A self-dual Hopf algebra on set partitions

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Abstract

The main contribution of this article is to provide a combinatorial Hopf algebra on set partitions, which can be seen as a Hopf subalgebra of the free quasi-symmetric functions, and which is free, cofree and self-dual. This algebraic construction comes naturally from a combinatorial algorithm, an analogous of the Robinson-Schensted correspondence on set partitions. We also introduce a new partial order on set partitions. We explain how these objects are related to each others through our Hopf algebra.

1 Introduction

In the past years several authors mixed Hopf algebras, partial orders, and combinatorial algorithms. Such a construction has been made on Young tableaux by Poirier and Reutenauer [11] for the Hopf algebra part, then Reiner and Taskin [12] for the connection with partial orders and with the Robinson-Schensted correspondence as combinatorial algorithm. A similar construction has also been made on binary trees by Hivert, Novelli and Thibon [6] with the Hopf algebra of Loday and Ronco [8], involving the Tamari order and the well known binary search tree insertion as combinatorial algorithm. Moreover, Hopf algebras and partial orders have also been mixed for set compositions, plane trees and segmented compositions in [10].

The main contribution of this article is to provide a new combinatorial Hopf algebra on set partitions. Indeed, there already exists a combinatorial Hopf algebra on set partitions defined by Rosas-Sagan [13] and Bergeron-Zabrocki [1], which is cocommutative. Our algebra is clearly different since it is noncommutative, non cocommutative and self-dual. This algebraic construction comes naturally from a simple combinatorial algorithm, an analogous of the Robinson-Schensted correspondence on set partitions, studied by Burstein and Lankham in [2] and described in Section 3. This insertionnal algorithm allows us to define equivalence classes on permutations; we study their structure in Section 4. In Section 5, we provide a new partial order on set partitions. All these tools allow us to build a Hopf algebra on set partitions which can be seen as a Hopf subalgebra of the Free Quasi-Symmetric functions, as explained in Section 6. Then, we explain how the product of this algebra is described by specific intervals of our partial order on set partitions. At last, we prove that this Hopf algebra is free, cofree and self-dual, thanks to the bidendriform bialgebra structure defined by Foissy in [4].

This paper hence reinforce the belief of a more general theory involving such mathematical objects.

2 Preliminaries and notations

Let $\mathfrak{S}$ be the set of all permutations; $\mathfrak{S}_n$ stands for the permutations of size n. In the sequel, we use on-line notation for permutations, writing $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, where $\sigma_i = \sigma(i)$. Thus, we consider a permutation of size n as a word without repetition of letter over the alphabet $[n]$. An *inversion* of a permutation is a pair (i, j) such that $i < j$ and $\sigma_i > \sigma_j$; the set of all the inversions of a permutation σ is denoted by $inv(\sigma)$. The *length* of a permutation is its number of inversions.

We say that $\delta = \{\delta^1, \delta^2, \cdots, \delta^k\}$, where δ^i stands for a subset of $[n]$, is a *set partition* of n, also written $\delta \vdash [n]$, if $\delta^i \cap \delta^j = \varnothing$ for $i \neq j$ and $\delta^1 \cup \delta^2 \cup \cdots \cup \delta^k = [n]$. We denote by SP the set of set partitions. An *integer composition*, or *composition* for short, $c = (c_1, c_2, \cdots, c_k)$ of n is a tuple of positive integers such that $c_1 + c_2 + \cdots + c_k = n$. The *shape* of a set partition $\delta = \{\delta^1, \delta^2, \cdots, \delta^k\}$ is the composition $shape(\delta) := (|\delta^1|, |\delta^2|, \cdots, |\delta^k|)$.

A *rewriting system* is a set S, together with a relation $\rhd$ which is a subset of $S \times S$. We call the relation $\rhd$ a *rewriting rule*. Along this paper, we consider several rewriting systems where set S is always $\mathfrak{S}$. A permutation π is said to be reachable by σ if $\sigma \overset{*}{\rhd} \pi$, where $\overset{*}{\rhd}$ stands for the reflexive-transitive closure of $\rhd$. A rewriting rule is said to be *terminating* if there exists no infinite sequence of permutations $\sigma \rhd \sigma^{(1)} \rhd \cdots$. A rewriting rule $\rhd$ is *locally confluent* if for every permutations σ, π and γ such that $\sigma \rhd \pi$ and $\sigma \rhd \gamma$, there exists a permutation ω

such that $\pi \overset{*}{\rhd} \omega$ and $\gamma \overset{*}{\rhd} \omega$. A rewriting rule $\rhd$ is *confluent* if for every permutations σ, π and γ such that $\sigma \overset{*}{\rhd} \pi$ and $\sigma \overset{*}{\rhd} \gamma$, there exists a permutation ω such that $\pi \overset{*}{\rhd} \omega$ and $\gamma \overset{*}{\rhd} \omega$. We recall from [7] the following result.

Proposition 2.1 *Assume that $\rhd$ is terminating. Then, $\rhd$ is confluent if and only if it is locally confluent.*

We rely on [5] for results and notations about posets. A *poset* $\langle S; \varrho\rangle$ consists of a nonempty set S and a binary relation ϱ on S such that, for all $a, b, c \in S$, relation ϱ satisfies the following properties:

- $a\,\varrho\,a$ (reflexivity),
- $a\,\varrho\,b$ and $b\,\varrho\,a$ imply that $a = b$ (antisymmetry),
- $a\,\varrho\,b$ and $b\,\varrho\,c$ imply that $a\,\varrho\,c$ (transitivity).

Let $\langle P; \leqslant\rangle$ be a poset and a and b be two elements of P. Then, b *covers* a or a *is covered by* b if $b \geqslant a$ such that there is no x such that $b \geqslant x \geqslant a$. A subset K of P is *convex* if, for all a and c in K and $b \in P$ such that $a \leqslant b \leqslant c$, we have $b \in K$. An *interval* $[a, c]$ of $\langle P; \leqslant\rangle$ is the convex set $\{b \,|\, a \leqslant b \leqslant c\}$. In this paper we consider the weak order $\langle \mathfrak{S}; \leqslant\rangle$ over permutations; we have $\sigma \leqslant \pi$ if $inv(\sigma) \subseteq inv(\pi)$. If π covers σ, we denote it by $\sigma \lessdot \pi$. We also consider the lexicographical order over compositions, also denoted by $\leqslant$.

Let $A = \{a < b < \cdots\}$ be a totally ordered alphabet. Then, for each word w of A^* of length n, we associate a permutation $Std(w) \in \mathfrak{S}_n$ called the *standardized* of w defined as the permutation obtained by iteratively scanning w from left to right, and labelling 1,2,... the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. For example, $Std(abcadbcaa) = 157296834$. Let w be a word of A^* and I a subset of A, then $w \backslash I$ stands for the word w where letters belonging to I have been deleted. Let w be a word of A^* and k be a positive integer, then $w[k]$ stands for the word w where each letter have been increased by k. For example, $231[2] = 453$. We define over $\mathbb{K}\langle A\rangle$ the *shuffle product* as follows. Let w_1 and w_2 be two words, then $w_1 \sqcup\!\sqcup\, w_2$ is recursively define by

- $w_1 \sqcup\!\sqcup\, \epsilon = w_1$ and $\epsilon \sqcup\!\sqcup\, w_2 = w_2$,
- $au \sqcup\!\sqcup\, bv = a(u \sqcup\!\sqcup\, bv) + b(au \sqcup\!\sqcup\, v)$,

where a and b are letters, and u, v are words. We define the *shifted shuffle* between two words w_1 of size n and w_2 of size m by $w_1 \Cup w_2 := w_1 \sqcup\!\sqcup\, w_2[n]$. For example,

$$12 \Cup 21 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312.$$

Remark 2.2 *It is easy to obtain that every permutation appear only once in the shifted shuffle of two permutations.*

Remark 2.3 *Let σ, σ', π and π' be permutations such that $\sigma \neq \sigma'$ and $\pi \neq \pi'$. Then, it is clear that $\sigma \Cup \pi$, $\sigma \Cup \pi'$, $\sigma' \Cup \pi$ and $\sigma' \Cup \pi'$ do not share any permutation in common.*

3 Patience sorting

We first define a simple insertionnal algorithm which from a permutation gives a set partition. We then define the equivalence relation this algorithm naturally induces and provide its canonical elements. On the way, we provide basic results of this algorithm. All the content of this section comes from the article of Burstein and Lankham [2] on combinatorics of the patience sorting piles.

Algorithm 3.1 *Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, we inductively build the set of piles $R(\sigma) = \{r_1, r_2, \cdots, r_m\}$ as follows:*

- *Put the first letter σ_1 into a pile r_1 by itself.*
- *For each remaining letter σ_i ($i = 2, \cdots, n$), consider the letters $d_1, d_2, \cdots, d_k$ atop the piles $r_1, r_2, \cdots, r_k$ that have already been formed.*
  - *If $\sigma_i > max\{d_1, d_2, \cdots, d_k\}$, then put σ_i into a new right-most pile r_{k+1} by itself.*
  - *Otherwise, find the left-most letter σ_j that is larger than σ_i and put the letter σ_i atop pile r_j.*

We denote by $R^{(i)}$ the pile configuration made at the step i of the Algorithm 3.1. Then, for $\sigma = 24867153$ we have:

$$R^{(1)} = \begin{array}{c} 2 \end{array} \qquad R^{(2)} = \begin{array}{cc} 2 & 4 \end{array} \qquad R^{(3)} = \begin{array}{ccc} 2 & 4 & 8 \end{array} \qquad R^{(4)} = \begin{array}{ccc} & & 6 \\ 2 & 4 & 8 \end{array}$$

$$R^{(5)} = \begin{array}{cccc} & & 6 & \\ 2 & 4 & 8 & 7 \end{array} \qquad R^{(6)} = \begin{array}{cccc} 1 & & 6 & \\ 2 & 4 & 8 & 7 \end{array} \qquad R^{(7)} = \begin{array}{cccc} & & 5 & \\ 1 & & 6 & \\ 2 & 4 & 8 & 7 \end{array} \qquad R^{(8)} = \begin{array}{cccc} & & 5 & \\ 1 & 3 & 6 & \\ 2 & 4 & 8 & 7. \end{array}$$

Remark 3.2 *Note that the $d_1, d_2, \cdots, d_k$ elements, which are on the top of their pile, are in increasing order in their pile configuration. In the previous example, we have $1 < 3 < 5 < 7$.*

In order to state that Algorithm 3.1 induces an equivalence relation over permutations, we need the following result.

Proposition 3.3 *Let σ be a permutation. Then, $R(\sigma)$ is a set partition of $\{1, 2, \cdots, n\}$. Moreover, for every set partition S, there is a permutation π such that $R(\pi) = S$.*

To see that outputs of Algorithm 3.1 are set partitions is straightforward when we consider that a set partition can be represented in a canonical way by ordering sets according to their minimal elements and ordering the sets themselves according to the natural order on integers. For example, set partition $\{\{1,5\},\{9,3,7\},\{6,8,2,4\}\}$ can be ordered as follows: $\{\{5 > 1\},\{8 > 6 > 4 > 2\},\{9 > 7 > 3\}\}$. Thus, in the sequel, we will transparentely use pile configuration as set partitions.

Definition 3.4 *Two permutations σ and π, are said to be patience sorting equivalent, written $\sigma \overset{PS}{\sim} \pi$, if they have the same pile configuration $R(\sigma) = R(\pi)$ under Algorithm 3.1.*

It is straightforward to check that $\overset{PS}{\sim}$ is an equivalence relation. We now defines the canonical elements of this equivalence relation.

Definition 3.5 *Let σ be a permutation. The reverse patience word $RPW(R(\sigma))$ for a pile configuration $R(\sigma)$, is the permutation formed by concatening the piles $r_1, r_2, \cdots, r_k$ together, with each pile r_j written in decreasing order.*

For example, we have $RPW(R(24867153)) = 21438657$.

Remark 3.6 *Let σ be a permutation. Then, $R(RPW(R(\sigma))) = R(\sigma)$.*

At last, we use a somewhere alternative definition of Algorithm 3.1 which will be useful in the sequel of this paper.

Definition 3.7 *Let $\pi = \pi_1\pi_2\cdots\pi_l$ be a partial permutation on $\{1, 2, \cdots, n\}$. Then, the left-to-right minima subsequence of π consists of those components π_j of π such that $\pi_j = min\{\pi_i \,|\, 1 \leqslant i \leqslant j\}$.*

We then inductively define the left-to-right minima subsequences $s_1, s_2, ..., s_k$ of a permutation σ by taking s_1 to be the left-to-right minima subsequence for σ itself and then each subsequent subsequence s_i to be the left-to-right minima subsequence for the partial permutation obtained by removing the elements of $s_1, s_2, ..., s_{i-1}$ from σ. The subsequence s_j is called the j^{th} *left-to-right minima subsequence* of σ. For example, the left-to-right minima subsequences of 24867153 are 21, 43, 865 and 7. Note that this is precisely the elements of the piles of $R(24867153)$; this holds on general.

Proposition 3.8 *Let σ be a permutation. Then, the k^{th} pile of $R(\sigma)$ contains exactly the elements of the k^{th} left-to-right minima subsequence of σ.*

4 Equivalence classes

In this section we study the structure of the equivalence classes induced by the insertionnal Patience Sorting algorithm. First, we introduce and study two rewriting rules on permutations that both describe these equivalence classes. These two rewriting rules allow us to prove, at the end of this section, that the equivalence classes are intervals, according to the weak order on permutations.

4.1 A first rewriting rule

Definition 4.1 *Let $\rightarrow$ be the rewriting rule on permutations such that:*

$$\sigma := \cdots bwac \cdots \rightarrow \cdots bwca \cdots , \tag{1}$$

with $a < b < c$ and where every letter of the word w, possibly empty, is greater than b.

For example, we have 25314 $\rightarrow$ 25341 (with $a = 1$, $b = 2$, $c = 4$ and $w = 53$) and 21435 $\rightarrow$ 24135 (with $a = 1$, $b = 2$, $c = 4$ and $w = \epsilon$).

Remark 4.2 *We easily notice that if $\sigma \rightarrow \pi$, then $l(\sigma) < l(\pi)$. This implies that the rewriting rule $\rightarrow$ is terminating.*

Remark 4.3 *We could have equivalentely defined the rewriting rule of Definition 4.1 with the condition that every letter of w is greater than c, instead of b, thus defining the rewriting rule $\rightarrow_{bis}$. Indeed, for two permutations σ and π, if we have $\sigma \rightarrow_{bis} \pi$, then we immediately obtain that $\sigma \rightarrow \pi$ since $c > b$. Conversely, if we have $\sigma \rightarrow \pi$ then we have the following two cases.*

1. *Every letter of w is greater than c, then it immediately yields by definition that $\sigma \rightarrow_{bis} \pi$.*
2. *It exists at least one letter b' in w such that $c > b'$. By choosing the right-most letter b' in w, it follows that all letters of w remaining between b' and a are greater than c. Hence, we have by definition that $\sigma \rightarrow_{bis} \pi$.*

We naturally denote by $\overset{*}{\longleftrightarrow}$ the reflexive-symmetric-transitive closure of $\rightarrow$.

Lemma 4.4 *Let σ be a permutation. Then, $RPW(R(\sigma)) \overset{*}{\rightarrow} \sigma$.*

Proof – Let σ be a permutation such that there is no permutation π such that $\pi \leftarrow \sigma$. By Remark 4.2, such a permutation exists. For the sequel of the proof, we set $\alpha := RPW(R(\sigma))$; we recall that $\sigma \overset{PS}{\sim} \alpha$ since $R(\alpha) = R(\sigma)$ by Remark 3.6. We are going to prove by induction that $\alpha = \sigma$.

The initial case of our induction, that is $\sigma_1 = \alpha_1$, is immediate by Algorithm 3.1. We now assume that $\sigma_1 \cdots \sigma_n = \alpha_1 \cdots \alpha_n$.

- First, we consider the case where $\sigma_{n+1} < \sigma_n$. Then, assuming that σ_n belongs to the pile r_k of $R(\sigma)$, we have that σ_{n+1} belongs to the r_l pile $R(\sigma)$ with $l \leqslant k$, by Algorithm 3.1. If $l < k$, by Definition 3.5 and by inductive assumption, σ_{n+1} would appear before σ_n in σ which is obviously a contradiction. Hence, we have obtained that $k = l$, that is σ_n and σ_{n+1} belongs to the same pile of $R(\sigma)$; moreover note that σ_n and σ_{n+1} are consecutive letters in their pile. Thus, by Definition 3.5, we have $\sigma_{n+1} = \alpha_{n+1}$.
- Otherwise, we consider the case $\sigma_{n+1} > \sigma_n$. By Definition 3.5 and by inductive assumption, we easily have that σ_n is on the top of the right-most pile r_k of $R^{(n)}(\sigma)$. Then, by Algorithm 3.1, σ_{n+1} is the first element of the pile r_{k+1} in $R^{(n+1)}(\sigma)$ and hence in $R(\sigma)$. We now distinguish two cases.
  - Consider that $\alpha_{n+1} > \alpha_n$. By Definition 3.5, we also have that α_{n+1} is the first element of the pile r_{k+1} in $R(\alpha)$. Thus, since $\sigma \overset{PS}{\sim} \alpha$, we obtain immediately that $\alpha_{n+1} = \sigma_{n+1}$.
  - Otherwise, consider that $\alpha_{n+1} < \alpha_n$. Then, we can straightforwardely prove by contradiction from inductive assumption that there exists $p > n+1$ such that $\sigma_p = \alpha_{n+1}$. Moreover, by Definition 3.5, we have that α_n and α_{n+1} are consecutive letters of the same pile and hence, by Proposition 3.8, of the same left-to-right minima subsequence s_k. Let us now consider the letters σ_i for $i \in [n+1, k-1]$.
    - If $\sigma_i \in s_k$, then σ_n and σ_p are not consecutive letters in s_k but σ_n and σ_i are, which is a contradiction since $\sigma \overset{PS}{\sim} \alpha$.
    - If $\sigma_i \in s_j$ with $j < k$, then σ_i should appear in $\sigma_1 \cdots \sigma_n$, by inductive assumption, which is obviously a contradiction.

  Hence, we have that for all $i \in [n+1, k-1]$, σ_i belongs to s_j with $j > k$. Then, it follows that we must have $\sigma_i > \sigma_n$ for all $i \in [n+1, k-1]$. In order to summarize, we have $\cdots \sigma_n w \sigma_{p-1} \sigma_p \cdots$ in σ with $\sigma_{p+1} > \sigma_n > \sigma_p$ and w a possibly empty word such that every letter is greater than σ_n. Hence, by Definition 4.1, there is a permutation π such that $\pi \leftarrow \sigma$. But this is a contradiction.

Hence, we obtain $\sigma = \alpha$. Since $\rightarrow$ is terminating, by Remark 4.2, it yields that for every permutation γ, we have $RPW(R(\gamma)) \overset{*}{\leftarrow} \gamma$.

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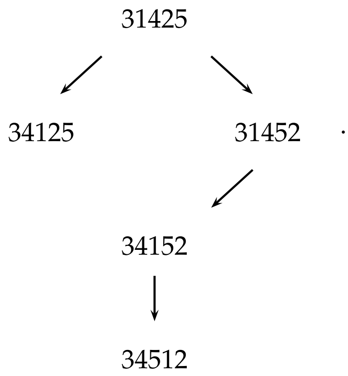
For example, the equivalence class of the canonical permutation 31425 can be drawn as:



(2)

Theorem 4.5 *Let σ, π be two permutations. Then, $\sigma \overset{PS}{\sim} \pi$ if and only if $\sigma \overset{*}{\longleftrightarrow} \pi$.*

Proof – First, let us prove that $\sigma \rightarrow \pi$ implies $\sigma \overset{PS}{\sim} \pi$. Assuming $\sigma \rightarrow \pi$, we can write σ and π as follows:

$$\sigma = \cdots bwac \cdots , \quad (3)$$

$$\pi = \cdots bwca \cdots , \quad (4)$$

according to Definition 4.1. Let i (resp. j) be the index of the letter b (resp. a) in σ. We straightforwardely have that

$$R^{(j-1)}(\sigma) = R^{(j-1)}(\pi) \,. \quad (5)$$

We consider now $R^i(\sigma)$, where b is inserted at the top of the pile r_k. Then, all letters of word w are inserted into piles located to the right of pile r_k, directely from Algorithm 3.1 since letters of w are greater than b by Definition 4.1. Hence, b is on the top of pile r_k in $R^{(j-1)}(\sigma)$, and so in $R^{(j-1)}(\pi)$ by Equation (5). Then, it follows by Algorithm 3.1 that a is inserted on the top of a pile r_l in $R^{(j)}(\sigma)$ with $l \leqslant k$, since $a < b$. Moreover, we have that c is inserted on the top of a pile r_m in $R^{(j+1)}(\sigma)$ with $m > k$, since $c > b$, or since $c > a$ if a is put on the top of the pile r_k in $R^{(j)}(\sigma)$. Consider now $R^{(j)}(\pi)$. By Equation (5) and since a in $R^{(j)}(\sigma)$ belongs to r_l with $l \leqslant k$, c in $R^{(j)}(\pi)$ is also inserted on the top of the pile r_m, on the right of the pile r_k. Then, considering now $R^{(j+1)}(\pi)$, by Equation (5) and since c in $R^{(j+1)}(\sigma)$ belongs to r_m with $m > k$, a in $R^{(j+1)}(\pi)$ is also inserted on the top of the pile r_l, on the left of the pile r_k. Hence, we have $R^{(j+1)}(\sigma) = R^{(j+1)}(\pi)$ and $R(\sigma) = R(\pi)$ is then straightforward. At last, since $\overset{PS}{\sim}$ is an equivalence relation, $\sigma \overset{*}{\longleftrightarrow} \pi$ implies $\sigma \overset{PS}{\sim} \pi$.

Conversely, assume that $\sigma \overset{PS}{\sim} \pi$. By Lemma 4.4, we have $RPW(R(\sigma)) \overset{*}{\rightarrow} \sigma$ and $RPW(R(\pi)) \overset{*}{\rightarrow} \pi$. Then, we obtain $\sigma \overset{*}{\leftarrow} RPW(R(\sigma)) \overset{*}{\rightarrow} \pi$; which means for short that we have $\sigma \overset{*}{\longleftrightarrow} \pi$.

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4.2 A second rewriting rule

Definition 4.6 *Let $\multimap$ be the rewriting rule on permutations such that:*

$$\sigma := \cdots b \cdots ac \cdots \multimap \cdots b \cdots ca \cdots , \quad (6)$$

with $a < b < c$ and where a and b both belong to the same left-to-right minima subsequence in σ.

For example, we have $34125 \multimap 34152$ (with $b = 4$, $a = 2$ and $c = 5$) since 4 and 2 belong to the same left-to-right minima subsequence 42. This rewriting rule also allows to describe equivalence classes of relation $\overset{PS}{\sim}$. We naturally denote by $\overset{*}{\circ\!\!-\!\!\circ}$ the reflexive-symmetric-transitive closure of $\multimap$.

Theorem 4.7 *Let σ, π be two permutations. Then, $\sigma \overset{PS}{\sim} \pi$ if and only if $\sigma \overset{*}{\multimap\!\!\circ} \pi$.*

Proof – First, let us prove that if $\sigma \rightarrow \pi$, then $\sigma \multimap \pi$. Assuming $\sigma \rightarrow \pi$, according to Definition 4.1, we can write σ and π as follows:

$$\sigma = \cdots bwac \cdots, \tag{7}$$
$$\pi = \cdots bwca \cdots. \tag{8}$$

By Algorithm 3.1, all letters of w in σ are inserted on the top of piles located to the right of the pile r_k of b in $R(\sigma)$. Then, also by Algorithm 3.1 and since $a < b$, a in σ is inserted in $R(\sigma)$ on the top of r_k or on the top of a pile located to the left of r_k. Hence, by Proposition 3.8, conditions of Definition 4.6 are satisfied, which means that we have $\sigma \multimap \pi$. Hence, by Theorem 4.5, if $\sigma \overset{PS}{\sim} \pi$ then we have $\sigma \overset{*}{\multimap\!\!\circ} \pi$.

Conversely, let us first prove now that if we have $\sigma \multimap \pi$, then we have $R(\sigma) = R(\pi)$. By Definition 4.6, we can write σ and π as follows:

$$\sigma = \cdots b \cdots ac \cdots, \tag{9}$$
$$\pi = \cdots b \cdots ca \cdots. \tag{10}$$

Let i be the index of a in σ (and then of c in π), and r_k the pile of b in $R(\sigma)$. First, we obviously have that:

$$R^{(i-1)}(\sigma) = R^{(i-1)}(\pi). \tag{11}$$

We consider now $R(\sigma)$. Since b and a in σ belong to the same left-to-right minima subsequence according to Definition 4.6 and by Proposition 3.8, a is inserted in r_k. Moreover, by Algorithm 3.1 and since $c > a$, c in $R(\sigma)$ is inserted of the top of pile r_j to the right of r_k. We consider now $R(\pi)$. Then, by Algorithm 3.1, c is inserted into the pile $r_{j'}$ on the right of pile r_k. Thus, since we have Equation (11) and a is in r_k in $R(\sigma)$, we obtain $j = j'$. Moreover, since $j > k$ and by Proposition 3.8, we have that a and b belong to the same left-to-right minima subsequence in π. Hence, we have $R^{(i+1)}(\sigma) = R^{(i+1)}(\pi)$ and then immediately we have $R(\sigma) = R(\pi)$. At last, since $\overset{PS}{\sim}$ is an equivalence relation, $\sigma \overset{*}{\multimap\!\!\circ} \pi$ implies $\sigma \overset{PS}{\sim} \pi$.

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Remark 4.8 *We easily notice that if $\sigma \multimap \pi$, then $l(\sigma) < l(\pi)$. This implies that the rewriting rule $\multimap$ is terminating.*

Note that on example (2), for all permutations σ, π such that $\sigma \rightarrow \pi$ we have $\sigma \multimap \pi$. Moreover, 34152 is not reachable by 34125 with $\rightarrow$, whereas we have 34125 $\multimap$ 34152. This gives us the intuition of the following result.

Proposition 4.9 *Rewriting rule $\multimap$ is confluent.*

Proof – Since $\multimap$ is terminating, according to Remark 4.8, we need only to prove that $\multimap$ is locally confluent by Proposition 2.1. Let σ, π and γ be permutations such that $\sigma \multimap \pi$ and $\sigma \multimap \gamma$. Let us prove there exists a permutation ω such that $\pi \overset{*}{\multimap} \omega$ and $\gamma \overset{*}{\multimap} \omega$. We consider two cases:

- If $\sigma = \cdots ac \cdots a'c' \cdots$ with $\pi = \cdots ca \cdots a'c' \cdots$ and $\gamma = \cdots ac \cdots c'a' \cdots$, then let us set $\omega := \cdots ca \cdots c'a' \cdots$. Since $\sigma \multimap \gamma$, by Definition 4.6, there exists b' in σ such that $b' > a'$, b' is in the same left-to-right minima subsequence of a' and such that b' is located to the left of a' in σ. Moreover, since $\sigma \multimap \pi$ and by Theorem 4.7, we have $\sigma \overset{PS}{\sim} \pi$. Thus, by Proposition 3.8, b' and a' belong to the same left-to-right minima subsequence in π. Hence, by Definition 4.6, we have $\pi \multimap \omega$. Similarly, we obtain $\gamma \multimap \omega$.
- If $\sigma = \cdots ace \cdots$ with $\pi = \cdots cae \cdots$ and $\gamma = \cdots aec \cdots$, then let us set $\omega = \cdots eca \cdots$. Since $\sigma \multimap \pi$ (resp. $\sigma \multimap \gamma$), by Definition 4.6, there exists b (resp. d) in σ such that $b > a$ (resp. $d > c$), b (resp. d) is in the same left-to-right minima subsequence of a (resp. c) and such that b (resp. d) is located to the left of a (resp. c) in σ. Then, since we have $\sigma \multimap \pi$, by Theorem 4.7, we have $\sigma \overset{PS}{\sim} \pi$. Thus, by Proposition 3.8 ,b and a belong to the same left-to-right minima subsequence in π. Hence, we have $\pi \multimap \pi'$ with $\pi' = \cdots cea \cdots$. Then, since $\sigma \overset{*}{\multimap} \pi'$, we have $\sigma \overset{PS}{\sim} \pi'$. Thus, by Proposition 3.8, d and c belong to the same left-to-right minima subsequence in π'. Hence, we have $\pi' \multimap \omega$ and thus we have obtained $\pi \overset{*}{\multimap} \omega$. Similarly, we have $\gamma \overset{*}{\multimap} \omega$.

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4.3 Structure of the equivalence classes

We provide the basic results to prove that equivalence classes are intervals, according to the weak order on permutations.

Proposition 4.10 *Let σ be a permutation and $\widetilde{\sigma}$ its equivalence class by relation $\overset{PS}{\sim}$. Then,*

1. *$RPW(R(\sigma))$ is the unique permutation of minimal length in $\widetilde{\sigma}$,*
2. *there exists a unique permutation of maximal length in $\widetilde{\sigma}$.*

Proof – We assume that π and γ are distinct permutations of minimal length in $\widetilde{\sigma}$. Then, by Lemma 4.4, we have $RPW(R(\pi)) \overset{*}{\rightarrow} \pi$ and $RPW(R(\gamma)) \overset{*}{\rightarrow} \gamma$. Since $\pi \overset{PS}{\sim} \gamma$, we have

$$RPW(R(\pi)) \overset{*}{\rightarrow} \gamma. \tag{12}$$

Moreover, by Remark 4.2, it follows that $l(RPW(R(\pi))) \leqslant l(\gamma)$. Then, since γ is of minimal length we have $l(RPW(R(\pi))) = l(\gamma)$, which implies $RPW(R(\pi)) = \gamma$, by Formula (12). Since π is a permutation of minimal length $l(\pi) = l(\gamma)$ but since $\gamma \overset{*}{\rightarrow} \pi$, this implies $\gamma = \pi$ which is a contradiction. Hence, $RPW(R(\sigma))$ is the unique permutation of minimal length of its equivalence class $\widetilde{\sigma}$.

Let ω and ω' be distinct permutations of maximal length of $\widetilde{\sigma}$. By Lemma 4.4, we have $RPW(R(\omega)) \overset{*}{\rightarrow} \omega$ and $RPW(R(\omega')) \overset{*}{\rightarrow} \omega'$. Then, as stated in the proof of Theorem 4.7, if $\sigma \rightarrow \pi$ then $\sigma \multimap \pi$; thus, we have $RPW(R(\omega)) \overset{*}{\multimap} \omega$ and $RPW(R(\omega')) \overset{*}{\multimap} \omega'$. Then, since $\omega \overset{PS}{\sim} \omega'$, we have $\omega' \overset{*}{\multimapinv} RPW(R(\omega)) \overset{*}{\multimap} \omega$. Thus, since $\multimap$ is confluent by Proposition 4.9, there exists a permutation ω'' such that $\omega \overset{*}{\multimap} \omega''$ and $\omega' \overset{*}{\multimap} \omega''$. But since $\omega \nparallel \omega'$ and by Remark 4.8, there is a contradiction. Hence, there is a unique permutation of maximal length in $\widetilde{\sigma}$.

■

For example, permutations 3164275 and 3674512 are respectively the minimal and maximal element of their equivalence class.

Lemma 4.11 *Let σ and π be permutations such that $\sigma \lessdot \pi$ and $\sigma \overset{PS}{\nsim} \pi$. Then, $shape(R(\sigma)) < shape(R(\pi))$.*

Proof – We clearly can write σ and π as follows: $\sigma = \cdots ab \cdots$ and $\pi = \cdots ba \cdots$, with $a < b$ and i the position of a in σ (and then of b in π). Then, it follows that we have

$$R^{(i-1)}(\sigma) = R^{(i-1)}(\pi). \tag{13}$$

We consider two cases.

- Let a be the first letter of its pile r_k in $R^{(i)}(\sigma)$. This means, by Algorithm 3.1, that a is in the right-most pile r_k in $R^{(i)}(\sigma)$ and is the unique element of r_k in $R^{(i)}(\sigma)$. Consider now $R^{(i)}(\pi)$. Then, since $b > a$ and by Equation (13), b is inserted in the right-most pile r_k in $R^{(i)}(\pi)$ and is the unique element of r_k in $R^{(i)}(\pi)$. Then, by Equation (13) and since a is the first letter of its pile in $R^{(i)}(\sigma)$, we have that a in $R^{(i+1)}(\pi)$ is inserted on the top of b in r_k.
- There exists b' in σ such that b' is located to the left of a in σ and such that b' and a are consecutive letters in the pile r_k of $R(\sigma)$. First, we assume that $b' < b$. Then, immediately by Definition 4.6 and Proposition 3.8, we have that $\sigma \multimap \pi$. Thus, by Theorem 4.7, we have $\sigma \overset{PS}{\sim} \pi$, but this is a contradiction. Hence, we have that $b' > b$. Hence, by Equation (13) and since b' and a belong to the same pile, we must have that b is inserted on the top of b' in r_k of $R^{(i)}(\pi)$. Similarly, we have that a is inserted on the top of b in r_k of $R^{(i+1)}(\pi)$.

Then, in both cases, we have $shape(R(\pi)) = (|r_1|, |r_2|, \cdots, |r_{k-1}|, |r_k| + 1, \cdots)$, where r_j stands for the j-th pile of $R(\sigma)$. Hence, it yields that $shape(R(\sigma)) < shape(R(\pi))$.

■

Remark 4.12 *From the proof of Lemma 4.11, it is straightforward to have the following result. Let σ and π be permutations such that $\sigma \lessdot \pi$ and $\sigma \overset{PS}{\nsim} \pi$. Then, at least one element of a pile r_k of $R(\sigma)$ is moved to the left on a pile r_j, with $j < k$, in order to obtain $R(\pi)$.*

For example, for $\sigma = 34125$ and $\pi = 34215$ we have $\sigma \lessdot \pi$; then, we have

$$R(\sigma) = \begin{array}{l} 1\;2 \\ 3\;4\;5 \end{array} \qquad \text{and} \qquad R(\pi) = \begin{array}{l} 1 \\ 2 \\ 3\;4\;5 \end{array},$$

where 2 has been moved from the second pile of $R(\sigma)$ to the first pile to obtain $R(\pi)$. Moreover, we have $shape(R(\sigma)) = (2,2,1)$ and $shape(R(\pi)) = (3,1,1)$; then we clearly have $(2,2,1) < (3,1,1)$.

Theorem 4.13 *Let σ be a permutation, α (resp. ω) the unique element of minimal (resp. maximal) length of $\widetilde{\sigma}$. Then, $[\alpha, \omega] = \widetilde{\sigma}$.*

Proof – Let π be a permutation of $\widetilde{\sigma}$. First, we have by Lemma 4.4 and Proposition 4.10 that $\alpha \stackrel{*}{\to} \pi$. Then, it is immediate that $\alpha \leqslant \pi$. Second, we have $\alpha \stackrel{*}{\to} \omega$ by Lemma 4.4 and Proposition 4.10. Thus, as stated in the proof of Theorem 4.7, we have

$$\alpha \stackrel{*}{\multimap} \omega. \tag{14}$$

Moreover, since we have $\alpha \stackrel{*}{\to} \pi$, then immediately we have

$$\alpha \stackrel{*}{\multimap} \pi. \tag{15}$$

Hence, combining Equations (14) and (15), and by Proposition 4.9, we have $\pi \stackrel{*}{\multimap} \omega$ since ω is the unique element of maximal length in $\widetilde{\sigma}$. Then, it is immediate that we have $\pi \leqslant \omega$. It follows that π belongs to $[\alpha, \omega]$.

Conversely, we now assume that $\pi \in [\alpha, \omega]$ and $\pi \notin \widetilde{\sigma}$. Then, there must exists two permutations κ and γ in the chain $\alpha \lessdot \cdots \lessdot \kappa \lessdot \gamma \lessdot \cdots \lessdot \pi$ such that $\kappa \stackrel{PS}{\nsim} \gamma$ and $\kappa \stackrel{PS}{\sim} \alpha$. Then, by Lemma 4.11, we have $shape(R(\kappa)) < shape(R(\gamma))$. Thus, since $\kappa \stackrel{PS}{\sim} \alpha$, we have

$$shape(R(\alpha)) < shape(R(\gamma)). \tag{16}$$

Similarly, there must exists two permutations κ' and γ' in the chain $\pi \lessdot \cdots \lessdot \kappa' \lessdot \gamma' \lessdot \cdots \lessdot \omega$ such that $\kappa' \stackrel{PS}{\nsim} \gamma'$ and $\gamma' \stackrel{PS}{\sim} \omega$. Then, by Lemma 4.11, we have $shape(R(\kappa')) < shape(R(\gamma'))$. Thus, since $\gamma' \stackrel{PS}{\sim} \omega$, we have

$$shape(R(\kappa')) < shape(R(\omega)). \tag{17}$$

Then, by induction on both chains $\gamma \lessdot \cdots \lessdot \pi$ and $\pi \lessdot \cdots \lessdot \kappa'$, we obtain $shape(R(\gamma)) \leqslant shape(R(\pi))$ and $shape(R(\pi)) \leqslant shape(R(\kappa'))$. Hence, we immediately have

$$shape(R(\gamma)) \leqslant shape(R(\kappa')). \tag{18}$$

Hence, combining Inegalities (16), (17) and (18), it follows that $shape(R(\alpha)) < shape(R(\omega))$ which is a contradiction since $\alpha \stackrel{PS}{\sim} \omega$. Hence, we have $\pi \in \widetilde{\sigma}$.

■

5 Poset on set partitions

In this section, we provide a poset over set partitions.

Definition 5.1 *Let δ, ς be set partitions. Then, we have $\delta \succeq \varsigma$ if there exists two permutations σ and π such that $R(\sigma) = \delta$, $R(\pi) = \varsigma$ and $\sigma \geqslant \pi$.*

Remark 5.2 *One can check that the relation $\succeq$ is not a poset over set partitions since it is not transitive. Indeed, we have*

$$\begin{array}{r} 2\;3 \\ 1\;4\;5 \end{array} \preceq \begin{array}{l} 2 \\ 3 \\ 1\;4\;5 \end{array} \qquad \textit{and} \qquad \begin{array}{l} 2 \\ 3 \\ 1\;4\;5 \end{array} \preceq \begin{array}{l} 1\;2 \\ 4\;3\;5 \end{array},$$

but we do not have

$$\begin{array}{c} 2\,3 \\ 1\,4\,5 \end{array} \preceq \begin{array}{c} 1\,2 \\ 4\,3\,5 \end{array} .$$

Indeed, we have $\widetilde{14253} = \{14253, 14523\}$, $\widetilde{41325} = \{41325, 41352\}$ *and* $\widetilde{14325} = \{14325, 14352, 14532\}$*. Clearly, we have* $14325 \lessdot 41325$ *and* $14523 \lessdot 14532$*, but no permutations of* $\widetilde{14253}$ *are smaller than any permutation of* $\widetilde{41325}$*.*

Definition 5.3 *Let* δ, ς *be set partitions. Then, we have* $\delta \geqslant \varsigma$ *if there exists a sequence* $\delta \succeq \cdots \succeq \varsigma$*.*

Proposition 5.4 *The pair* $\langle SP; \geqslant \rangle$ *is a poset.*

Proof – Let δ be a set partition. First, $\geqslant$ is reflexive since we clearly have $\delta \succeq \delta$. The transitivity of $\geqslant$ is obvious, by Definition 5.3. Let us prove the antisymmetry of $\geqslant$.

Let δ and ς be set partitions such that $\delta \leqslant \varsigma$ and $\delta \geqslant \varsigma$. We assume that $\delta \neq \varsigma$. Then, by Definition 5.3 and since $\delta \geqslant \varsigma$, there exists a sequence such that:

$$\delta = \delta^{(0)} \succeq \delta^{(1)} \succeq \delta^{(2)} \succeq \cdots \succeq \delta^{(m)} = \varsigma . \tag{19}$$

Futhermore, since $\delta \neq \varsigma$, there exists $k \in [0, m-1]$ such that $\delta^{(k)} \neq \delta^{(k+1)}$. Hence, by Lemma 4.11 and Definition 5.1, we have:

$$shape(\delta) \geqslant shape(\delta^{(1)}) \geqslant \cdots \geqslant shape(\delta^{(k)}) > shape(\delta^{(k+1)}) \geqslant \cdots \geqslant shape(\varsigma) . \tag{20}$$

Hence, we have $shape(\delta) > shape(\varsigma)$. Similarly, since $\delta \leqslant \varsigma$, we can obtain $shape(\delta) < shape(\varsigma)$. Thus we have a contradiction; hence, we have $\delta = \varsigma$. ■

6 Hopf algebra on set partitions

In this section, we first recall the definition of the Free Quasi-Symmetric functions. Then, we prove that equivalence classes define elements over the Free Quasi-Symmetric functions that form a Hopf subalgebra. Then, we describe the product of this subalgebra thanks to the partial order defined in Section 5. At last, we prove that this Hopf subalgebra is free, cofree and self-dual.

6.1 Free Quasi-Symmetric functions

In [9], Malvenuto and Reutenauer endow the space $\mathbb{K}[\mathfrak{S}] := \bigoplus_{n \geqslant 0} \mathbb{K}[\mathfrak{S}_n]$ with a Hopf algebra structure. The image of the embedding which sends $\mathbb{K}[\mathfrak{S}]$ on $\mathbb{K}\langle A \rangle$ is called the algebra of *Free Quasi-Symmetric functions*, denoted **FQSym**, introduced in [3]. Its natural basis $(\mathbf{F}_\sigma)_{\sigma \in \mathfrak{S}}$ is defined as follows.

Definition 6.1 *[3] Let* σ *be a permutation. The free quasi-ribbon* $\mathbf{F}_\sigma$ *is the noncommutative polynomial*

$$\mathbf{F}_\sigma := \sum_{Std(w) = \sigma^{-1}} w , \tag{21}$$

where w *runs over the free associative algebra* $\mathbb{K}\langle A \rangle$*.*

For example, over the alphabet $A = \{1, 2, 3\}$, one has:

$$\mathbf{F}_{123} = 111 + 112 + 113 + 122 + 123 + 133 + 222 + 223 + 233 + 333, \tag{22}$$

$$\mathbf{F}_{312} = 211 + 311 + 312 + 322. \tag{23}$$

The product and coproduct of free quasi-ribbons is easily described respectively with the shifted shuffle and the deconcatenation.

Definition 6.2 *[3] Let* σ *and* π *be two permutations. Then,*

$$\mathbf{F}_\sigma \mathbf{F}_\pi = \sum_{\gamma \in \sigma \Cup \pi} \mathbf{F}_\gamma . \tag{24}$$

For example,

$$\mathbf{F}_{12}\mathbf{F}_{21} = \mathbf{F}_{1243} + \mathbf{F}_{1423} + \mathbf{F}_{1432} + \mathbf{F}_{4123} + \mathbf{F}_{4132} + \mathbf{F}_{4312} . \tag{25}$$

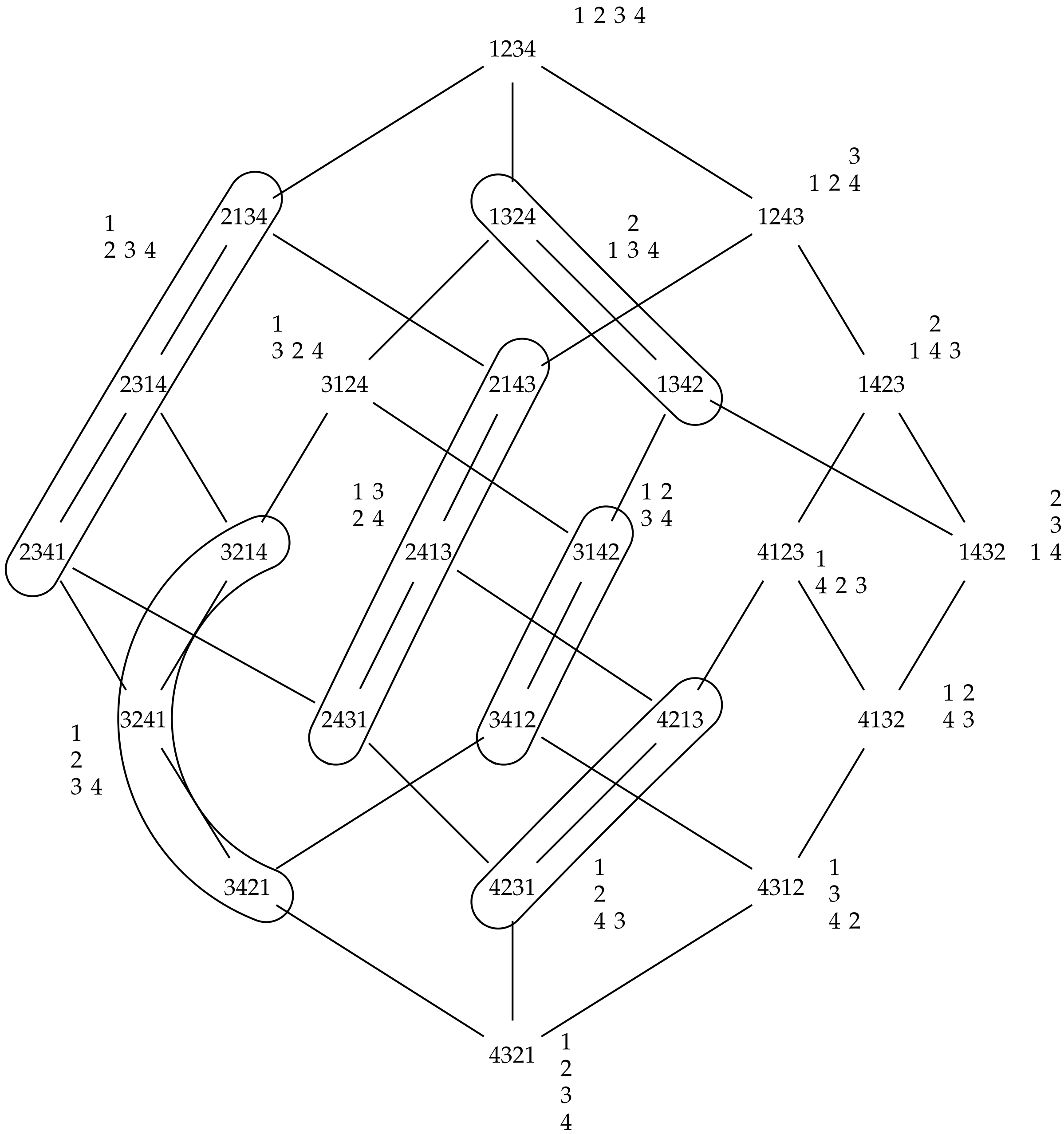

Figure 1: Equivalence classes drawn over the weak order of $\mathfrak{S}_4$

Definition 6.3 *[3] Let σ be a permutation. Then, we have*

$$\Delta(\mathbf{F}_\sigma) = \sum_{u \cdot v = \sigma} \mathbf{F}_{Std(u)} \otimes \mathbf{F}_{Std(v)} \,. \tag{26}$$

For example,

$$\Delta(\mathbf{F}_{4132}) = 1 \otimes \mathbf{F}_{4132} + \mathbf{F}_{1} \otimes \mathbf{F}_{132} + \mathbf{F}_{21} \otimes \mathbf{F}_{21} + \mathbf{F}_{312} \otimes \mathbf{F}_{1} + \mathbf{F}_{4132} \otimes 1 \,. \tag{27}$$

6.2 Hopf subalgebra of FQSym

We first define the elements of **FQSym** we choose to consider, using the combinatorial Algorithm 3.1, and then prove they form a subalgebra and a subcoalgebra of **FQSym**.

Definition 6.4 *We consider the following elements of* **FQSym***:*

$$\mathbf{P}_\delta := \sum_{\sigma;\, R(\sigma) = \delta} \mathbf{F}_\sigma \,, \tag{28}$$

for every set partitions δ.

For example,

$$\mathbf{P}_{\{\{3,1\},\{4,2\},\{5\}\}} = \mathbf{F}_{31425} + \mathbf{F}_{34125} + \mathbf{F}_{31452} + \mathbf{F}_{34152} + \mathbf{F}_{34512}\,. \tag{29}$$

Let us first state that equivalence classes are compatible with intervals.

Proposition 6.5 *Let σ, π be permutations of size n such that $\sigma \overset{PS}{\sim} \pi$. Then, $Std(\sigma\backslash I) \overset{PS}{\sim} Std(\pi\backslash I)$, with I equal to $[1,k]$ or $[k,n]$, for any positive integer k.*

Proof – By Theorem 4.5, we have $\sigma \overset{*}{\longleftrightarrow} \pi$. Then, we can easily prove that, for two permutations σ' and π', if $\sigma' \to \pi'$ then, by considering Definition 4.1, we have either $Std(\sigma'\backslash I) \to Std(\pi'\backslash I)$ or $Std(\sigma'\backslash I) = Std(\pi'\backslash I)$. Thus, it follows that $Std(\sigma\backslash I) \overset{*}{\longleftrightarrow} Std(\pi\backslash I)$; hence, by Theorem 4.5, we obtain $Std(\sigma\backslash I) \overset{PS}{\sim} Std(\pi\backslash I)$. ∎

For example, we have $3164275 \overset{PS}{\sim} 3674512$; one can check that $3164275\backslash[1,2] = 36475$ and $3674512\backslash[1,2] = 36745$ and then $36475 \overset{PS}{\sim} 36745$. One can also check that $3164275\backslash[5,7] = 3142$ and $3674512\backslash[5,7] = 3412$ and then $3142 \overset{PS}{\sim} 3412$. But one check that for $3164275\backslash\{3\} = 164275$ and $3674512\backslash\{3\} = 674512$, we clearly do not have that 164275 and 674512 belong to the same equivalence class.

Lemma 6.6 *The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ form a subalgebra of* **FQSym**.

Proof – We have to prove the following equality:

$$\mathbf{P}_{\delta_1}\mathbf{P}_{\delta_2} = \sum_{\delta \in SP} c_\delta \mathbf{P}_\delta\,, \tag{30}$$

with δ_1 and δ_2 two set partitions. Futhermore, immediately by Remark 2.2 and Remark 2.3, we have that coefficients $(c_\delta)_{\delta\in SP}$ of Equation (30) are either equal to 0 or to 1. Considering now the product of $\mathbf{P}_{\delta_1}$ and $\mathbf{P}_{\delta_2}$, by Definition 6.4, we clearly can write:

$$\mathbf{P}_{\delta_1}\mathbf{P}_{\delta_2} = \sum_{R(\sigma)=\delta_1}\ \sum_{R(\pi)=\delta_2} \mathbf{F}_\sigma \mathbf{F}_\pi\,. \tag{31}$$

Then, by Definition 6.2, we have:

$$\mathbf{P}_{\delta_1}\mathbf{P}_{\delta_2} = \sum_{R(\sigma)=\delta_1}\ \sum_{R(\pi)=\delta_2}\ \sum_{\gamma\in\sigma\Cup\pi} \mathbf{F}_\gamma\,. \tag{32}$$

Hence, we only have to prove that if $\mathbf{F}_\gamma$ appears in the product of $\mathbf{P}_{\delta_1}$ and $\mathbf{P}_{\delta_2}$, then every $\mathbf{F}_{\gamma'}$, such that $\gamma' \overset{PS}{\sim} \gamma$, also appear in Formula (32).

First, let us prove that if $\mathbf{F}_\gamma$ appears in the product of Equation (32) then $\mathbf{F}_{\gamma'}$, such that $\gamma \multimap \gamma'$, also appears in the product of Formula (32). By Definition 4.6, we write $\gamma = \cdots b \cdots ac \cdots$ and $\gamma' = \cdots b \cdots ca \cdots$. Then, since γ appears in the shifted shuffle of two permutations σ of size n, and π of size m, we distinguish three cases.

- We consider that a and c both belong to $\gamma\backslash[n+1, n+m]$ (that is σ). Then, since $\gamma \overset{PS}{\sim} \gamma'$ (by Theorem 4.7) and by Lemma 6.5, we have $\sigma \overset{PS}{\sim} \gamma'\backslash[n+1, n+m]$. Thus, $\mathbf{F}_{\gamma'\backslash[n+1,n+m]}$ appears in $\mathbf{P}_{\delta_1}$. Hence, γ' appears in $(\gamma'\backslash[n+1, n+m]) \Cup \pi$.

- We consider that a and c both belong to $\gamma\backslash[1, n]$ (that is π). Then, since $\gamma \overset{PS}{\sim} \gamma'$ (by Theorem 4.7) and by Lemma 6.5, we have $\pi \overset{PS}{\sim} Std(\gamma'\backslash[1, n])$. Thus, $\mathbf{F}_{Std(\gamma'\backslash[1,n])}$ appears in $\mathbf{P}_{\delta_2}$. Hence, γ' appears in $\sigma \Cup Std(\gamma'\backslash[1, n])$.

- We consider that a and c belong respectively to σ and $\pi[n]$. Then, by definition of the shifted shuffle, γ' appears in $\sigma \Cup \pi$.

Second, with very similar arguments, we prove that if $\mathbf{F}_\gamma$ appears in the product of Equation (32) then $\mathbf{F}_{\gamma'}$, such that $\gamma' \multimap \gamma$, also appears in the product of Formula (32).

Hence, by Theorem 4.7, we end the proof.

∎

For example,

$$\mathbf{P}_{\{\{3,1\},\{2\}\}}\mathbf{P}_{\{1\}} = \mathbf{P}_{\{\{4,3,1\},\{2\}\}} + \mathbf{P}_{\{\{3,1\},\{4,2\}\}} + \mathbf{P}_{\{\{3,1\},\{2\},\{4\}\}},$$

$$\mathbf{P}_{\{\{1\},\{2\}\}}\mathbf{P}_{\{\{1\},\{2\}\}} = \mathbf{P}_{\{\{1\},\{2\},\{3\},\{4\}\}} + \mathbf{P}_{\{\{1\},\{3,2\},\{4\}\}} + \mathbf{P}_{\{\{3,1\},\{2\},\{4\}\}} + \mathbf{P}_{\{\{3,1\},\{4,2\}\}},$$

$$\mathbf{P}_{\{\{1\},\{2\}\}}\mathbf{P}_{\{\{2,1\}\}} = \mathbf{P}_{\{\{1\},\{2\},\{4,3\}\}} + \mathbf{P}_{\{\{1\},\{4,2\},\{3\}\}} + \mathbf{P}_{\{\{1\},\{4,3,2\}\}} + \mathbf{P}_{\{\{4,1\},\{3,2\}\}} + \mathbf{P}_{\{\{4,1\},\{2\},\{3\}\}} + \mathbf{P}_{\{\{4,3,1\},\{2\}\}}.$$

Then, let us state that equivalence classes are compatible with some kind of concatenation.

Proposition 6.7 *Let σ, σ', π and π' be permutations such that $\sigma \overset{PS}{\sim} \sigma'$ and $\pi \overset{PS}{\sim} \pi'$. Let $\hat{\sigma}$ and $\hat{\pi}$ be the de-standardisated words of respectively σ and π such that $\hat{\sigma} \cdot \hat{\pi}$ is a permutation; let $\hat{\sigma}'$ and $\hat{\pi}'$ be words with the same de-standardisation process. Then, $\hat{\sigma} \cdot \hat{\pi} \overset{PS}{\sim} \hat{\sigma}' \cdot \hat{\pi}'$.*

Proof – By Theorem 4.5 and considering the Definition 4.1 of rewriting rule $\rightarrow$, we obtain the result since the de-standardisation process conserve the order between letters.

■

For example, we set $\sigma = 13425$, $\sigma' = 13452$, $\pi = 2413$ and $\pi' = 2431$; one can check we have $\sigma \rightarrow \sigma'$ and $\pi \rightarrow \pi'$. Then, we de-standardisate the permutations σ and π to obtain $\hat{\sigma} = 15729$ and $\hat{\pi} = 4836$. With the same process of de-standardisation applied on σ' and π' we obtain $\hat{\sigma}' = 15792$ and $\hat{\pi}' = 4863$. One can check that $\hat{\sigma} \cdot \hat{\pi} = 157294836 \rightarrow 157924836 \rightarrow 157924863 = \hat{\sigma}' \cdot \hat{\pi}'$, so that $\hat{\sigma} \cdot \hat{\pi} \overset{PS}{\sim} \hat{\sigma}' \cdot \hat{\pi}'$.

Lemma 6.8 *The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ form a subcoalgebra of* **FQSym**.

Proof – We have to prove:

$$\Delta(\mathbf{P}_\delta) = \sum_{\delta_1,\delta_2} \mathbf{P}_{\delta_1} \otimes \mathbf{P}_{\delta_2}\,, \tag{33}$$

for every set partition δ. Let δ be a set partition, then, by Definition 6.4 and Definition 6.3, we have the following:

$$\Delta(\mathbf{P}_\delta) = \Delta(\sum_{R(\sigma)=\delta} \mathbf{F}_\sigma) = \sum_{R(\sigma)=\delta} \Delta(\mathbf{F}_\sigma) = \sum_{R(\sigma)=\delta} \sum_{u\cdot v=\sigma} \mathbf{F}_{Std(u)} \otimes \mathbf{F}_{Std(v)}\,. \tag{34}$$

Hence, we have:

$$\Delta(\mathbf{P}_\delta) = \sum_{R(u\cdot v)=\delta} \mathbf{F}_{Std(u)} \otimes \mathbf{F}_{Std(v)}\,. \tag{35}$$

Then, let σ, π and π' be permutations such that $\pi \overset{PS}{\sim} \pi'$; assume that $\mathbf{F}_\sigma \otimes \mathbf{F}_\pi$ belongs to the right hand-side of Equation (35). Let us prove that $\mathbf{F}_\sigma \otimes \mathbf{F}_{\pi'}$ also belongs to this sum. We consider the de-standardisation process of σ and π to respectively obtain words $\hat{\sigma}$ and $\hat{\pi}$ such that $R(\hat{\sigma} \cdot \hat{\pi}) = \delta$. Then, we apply the same process on permutations σ and π' to respectively obtain words $\hat{\sigma}$ and $\hat{\pi}'$. Thus, by Proposition 6.7, $R(\hat{\sigma} \cdot \hat{\pi}) = R(\hat{\sigma} \cdot \hat{\pi}')$; hence, $\mathbf{F}_\sigma \otimes \mathbf{F}_{\pi'}$ appears in the right hand-side of Equation (35). Similarly, one can prove that if $\mathbf{F}_\sigma \otimes \mathbf{F}_\pi$ appears in the right hand-side of Equation (35), then $\mathbf{F}_{\sigma'} \otimes \mathbf{F}_\pi$ with $\sigma \overset{PS}{\sim} \sigma'$ also belong to this sum. Hence, we obtain Equation (33).

■

For example,

$$\begin{aligned}\Delta(\mathbf{P}_{\{\{4,2,1\},\{3\}\}}) = \quad & 1 \otimes \mathbf{P}_{\{\{4,2,1\},\{3\}\}} + \mathbf{P}_{\{1\}} \otimes \mathbf{P}_{\{\{2,1\},\{3\}\}} + \mathbf{P}_{\{\{2,1\}\}} \otimes \mathbf{P}_{\{\{1\},\{2\}\}} + \\ & \mathbf{P}_{\{\{2,1\}\}} \otimes \mathbf{P}_{\{\{2,1\}\}} + \mathbf{P}_{\{\{3,2,1\}\}} \otimes \mathbf{P}_{\{1\}} + \mathbf{P}_{\{\{3,1\},\{2\}\}} \otimes \mathbf{P}_{\{1\}} + \mathbf{P}_{\{\{4,2,1\},\{3\}\}} \otimes 1\,.\end{aligned}$$

At last, we then obtain immediately from Lemma 6.6 and Lemma 6.8 the following result.

Theorem 6.9 *The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ form a Hopf subalgebra of* **FQSym**. *We denote by* **SP** *this Hopf algebra on set partitions.*

6.3 Combinatorial description of the product

First, we define two operations on set partitions and then state the main result of this subsection.

Definition 6.10 *Let $\delta = \{\delta_1, \delta_2, \cdots, \delta_k\} \vdash [n]$ and $\varsigma = \{\varsigma_1, \varsigma_2, \cdots, \varsigma_l\} \vdash [m]$ be two set partitions.*

1. *We denote by $\delta|\varsigma$ the set partition $\{\delta_1, \delta_2, \cdots, \delta_k, \varsigma_1 + n, \varsigma_2 + n, \cdots, \varsigma_l + n\} \vdash [n+m]$, where $\varsigma_i + n$ stands for the set ς_i where elements have been increased by n, for $i \in [1, l]$.*
2. *We denote by $\delta \sqcup \varsigma$ the set partition of $[n+m]$ equal to:*
   - $\{\delta_1 \cup (\varsigma_1 + n), \delta_2 \cup (\varsigma_2 + n), \cdots, \delta_l \cup (\varsigma_l + n), \delta_{l+1}, \cdots, \delta_k\}$, *if $l < k$,*
   - $\{\delta_1 \cup (\varsigma_1 + n), \delta_2 \cup (\varsigma_2 + n), \cdots, \delta_k \cup (\varsigma_k + n), \varsigma_{k+1} + n, \cdots, \varsigma_l + n\}$, *if $l > k$,*
   - $\{\delta_1 \cup (\varsigma_1 + n), \delta_2 \cup (\varsigma_2 + n), \cdots, \delta_k \cup (\varsigma_k + n)\}$, *if $l = k$,*

   where $\varsigma_i + n$ stands for the set ς_i where elements have been increased by n, for $i \in [1, l]$.

For example, with

$$\delta = \begin{array}{ccc} & 2 & 3 \\ 1 & 4 & 5 \end{array} \qquad \text{and} \qquad \varsigma = \begin{array}{cc} & 2 \\ & 3 \\ 1 & 4 \end{array},$$

we have

$$\delta|\varsigma = \begin{array}{ccccc} & & & & 7 \\ & 2 & 3 & & 8 \\ 1 & 4 & 5 & 6 & 9 \end{array} \qquad \text{and} \qquad \delta \sqcup \varsigma = \begin{array}{ccc} & 2 & \\ & 4 & \\ & 7 & \\ 1 & 8 & 3 \\ 6 & 9 & 5 \end{array}.$$

Then, we state the main contribution of this section.

Theorem 6.11 *Let δ_1 and δ_2 be two set partitions. Then,*

$$\mathbf{P}_{\delta_1} \mathbf{P}_{\delta_2} = \sum_{\delta} \mathbf{P}_{\delta}, \tag{36}$$

where δ runs over set partitions greater than or equal to $\delta_1|\delta_2$ and smaller than or equal to $\delta_1 \sqcup \delta_2$, according to the order on set partitions of Definition 5.3.

Proof – Let $\delta_1 \vdash n$ and $\delta_2 \vdash m$ be set partitions. We recall from the proof of Lemma 6.6 the following equation:

$$\mathbf{P}_{\delta_1} \mathbf{P}_{\delta_2} = \sum_{R(\sigma)=\delta_1} \sum_{R(\pi)=\delta_2} \sum_{\gamma \in \sigma \Cup \pi} \mathbf{F}_{\gamma}. \tag{37}$$

Let α_1 and α_2 be the smallest permutations of their respective equivalence classe such that $R(\alpha_1) = \delta_1$ and $R(\alpha_2) = \delta_2$. Both permutations exist by Proposition 4.10. Then, the permutation $\alpha_1 \cdot \alpha_2[n]$ clearly is the smallest permutation which appears among all permutations of the right hand-side of Equation (37). Futhermore, by Algorithm 3.1, it is easy to check that $R(\alpha_1 \cdot \alpha_2[n])$ is the set partition $\delta_1|\delta_2$. Let ω_1 and ω_2 be the greatest permutations of their equivalence classe such that $R(\omega_1) = \delta_1$ and $R(\omega_2) = \delta_2$. Both permutations exist by Proposition 4.10. Then, the permutation $\omega_2[n] \cdot \omega_1$ clearly is the greatest permutation which appears among all permutations of the right hand-side of Equation (37). Moreover, $R^{(m)}(\omega_2[n] \cdot \omega_1)$ is the set partition $R(\omega_2)$ where elements have been increased by n. Thus, for $R^{(i)}(\omega_2[n] \cdot \omega_1)$ with $i \in [m+1, m+n]$, the elements of w_1 are inserted on the top of piles of $R^{(m)}(\omega_2[n] \cdot \omega_1)$; since all elements of $R^{(m)}(\omega_2[n] \cdot \omega_1)$ are greater than those of ω_1, by Algorithm 3.1, the piles of $R(\omega_1)$ are preserved in $R(\omega_2[n] \cdot \omega_1)$. Hence, by Definition 6.10, we have $R(\omega_2[n] \cdot \omega_1) = \delta_1 \sqcup \delta_2$.

Let δ be a set partition such that $\mathbf{P}_{\delta}$ appears in the product $\mathbf{P}_{\delta_1} \mathbf{P}_{\delta_2}$. Then, let us prove that we have $\delta_1|\delta_2 \leqslant \delta \leqslant \delta_1 \sqcup \delta_2$. By Theorem 4.13, the equivalence classe of δ_1 (resp. δ_2) is $[\alpha_1, omega_1]$ (resp. $[\alpha_2, omega_2]$). Thus, since the shifted shuffle of intervals of permutations is also an interval of permutations, all permutations that appear in the right hand-side of Equation (37) form the interval $[\alpha_1 \cdot \alpha_2[n], \omega_2[n] \cdot \omega_1]$. Then, let γ be a permutation such that $\mathbf{F}_{\gamma}$ appears in the sum of Equation (37); thus we have $\alpha_1 \cdot \alpha_2[n] \leqslant \gamma$ and $\gamma \leqslant \omega_2[n] \cdot \omega_1$. Hence, by Definition 5.1, we have $\delta_1|\delta_2 \preceq \delta$ and $\delta \preceq \delta_1 \sqcup \delta_2$. Thus, by Definition 5.3, we have $\delta_1|\delta_2 \leqslant \delta \leqslant \delta_1 \sqcup \delta_2$.

Let δ be a set partition such that $\delta_1|\delta_2 \leqslant \delta \leqslant \delta_1 \sqcup \delta_2$. Let us prove that $\mathbf{P}_\delta$ appears in the product $\mathbf{P}_{\delta_1}\mathbf{P}_{\delta_2}$. We have

$$\delta^{(0)} = \delta_1|\delta_2 \preceq \delta^{(1)} \preceq \cdots \preceq \delta^{(k)} = \delta \preceq \cdots \preceq \delta^{(p)} = \delta_1 \sqcup \delta_2 \,. \tag{38}$$

Let us assume, without loss of generality, that $\delta^{(i)} \neq \delta^{(j)}$ for every i and j in $[0,p]$ and $i \neq j$ and that, for every $i \in [0, p-1]$, we have $\delta^{(i)} \preceq \delta^{(i+1)}$ only if there exist two permutations σ and π such that $R(\sigma) = \delta^{(i)}$, $R(\pi) = \delta^{(i+1)}$ and $\sigma \lessdot \pi$. We proceed by induction on the Sequence (38). For the initial case, it is clear, by Algorithm 3.1, that $R(\alpha_1 \cdot \alpha_2[n]) = \delta_1|\delta_2$. Then, by Lemma 6.6, $\mathbf{P}_{\delta_1|\delta_2}$ belongs to $\mathbf{P}_{\delta_1}\mathbf{P}_{\delta_2}$. Assume that $\mathbf{P}_{\delta^{(i)}}$ appears in the product $\mathbf{P}_{\delta_1}\mathbf{P}_{\delta_2}$ and let σ and π be permutations such that $R(\sigma) = \delta^{(i)}$, $R(\pi) = \delta^{(i+1)}$ and $\sigma \lessdot \pi$. Then, we write $\sigma = \cdots ab \cdots$ and $\pi = \cdots ba \cdots$. We consider two cases:

1. Let $a \in [1, n]$ and $b \in [n+1, n+m]$. Then, it is clear that π also belong to the shifted shuffle where σ appears in sum of Equation (37). Thus, by Lemma 6.6, $\mathbf{P}_{\delta^{(i+1)}}$ appears in product $\mathbf{P}_{\delta_1}\mathbf{P}_{\delta_2}$.
2. Let a and b both belong to $[1,n]$ or $[n+1, n+m]$. Then, by Remark 4.12, piles of $R(\alpha_1)$ or $R(\alpha_2)$ are not preserved in $\delta^{(i+1)}$, whereas they are preserved in $\delta_1 \sqcup \delta_2$, according to Definition 6.10. Hence, set partition $\delta_1 \sqcup \delta_2$ clearly cannot be reached from $\delta^{(i+1)}$ by the sequence $\delta^{(i+1)} \preceq \cdots \preceq \delta^{(p)} = \delta_1 \sqcup \delta_2$; but this is a contradiction.

■

One can check this result on the previous examples of product between $\mathbf{P}$ elements with Figure 1.

6.4 Bidendriform bialgebra

In [4], Foissy has introduced the definition of *bidendriform bialgebra*; he also proved that **FQSym** is a bidendriform bialgebra and that a bidendriform bialgebra is always free, cofree and self-dual.

In this section, we prove that our Hopf algebra on set partitions is also a bidendriform bialgebra, using the useful result that **FQSym** is itself a bidendriform bialgebra.

As define in [4], a *dendriform algebra* is a family $(D, \leftharpoonup, \rightharpoonup)$ such that:

1. D is a vector space and:

$$\leftharpoonup : \begin{cases} D \otimes D & \longrightarrow & D \\ a \otimes b & \longrightarrow & a \leftharpoonup b, \end{cases} \qquad \rightharpoonup : \begin{cases} D \otimes D & \longrightarrow & D \\ a \otimes b & \longrightarrow & a \rightharpoonup b. \end{cases}$$

2. For all $a, b, c \in D$:

$$(a \leftharpoonup b) \leftharpoonup c = a \leftharpoonup (b \leftharpoonup c + b \rightharpoonup c), \tag{39}$$
$$(a \rightharpoonup b) \leftharpoonup c = a \rightharpoonup (b \leftharpoonup c), \tag{40}$$
$$(a \leftharpoonup b + a \rightharpoonup b) \rightharpoonup c = a \rightharpoonup (b \rightharpoonup c). \tag{41}$$

Moreover, for a dendriform algebra D, we put:

$$m : \begin{cases} D \otimes D & \longrightarrow & D \\ a \otimes b & \longrightarrow & ab = a \leftharpoonup b + a \rightharpoonup b. \end{cases}$$

A *dendriform coalgebra* is a family $(C, \Delta_\leftharpoonup, \Delta_\rightharpoonup)$ such that:

1. C is a vector space and:

$$\Delta_\leftharpoonup : \begin{cases} C & \longrightarrow & C \otimes C \\ a & \longrightarrow & \Delta_\leftharpoonup(a) = a'_\leftharpoonup \otimes a''_\leftharpoonup, \end{cases} \qquad \Delta_\rightharpoonup : \begin{cases} C & \longrightarrow & C \otimes C \\ a & \longrightarrow & \Delta_\rightharpoonup(a) = a'_\rightharpoonup \otimes a''_\rightharpoonup. \end{cases}$$

2. For all $a \in C$:

$$(\Delta_\leftharpoonup \otimes Id) \circ \Delta_\leftharpoonup(a) = (Id \otimes \Delta_\leftharpoonup + Id \otimes \Delta_\rightharpoonup) \circ \Delta_\leftharpoonup(a), \tag{42}$$
$$(\Delta_\rightharpoonup \otimes Id) \circ \Delta_\leftharpoonup(a) = (Id \otimes \Delta_\leftharpoonup) \circ \Delta_\rightharpoonup(a), \tag{43}$$
$$(\Delta_\leftharpoonup \otimes Id + \Delta_\rightharpoonup \otimes Id) \circ \Delta_\rightharpoonup(a) = (Id \otimes \Delta_\rightharpoonup) \circ \Delta_\rightharpoonup(a). \tag{44}$$

Moreover, for a dendriform coalgebra C, we put:

$$\tilde{\Delta} : \begin{cases} C & \longrightarrow & C \otimes C \\ a & \longrightarrow & \tilde{\Delta}(a) = \Delta_{\leftharpoonup}(a) + \Delta_{\rightharpoonup}(a) = a' \otimes a''. \end{cases}$$

Then, a *bidendriform bialgebra* $(B, \leftharpoonup, \rightharpoonup, \Delta_{\leftharpoonup}, \Delta_{\rightharpoonup})$ is both a dendriform algebra $(B, \leftharpoonup, \rightharpoonup)$ and a dendriform coalgebra $(B, \Delta_{\leftharpoonup}, \Delta_{\rightharpoonup})$ such that the following relations are satisfied for all $a, b \in B$:

$$\Delta_{\rightharpoonup}(a \rightharpoonup b) = a'b'_{\rightharpoonup} \otimes a'' \rightharpoonup b''_{\rightharpoonup} + a' \otimes a'' \rightharpoonup b + b'_{\rightharpoonup} \otimes a \rightharpoonup b''_{\rightharpoonup} + ab'_{\rightharpoonup} \otimes b''_{\rightharpoonup} + a \otimes b, \tag{45}$$
$$\Delta_{\rightharpoonup}(a \leftharpoonup b) = a'b'_{\rightharpoonup} \otimes a'' \leftharpoonup b''_{\rightharpoonup} + a' \otimes a'' \leftharpoonup b + b'_{\rightharpoonup} \otimes a \leftharpoonup b''_{\rightharpoonup}, \tag{46}$$
$$\Delta_{\leftharpoonup}(a \rightharpoonup b) = a'b'_{\leftharpoonup} \otimes a'' \rightharpoonup b''_{\leftharpoonup} + ab'_{\leftharpoonup} \otimes b''_{\leftharpoonup} + b'_{\leftharpoonup} \otimes a \rightharpoonup b''_{\leftharpoonup}, \tag{47}$$
$$\Delta_{\leftharpoonup}(a \leftharpoonup b) = a'b'_{\leftharpoonup} \otimes a'' \leftharpoonup b''_{\leftharpoonup} + a'b \otimes a'' + b'_{\leftharpoonup} \otimes a \leftharpoonup b''_{\leftharpoonup} + b \otimes a. \tag{48}$$

By considering the augmentation ideal $(\mathbf{FQSym})_+ = Vect(\mathbf{F}_\sigma \,|\, \sigma \in \mathfrak{S}_n, n \geqslant 1)$ and over the following defined operations, **FQSym** is bidendriform. For all permutations σ of size n and π of size m we have:

- $\mathbf{F}_\sigma \leftharpoonup \mathbf{F}_\pi$ is the sum of elements appearing in the product $\mathbf{F}_\sigma \mathbf{F}_\pi$, such that their last letter is the last letter of σ.
- $\mathbf{F}_\sigma \rightharpoonup \mathbf{F}_\pi$ is the sum of elements appearing in the product $\mathbf{F}_\sigma \mathbf{F}_\pi$, such that their last letter is the last letter of $\pi[n]$.
- $\Delta_{\leftharpoonup}(\mathbf{F}_\sigma)$ is the sum of elements of $\Delta(\mathbf{F}_\sigma)$, not of the form $1 \otimes \mathbf{F}_\sigma$ nor $\mathbf{F}_\sigma \otimes 1$, where the maximal letter of σ appears on the left side of $\otimes$.
- $\Delta_{\rightharpoonup}(\mathbf{F}_\sigma)$ is the sum of elements of $\Delta(\mathbf{F}_\sigma)$, not of the form $1 \otimes \mathbf{F}_\sigma$ nor $\mathbf{F}_\sigma \otimes 1$, where the maximal letter of σ appears on the right side of $\otimes$.

For example,

$$\mathbf{F}_{12} \leftharpoonup \mathbf{F}_{123} = \mathbf{F}_{13452} + \mathbf{F}_{31452} + \mathbf{F}_{34152} + \mathbf{F}_{34512},$$
$$\mathbf{F}_{12} \rightharpoonup \mathbf{F}_{123} = \mathbf{F}_{12345} + \mathbf{F}_{13245} + \mathbf{F}_{13425} + \mathbf{F}_{31245} + \mathbf{F}_{31425} + \mathbf{F}_{34125},$$

$$\Delta_{\leftharpoonup}(\mathbf{F}_{12543}) = \mathbf{F}_{123} \otimes \mathbf{F}_{21} + \mathbf{F}_{1243} \otimes \mathbf{F}_1,$$
$$\Delta_{\rightharpoonup}(\mathbf{F}_{12543}) = \mathbf{F}_1 \otimes \mathbf{F}_{1432} + \mathbf{F}_{12} \otimes \mathbf{F}_{321}.$$

A way to prove quickly that our Hopf algebra is also bidendriform is to consider the morphism *rev* on **FQSym** that sends $\mathbf{F}_\sigma$ to $\mathbf{F}_{\sigma^r}$, with σ^r the mirror image of σ, for every permutations σ. This is an isomorphism of associative algebra and an anti-isomorphism of coalgebra. It can be proved that $rev(\mathbf{FQSym}) = \mathbf{FQSym}$ and more generally that any Hopf subalgebra of **FQSym** is the same that its image through map *rev*. Thus, the family of elements $(\mathbf{P}^r_\delta)_{\delta \in SP}$ defined as follows:

$$\mathbf{P}^r_\delta := \sum_{\sigma; R(\sigma) = \delta} \mathbf{F}_{\sigma^r}, \tag{49}$$

induces a Hopf subalgebra $\mathbf{SP}^r$ on **FQSym**, identical to our Hopf algebra **SP**. It is obvious that the **F** elements involved in sum (49) can be seen as those elements indexed by permutations where the image through Algorithm 3.1 is equal to δ, by inserting letters from right to left.

Dendriform algebra First, immediately by Algorithm 3.1, note that for two permutations σ and π such that $\sigma \overset{PS}{\sim} \pi$, we have that σ^r and π^r end by the same letter. Thus, all permutations of elements **F** belonging to an element $\mathbf{P}^r_\delta$ end by the same letter. Hence, if $\mathbf{F}_\sigma$ appears in $\mathbf{P}^r_{\delta_1} \leftharpoonup \mathbf{P}^r_{\delta_2}$, then $\mathbf{F}_\pi$ such that $\sigma \overset{PS}{\sim} \pi$ also appears in $\mathbf{P}^r_{\delta_1} \leftharpoonup \mathbf{P}^r_{\delta_2}$ and not in $\mathbf{P}^r_{\delta_1} \rightharpoonup \mathbf{P}^r_{\delta_2}$. By Lemma 6.6, this implies that

$$\mathbf{P}^r_{\delta_1} \leftharpoonup \mathbf{P}^r_{\delta_2} = \sum_\delta \mathbf{P}_\delta, \tag{50}$$

and similarly we have

$$\mathbf{P}^r_{\delta_1} \rightharpoonup \mathbf{P}^r_{\delta_2} = \sum_\delta \mathbf{P}_\delta, \tag{51}$$

where δ runs over some set partitions. Then, Relations (39), (40), (41) are satisfied for the family $(\mathbf{P}^r_\delta)_{\delta \in SP}$ by distributivity, since they are satisfied for **FQSym**. Hence, $\mathbf{SP}^r$ is a dendriform algebra.

Dendriform coalgebra It is easy to prove by contradiction that $\Delta_{\leftarrow}(\mathbf{P}^r_\delta) = \sum_{\delta_1,\delta_2} \mathbf{P}^r_{\delta_1} \otimes \mathbf{P}^r_{\delta_2}$ and $\Delta_{\rightarrow}(\mathbf{P}^r_\delta) = \sum_{\delta_1,\delta_2} \mathbf{P}^r_{\delta_1} \otimes \mathbf{P}^r_{\delta_2}$, using Lemma 6.8. Then, Relations (42), (43), (44) are satisfied for the family $(\mathbf{P}^r_\delta)_{\delta \in SP}$ by linearity of $\Delta_{\leftarrow}$ and $\Delta_{\rightarrow}$, since they are satisfied for **FQSym**. Hence, $\mathbf{SP}^r$ is a dendriform coalgebra.

At last, compatibility relations (45), (46), (47), (48) are respected by distributivity and linearity and since **FQSym** is a bidendriform bialgebra. Hence, $\mathbf{SP}^r$ is a bidendriform bialgebra, as **SP** which is by consequence free, cofree and self-dual.

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