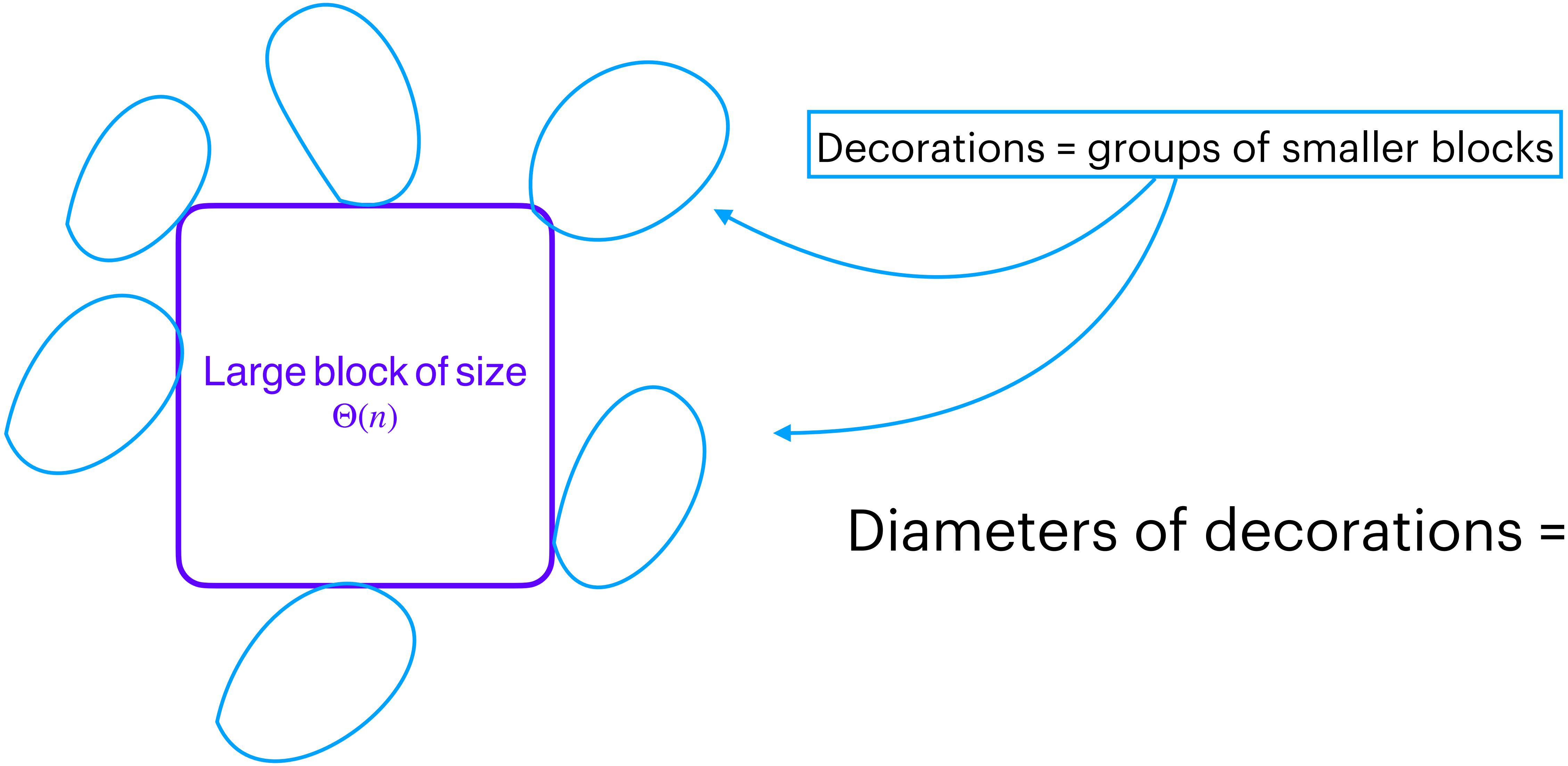
Brownian Sphere $\mathcal{S}_e$

So, *if* $cn^{-1/4}B_n \to \mathcal{S}_e$, then

$$\frac{C_1(u)}{cn^{1/4}}M_n \to \mathcal{S}_e.$$

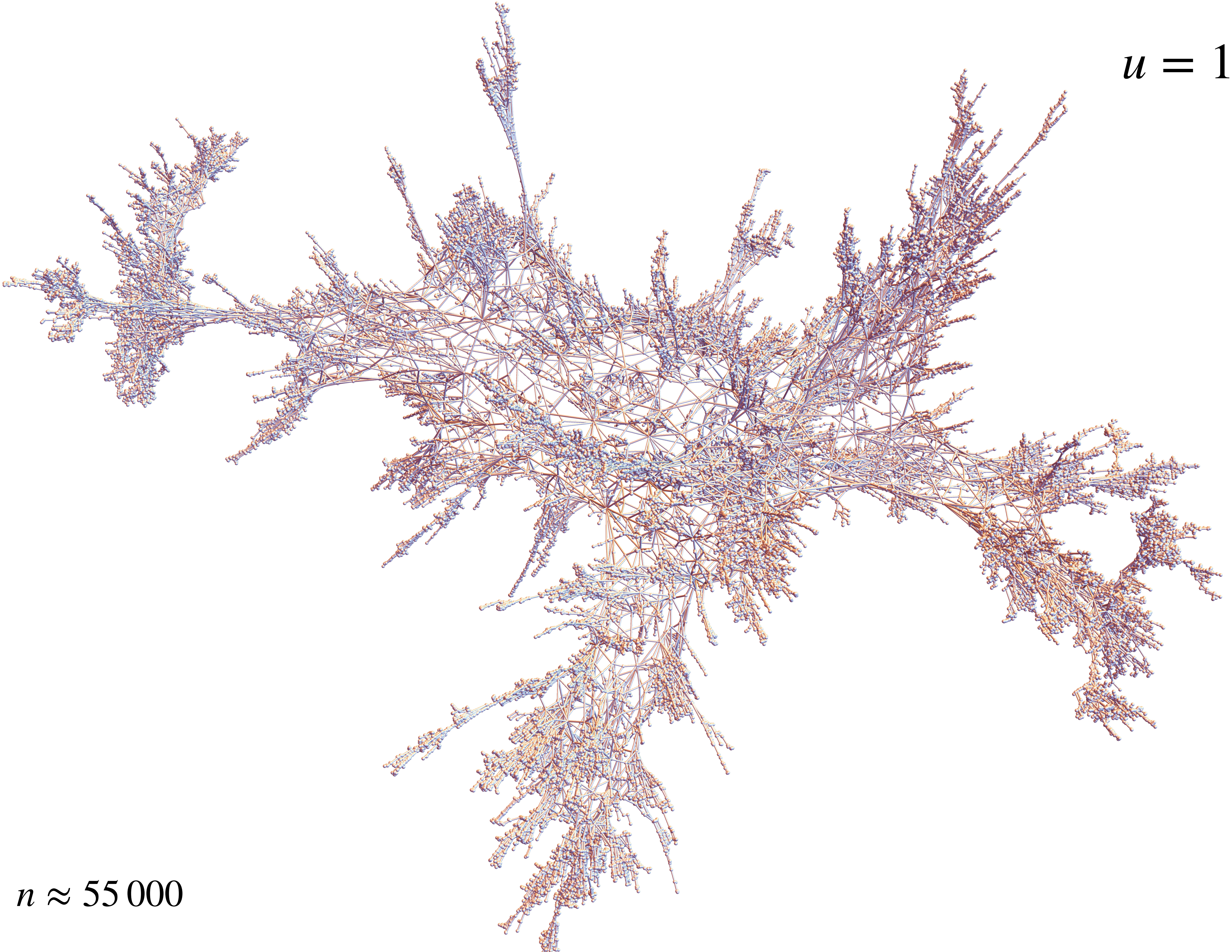
See [Addario-Berry, Wen 2019] for a similar result and method.

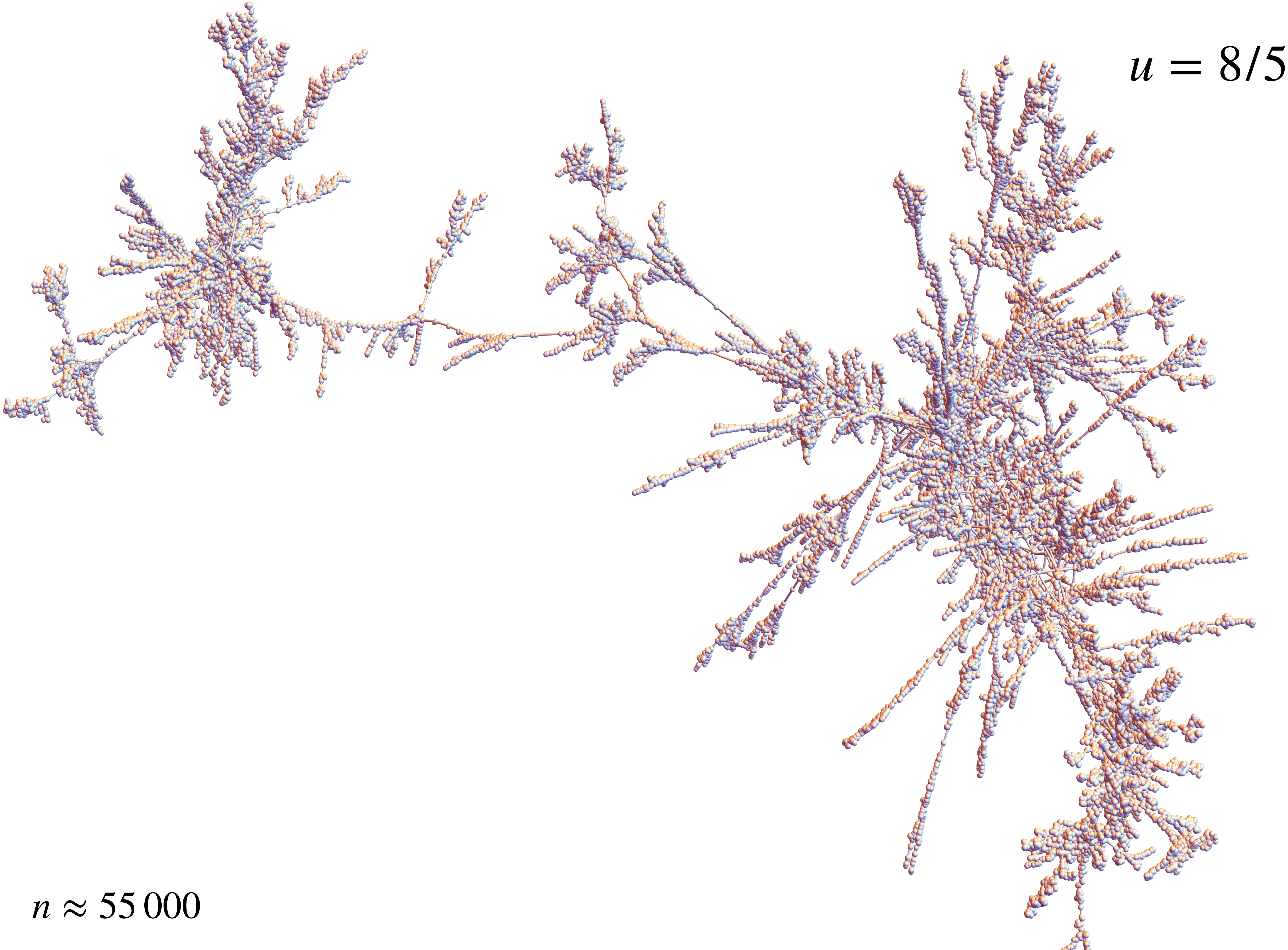
Subcritical case = “general map” case



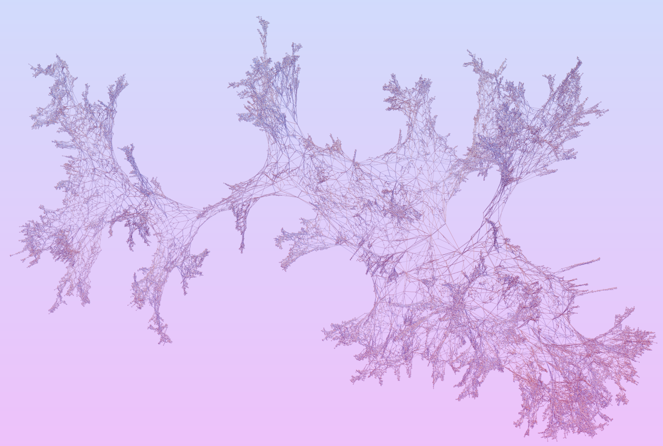
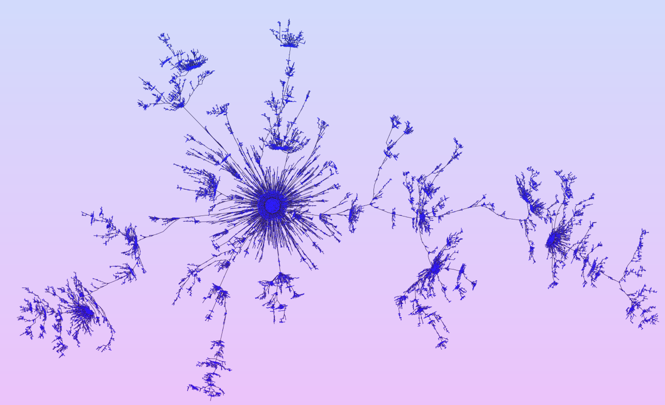
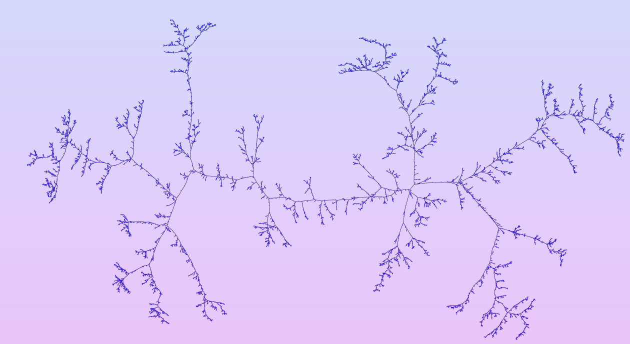
Diameters of decorations = $o(n^{1/4})$.

The scaling limit of M_n (rescaled by $n^{1/4}$) is the scaling limit of uniform blocks!


$u = 1$
$n \approx 55\,000$


$u = 8/5$
$n \approx 55\,000$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{y,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n	$\frac{C_1(u)}{n^{1/4}} M_n \to \mathcal{S}_e$  Assuming the convergence of 2-connected maps towards the Brownian sphere	$\frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$ 	$\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e$ [Stufler 2020] 

III. Extension to other families of maps

Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1 - z(1+M))^2$	$z(1+M)^2$
loopless $M_2(z)$	simple $M_3(z)$	$z(1+M)$	–
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1+M)^2$	–
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1+M)$	–
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1+M)$
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1+M)$	–
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1 - z(1+M))^2$	$z(1+M)^2$
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1+M)^2$	–
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1+M)$	–
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1+M)^3$	–
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1+M)^2$	–

→ *Unified study of the phase transition for block-weighted random planar maps* Z. Salvy (EUROCOMB'23)

Statement of the results

Theorem [S. 23] Model of the preceding table without coreless maps exhibits a phase transition at some explicit u_C. When $n \to \infty$:

- Subcritical phase $u < u_C$: "general map phase" one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: "tree phase" only small blocks.

We obtain explicit results on enumeration and size of blocks in each case.

Perspectives

- Extension to decompositions with coreless maps;
- Study of the critical window(s);
- Extension to spanning-tree decorated maps.

Thank you!