A phase transition in block-weighted random maps

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Planar maps

Planar map $\mathbf{m} =$ embedding on the sphere of a connected planar graph, considered up to homeomorphisms.

- **Rooted** planar map = map endowed with a marked oriented edge (represented by an arrow);
- **Size** $|\mathbf{m}| =$ number of edges;
- **Corner** (does not exist for graphs !) = space between an oriented edge and the next one for the trigonometric order.

Map = graph + cyclic order on neighbours.
Universality results for planar maps

- Enumeration: $\kappa \rho^{-n} n^{-5/2}$ [Tutte 1963, Drmota, Noy, Yu 2020];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for arbitrary maps [Bettinelli, Jacob, Miermont 2014];
- Universality:
  - Same enumeration;
  - Same scaling limit, e.g. for quadrangulations [Miermont 2013], triangulations & 2q-angulations [Le Gall 2013], simple quadrangulations [Addario-Berry, Albenque 2017].
Universality results for planar trees

- Enumeration: $\kappa \rho^{-n} n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];
- Universality:
  - Same enumeration,
  - Same scaling limit, even for some classes of maps; e.g. outerplanar maps [Caraceni 2016], maps with a boundary of size $\gg n^{1/2}$ [Bettinelli 2015].

Models with (very) constrained boundaries
Motivation

Interpolating model?
2-connectivity

Cut vertex: vertex that when removed disconnects the map

2-connected: no cut vertex (= to be able to disconnect, at least two vertices must be removed)

Block = maximal (for inclusion) 2-connected submap
2-connectivity

Cut vertex: vertex that when removed disconnects the map

2-connected: no cut vertex (=to be able to disconnect, at least two vertices must be removed)

Block = maximal (for inclusion) 2-connected submap
Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

Motivation

Only small blocks.

Interpolating model?
Outline of the talk

A phase transition in block-weighted random maps

I. Approach
II. Largest blocks
III. Similar model: quadrangulations
IV. Scaling limits
V. Perspectives
I. Approach
Model

Goal: parameter that affects the typical number of blocks.

We choose: \( P_{n,u}(m) = \frac{u \#\text{blocks}(m)}{Z_{n,u}} \) where

\( u > 0, \)
\( M_n = \{\text{maps of size } n\}, \)
\( m \in M_n, \)
\( Z_{n,u} = \text{normalisation}. \)

Inspired by [Bonzom 2016].

- \( u = 1 \): uniform distribution on maps of size \( n \);
- \( u \to 0 \): minimising the number of blocks (=2-connected maps);
- \( u \to \infty \): maximising the number of blocks (= trees!).

Given \( u \), asymptotic behaviour when \( n \to \infty \)?
### Results

For $M_n \leftrightarrow P_{n,u}$

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Scaling limit of $M_n$
Inspiration from [Tutte 1963]

Decomposition of a map into blocks

\[ M(z, u) = \sum_{m \in M} z^{|m|} u^{\text{blocks}(m)} \]
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\[ M(z, u) = \sum_{m \in \mathcal{M}} z^{|m|} u^{\#\text{blocks}(m)} \]

With a weight \( u \) on blocks: 
\[ M(z, u) = u B(zM^2(z, u)) + 1 - u \]

GS of 2-connected maps
Decomposition of a map into blocks

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\[ \Rightarrow \text{Underlying block tree structure, made explicit by [Addario-Berry 2019].} \]

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### Results

For $M_n \leftrightarrow \mathbb{P}_{n,u}$

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Decomposition of a map into blocks: properties

- Internal node (with $k$ children) of $T_m$ ↔ block of $m$ of size $k/2$;
- $m$ is entirely determined by $T_m$ and $(b_v, v \in T_m)$ where $b_v$ is the block of $m$ represented by $v$ in $T_m$.

$T_{M_n}$ gives the block sizes of a random map $M_n$. 
**Galton-Watson trees for map blocks**

**μ-Galton-Watson tree** : random tree where the number of children of each node is given by $\mu$ independently, with $\mu = \mu$ probability law on $\mathbb{N}$. 
Galton-Watson trees for map blocks

\( \mu \)-Galton-Watson tree: random tree where the number of children of each node is given by \( \mu \) independently, with \( \mu = \) probability law on \( \mathbb{N} \).

**Theorem**

If \( M_n \hookrightarrow \mathcal{P}_{n,u} \), then \( T_{M_n} \) has the law of a Galton-Watson tree of reproduction law \( \mu^{y,u} \) conditioned to be of size \( 2n \), with

\[
\mu^{y,u}(\{2k\}) = \frac{B_k y^k u^{1_{k \neq 0}}}{uB(y) + 1 - u}.
\]

\( u > 0 \) \quad y \in (0,4/27]
Galton-Watson trees for map blocks

**μ-Galton-Watson tree**: random tree where the number of children of each node is given by $\mu$ independently, with $\mu = \mu$ probability law on $\mathbb{N}$.

**Theorem**

If $M_n \sim \mathbb{P}_{n,\mu}$, then $T_{M_n}$ has the law of a Galton-Watson tree of reproduction law $\mu^{y,u}$ conditioned to be of size $2n$, with

$$
\mu^{y,u}(\{2k\}) = \frac{B_k y^k u^1_{k \neq 0}}{uB(y) + 1 - u}.
$$

$u > 0, \quad y \in (0,4/27]$  

$\Rightarrow$ Choice of $y$?
Phase transition

When is \( \mu^{y,u} \) critical? (= \( \mathbb{E}(\mu) = 1 \)?)

\[
\mathbb{E}(\mu^{y,u}) = 1 \iff u = \frac{1}{2yB'(y) - B(y) + 1}
\]

covers \([9/5, + \infty)\) when \(y\) covers \((0, \rho_B = 4/27)]\).

**Theorem**

- If \( u < 9/5 \), then \( \mathbb{E}(\mu^{y,u}) < 1 \). The mean is maximal for \( y = 4/27 \) and then \( \mu^{y,u}(2k) \sim c_u k^{-5/2} \);
- If \( u = 9/5 \) and \( y = 4/27 \), then \( \mathbb{E}(\mu^{y,u}) = 1 \) and \( \mu^{y,u}(2k) \sim c_u k^{-5/2} \);
- If \( u > 9/5 \) and \( y \) is well chosen, then \( \mathbb{E}(\mu^{y,u}) = 1 \) and \( \mu^{y,u}(2k) \sim c_u \pi_u^k k^{-5/2} \).
Phase transition

\[ u_C = \frac{9}{5} \]

Critical GW

\[ y = \frac{4}{27} \]

Subcritical GW

\[ y \text{ s.t. } u = \frac{1}{2yB'(y) - B(y) + 1} \]

“Map regime”

“Tree regime”
II. Largest blocks
**Properties of $T_{M_n}$**

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**Tool:** [Janson 2012] = extensive study of the degrees in Galton-Watson trees

**Properties on trees give properties of maps.**
Size $L_{n,k}$ of the $k$-th largest block

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<td>$L_{n,1}$</td>
<td>$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$</td>
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<td>$\frac{\ln(n)}{2 \ln \left( \frac{4}{27} \right)} - \frac{5 \ln(\ln(n))}{4 \ln \left( \frac{4}{27} \right)} + O(1)$</td>
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[Stufler 2020]

Size of the linear block $\times n^{-1}$
### Rough intuition

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**Dichotomy between situations:**

- **Subcritical:** condensation, cf [Jonsson Stefánsson 2011];
- **Supercritical:** behaves as maximum of independent variables.
## Results

For $M_n \leftrightarrow \mathbb{P}_{n,u}$

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**Scaling limit of $M_n$**

- $\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$
- $\Theta(n^{2/3})$
- $\Theta(n^{2/3})$

- $\frac{\ln(n)}{2 \ln \left( \frac{4}{27y} \right)} - \frac{5 \ln(\ln(n))}{4 \ln \left( \frac{4}{27y} \right)} + O(1)$
III. Similar model: quadrangulations
**Quadrangulations**

Def: map with all faces of degree 4.

Simple quadrangulation = no multiple edges.

Size $|q|$ = number of faces.

$$|V(q)| = |q| + 2, \quad |E(q)| = 2|q|.$$
Construction of a quadrangulation from a simple core
Construction of a quadrangulation from a simple core
Construction of a quadrangulation from a simple core
Block tree for a quadrangulation

With a weight \( u \) on blocks: \( Q(z, u) = uS(zQ^2(z, u)) + 1 - u \)

Remember: \( M(z, u) = uB(zM^2(z, u)) + 1 - u \)
Tutte’s bijection

Map

Quadrangulation

[Tutte 1963]
Tutte’s bijection for 2-connected maps

Cut vertex => multiple edge

2-connected maps <=> simple quadrangulations

[Brown 1965]
Block trees under Tutte’s bijection
Implications on results

We choose: \( P_{n,u}(q) = \frac{u^{\# \text{blocks}(q)}}{Z_{n,u}} \) where

- \( u > 0 \),
- \( \mathcal{Q}_n = \{ \text{quadrangulations of size } n \} \),
- \( q \in \mathcal{Q}_n \),
- \( Z_{n,u} = \text{normalisation} \).

Results on the size of (2-connected) blocks can be transferred immediately for quadrangulations and their simple blocks.
# Results

For $M_n \leftrightarrow \mathbb{P}_{n,u}$

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- $\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$
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[Stufler 2020]

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\frac{\ln(n)}{2 \ln \left( \frac{4}{27y} \right)} - \frac{5 \ln(\ln(n))}{4 \ln \left( \frac{4}{27y} \right)} + O(1)
\]

\[
\ln(n) - 5 \ln(\ln(n)) + O(1)
\]
IV. Scaling limits
Scaling limits

Convergence of the whole object considered as a metric space (with the graph distance), after renormalisation.

What is the limit of the sequence of metric spaces \((M_n, d/n^?)\) for \(n \in \mathbb{N}\)?

(Convergence for Gromov-Hausdorff metric)
Scaling limits of Galton-Watson trees

Theorem

For $M_n \leftrightarrow \mathbb{P}_{n,u'}$

- If $u > 9/5$, \( \frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e \).
- If $u = 9/5$, \( \frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2} \).

Stable tree of index 3/2 $\mathcal{T}_{3/2}$

Brownian tree $\mathcal{T}_e$ (Aldous's CRT)
Scaling limits of Galton-Watson trees

Theorem
For \( M_n \leftrightarrow \mathbb{P}_{n,u'} \)

1. If \( u > 9/5 \), \( \frac{c_3(u)}{n^{1/2}} T_{M_n} \rightarrow \mathcal{T}_e \).
2. If \( u = 9/5 \), \( \frac{c_2}{n^{1/3}} T_{M_n} \rightarrow \mathcal{T}_{3/2} \).

Proof

- Scaling limit of critical Galton-Watson trees with finite variance [Aldous 1993, Le Gall 2006];
- Scaling limit of critical Galton-Watson with infinite variance and nice tails [Duquesne 2003].
Scaling limit of supercritical and critical maps

Theorem: For $M_n \leftrightarrow P_{n,u}$,

- If $u > 9/5$,
  \[
  \frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e.
  \]

- If $u = 9/5$,
  \[
  \frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}.
  \]

[Stufler 2020]
Difficult part = show that distances in $\mathfrak{m}$ behave like distances in $T_{\mathfrak{m}}$. We show

$$\forall e_1, e_2 \in \overrightarrow{E}(M_n), d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2).$$
Let $\kappa = \mathbb{E}(\text{"diameter" bipointed block})$. By a "law of large numbers"-type argument

$$d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2).$$

Difficult for the critical case => use diameter estimates
Theorem If $u < 9/5$, for $M_n \leftrightarrow \mathbb{P}_{n,u}$ a quadrangulation, 
\[
\frac{C_1(u)}{n^{1/4}} M_n \rightarrow \mathcal{S}_e.
\]
Moreover, $M_n$ and its simple core converge jointly to the same Brownian sphere.

We expect the same scaling limits for maps but the scaling limit of 2-connected maps is not yet proved.

See [Addario-Berry, Wen 2019] for a similar result and method.
Diameter of a decoration ≤ number of blocks × max diameter of blocks

\[ \leq \text{diam}(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = \text{diam}(T_{M_n}) \times O(n^{1/6+\delta}) \]

\[ = O(n^{1/6+2\delta}) = o(n^{1/4}). \]

[Chapuy Fusy Giménez Noy 2015]
Subcritical case (1)

Diameters of decorations = $\Theta(n^{1/4})$.

Decorations = groups of smaller blocks

Diameters of decorations = $o(n^{1/4})$.

Diameter of a decoration $\leq$ number of blocks $\times$ max diameter of blocks

$$\leq \text{diam}(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = \text{diam}(T_{M_n}) \times O(n^{1/6+\delta})$$

$$= O(n^{1/6+2\delta}) = o(n^{1/4}).$$

$T_{M_n}$ is a subcritical Galton-Watson tree

[Chapuy Fusy Giménez Noy 2015]
Subcritical case (2)

The scaling limit of $M_n$ (rescaled by $n^{1/4}$) is the scaling limit of uniform blocks!

Decorations = groups of smaller blocks

Large block of size $\Theta(n)$

The scaling limit of uniform $\sim$ (rescaled by $n^{1/4}$)

- 2-connected maps = brownian sphere (assumed);
- Simple quadrangulations = Brownian sphere [Addario-Berry Albenque 2017].
### Results

For $M_n \leftrightarrow \mathbb{P}_{n,u}$:

<table>
<thead>
<tr>
<th>$u$</th>
<th>$u &lt; 9/5$</th>
<th>$u = 9/5$</th>
<th>$u &gt; 9/5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Enumeration</td>
<td>$\rho(u)^{-n}n^{-5/2}$</td>
<td>$\rho(u)^{-n}n^{-5/3}$</td>
<td>$\rho(u)^{-n}n^{-3/2}$</td>
</tr>
<tr>
<td>Size of - the largest block</td>
<td>$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$</td>
<td>$\Theta(n^{2/3})$</td>
<td>$\frac{\ln(n)}{2 \ln \left(\frac{4}{27y}\right)} - \frac{5 \ln(\ln(n))}{4 \ln \left(\frac{4}{27y}\right)} + O(1)$</td>
</tr>
<tr>
<td>- the second one</td>
<td>$\Theta(n^{2/3})$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Scaling limit of $M_n$

- **For $u < 9/5$**:
  
  \[ \frac{C_1(u)}{n^{1/4}} M_n \to \mathcal{S}_e \]

- **For $u > 9/5$**:
  
  \[ \frac{C_2(u)}{n^{1/3}} M_n \to \mathcal{T}_{3/2} \]

- **For $u = 9/5$**:
  
  \[ \frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e \]

*Assuming the convergence of 2-connected maps towards the brownian sphere*
V. Perspectives
Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

Table 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

<table>
<thead>
<tr>
<th>maps, $M(z)$</th>
<th>cores, $C(z)$</th>
<th>submaps, $H(z)$</th>
<th>coreless, $D(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>all, $M_1(z)$</td>
<td>bridgeless, or loopless $M_2(z)$</td>
<td>$z/(1 - z(1 + M))^2$</td>
<td>$z(1 + M)^2$</td>
</tr>
<tr>
<td>loopless $M_2(z)$</td>
<td>simple $M_3(z)$</td>
<td>$z(1 + M)$</td>
<td>-</td>
</tr>
<tr>
<td>all, $M_1(z)$</td>
<td>nonsep., $M_4(z)$</td>
<td>$z(1 + M)^2$</td>
<td>-</td>
</tr>
<tr>
<td>nonsep. $M_4(z) - z$</td>
<td>nonsep. simple $M_5(z)$</td>
<td>$z(1 + M)$</td>
<td>-</td>
</tr>
<tr>
<td>nonsep. $M_4(z)/z - 2$</td>
<td>3-connected $M_6(z)$</td>
<td>$M$</td>
<td>$z + 2M^2/(1 + M)$</td>
</tr>
<tr>
<td>bipartite, $B_1(z)$</td>
<td>bip. simple, $B_2(z)$</td>
<td>$z(1 + M)$</td>
<td>-</td>
</tr>
<tr>
<td>bipartite, $B_1(z)$</td>
<td>bip. bridgeless, $B_3(z)$</td>
<td>$z/(1 - z(1 + M))^2$</td>
<td>$z(1 + M)^2$</td>
</tr>
<tr>
<td>bipartite, $B_1(z)$</td>
<td>bip. nonsep., $B_4(z)$</td>
<td>$z(1 + M)^2$</td>
<td>-</td>
</tr>
<tr>
<td>bip. nonsep., $B_4(z)$</td>
<td>bip. ns. smpl, $B_5(z)$</td>
<td>$z(1 + M)$</td>
<td>-</td>
</tr>
<tr>
<td>singular tri., $T_1(z)$</td>
<td>triang., $z + zT_2(z)$</td>
<td>$z(1 + M)^3$</td>
<td>-</td>
</tr>
<tr>
<td>triangulations, $T_2(z)$</td>
<td>irreducible tri., $T_3(z)$</td>
<td>$z(1 + M)^2$</td>
<td>-</td>
</tr>
</tbody>
</table>
Critical window?

Phase transition very sharp => what if $u = 9/5 \pm \varepsilon(n)$?

- Block size results still hold if $u_n = 9/5 - \varepsilon(n), \varepsilon^3 n \to \infty$;
- For $u_n = 9/5 + \varepsilon(n)$, conjecture $L_{n,1} \sim 2.7648 \varepsilon^{-2} \ln(\varepsilon^3 n)$ when $\varepsilon^3 n \to \infty$ (analogous to [Bollobás 1984]'s result for Erdős-Rényi graphs!);
- Results exist for scaling limits in ER graphs [Addario-Berry, Broutin, Goldschmidt 2010], open question in our case.

Is there a critical window? If so, what is its width?
<table>
<thead>
<tr>
<th>$M_n \leftrightarrow \mathbb{P}_{n,u}$</th>
<th>$u &lt; 9/5$</th>
<th>$u = 9/5$</th>
<th>$u &gt; 9/5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{n,1}$</td>
<td>$u_n = 9/5 - \varepsilon(n)$</td>
<td>$u_n = 9/5 + \varepsilon(n)$</td>
<td>$u &gt; 9/5$</td>
</tr>
<tr>
<td>$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$</td>
<td>$\Theta(n^{2/3})$</td>
<td>$\sim 2.7648 \varepsilon^{-2} \ln(e^3 n)$</td>
<td></td>
</tr>
<tr>
<td>$L_{n,2}$</td>
<td>$\Theta(n^{2/3})$</td>
<td>$\Theta(n^{2/3})$</td>
<td>$\Theta(n^{2/3})$</td>
</tr>
<tr>
<td>Scaling limit of $M_n$</td>
<td>$\varepsilon(n) = n^{-\alpha}$</td>
<td>$\varepsilon(n) = n^{-\alpha}$</td>
<td>$\varepsilon(n) = n^{-\alpha}$</td>
</tr>
<tr>
<td>$\frac{C_1(u)}{n^{1/4}} M_n \rightarrow \mathcal{S}_e$</td>
<td>$\frac{C_4}{n^{(1-\alpha)/4}} M_n \rightarrow \mathcal{S}_e$</td>
<td>$\frac{C_2}{n^{1/3}} M_n \rightarrow \mathcal{T}_{3/2}$</td>
<td>$\frac{C_3(u)}{n^{1/2}} M_n \rightarrow \mathcal{T}_e$</td>
</tr>
<tr>
<td>Admitting the convergence of 2-connected maps towards the brownian map</td>
<td></td>
<td>stable tree?</td>
<td></td>
</tr>
</tbody>
</table>

Pink = work in progress
Thank you!