A phase transition in block-weighted random maps

Journée cartes à l’IPhT
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Planar maps

Planar map $m$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms.

- **Rooted** planar map = map endowed with a marked oriented edge (represented by an arrow);
- **Size** $|m| = $ number of edges;
- **Corner** (does not exist for graphs !) = space between an oriented edge and the next one for the trigonometric order.

Map = graph + cyclic order on neighbours.
Universality results for planar maps

- Enumeration: $\kappa \rho^{-n} n^{-5/2}$ [Tutte 1963, Drmota, Not, Yu 2020];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for quadrangulations [Miermont 2013], triangulations & 2q-angulations [Le Gall 2013];
- Universality:
  - Same enumeration;
  - Same scaling limit, e.g. for simple quadrangulations [Addario-Berry Albenque 2017], arbitrary maps [Bettinelli, Jacob Miermont 2014].

\[ \kappa \rho^{-n} n^{-5/2} \]
Universality results for planar trees

- Enumeration: $\kappa \rho^{-n} n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];
- Universality:
  - Same enumeration,
  - Same scaling limit, even for some classes of maps; e.g. outerplanar maps [Caraceni 2016], maps with a boundary of size $\gg n^{1/2}$ [Bettinelli 2015].

Models with (very) constrained boundaries

Brownian tree $\mathcal{T}_e$
Motivation

Interpolating model?
**2-connectivity**

**Cut vertex**: vertex that when removed deconnects the map

**2-connected**: no cut vertex (=to be able to disconnect, at least two vertices must be removed)

**Block** = maximal (for inclusion) 2-connected submap
2-connectivity

**Cut vertex**: vertex that when removed deconnects the map

**2-connected**: no cut vertex (=to be able to disconnect, at least two vertices must be removed)

**Block** = maximal (for inclusion) 2-connected submap
Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

Motivation

Interpolating model?
Outline of the talk

A phase transition in block-weighted random maps

I. Approach
II. Largest blocks
III. Similar model: quadrangulations
IV. Scaling limits
V. Perspectives
I. Approach
Model

Goal: parameter that affects the typical number of blocks.

We choose: \( \mathbb{P}_{n,u}(m) = \frac{u \#\text{blocks}(m)}{Z_{n,u}} \) where

\[
\begin{align*}
\mathcal{M}_n &= \{\text{maps of size } n\}, \\
Z_{n,u} &= \text{normalisation}.
\end{align*}
\]

Inspired by [Bonzom 2016].

- \( u = 1 \): uniform distribution on maps of size \( n \);
- \( u \to 0 \): minimising the number of blocks (=2-connected maps);
- \( u \to \infty \): maximising the number of blocks (= trees!).

Given \( u \), asymptotic behaviour when \( n \to \infty \)?
<table>
<thead>
<tr>
<th>For $M_n \leftrightarrow \mathcal{P}_{n,u}$</th>
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Decomposition of a map into blocks

Inspiration from [Tutte 1963]

\[ M(z, u) = \sum_{m \in \mathcal{M}} z^{|m|} u^{|\text{blocks}(m)|} \]
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\[ M(z, u) = \sum_{m \in \mathcal{M}} z^{m_1} u^{\#blocks(m)} \]
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With a weight \( u \) on blocks: 
\[ M(z, u) = uB(zM^2(z, u)) + 1 - u \]

GS of 2-connected maps
Decomposition of a map into blocks

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⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

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</table>
Decomposition of a map into blocks: properties

- Internal node (with $k$ children) of $T_m \leftrightarrow$ block of $\mathbf{m}$ of size $k/2$;
- $\mathbf{m}$ is entirely determined by $T_m$ and $(b_v, v \in T_m)$ where $b_v$ is the block of $\mathbf{m}$ represented by $v$ in $T_m$.

$T_{M_n}$ gives the block sizes of a random map $M_n$. 
Galton-Watson trees for map blocks

$\mu$-Galton-Watson tree: random tree where the number of children of each node is given by $\mu$ independently, with $\mu = \mu$ probability law on $\mathbb{N}$.

=> Choice of $y$?

$u > 0$

$y \in (0,4/27]$
Galton-Watson trees for map blocks

$\mu$-Galton-Watson tree: random tree where the number of children of each node is given by $\mu$ independently, with $\mu = \text{probability law on } \mathbb{N}$.

**Theorem**

If $M_n \hookrightarrow \mathbb{P}_{n,u}$, then $T_{M_n}$ has the law of a Galton-Watson tree of reproduction law $\mu^{y,u}$ conditioned to be of size $2n$, with

$$
\mu^{y,u}([2k]) = \frac{B_k y^k u^{1_k \neq 0}}{u B(y) + 1 - u}.
$$

$\Rightarrow$ Choice of $y$?

$u > 0$

$y \in (0,4/27]$
Phase transition

When is $\mu^{y,u}$ critical? (= $\mathbb{E}(\mu) = 1$?)

$$\mathbb{E}(\mu^{y,u}) = 1 \iff u = \frac{1}{2yB'(y) - B(y) + 1}$$

covers $[9/5, + \infty)$ when $y$ covers $(0, \rho_B = 4/27]$.

Theorem

- If $u < 9/5$, then $\mathbb{E}(\mu^{y,u}) < 1$. The mean is maximal for $y = 4/27$ and then $\mu^{y,u}(2k) \sim c_u k^{-5/2}$;

- If $u = 9/5$ and $y = 4/27$, then $\mathbb{E}(\mu^{y,u}) = 1$ and $\mu^{y,u}(2k) \sim c_u k^{-5/2}$;

- If $u > 9/5$ and $y$ is well chosen, then $\mathbb{E}(\mu^{y,u}) = 1$ and $\mu^{y,u}(2k) \sim c_u \pi_u k^{-5/2}$. 

Phase transition

\[ u_C = \frac{9}{5} \]

y s.t. \( u = \frac{1}{2yB'(y) - B(y) + 1} \)

Subcritical GW

Critical GW

"Map regime"  "Tree regime"
II. Largest blocks
## Properties of $T_{M_n}$

<table>
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<tr>
<td>Variance</td>
<td>$\infty$</td>
<td></td>
<td>$&lt; \infty$</td>
</tr>
<tr>
<td>Galton-Watson tree</td>
<td>subcritical</td>
<td>critical</td>
<td></td>
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</tbody>
</table>

**Tool:** [Janson 2012] = extensive study of the degrees in Galton-Watson trees

Properties on trees give properties of maps.
### Size $L_{n,k}$ of the $k$-th largest block

For $M_n \hookrightarrow \mathcal{P}_{n,u}$

<table>
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<tr>
<th>$u$</th>
<th>$L_{n,1}$</th>
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<td>$u &lt; 9/5$</td>
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Stufler 2020

**Size of the linear block $\times n^{-1}$**
### Rough intuition

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**Galton-Watson tree**

- **Subcritical**: condensation, cf [Jonsson Stefánsson 2011];
- **Supercritical**: behaves as maximum of independent variables.

**Dichotomy between situations:**

- Subcritical: condensation, cf [Jonsson Stefánsson 2011];
- Supercritical: behaves as maximum of independent variables.
### Results

For $M_n \leftrightarrow P_{n,u}$

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III. Similar model: quadrangulations
Quadrangulations

Def: map with all faces of degree 4.

Simple quadrangulation = no multiple edges.

Size $|q|$ = number of faces.

$$|V(q)| = |q| + 2, \quad |E(q)| = 2|q|.$$
Construction of a quadrangulation from a simple core
Construction of a quadrangulation from a simple core
Construction of a quadrangulation from a simple core
With a weight \( u \) on blocks: \( Q(z, u) = uS(zQ^2(z, u)) + 1 - u \)

Remember: \( M(z, u) = uB(zM^2(z, u)) + 1 - u \)
Tutte’s bijection

Map

Quadrangulation

[Tutte 1963]
Tutte’s bijection for 2-connected maps

Cut vertex => multiple edge

2-connected maps <=> simple quadrangulations

[Brown 1965]
Block trees under Tutte’s bijection
Implications on results

We choose: \( P_{n,u}(q) = \frac{u \# \text{blocks}(q)}{Z_{n,u}} \) where

\( u > 0, \)
\( \mathcal{Q}_n = \{ \text{quadrangulations of size } n \}, \)
\( q \in \mathcal{Q}_n, \)
\( Z_{n,u} = \text{normalisation}. \)

Results on the size of (2-connected) blocks can be transferred immediately for quadrangulations and their simple blocks.
## Results

For $M_n \leftrightarrow \mathbb{P}_{n,u}$:

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IV. Scaling limits
Scaling limits

Convergence of the whole object considered as a metric space (with the graph distance), after renormalisation.

What is the limit of the sequence of metric spaces \(((M_n, d/n^?)_n \in \mathbb{N})\)?

(Convergence for Gromov-Hausdorff metric)
Scaling limits of Galton-Watson trees

Theorem For $M_n \sim \mathcal{P}_{n,u}$

- If $u > 9/5$, $\frac{c_3(u)}{n^{1/2}}T_{M_n} \rightarrow \mathcal{T}_e$.
- If $u = 9/5$, $\frac{c_2}{n^{2/3}}T_{M_n} \rightarrow \mathcal{T}_{3/2}$.

Stable tree of index 3/2 $\mathcal{T}_{3/2}$

Brownian tree $\mathcal{T}_e$ (Aldous’s CRT)
# Scaling limits of Galton-Watson trees

## Theorem

For $M_n \leftrightarrow \mathcal{P}_{n,u}$

- If $u > 9/5$, $\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e$.
- If $u = 9/5$, $\frac{c_2}{n^{2/3}} T_{M_n} \to \mathcal{T}_{3/2}$.

## Proof

- Scaling limit of **critical** Galton-Watson trees with finite variance [Aldous 1993, Le Gall 2006];
- Scaling limit of **critical** Galton-Watson with infinite variance and nice tails [Duquesne 2003].
Theorem
For $M_n \leftrightarrow \mathbb{P}_{n,u}$,

- If $u > 9/5$, 
  \[ \frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e. \]

- If $u = 9/5$, 
  \[ \frac{C_2}{n^{2/3}} M_n \to \mathcal{T}_{3/2}. \]

[Stufler 2020]
Difficult part = show that distances in $\mathfrak{m}$ behave like distances in $T_\mathfrak{m}$. We show

$$\forall e_1, e_2 \in \overrightarrow{E}(M_n), d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2).$$
Let $\kappa = \mathbb{E}(\text{"diameter" bipointed block}).$ By a "law of large numbers"-type argument

$$d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2).$$

Difficult for the critical case => use diameter estimates
Scaling limits of subcritical maps

Theorem If $u < 9/5$, for $M_n \leftrightarrow \mathcal{P}_{n,u}$ a quadrangulation,

$$\frac{C_1(u)}{n^{1/4}} M_n \to \mathcal{S}_e.$$ 

Moreover, $M_n$ and its simple core converge jointly to the same Brownian sphere.

We expect the same scaling limits for maps but the scaling limit of 2-connected maps is not yet proved.

See [Addario-Berry, Wen 2019] for a similar result and method.
Subcritical case (1)

Diameters of decorations = \( \Theta(n^{1/4}) \).

Diameters of decorations = \( o(n^{1/4}) \).

Diameter of a decoration \( \leq \) number of blocks \( \times \) max diameter of blocks

\[
\leq \text{diam}(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = \text{diam}(T_{M_n}) \times O(n^{1/6+\delta})
\]

\[
= O(n^{1/6+2\delta}) = o(n^{1/4}).
\]

\( T_{M_n} \) is a subcritical Galton-Watson tree

[Chapuy Fusy Giménez Noy 2015]
Subcritical case (2)

The scaling limit of $M_n$ (rescaled by $n^{1/4}$) is the scaling limit of uniform blocks!

Decorations = groups of smaller blocks

Scaling limit of uniform $\sim$ (rescaled by $n^{1/4}$)

- 2-connected maps = brownian sphere (assumed);
- Simple quadrangulations = Brownian sphere [Addario-Berry Albenque 2017].
## Results

For $M_n \leftrightarrow P_{n,u}$

<table>
<thead>
<tr>
<th>$u$</th>
<th>$u &lt; 9/5$</th>
<th>$u = 9/5$</th>
<th>$u &gt; 9/5$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Enumeration</strong></td>
<td>$\rho(u)^{-n}n^{-5/2}$</td>
<td>$\rho(u)^{-n}n^{-5/3}$</td>
<td>$\rho(u)^{-n}n^{-3/2}$</td>
</tr>
<tr>
<td>Bonzom 2016 for 2-c case</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Size of</strong></td>
<td>$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$</td>
<td>$\Theta(n^{2/3})$</td>
<td></td>
</tr>
<tr>
<td>- the largest block</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>- the second one</td>
<td>$\Theta(n^{2/3})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Scaling limit of $M_n$</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{C_1(u)}{n^{1/4}} M_n \to S_e$</td>
<td>$\frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$</td>
<td>$\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e$</td>
<td></td>
</tr>
</tbody>
</table>

Assuming the convergence of 2-connected maps towards the brownian sphere

[Stufler 2020]
V. Perspectives
Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

Table 3. Composition schemas, of the form \( \mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D} \), except the last one where \( \mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H}) \).

<table>
<thead>
<tr>
<th>maps, ( M(z) )</th>
<th>cores, ( C(z) )</th>
<th>submaps, ( H(z) )</th>
<th>coreless, ( D(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>all, ( M_1(z) )</td>
<td>bridgeless,</td>
<td>( z/(1 - z(1 + M))^2 )</td>
<td>( z(1 + M)^2 )</td>
</tr>
<tr>
<td></td>
<td>or loopless</td>
<td></td>
<td></td>
</tr>
<tr>
<td>loopless ( M_2(z) )</td>
<td>simple ( M_3(z) )</td>
<td>( z(1 + M) )</td>
<td></td>
</tr>
<tr>
<td>all, ( M_1(z) )</td>
<td>nonsep., ( M_4(z) )</td>
<td>( z(1 + M)^2 )</td>
<td></td>
</tr>
<tr>
<td>nonsep. ( M_4(z) - z )</td>
<td>nonsep. simple ( M_5(z) )</td>
<td>( z(1 + M) )</td>
<td></td>
</tr>
<tr>
<td>nonsep. ( M_4(z)/z - 2 )</td>
<td>3-connected ( M_6(z) )</td>
<td>( M )</td>
<td>( z + 2M^2/(1 + M) )</td>
</tr>
<tr>
<td>bipartite, ( B_1(z) )</td>
<td>bip. simple, ( B_2(z) )</td>
<td>( z(1 + M) )</td>
<td></td>
</tr>
<tr>
<td>bipartite, ( B_1(z) )</td>
<td>bip. bridgeless, ( B_3(z) )</td>
<td>( z/(1 - z(1 + M))^2 )</td>
<td>( z(1 + M)^2 )</td>
</tr>
<tr>
<td>bipartite, ( B_1(z) )</td>
<td>bip. nonsep., ( B_4(z) )</td>
<td>( z(1 + M)^2 )</td>
<td></td>
</tr>
<tr>
<td>bip. nonsep., ( B_4(z) )</td>
<td>bip. ns. smpl, ( B_5(z) )</td>
<td>( z(1 + M) )</td>
<td></td>
</tr>
<tr>
<td>singular tri., ( T_1(z) )</td>
<td>triang., ( z + zT_2(z) )</td>
<td>( z(1 + M)^3 )</td>
<td></td>
</tr>
<tr>
<td>triangulations, ( T_2(z) )</td>
<td>irreducible tri., ( T_3(z) )</td>
<td>( z(1 + M)^2 )</td>
<td></td>
</tr>
</tbody>
</table>
Critical window?

Phase transition very sharp => what if $u = 9/5 \pm \varepsilon(n)$?

- Block size results still hold if $u_n = 9/5 - \varepsilon(n), \varepsilon^3 n \to \infty$;
- For $u_n = 9/5 + \varepsilon(n)$, conjecture $L_{n,1} \sim 2.7648 \varepsilon^{-2} \ln(\varepsilon^3 n)$ when $\varepsilon^3 n \to \infty$ (analogous to [Bollobás 1984]’s result for Erdős-Rényi graphs!);
- Results exist for scaling limits in ER graphs [Addario-Berry, Broutin, Goldschmidt 2010], open question in our case.

Is there a critical window? If so, what is its width?
### Perspectives

<table>
<thead>
<tr>
<th>$M_n \leftrightarrow \mathbb{P}_{n,u}$</th>
<th>$u &lt; 9/5$</th>
<th>$u = 9/5$</th>
<th>$u &gt; 9/5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_n = 9/5 - \varepsilon(n)$</td>
<td>$\varepsilon^3 n \to \infty$</td>
<td>$u_n = 9/5 + \varepsilon(n)$</td>
<td>$\varepsilon^3 n \to \infty$</td>
</tr>
</tbody>
</table>

| $L_{n,1}$ | $\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ | $\Theta(n^{2/3})$ | $\sim 2.7648 \varepsilon^{-2} \ln(\varepsilon^3 n)$ |

| $L_{n,2}$ | $\Theta(n^{2/3})$ | $\Theta(n^{2/3})$ | $\Theta(n^{2/3})$ |

| Scaling limit of $M_n$ | $\varepsilon(n) = n^{-\alpha}$ | $\varepsilon(n) = n^{-\alpha}$ | $\varepsilon(n) = n^{-\alpha}$ |

| $\frac{C_1(u)}{n^{1/4}}M_n \to \mathcal{S}_e$ | $\frac{C_4}{n^{(1-\alpha)/4}}M_n \to \mathcal{S}_e$ | $\frac{C_2}{n^{1/3}}M_n \to \mathcal{T}_{3/2}$ | stable tree? |

Admitting the convergence of 2-connected maps towards the brownian map

$\frac{C_3(u)}{n^{1/2}}M_n \to \mathcal{T}_e$

Pink = work in progress
Thank you!