On some Hopf algebras of type B

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Abstract

It is well known that permutations, binary trees and compositions are strongly related together through mathematical objects such that Hopf algebras (free quasi-symmetric functions or Malvenuto-Reutenaeur algebra, Loday-Ronco algebra, noncommutative symmetric functions) equipped with their posets (weak order, Tamari lattice, boolean lattice inclusion). This article is an attempt to reproduce this theory for type B.

1 Introduction

In [10], Novelli and Thibon introduced Hopf algebras for colored combinatorial objects such that colored permutations and colored binary trees. Hence, free quasi-symmetric functions have their colored analogs denoted by $\mathbf{FQSym}^{(l)}$, namely the free quasi-symmetric functions of level l. A simple observation to make is to notice that colored permutations with only two colors are obviously in bijection with signed permutations. Hence, we shall consider $\mathbf{FQSym}^{(2)}$ as the Hopf algebra of signed permutations. We should emphasize here that the product and the coproduct of $\mathbf{FQSym}^{(2)}$ are very natural and hence nicely generalize the algebraic structure of **FQSym** (see [3]) for the type B case. Then, similarly than in the case of type A, we naturally consider $\mathbf{FQSym}^{(2)}$ together with its poset, that is the weak order of the Coxeter group of signed permutations. One main contribution of this article is to investigate relations between the Hopf algebra of signed permutations and its poset.

Another Hopf algebra we shall consider in this paper is an analog of **PBT**, which is well known to be isomorphic to the Loday-Ronco Hopf algebra (see [6] [7]). Indeed, once again in [10], Novelli and Thibon introduced the colored analogs of planar binary trees, written $\mathbf{PBT}^{(l)}$. In this paper, we will focus only on $\mathbf{PBT}^{(2)}$, which is indexed by bicolored binary trees, and also virtually consider it as a Hopf algebra of type B. Let us mention that, similarly that in the type A case where **PBT** is a Hopf subalgebra of **FQSym**, $\mathbf{PBT}^{(2)}$ is a Hopf subalgebra of $\mathbf{FQSym}^{(2)}$.

Let us now recall some useful facts of the case of type A theory. As mentioned above, it is natural to consider **FQSym**, or the Malvenuto-Reutenaeur Hopf algebra [9], together with the weak order of permutations of type A. Indeed, the algebraic structure is related to its poset through a key notion of poset theory: the interval notion. First, a well known result is that product of intervals of permutations is an interval of permutations itself. Second, elements themselves of interesting subalgebras of **FQSym** - such that **PBT** (or Loday-Ronco Hopf algebra), **SP** (see [11]), **Baxter** (see [5]), or the noncommutative symmetric functions **Sym** (see [4])- are intervals over the elements of **FQSym**. Moreover, the interval notion helps to define posets associated to the subalgebras of **FQSym** and also often allow to describe the product of these subalgebras over their posets.

In type B theory, the interval notion does not play this key function. Indeed, a quick check allow us to see that the product in $\mathbf{FQSym}^{(2)}$ of two intervals of signed permutations is generally not an interval itself of signed permutations. Hence, in order to reach our goal, we have to find the analogous notion of interval - which we recall to be the key notion for type A theory - for type B theory; in other words, we have to find a kind of interval of type B.

In this paper, we propose a candidate notion of poset theory to fulfill this function: we introduce the multi-interval notion. We believe that the multi-interval notion is the key poset theory notion for type B theory for at least two reasons we state next and which are the main contributions of this paper.

1. The product in the Hopf algebra of signed permutations $\mathbf{FQSym}^{(2)}$ of two multi-intervals gives as a result a multi-interval of signed permutations.

2. The elements of $\mathbf{PBT}^{(2)}$, which we recall to be the type B analog of **PBT** and a subalgebra of $\mathbf{FQSym}^{(2)}$, are multi-intervals of signed permutations.

Moreover, it is well known (see [6] [8]) that **PBT**, or the Loday-Ronco Hopf algebra, has the Tamari lattice as poset. Similarly, in order to reinforce the consideration for what we call the type B case, based on several computations, we state a conjecture of the existence of a Tamari lattice of type B which is naturally associated to $\mathbf{PBT}^{(2)}$.

In Section 2, we provide basic material used in the sequel. Then, in Section 3, we provide definitions and examples for the Hopf algebra of signed permutations, that is $\mathbf{FQSym}^{(2)}$, and for $\mathbf{PBT}^{(2)}$; we also briefly mention the possibility of the existence of noncommutative symmetric functions of type B. In the following Section 4, we define multi-intervals and prove the main result of this article: the product in $\mathbf{FQSym}^{(2)}$ of multi-intervals gives a multi-interval again. Then, in order to strengthen the multi-interval notion, we prove in Section 5 that elements of $\mathbf{PBT}^{(2)}$ are multi-intervals of signed permutations. Moreover, we state the conjecture about the Tamari lattice of type B in Section 6. Finally, due to the size of the studied combinatorial objects, the computer exploration associated to this mathematical work could not have been possible without efficient algorithms; hence, in Appendix A, we provide an algorithm to efficiently compute intervals in the weak order of the Coxeter group of signed permutations.

2 Preliminaries

A *signed permutation* σ of size n is a word of size n over the alphabet $[-n,-1] \cup [1,n]$ such that if i appears in σ then $-i$ does not appear in σ and with the additional constraint to have no repetitions of letters. Thus, we denote by B_n the set of the signed permutations of size n and by B the set of all signed permutations. An *inversion* of a signed permutation σ is a pair (σ_i, σ_j) such that $\sigma_i > \sigma_j$ with $i < j$; moreover, every signed permutation which the first letter l is negative has the inversion denoted by $(l, -l)$. We denote by $I(\sigma)$ the set of inversions of the signed permutation σ. The *weak order* over signed permutations is the partial order defined by the following covering relation: we have $\sigma \lessdot \pi$ if $I(\sigma) \subset I(\pi)$ and $|I(\pi)| = |I(\sigma)| + 1$. For example, we have 13-2 $\lessdot$ -13-2 and 3-41-2 $\lessdot$ 31-4-2.

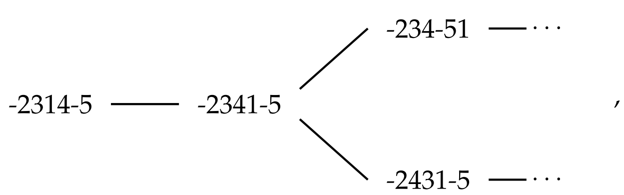
The map *perm* associates to a signed permutation σ its permutation of type A by associating to each letter of σ its absolute value and preserving the order between letters of σ. For example, we have *perm*(3-41-2) = 3412. The map *sign* associates to a signed permutation σ a word made by reading σ from left to right and concatening 1 if the letter is positive and -1 if the letter is negative. For example, we have *sign*(3-41-2) = 1-11-1.

Let w_1 and w_2 be two words. Then the *shuffle* of w_1 and w_2 denoted by $w_1 \sqcup\!\sqcup w_2$ is recursively defined by

- $w_1 \sqcup\!\sqcup \epsilon = w_1, \quad \epsilon \sqcup\!\sqcup w_2 = w_2,$
- $au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v),$

where a, b are letters, and u, v are words. For example, we have 12 $\sqcup\!\sqcup$ -43 = 12-43 + 1-423 + 1-432 + -4123 + -4132 + -4312. The *shift* of a signed permutation σ by n is word obtained by deleting the signs for every letter of σ, then adding n to every letter and setting the sign for every letter as initial. For example, we have 3-41-2[2] = 5-63-4. The *shifted shuffle* of two signed permutations σ and π is the operation $\sigma \sqcup\!\sqcup \pi[n]$ where n is the size of σ and is denoted $\sigma \Cup \pi$. Let γ be a signed permutation that appears in the shifted shuffle of two signed permutations σ and π, then we set $\gamma|_\sigma = \sigma$ and $\gamma|_\pi = \pi$; if σ belongs to a set X and π to a set Y we use without ambiguity the equivalent notation $\gamma|_X = \sigma$ and $\gamma|_Y = \pi$.

Remark 2.1 *We notice that the result of the shifted shuffle of two signed permutations (as well as for permutations of type A) can be structured as a tree. Indeed, for example* -231 $\Cup$ 1-2 *can be represented as follows:*



where every pair (σ, π) of adjacent signed permutations of the tree have the following property: π can be obtained from σ by transposing two consecutive letters of σ; hence, we have either $\sigma \lessdot \pi$ or $\sigma \gtrdot \pi$.

A *colored permutation* is a pair (σ, c) where σ is a permutation and c a word named a *color*, over an alphabet C, and which the size is identical to the size of σ. Since clearly a signed permutation π is defined by the pair $(perm(\pi), sign(\pi))$, it follows that signed permutations are in bijection with *bicolored permutations* that are colored permutation which the color is defined over $\{-1,1\}$. We denote by $(\sigma,c)^{\uparrow}$ the signed permutation associated to the bicolored permutation (σ,c).

The *standardization* process sends a word v of length n to a permutation $Std(v)$ of size n, called the standardized of v, defined as the permutation obtained by iteratively scanning v from left to right, and labeling $1,2,\ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. For example, $Std(abcadbdaa) = 157286934$. Let (v,c) be a pair such that v is a word and c a color; then the colored permutation $\mathbf{Std}(v,c)$ is set as $(Std(v),c)$.

Let σ be a permutation; its *binary search tree* denoted $\mathcal{P}(\sigma)$ is obtained as follows: reading σ from right to left, one inserts each letter in a binary search tree in the following way: if the tree is empty, one creates a node labeled by the letter; otherwise, this letter is recursively inserted in the left (resp. right) subtree if it is smaller than or equal to (resp. strictly greater than) the root. The *shape* of a binary search tree T is the binary tree structure of T. A *bicolored binary tree* is the pair (T,c) where T is a binary tree and c a color.

An *interval* denoted $[\alpha,\omega]$ over the poset $(B,\leqslant)$ is the set $\{\sigma \in B \mid \alpha \leqslant \sigma \leqslant \omega\}$. A *path* between two signed permutations σ and π, denoted by $\sigma \rightsquigarrow \pi$, is a sequence $\sigma \lessdot \cdots \gtrdot \pi$ where the relation between two signed permutations of the sequence can indifferently be either $\lessdot$ or $\gtrdot$. A set X of signed permutations is *connected* if for every pair of signed permutations σ and π of X there is a path $\sigma \rightsquigarrow \pi$.

Remark 2.2 *Let X,Y be connected sets of signed permutations such that $X \cap Y \neq \emptyset$. Then, clearly, $X \cup Y$ is connected.*

3 Two Hopf algebras of type B

In this Section, we define the Hopf algebra of signed permutations straightforwardly from $\mathbf{FQSym}^{(2)}$. As the reader should notice, it admits a very natural definition, similar to **FQSym**. We then provide the definition of its Hopf subalgebra $\mathbf{PBT}^{(2)}$, which is indexed by bicolored binary trees. The proofs that these objects are effectively Hopf algebras are not difficult and done in [10]. Finally, in order to fully reproduce the type A theory, we quickly discuss about potential candidate for noncommutative symmetric functions of type B.

3.1 Hopf algebra of signed permutations

Similarly than **FQSym**, we define an algebraic structure over the signed permutations.

Definition 3.1 *We set $\mathbb{K}[B] := \bigoplus_{n\geqslant 0}\mathbb{K}[B_n]$.*

1. *The space $\mathbb{K}[B]$ is endowed with an algebra structure by providing the shuffle product $\Cup$ over this space.*
2. *Moreover, we endow the space $\mathbb{K}[B]$ with a coalgebra structure by providing the following coproduct Δ over this space.*
$$\Delta(\sigma) := \sum_{u\cdot v = perm(\sigma)} \mathbf{Std}(u, sign(\sigma))^{\uparrow} \otimes \mathbf{Std}(v, sign(\sigma))^{\uparrow},$$
where σ is a signed permutation.

We denote by the triple $(\mathbb{K}[B],\Cup,\Delta)$ these algebra and coalgebra structures.

Example 3.2

$$\begin{aligned}\text{-231} \Cup \text{1-2} &= \text{-2314-5} + \text{-2341-5} + \text{-234-51} + \text{-2431-5} + \text{-243-51} + \text{-24-531} + \text{4-231-5} + \text{4-23-51} \\ &\quad + \text{4-2-531} + \text{4-5-231}\end{aligned}$$

$$\Delta(\text{2-41-3}) = \mathbf{1} \otimes \text{2-41-3} + 1 \otimes \text{-31-2} + \text{1-2} \otimes \text{1-2} + \text{2-31} \otimes \text{-1} + \text{2-41-3} \otimes \mathbf{1}$$

We state the following Proposition without proof.

Proposition 3.3 *[10] $(\mathbb{K}[B],\Cup,\Delta)$ is a Hopf algebra written* $\mathbf{FQSym}^{(2)}$.

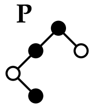
3.2 Hopf algebra of binary trees of type B

Definition 3.4 *Let* $(\mathbf{P}_b)_{b\in BBT}$ *a family of elements of* $\mathbf{FQSym}^{(2)}$ *indexed by bicolored binary trees such that:*

$$\mathbf{P}_{(T,c)} := \sum_{\sigma;\ shape(\mathcal{P}(\sigma))=T} (\sigma, c)^{\uparrow} \ , \tag{1}$$

where σ runs over permutations.

Example 3.5

$\mathbf{P}$  $=$ 2-13-54 + 2-1-534 + 2-5-134 + -52-134

We remind to the reader that binary search trees are in bijection with binary trees. Hence, a color associated to a node of a binary tree can be associated to an element of a permutation.

Then, we remind first a construction coming from the type A case in order to formulate another useful definition of the elements of Definition 3.4 through Proposition 3.9.

Definition 3.6 *Let* $\rightarrow_{sylv}$ *be the rewriting rule on permutations such that:*

$$u\,ac\,v\,b\,w \ \rightarrow_{sylv}\ u\,ca\,v\,b\,w, \tag{2}$$

where u,v and w are words and a,b and c are letters such that $a < b < c$*. Moreover, let* $\overset{*}{\longleftrightarrow}_{sylv}$ *the reflexive-symmetric-transitive closure of* $\rightarrow_{sylv}$*.*

Then, we define $\equiv_{sylv}$ *as the equivalence relation over permutations such that* $\sigma \equiv_{sylv} \pi$ *if* $\sigma \overset{*}{\longleftrightarrow}_{sylv} \pi$*, for any permutations σ and π. It is named the sylvester equivalence relation.*

For example, we have $21534 \rightarrow_{sylv} 25134$ and $21354 \equiv_{sylv} 21534 \equiv_{sylv} 25134 \equiv_{sylv} 52134$. Then, Definition 3.6 allows us to state the following useful Proposition from [6].

Proposition 3.7 *[6] Let* σ, π *be permutations. Then,* $shape(\mathcal{P}(\sigma)) = shape(\mathcal{P}(\pi))$ *if and only if* $\sigma \equiv_{sylv} \pi$*.*

Next, we give the analog type B definition of Definition 3.6.

Definition 3.8 *We define* $\equiv^B_{sylv}$ *as the equivalence relation over signed permutations such that* $\sigma \equiv^B_{sylv} \pi$ *if* $perm(\sigma) \equiv_{sylv} perm(\pi)$ *and* $sign(\sigma) = sign(\pi)$*.*

For example, one can check that the set of signed permutations {2-13-54, 2-1-534, 2-5-134, -52-134} of Example 3.5 is an equivalence classe of $\equiv^B_{sylv}$. More generally, we have the following result.

Proposition 3.9 *The* $(\mathbf{P}_b)_{b\in BBT}$ *elements are sum of the elements of equivalence classes of* $\equiv^B_{sylv}$ *over signed permutations.*

Proof – The result comes straightforwardly from Definition 3.4, Definition 3.8 and Proposition 3.7. ■

At last, we state the following Proposition without proof.

Proposition 3.10 *[10] The family* $(\mathbf{P}_b)_{b\in BBT}$ *spans a Hopf subalgebra of* $\mathbf{FQSym}^{(2)}$ *written* $\mathbf{PBT}^{(2)}$*.*

3.3 Noncommutative symmetric functions of type B

Noncommutative symmetric functions of type B, in the meaning discussed in this paper, have still to be discovered.

Nevertheless, the author believes that by defining hypoplactic relations over signed permutations then, similarly than Proposition 3.9 and Proposition 3.10, one can defines elements over $\mathbf{FQSym}^{(2)}$ which span a Hopf subalgebra indexed by bicolored compositions. Some arguments of the Proof of Theorem 5.3 could potentially be used to prove that elements of this hypothetical Hopf algebra describe multi-intervals over signed permutations. Moreover, one could check we may have a kind of hypercube lattice of type B naturally defined from the weak order of type B.

However, even if such a mathematical object exists and fulfills all the conditions mentionned above, some additional work has to be done. In particular, noncommutative symmetric functions of type B already exist and are introduced in [2]; then it would be appropriate to check either there is a link between these objects.

4 Intervals of type B

In this Section, we provide the definition of intervals of type B which are called *multi-intervals*. Then, we state the main Theorem of this Section which describes the behavior of the multi-intervals into **FQSym**$^{(2)}$, with all the necessary Lemmas.

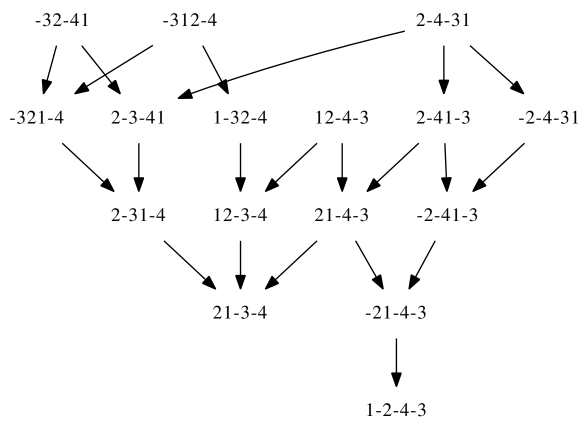
Definition 4.1 *Let X be a set of signed permutations. Then, X is a multi-interval if it is a connected set such that:*

1. *X has at least one minimal element (we note S_X the set of minimal elements),*
2. *X has at least one maximal element (we note G_X the set of maximal elements),*
3. *For all $s \in S_X$ and for all $g \in G_X$, we have $[s, g] \subset X$.*

It obviously follows from Definition 4.1 that intervals of signed permutations are multi-intervals with a unique minimal element and a unique maximal element.

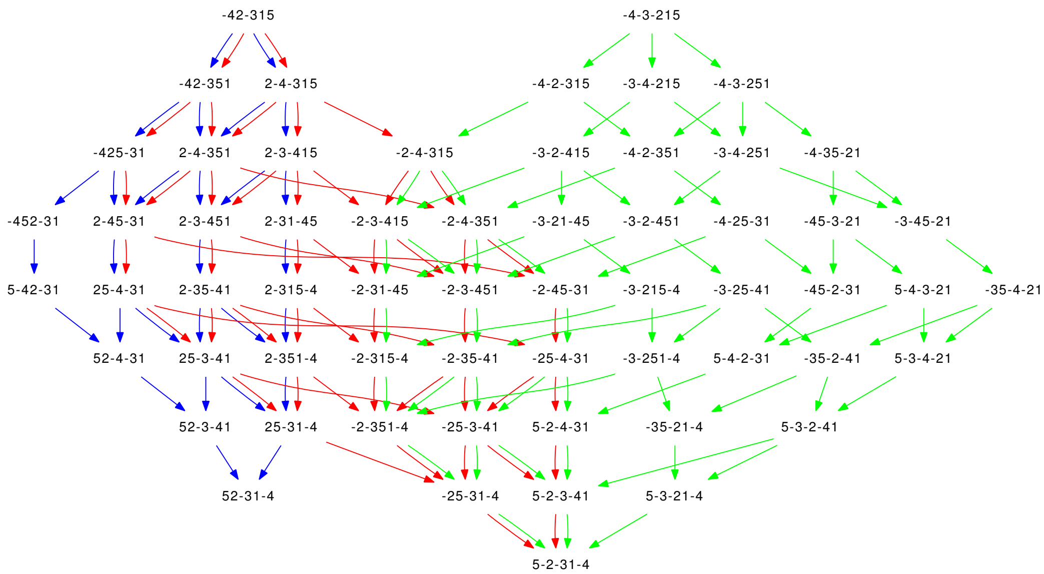
Example 4.2

$X =$



We have S_X = {-32-41, -312-4, 12-4-3, 2-4-31} and G_X = {21-3-4, 1-2-4-3}. In particular, one can check that we have [12-4-3, 1-2-4-3] = {12-4-3, 21-4-3, -21-4-3, 1-2-4-3}. *We notice that an empty interval like* [-312-4, 1-2-4-3] *is included in X.*

Example 4.3



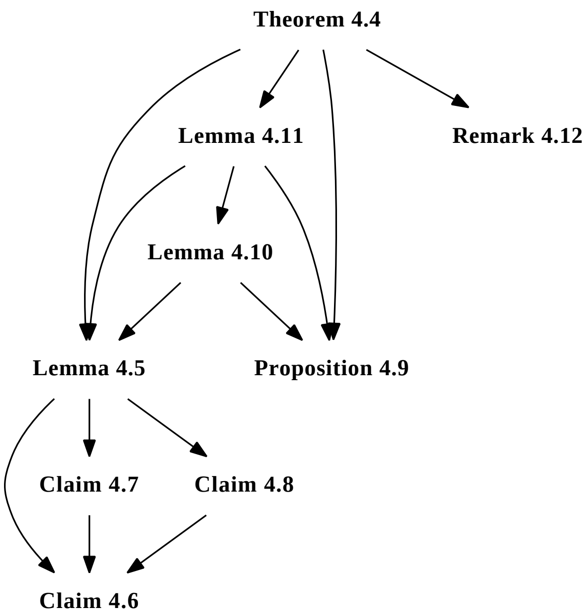
This multi-interval is made of three non-trivial intervals of signed permutations.

4.1 The main Theorem

In this Subsection we only state the main result of this Section, and provide the proof in Section 4.5.

Theorem 4.4 *Let X, Y be two multi-intervals of signed permutations. Then, $Z = X \Cup Y$ is a multi-interval.*

In order to help to reader to browse the whole proof, we display the graph of dependencies of Theorem 4.4.



4.2 Around connected property

Lemma 4.5 *Let X, Y be two multi-intervals. Then, $Z = X \Cup Y$ is connected.*

Proof – Let γ and λ be signed permutations which belongs to Z. Let σ, σ' be signed permutations of X and π, π' be signed permutations of Y such that $\gamma \in \sigma \Cup \pi$ and $\lambda \in \sigma' \Cup \pi'$. Since X and Y are connected by Definition 4.1, we have $\sigma \rightsquigarrow \sigma'$ and $\pi \rightsquigarrow \pi'$. It follows, by Claim 4.7, that $(\sigma \Cup \pi) \cup \cdots \cup (\sigma' \Cup \pi)$ is connected and, by Claim 4.8, that $(\sigma' \Cup \pi) \cup \cdots \cup (\sigma' \Cup \pi')$ is connected. Hence, for all $x \in \sigma \Cup \pi$ and for all $y \in \sigma' \Cup \pi$ we have $x \rightsquigarrow y$ and for all $x' \in \sigma' \Cup \pi$ and for all $y' \in \sigma' \Cup \pi'$ we have $x' \rightsquigarrow y'$. Moreover, since $\sigma' \Cup \pi$ is connected by Claim 4.6, we have $y \rightsquigarrow x'$. It follows that, for all $x \in \sigma \Cup \pi$ and for all $y' \in \sigma' \Cup \pi'$ we have $x \rightsquigarrow y'$. Hence, in particular, we obtain $\gamma \rightsquigarrow \lambda$. ■

Claim 4.6 *For every signed permutations σ and π, we have that $\sigma \Cup \pi$ is connected.*

Proof – Let σ, π be two signed permutations. By Remark 2.1, we have that $\sigma \Cup \pi$ can be represented as a tree where every adjacent signed permutations respectively cover each other. Then, since a tree is connected (in the graph theory meaning), as a result there exists a path between every pair of signed permutations of $\sigma \Cup \pi$. Hence, it follows that $\sigma \Cup \pi$ is connected. ■

Claim 4.7 *For every signed permutations σ, π and γ such that $\sigma \rightsquigarrow \pi$, we have that $(\sigma \Cup \gamma) \cup \cdots \cup (\pi \Cup \gamma)$ is connected.*

Proof – Let us prove that for every signed permutations σ, π and γ such that $\sigma > \pi$ we have that $(\sigma \Cup \gamma) \cup (\pi \Cup \gamma)$ is connected. We distinguish two cases.

- Let us assume that the first letter of π is positive and the first letter of σ is negative. Then, we have

$$\sigma \cdot \gamma[n] > \pi \cdot \gamma[n] \,, \tag{3}$$

  where n stand for the size of σ. Obviously, we have that $\sigma \cdot \gamma[n]$ belongs to $\sigma \Cup \gamma$ and that $\pi \cdot \gamma[n]$ belongs to $\pi \Cup \gamma$. Let now σ' and π' be signed permutations such that $\sigma' \in \sigma \Cup \gamma$ and $\pi' \in \pi \Cup \gamma$. Since $\sigma \Cup \gamma$ is connected by Claim 4.6, we have $\sigma' \rightsquigarrow \sigma \cdot \gamma[n]$; and since $\pi \Cup \gamma$ is connected again by Claim 4.6, we have $\pi' \rightsquigarrow \pi \cdot \gamma[n]$. Hence, by Equation (3), we have $\sigma' \rightsquigarrow \pi'$. It follows that for every signed permutations σ' and π' such that $\sigma' \in \sigma \Cup \gamma$ and $\pi' \in \pi \Cup \gamma$, we have $\sigma' \rightsquigarrow \pi'$. Thus, we have that $(\sigma \Cup \gamma) \cup (\pi \Cup \gamma)$ is connected.

- If we have $\sigma = u\,ba\,v$ and $\pi = u\,ab\,v$ where u, v stand for words and a, b for letters, we also have $\sigma \cdot \gamma[n] > \pi \cdot \gamma[n]$, where n stand for the size of σ. Hence, by the same arguments than for the previous case, we have that $(\sigma \Cup \gamma) \cup (\pi \Cup \gamma)$ is connected.

Thus, by induction and Remark 2.2, it follows that for every signed permutations σ, π and γ such that $\sigma \rightsquigarrow \pi$ we have that $(\sigma \Cup \gamma) \cup \cdots \cup (\pi \Cup \gamma)$ is connected. ∎

Claim 4.8 *For every signed permutations σ, π and γ such that $\sigma \rightsquigarrow \pi$, we have that $(\gamma \Cup \sigma) \cup \cdots \cup (\gamma \Cup \pi)$ is connected.*

Proof – Let us prove that for every signed permutations σ, π and γ such that $\sigma > \pi$ we have that $(\gamma \Cup \sigma) \cup (\gamma \Cup \pi)$ is connected. We distinguish two cases.

- Let us assume that the first letter of π is positive and the first letter of σ is negative. Then, let us write $\gamma \cdot \sigma[n] = \gamma \cdot \text{-}a\,v$ and $\gamma \cdot \pi[n] = \gamma \cdot a\,v$ where n is the size of γ, a a letter and v a word. Clearly, we have $\gamma \cdot \text{-}a\,v \rightsquigarrow \text{-}a \cdot \gamma \cdot v$ and $\gamma \cdot a\,v \rightsquigarrow a \cdot \gamma \cdot v$. Moreover, we obviously have $\text{-}a \cdot \gamma \cdot v > a \cdot \gamma \cdot v$.

$$\text{-}a \cdot \gamma \cdot v > a \cdot \gamma \cdot v\,, \tag{4}$$

  Then, we have that $\text{-}a \cdot \gamma \cdot v$ clearly belongs to $\gamma \Cup \sigma$ and that $a \cdot \gamma \cdot v$ belongs to $\gamma \Cup \pi$. Let now σ' and π' be signed permutations such that $\sigma' \in \gamma \Cup \sigma$ and $\pi' \in \gamma \Cup \pi$. Since $\gamma \Cup \sigma$ is connected by Claim 4.6, we have $\sigma' \rightsquigarrow \text{-}a \cdot \gamma \cdot v$; and since $\gamma \Cup \pi$ is connected again by Claim 4.6, we have $\pi' \rightsquigarrow a \cdot \gamma \cdot v$. Hence, by Equation (4), we have $\sigma' \rightsquigarrow \pi'$. It follows that for every signed permutations σ' and π' such that $\sigma' \in \gamma \Cup \sigma$ and $\pi' \in \gamma \Cup \pi$, we have $\sigma' \rightsquigarrow \pi'$. Thus, we have that $(\gamma \Cup \sigma) \cup (\gamma \Cup \pi)$ is connected.

- If we have $\sigma = u\,ba\,v$ and $\pi = u\,ab\,v$ where u, v stand for words and a, b for letters, we have $\gamma \cdot \sigma[n] > \gamma \cdot \pi[n]$, where n stand for the size of σ. Hence, by the same arguments than for the previous case, we have that $(\sigma \Cup \gamma) \cup (\pi \Cup \gamma)$ is connected.

Thus, by induction and Remark 2.2, the result follows. ∎

4.3 Technical results

Let X, Y be two multi-intervals of signed permutations; we set $Z = X \Cup Y$. We know from Lemma 4.5 that Z is connected. Thus, we can consider the minimal and maximal elements of Z (respectively S_Z and G_Z). We do not give a way to compute the minimal and maximal elements of Z from the elements X and Y, instead we state the following Proposition.

Proposition 4.9 *Let X, Y be two multi-intervals of signed permutations. Let $Z = X \Cup Y$, we denote by S_Z the minimal elements and G_Z the maximal elements of the connected set Z. Then,*

1. *for all $s_z \in S_Z$, it exists $e_x^1 \in X$ and $s_y \in S_Y$ such that $s_z \in e_x^1 \Cup s_y$,*
2. *for all $g_z \in G_Z$, it exists $e_x^2 \in X$ and $g_y \in G_Y$ such that $g_z \in e_x^2 \Cup g_y$.*

Proof – We start to prove the first assertion. Let $s_z \in S_Z$; then it exists e_x^1 and e_y such that $s_z \in e_x^1 \Cup e_y$, since $S_Z \subset Z$ and by definition of Z. Let us assume that there exists $e'_y \in Y$ such that $e'_y < e_y$. We consider two cases.

- If the first letter of $e_y = \text{-}l \cdot v$ is negative and the first letter of $e'_y = l \cdot v$ is positive, then we distinguish two cases again.
  1. If the first letter of e_y is equal to the first letter of s_z, we can write $s_z = \text{-}l \cdot w$. Then, let us consider $s'_z = l \cdot w$; clearly we have $s'_z < s_z$ and $s'_z \in e_x^1 \Cup e'_y$. Hence, we have a contradiction since s_z is a minimal element of Z.

2. If the first letter of e_y is not equal to the first letter of s_z, then we can write $s_z = v \cdot b \cdot \text{-}(l+n) \cdot w$ where v, w are possibly empty words such that v is made of letters of e_x^1, b a letter of e_x^1 and n the size of e_x^1. Clearly, $\text{-}(l+n)$ is smaller than any letter of e_x^1; hence, in particular, $b > \text{-}(l+n)$. Thus, we set $s'_z = v \cdot \text{-}(l+n) \cdot b \cdot w$. Obviously, we have $s'_z \lessdot s_z$ and $s'_z \in e_x^1 \Cup e_y$. Hence, we have a contradiction since s_z is a minimal element of Z.

- In this case, we write $e_y = \alpha \cdot a \cdot b \cdot \beta$ and $e'_y = \alpha \cdot b \cdot a \cdot \beta$ where α, β are words and a, b letters such that $a < b$. We set $a' = a[n]$ and $b' = b[n]$ where n is the size of e_x^1. We distinguish two cases.

  1. If we have $s_z = u \cdot b' \cdot a' \cdot v$, in other words if a' and b' are consecutive in s_z, then we can set $s'_z = u \cdot a' \cdot b' \cdot v$. Clearly, we have $s'_z \in Z$ since s'_z obviously belongs to $e_x^1 \Cup e'_y$. Moreover, we also have $s'z \lessdot s_z$. Hence, we have a contradiction since s_z is a minimal element of Z.
  2. Otherwise, we can write $s_z = u \cdot b' \cdot v \cdot a' \cdot w$ where u, v and w are words. Clearly, v in s_z is made only of letters of e_x^1. We consider two subcases.
     (i) If a is negative, then a' is smaller than every letters of v. Thus, we set $s'_z = u \cdot b' \cdot a' \cdot v \cdot w$; hence, we have $s'_z < s_z$. Moreover, since we obviously have $s'_z \in e_x^1 \Cup e_y$, we have $s'_z \in Z$. Hence, we have a contradiction since $s_z \in S_Z$.
     (ii) Otherwise, a is positive, then b is also positive since $b > a$. Moreover, b' is greater than every letters of v. Thus, we set $s'_z = u \cdot v \cdot b' \cdot a' \cdot w$; hence, we have $s'_z < s_z$. Moreover, since we obviously have $s'_z \in e_x^1 \Cup e_y$, we have $s'_z \in Z$. Hence, we have a contradiction since $s_z \in S_Z$.

The proof of the second assertion is similar. ∎

In order to state the following Lemma, we use the Lemma 4.5 and Proposition 4.9.

Lemma 4.10

- *Let X, Y be two intervals of signed permutations; we set $Z = X \Cup Y$.*
- *Let $s_z \in S_Z$ and $g_z \in G_Z$ such that $s_z < g_z$.*
- *Let $e_x^1 \in X$ and s_y be the minimal element of Y such that $s_z \in e_x^1 \Cup s_y$.*
- *Let $e_x^2 \in X$ and g_y be the maximal element of Y such that $s_z \in e_x^2 \Cup g_y$.*

Then, we have $e_x^1 \leqslant e_x^2$.

Proof – If $s_z < g_z$ then we have the chain $C = s_z \lessdot \cdots \lessdot g_z$. Let $e_z \in C$ and $e'_z \in C$ such that $e_z \lessdot e'_z$. Let us prove that we either have $e_z|_X = e'_z|_X$ or $e_z|_X \lessdot e'_z|_X$. We distinguish two cases.

- If the first letter of e'_z is negative and the first letter of e_z is positive, then we consider two cases again.

  1. If the first letter l of e_z is equal to the first letter of $e_z|_X$, then we easily have, by considering the range of values where l belongs, that the first letter of e'_z is equal to the first letter of $e'_z|_X$. Hence, since $e_z \lessdot e'_z$, we have $e_z|_X \lessdot e'_z|_X$.
  2. If the first letter l of e_z is not equal to the first letter of $e_z|_X$, then l is equal to a shifted letter of $e_z|_Y$. Thus, we easily have, by considering the range of values where l belongs, that the first letter of e'_z is also equal to a shifted letter of $e'_z|_Y$. Hence, we have $e_z|_X = e'_z|_X$.

- If $e_z = u \cdot a \cdot b \cdot v$ and $e'_z = u \cdot b \cdot a \cdot v$ where u, v are words and a, b letters, then we distinguish four cases.

  1. If $a \in e_z|_X$ and $b \in e_z|_Y[n]$ where n is the size of $e_z|_X$, then we have $a \in e'_z|_X$ and $b \in e'_z|_Y[n]$. Hence, we have $e_z|_X = e'_z|_X$.
  2. If $a \in e_z|_Y[n]$ and $b \in e_z|_X$, then we have $a \in e'_z|_Y[n]$ and $b \in e'_z|_X$. Hence, we have $e_z|_X = e'_z|_X$.
  3. If $a \in e_z|_Y[n]$ and $b \in e_z|_Y[n]$, then we have $a \in e'_z|_Y[n]$ and $b \in e'_z|_Y[n]$. Hence, we have $e_z|_X = e'_z|_X$.
  4. If $a \in e_z|_X$ and $b \in e_z|_X$, then we have $a \in e'_z|_X$ and $b \in e'_z|_X$. Hence, we clearly have $e_z|_X \lessdot e'_z|_X$.

Moreover, we have $e_x^1 = s_z|_X$ and $e_x^2 = g_z|_X$. Hence, by induction on C, we have $e_x^1 \leqslant e_x^2$. ∎

4.4 A smaller problem

In this Section, we prove an easier, but necessary, result than the main Theorem 4.4.

Lemma 4.11 *Let X, Y be two intervals of signed permutations. Then, $Z = X \Cup Y$ is a multi-interval.*

Proof – By Lemma 4.5, we have that Z have a set S_Z of minimal elements and a set G_Z of maximal elements. Let $s_z \in S_Z$ and $g_z \in G_Z$. Since s_z is a minimal element we cannot have $s_z > g_z$; thus we distinguish two cases.

- If s_z and g_z are not comparable, then we have $[s_z, g_z] = \emptyset$.
- If $s_z < g_z$, then let us prove that $[s_z, g_z] \subset Z$. Thus, by Proposition 4.9, there exists $e_x^1 \in X$ and $s_y \in Y$ such that $s_z \in e_x^1 \Cup s_y$; moreover, we also have $e_x^2 \in X$ and $g_y \in Y$ such that $g_z \in e_x^2 \Cup g_y$. Then, by Lemma 4.10, we have $e_x^1 \leqslant e_x^2$. Hence, since both e_x^1 and e_x^2 belong to X and since X is an interval by hypothesis, we easily have $[e_x^1, e_x^2] \subset X$.

  In order to prove that $[s_z, g_z] \subset Z$, we proceed by induction on the elements of $[s_z, g_z]$. Our inductive assumption is the following.

  For all $e_z \in [s_z, g_z]$ having n inversions, there exist $e_x \in [e_x^1, e_x^2]$ and $e_y \in [s_y, g_y]$ such that $e_z \in e_x \Cup e_y$.

  We notice, since Y is an interval by hypothesis, that $[s_y, g_y] = Y$. For the initial case, we have that $s_z \in e_x^1 \Cup s_y$; hence we have $s_z \in Z$.

  We now consider $e'_z \in [s_z, g_z]$ having $n+1$ inversions such that $e'_z > e_z$, with $e_z \in [s_z, g_z]$. By inductive assumption, we have $e_z|_X = e_x$ and $e_z|_Y = e_y$; let k be the size of e_x and m be the size of e_y. We denote by e'_x the signed permutation build by extracting the letters of e'_z which belong to the range $[-k, k]$ still preserving the order among them. Similarly, we denote by e'_y the signed permutation build by extracting the letters of e'_z which belong to the range $[-k-m, -k-1] \cup [k+1, k+m]$ still preserving the order among them. Then, since $e_z \in e_x \Cup e_y$, we clearly have

  $$e'_z \in e'_x \Cup e'_y \,. \tag{5}$$

  We distinguish two cases.

  1. If the first letter of e'_z is negative and the first letter of e_z is positive, then we consider two cases again.
     (i) If the first letter of e'_z is equal to the first letter of e'_x, then we have $e'_x > e_x$ and $e'_y = e_y$. Moreover, since $e'_z \leqslant g_z$, we have $e'_x \leqslant g_z|_X$, so $e'_x \leqslant e_x^2$. Moreover, since $e'_x > e_x$ and $e_x \geqslant e_x^1$ by inductive assumption, we have $e'_x > e_x^1$. Hence, we have $e'_x \in [e_x^1, e_x^2]$ and trivially $e'_y \in Y$. As a result, by Relation (5), we have $e'_z \in Z$.
     (ii) If the first letter of e'_z is equal to the first letter of e'_y shifted by the size of e'_x, then $e'_x = e_x$ and $e'_y > e_y$. Moreover, since $e'_z \leqslant g_z$, we have $e'_y \leqslant g_y$. Moreover, since $e'_y > e_y$ and $e_y \geqslant s_y$ by inductive assumption, we have $e'_y > s_y$. Hence, we have $e'_y \in Y$ and trivially $e'_x \in [e_x^1, e_x^2]$. As a result, by Relation (5), we have $e'_z \in Z$.
  2. If we can write $e_z = u \cdot a \cdot b \cdot v$ and $e'_z = u \cdot b \cdot a \cdot v$ where u, v are words and a, b letters, then we distinguish four cases.
     (i) If $a \in e'_x$ and $b \in e'_y$, then we have $a \in e_x$ and $b \in e_y$. Hence, we have $e'_z \in e_x \Cup e_y$, and as a result $e'_z \in Z$.
     (ii) If $a \in e'_y$ and $b \in e'_x$, then we have $a \in e_y$ and $b \in e_x$. Hence, we have $e'_z \in e_x \Cup e_y$, and as a result $e'_z \in Z$.
     (iii) If $a \in e_x$ and $b \in e_x$, then $e'_x > e_x$ and $e'_y = e_y$. Moreover, since $e'_z \leqslant g_z$, we have $e'_x \leqslant g_z|_X$, so $e'_x \leqslant e_x^2$. Moreover, since $e'_x > e_x$ and $e_x \geqslant e_x^1$ by inductive assumption, we have $e'_x > e_x^1$. Hence, we have $e'_x \in [e_x^1, e_x^2]$ and trivially $e'_y \in Y$. As a result, by Relation (5), we have $e'_z \in Z$.
     (iv) If $a \in e_y$ and $b \in e_y$, then $e'_x = e_x$ and $e'_y > e_y$. Moreover, since $e'_z \leqslant g_z$, we have $e'_y \leqslant g_y$. Moreover, since $e'_y > e_y$ and $e_y \geqslant s_y$ by inductive assumption, we have $e'_y > s_y$. Hence, we have $e'_y \in Y$ and trivially $e'_x \in [e_x^1, e_x^2]$. As a result, by Relation (5), we have $e'_z \in Z$.

■

4.5 Proof of the main Theorem

Remark 4.12 *Let σ, π, γ and λ be signed permutations.*

- *If σ and π are not comparable, then for all $x \in \sigma \Cup \gamma$ and for all $y \in \pi \Cup \lambda$, x and y are not comparable.*
- *If γ and λ are not comparable, then for all $x \in \sigma \Cup \gamma$ and for all $y \in \pi \Cup \lambda$, x and y are not comparable.*

Proof of the main Theorem – By Lemma 4.5, there is a set S_Z of minimal elements of Z and a set G_Z of maximal elements of Z; let $s_z \in S_Z$ and $g_z \in G_Z$. We distinguish two cases.

- If there exist $s_x \in S_X, g_x \in G_X$ and $s_y \in S_Y, g_y \in G_Y$, such that both s_z and g_z belong to $[s_x, g_x] \Cup [s_y, g_y]$, then $[s_z, g_z] \subset [s_x, g_x] \Cup [s_y, g_y]$, since by Lemma 4.11 $[s_x, g_x] \Cup [s_y, g_y]$ is a multi-interval. Hence, $[s_z, g_z] \subset Z$.
- Otherwise,
  - let $s_x^1 \in S_X, g_x^1 \in G_X$ and $s_y^1 \in S_Y, g_y^1 \in G_Y$ such that $s_z \in [s_x^1, g_x^1] \Cup [s_y^1, g_y^1]$,
  - let $s_x^2 \in S_X, g_x^2 \in G_X$ and $s_y^2 \in S_Y, g_y^2 \in G_Y$ such that $g_z \in [s_x^2, g_x^2] \Cup [s_y^2, g_y^2]$.

  By Proposition 4.9, it exists $e_x^1 \in [s_x^1, g_x^1]$ such that $s_z \in e_x^1 \Cup s_y^1$; it also exists $e_x^2 \in [s_x^2, g_x^2]$ such that $g_z \in e_x^2 \Cup g_y^2$. Let us assume $s_z < g_z$. Then, by Remark 4.12 (and by contraposition), we have $e_x^1 \leqslant e_x^2$ or $e_x^1 \geqslant e_x^2$, and $s_y^1 \leqslant g_y^2$ or $s_y^1 \geqslant g_y^2$. Since s_y^1 is a minimal element of Y, we cannot have $s_y^1 \geqslant g_y^2$. Hence, we have $s_y^1 \leqslant g_y^2$ and since Y is a multi-interval we have $[s_y^1, g_y^2] \subset Y$. We consider the two remaining cases.

  1. If $e_x^1 \geqslant e_x^2$, then since $g_x^1 > e_x^1$ and $e_x^2 > s_x^2$ we have $g_x^1 > s_x^2$. Then, since X is a multi-interval we have $[s_x^2, g_x^1] \subset X$. Hence, we have that s_z and g_z both belong to $[s_x^2, g_x^1] \Cup [s_y^1, g_y^2]$ but this a contradiction. As a result, we do not have $s_z < g_z$.
  2. If $e_x^1 \leqslant e_x^2$, then since $s_x^1 < e_x^1$ and $e_x^2 < g_x^2$ we have $s_x^1 < g_x^2$. Then, since X is a multi-interval we have $[s_x^1, g_x^2] \subset X$. Hence, we have that s_z and g_z both belong to $[s_x^1, g_x^2] \Cup [s_y^1, g_y^2]$ but this a contradiction. As a result, we do not have $s_z < g_z$.

  Thus, in both cases we do not have that $s_z < g_z$. Moreover, since s_z is a minimal element of Z, we cannot have $s_z > g_z$. Hence, s_z and g_z are not comparable and as a result we trivially have $[s_z, g_z] \subset Z$.

■

5 On elements of PBT$^{(2)}$

Proposition 5.1 *[1] Sylvester equivalence classes are intervals over the weak order of permutations of type A.*

Lemma 5.2 *Let S be a sylvester equivalence classe over permutations.*

1. *Let $e \in S$ such that $e = \cdots a\, c \cdots$ with $a < c$. If there is no b such that $e = \cdots a\, c \cdots b \cdots$ with $a < b < c$, then for every permutations of S we have that a is located before c.*
2. *Let $e \in S$ such that $e = \cdots c\, a \cdots$ with $a < c$. If there is no b such that $e = \cdots c\, a \cdots b \cdots$ with $a < b < c$, then for every permutations of S we have that c is located before a.*

Proof – We start to prove the first assertion. We proceed by induction; let $e = \cdots a\, c \cdots$ be a permutation such that there is no b such that $e = \cdots a\, c \cdots b \cdots$ with $a < b < c$, our inductive assumption is the following.

> *For all permutations obtained by applying indifferently n times the rewriting rules $\rightarrow_{sylv}$ or $\leftarrow_{sylv}$ from e, there is no b such that $a < b < c$ to the right of a.*

Let us notice that the induction works by Proposition 5.1, since an interval is a connected set. For the initial case, there is no b to the right of a in e by hypothesis.

Let x be a permutation satisfying the inductive assumption; then we notice that a is before c in x. We now consider y a permutation obtained from x by applying $\rightarrow_{sylv}$ or $\leftarrow_{sylv}$. We distinguish the two cases.

- If $x \rightarrow_{sylv} y$, then we consider four cases.

1. If the elementary transposition of the rewriting rule is located to the left of a, in other words if we can write $x = \cdots a' c' \cdots a \cdots c \cdots$ and $y = \cdots c' a' \cdots a \cdots c \cdots$, then obviously y satisfies the inductive assumption.
2. If the elementary transposition of the rewriting rule is located to the right of a, in other words if we can write $x = \cdots a \cdots a' c' \cdots$ and $y = \cdots a \cdots c' a' \cdots$ with a' or c' possibly equal to c, then obviously y satisfies the inductive assumption.
3. If we can write $x = \cdots a' a \cdots c \cdots$ and $y = \cdots a a' \cdots c \cdots$, then by Definition 3.6 of $\rightarrow_{sylv}$ we have $a' < a$. Hence, we have y satisfies the inductive assumption.
4. If we can write $x = \cdots a c' \cdots c \cdots$ and $y = \cdots c' a \cdots c \cdots$, then we obviously have that y satisfies the inductive assumption.

- If $y \rightarrow_{sylv} x$, then we also consider four cases.

1. If the elementary transposition of the rewriting rule is located to the left of a, in other words if we can write $x = \cdots c' a' \cdots a \cdots c \cdots$ and $y = \cdots a' c' \cdots a \cdots c \cdots$, then obviously y satisfies the inductive assumption.
2. If the elementary transposition of the rewriting rule is located to the right of a, in other words if we can write $x = \cdots a \cdots c' a' \cdots$ and $y = \cdots a \cdots a' c' \cdots$ with a' or c' possibly equal to c, then obviously y satisfies the inductive assumption.
3. If we can write $x = \cdots c' a \cdots c \cdots$ and $y = \cdots a c' \cdots c \cdots$, then by Definition 3.6 of $\rightarrow_{sylv}$ we have $c' > a$. Let us assume that $c' < c$. Then, since $y \rightarrow_{sylv} x$, it exists b' in x such that $a < b' < c'$. Hence, we have that $a < b' < c$ and that b is located to the right of a in x; but this a contradiction by inductive assumption. Hence, we have $c' > c$ and as a result y satisfies the inductive assumption.
4. If we can write $x = \cdots a a' \cdots c \cdots$ and $y = \cdots a' a \cdots c \cdots$, then we obviously have that y satisfies the inductive assumption.

The proof of the second assertion is similar. ∎

Theorem 5.3 *Let X be an equivalence classe of $\equiv^B_{sylv}$ over signed permutations. Then, X is a multi-interval.*

Proof – Let $X' = \{perm(\sigma) \mid \sigma \in X\}$; then, by Definition 3.8, X' is a sylvester equivalence classe over permutations. Then, by Proposition 5.1, X' is an interval; hence X' is connected. Let σ, π be two signed permutations of X; then by Definition 3.8, we have $perm(\sigma) \equiv_{sylv} perm(\pi)$. Then, since X' is connected, we have $perm(\sigma) \rightsquigarrow perm(\pi)$; hence, we clearly have $\sigma \rightsquigarrow \pi$. Hence, X is connected.

Since X is connected, we can consider S_X the set of minimal elements of X and G_X the set of maximal elements of X. Let $s_x \in S_X$ and $g_x \in G_X$. We distinguish two cases.

- If s_x and g_x are not comparable, then trivially $[s_x, g_x] \subset X$.
- If $s_x < g_x$, we proceed by induction in order to prove that $[s_x, g_x] \subset X$. For the initial case, we clearly have that $s_x \in X$. Then, let $e_x \in [s_x, g_x]$ such that $e_x \in X$ and consider $e'_x \in [s_x, g_x]$ such that $e'_x > e_x$. We write $e'_x = \cdots c a \cdots$ and $e_x = \cdots a c \cdots$ with $a < c$. We distinguish two cases.

1. If $|a| < |c|$, then let us assume we do not have $perm(e_x) \rightarrow_{sylv} perm(e'_x)$. Then, since $e'_x \leqslant g_x$, we write
$$g_x = \cdots c \cdots a \cdots . \tag{6}$$
Moreover, since $e_x \in X$ by inductive assumption and $g_x \in X$ by hypothesis, we have, by Definition 3.8, $perm(e_x) \equiv_{sylv} perm(g_x)$. Hence, we have $perm(e_x) \overset{*}{\longleftrightarrow}_{sylv} perm(g_x)$. Then, since $e_x = \cdots a c \cdots$ and by Equation (6), there clearly exists x and y in X' such that $perm(e_x) \overset{*}{\longleftrightarrow}_{sylv} x \rightarrow_{sylv} y \overset{*}{\longleftrightarrow}_{sylv} perm(g_x)$ with $x = \cdots |a||c| \cdots$ and $y = \cdots |c||a| \cdots$. Then, since we do not have $perm(e_x) \rightarrow_{sylv} perm(e'_x)$, there is no b satisfying Definition 3.6 in $perm(e_x)$. Hence, by Lemma 5.2, we have in particular that y which belong to X' has its letter $|a|$ located before $|c|$; but this is a contradiction. Hence, we have $perm(e_x) \rightarrow_{sylv} perm(e'_x)$ and then $e_x \equiv^B_{sylv} e'_x$ by Definition 3.8; in other words we have $e'_x \in X$.
2. If $|a| > |c|$, then we assume we do not have $perm(e_x) \leftarrow_{sylv} perm(e'_x)$. Thus, we have a similar proof than the previous case, involving the second assertion of Lemma 5.2.

∎

6 A Tamari lattice of type B

Definition 6.1 *For all equivalence classes X, Y of $\equiv^B_{sylv}$, we set the relation $X \leqslant_B Y$ if there exists a signed permutation $x \in X$ and a signed permutation $y \in Y$ such that $x \leqslant y$.*

Based on computations, we state the following conjecture.

Conjecture 6.2 *The set of all equivalence classes of $\equiv^B_{sylv}$ together with the relation $\leqslant_B$ is a lattice.*

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Appendix A An algorithm to efficiently compute intervals in a graded poset

The Algorithm provided in this Section can be applied to any graded poset, in particular to the weak order of type B. We recall the definition of a graded poset.

Definition A.1 *A graded poset is a poset P equipped with a rank function ρ from P to $\mathbb{N}$ satisfying the following two properties.*

- *The rank function is compatible with the ordering, meaning that for every x and y such that $x < y$, then we have $\rho(x) < \rho(y)$.*
- *The rank is consistent with the covering relation of the ordering, meaning that for every x and y such that y covers x, then we have $\rho(y) = \rho(x) + 1$.*

The Algorithm avoids the use of the traditional computation of an interval $[\alpha, \omega]$ by application of the following Formula :

$$[\alpha, \omega] = sup(\alpha) \cap inf(\omega) , \tag{7}$$

where $sup(\alpha)$ computes all the elements greater than α and $inf(\omega)$ all the elements smaller than ω. Formula (7) have a high computational complexity compare to our Algorithm, especially for graded poset of huge size.

Algorithm A.2

Input

min_elt : the minimal element of the interval to be computed
max_elt : the maximal element of the interval to be computed

Output

$result$: the set of the elements of the interval $[min\_elt, max\_elt]$

```
Interval(min_elt, max_elt):

    rank_min := ρ(min_elt)
    rank_max := ρ(max_elt)

    flag := True
    elts_up := { max_elt }
    elts_down := { min_elt }

    while rank_min < rank_max:
        if flag:
            elts_up := {x | ∃y ∈ elts_up, x < y}
            rank_min := rank_min +1
        else:
            elts_down := {x | ∃y ∈ elts_down, x > y}
            rank_max := rank_max −1

        flag := not flag

    inter := elts_up ∩ elts_down

    for elt ∈ inter:
        Interval(min_elt, elt)
        Interval(elt, max_elt)

    result := result ∪ inter
```

Basically, Algorithm A.2 works similarly than a dichotomic algorithm: it determines the elements which are at an intermediate rank between the two minimal and maximal elements. Once this is done, it applies recursively between several new and smaller intervals until the whole interval is explored.