Effective Asymptotics of Combinatorial Systems

Carine Pivoteau
joint work with Bruno Salvy

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LIGM - Université Gustave Eiffel

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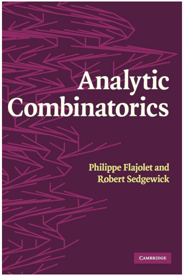
From book to algorithm



Analytic Combinatorics is a general approach:

- that starts from a **combinatorial description**
- translates this description into equations satisfied by **generating functions**
- views them as **analytic functions**
- and exploits their singular behavior to deduce **asymptotic** properties of the objects.

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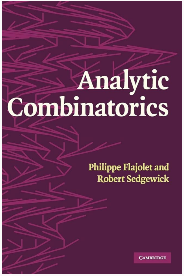

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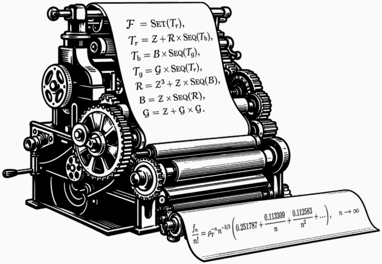
What we did: develop computational tools that **automate** large parts of this approach.

Today: outline of the main steps.



First part: Combinatorics

Combinatorial systems

The symbolic method with a touch of species

Definition (Combinatorial system)

A *combinatorial system* $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}})$ for an m–tuple $\boldsymbol{\mathcal{Y}} = (\mathcal{Y}_1, \cdots, \mathcal{Y}_m)$ of **species** (labeled classes) is a vector

$$\begin{cases} \mathcal{Y}_1 = \mathcal{H}_1(\mathcal{Z}, \mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_m), \\ \mathcal{Y}_2 = \mathcal{H}_2(\mathcal{Z}, \mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_m), \\ \qquad\qquad \vdots \\ \mathcal{Y}_m = \mathcal{H}_m(\mathcal{Z}, \mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_m), \end{cases}$$

where each $\mathcal{H}_i$ denotes a multisort species (constructions).

Definition (Specification)

A *specification* is a combinatorial system where the $\mathcal{H}_i$ are combinations of

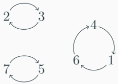
$$1, \mathcal{Z}, +, \times, \textsc{Seq}, \textsc{Cyc}, \textsc{Set}.$$

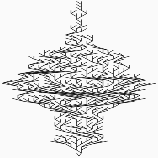
Let $\mathcal{S}$ be a species,

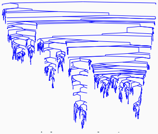
- the size of a $\mathcal{S}$-structure is the number of $\mathcal{Z}$ it contains;
- the number of $\mathcal{S}$-structures of any given size is finite.

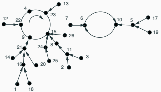
Simple examples(?)

- Permutations: $\mathcal{P} = \textsc{Set}(\textsc{Cyc}(\mathcal{Z}))$
- Binary trees: $\mathcal{B} = \mathcal{Z} + \mathcal{B} \times \mathcal{B}$
- General (Cayley) trees: $\mathcal{T} = \mathcal{Z} \times \textsc{Set}(\mathcal{T})$
- Functional graphs: $\mathcal{F} = \textsc{Set}(\mathcal{K}), \mathcal{K} = \textsc{Cyc}(\mathcal{T}), \mathcal{T} = \mathcal{Z} \times \textsc{Set}(\mathcal{T})$









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What if the system $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ looks like this?

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> {L = Union (F, Sorig),
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   FA = Epsilon,
   FA01 = Prod(t, Union(F, Prod(L, LA0))),
   FA0011 = Prod(t, Union(F01, Prod(L, LA011))),
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- **Sufficient** conditions for a system to define combinatorial species?

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- Are these **necessary** conditions?
  ▷ Yes, if we do not authorize structures of size 0 and zero-coordinates.

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   LA010 = Prod(t, Union(Union(S01, F01), Prod(L, Union(LA010, GA010)))),
   LA011 = Prod(t, Union(Union(S01, F01), Prod(L, Union(LA011, GA011)))),
   SA = Prod(Union(Epsilon, Epsilon), LA0),
   SA01 = Prod( t, Union(S0, Prod(L, Union(Union( SA01, GA001), GA010))) ),
   GA0 = Prod(t, Union(Union(Union(Union(Sorig, Prod(L, GA0)), Prod(L, GA010)), Prod(L, GA011)), Prod(L, GA001))),
   GA001 = Prod(t, Union(S00, Prod(L, GA001))),
   GA010 = Prod(t, Union(S01, Prod(L, GA010))),
   GA011 = Prod(t, Union(S01, Prod(L, GA011))),
   t=Atom}):
```

- **Sufficient** conditions for a system to define combinatorial species?
  ▷ $\boldsymbol{\mathcal{H}}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\boldsymbol{\partial\mathcal{H}/\partial\mathcal{Y}}(\mathbf{0}, \mathbf{0})$ is nilpotent [Joyal, 1981].
- Are these **necessary** conditions?
  ▷ Yes, if we do not authorize structures of size 0 and zero-coordinates.
- Can we give a characterization that allows structures of size 0?

What if the system $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ looks like this?

```
> {L = Union (F, Sorig),
   F = FA,
   F01 = FA01,
   Sorig = Prod(L,LA),
   S0 =  Union(LA0, Prod( Prod(Union(Epsilon,Epsilon), S0), LA0)),
   S00 =  Union(LA00, Prod( Prod(Union(Epsilon,Epsilon), S0), LA00)),
   S01 =  Union(SA01, Prod( Prod(Union(Epsilon,Epsilon), S0), LA01)),
   FA = Epsilon,
   FA01 = Prod(t, Union(F, Prod(L, LA0))),
   FA0011 = Prod(t, Union(F01, Prod(L, LA011))),
   FA0101 = Prod(t, Union(F01, Prod(L, LA010))),
   FA0110 = Prod(t, Prod(L, LA001)),
   LA = Prod(Union(Epsilon, Epsilon),LA0),
   LA0 = Prod(t, Union(Union(Sorig, F), Prod(L, Union(Union(Union(Union(LA0, GA0), GA010), GA011), GA001)))),
   LA00 = Prod(t, Union(S0, Prod(L, Union(LA00, GA011)))),
   LA01 = Prod(t, Union(Union(S0, F), Prod(L, Union(Union(Union(LA01, GA001), GA010), GA0)))),
   LA000 = Prod(t, Union(S00, Prod(L, LA000))),
   LA001 = Prod(t, Union(S00, Prod(L, Union(LA001, GA001)))),
   LA010 = Prod(t, Union(Union(S01, F01), Prod(L, Union(LA010, GA010)))),
   LA011 = Prod(t, Union(Union(S01, F01), Prod(L, Union(LA011, GA011)))),
   SA = Prod(Union(Epsilon, Epsilon), LA0),
   SA01 = Prod( t, Union(S0, Prod(L, Union(Union( SA01, GA001), GA010))) ),
   GA0 = Prod(t, Union(Union(Union(Union(Sorig, Prod(L, GA0)), Prod(L, GA010)), Prod(L, GA011)), Prod(L, GA001))),
   GA001 = Prod(t, Union(S00, Prod(L, GA001))),
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   t=Atom}):
```

- **Sufficient** conditions for a system to define combinatorial species?
  ▷ $\mathcal{H}(\mathbf{0},\mathbf{0}) = \mathbf{0}$ and $\partial\mathcal{H}/\partial\mathcal{Y}(\mathbf{0},\mathbf{0})$ is nilpotent [Joyal, 1981].
- Are these **necessary** conditions?
  ▷ Yes, if we do not authorize structures of size 0 and zero-coordinates.
- Can we give a characterization that allows structures of size 0?
  ▷ Yes, we have two different characterizations.

Well-founded systems

Definition (Well founded)

The combinatorial system $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}})$ is *well founded* when

$$\boldsymbol{\mathcal{Y}}^{[0]} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\mathcal{Y}}^{[n+1]} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}}^{[n]}), \quad n \geq 0$$

is well defined for all n and defines a sequence $(\boldsymbol{\mathcal{Y}}^{[n]})_{n\geq 0}$ that is convergent. Its limit $\boldsymbol{\mathcal{S}}$ is *the combinatorial solution* of the system.

- *well defined*: for each n and each size k, there are finitely many combinatorial structures of size k in $\boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}}^{[n]})$;
- *convergent*: there is a limit class $\boldsymbol{\mathcal{S}}$ such that for all $k \geq 0$, $\boldsymbol{\mathcal{Y}}^{[n]}$ is isomorphic to $\boldsymbol{\mathcal{S}}$ for all structures of size up to k, for large enough n.

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Theorem (Characterization by Leading Terms)

The system $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}})$ is well founded if and only if the **leading terms** *(smallest structures) of $\boldsymbol{\mathcal{Y}}^{[m]}$ and $\boldsymbol{\mathcal{Y}}^{[m+1]}$ are equal.*

Computing the leading terms

Algorithm WellFoundedAndLeadingTerm:

Input: $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\mathcal{Z}, \boldsymbol{\mathcal{Y}})$ a specification in normal form with m equations

Output: Vector of pairs (c, v) corresponding to the vector of leading terms $c\mathcal{Z}^v$ of $\boldsymbol{\mathcal{Y}}$; FAIL if and only if the system is not well founded

$w_{1:m} := ((0, \infty), \dots, (0, \infty))$
repeat $m{+}1$ **times**
  $v := w$
  for $i = 1$ *to* m **do**
    if $\mathcal{H}_i = 1$ **then** $w_i := (1, 0)$
    if $\mathcal{H}_i = \mathcal{Z}$ **then** $w_i := (1, 1)$
    if $\mathcal{H}_i = \textsc{Set}(\mathcal{Y}_j)$ *or* $\textsc{Seq}(\mathcal{Y}_j)$ **then** $w_i := (1, 0)$ if $v_{j,2} \neq 0$, otherwise **FAIL**
    if $\mathcal{H}_i = \textsc{Cyc}(\mathcal{Y}_j)$ **then** $w_i := v_j$ if $v_{j,2} \neq 0$, otherwise **FAIL**
    if $\mathcal{H}_i = \mathcal{Y}_{j_1} \times \cdots \times \mathcal{Y}_{j_k}$ **then** $w_i := (v_{j_1,1} \cdots v_{j_k,1}, v_{j_1,2} + \cdots + v_{j_k,2})$
    if $\mathcal{H}_i = \mathcal{Y}_{j_1} + \cdots + \mathcal{Y}_{j_k}$ **then**
      $w_{i,2} := \min(v_{j_1,2}, \dots, v_{j_k,2})$, $w_{i,1} := \sum_{\ell \in [1,k], v_{j_\ell,2} = w_{i,2}} v_{j_\ell,1}$
  if $v = w$ **then return** v ◁ `the system is well founded`
if $v \neq w$ **then FAIL**
return v

Improvement of Paul Zimmermann's algorithm (1991)

A system not too big, but not that small

Specification for unordered forests of trees with red, green and blue nodes and its normal form.

$$\begin{cases} \mathcal{F} = \textsc{Set}(\mathcal{T}_r), \\ \mathcal{T}_r = \mathcal{Z} + \mathcal{R} \times \textsc{Seq}(\mathcal{T}_b), \\ \mathcal{T}_b = \mathcal{B} \times \textsc{Seq}(\mathcal{T}_g), \\ \mathcal{T}_g = \mathcal{G} \times \textsc{Seq}(\mathcal{T}_r), \\ \mathcal{R} = \mathcal{Z}^3 + \mathcal{Z} \times \textsc{Seq}(\mathcal{B}), \\ \mathcal{B} = \mathcal{Z} \times \textsc{Seq}(\mathcal{R}), \\ \mathcal{G} = \mathcal{Z} + \mathcal{G} \times \mathcal{G}. \end{cases} \qquad \begin{cases} \mathcal{F} = \textsc{Set}(\mathcal{T}_r), \\ \mathcal{T}_r = \mathcal{Z} + \mathcal{N}_1, \quad \mathcal{N}_1 = \mathcal{R} \times \mathcal{N}_2, \ \mathcal{N}_2 = \textsc{Seq}(\mathcal{T}_b), \\ \mathcal{T}_b = \mathcal{B} \times \mathcal{N}_3, \quad \mathcal{N}_3 = \textsc{Seq}(\mathcal{T}_g), \\ \mathcal{T}_g = \mathcal{G} \times \mathcal{N}_4, \quad \mathcal{N}_4 = \textsc{Seq}(\mathcal{T}_r), \\ \mathcal{R} = \mathcal{N}_5 + \mathcal{N}_6, \ \mathcal{N}_5 = \mathcal{Z}^3, \ \mathcal{N}_6 = \mathcal{Z} \times \mathcal{N}_7, \ \mathcal{N}_7 = \textsc{Seq}(\mathcal{B}), \\ \mathcal{B} = \mathcal{Z} \times \mathcal{N}_8, \quad \mathcal{N}_8 = \textsc{Seq}(\mathcal{R}), \\ \mathcal{G} = \mathcal{Z} + \mathcal{N}_9, \quad \mathcal{N}_9 = \mathcal{G} \times \mathcal{G} \end{cases}$$

The algorithm produces the following vectors, the system is well founded:

$$\begin{gathered}
[1,\ \mathcal{Z},\ 0, 1,\ 0, 1,\ 0, 1,\ 0,\ \mathcal{Z}^3,\ 0, 1,\ 0, 1, \mathcal{Z},\ 0] \\
[1,\ \mathcal{Z},\ 0, 1,\ 0, 1, \mathcal{Z}, 1, \mathcal{Z}^3,\ \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^2] \\
[1,\ \mathcal{Z}, \mathcal{Z}^3, 1, \mathcal{Z}, 1, \mathcal{Z}, 1,\ \mathcal{Z},\ \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^2] \\
[1,\ \mathcal{Z},\ \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, 1,\ \mathcal{Z},\ \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^2] \\
[1, 2\mathcal{Z},\ \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, 1,\ \mathcal{Z},\ \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^2] \\
[1, 2\mathcal{Z},\ \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, 1,\ \mathcal{Z},\ \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^2]
\end{gathered}$$

A different system not too big, but not that small

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At some point, the leading term of $\mathcal{T}_g$ is 1. This breaks the condition of being well defined for $\mathcal{N}_3 = \textsc{Seq}(\mathcal{T}_g)$ in the next iteration. The system is not well founded anymore.

$$\begin{array}{c} [1,\ \mathcal{Z},\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \mathcal{Z}^3,\ 0,\ 1,\ 0,\ 1,\ 1,\ 0] \\ [1,\ \mathcal{Z},\ 0,\ 1,\ 0,\ 1,\ \underset{\mathcal{T}_g}{\underset{\triangle}{1}},\ 1,\ \mathcal{Z}^3,\ \mathcal{Z}^3,\ \mathcal{Z},\ 1,\ \mathcal{Z},\ 1,\ 1,\ \mathcal{Z}] \end{array}$$

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The previous condition is not broken anymore... But the last 2 vectors after $m + 1 = 18$ iterations are not equal. This system is (still) not well founded:

$$[\ 1,\ \mathcal{Z},\ 0, 1,\ 0,\ 1, 0, 0, 1,\ 0, \mathcal{Z}^3,\ 0, 1,\ 0, 1, 1,\ 0\ ]$$

$$\dots$$

$$[\ 1, 2\mathcal{Z}, \mathcal{Z}, 1, 8\mathcal{Z},\ 8, 8, 1, 1, \mathcal{Z}, \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, 1, \mathcal{Z}\ ]$$
$$[\ 1, 2\mathcal{Z}, \mathcal{Z}, 1, 8\mathcal{Z},\ 9, 8, 1, 1, \mathcal{Z}, \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, 1, \mathcal{Z}\ ]$$

$$\triangle$$
$$\mathcal{N}_3$$

Generating functions

Exponential generating functions for labeled structures

The exponential generating function of a species $\boldsymbol{\mathcal{Y}}_{1:m}$ is the power series

$$\boldsymbol{Y}(z) = \sum_{n\geq 0} \boldsymbol{Y}_n \frac{z^n}{n!}$$

where $\boldsymbol{Y}_n$ is the vector of numbers of $\mathcal{Y}_k$-structures of size n, for $k = 1..m$.

species	constr.	exponential g.f.	derivative $\partial H/\partial z$
Union	$\mathcal{F} + \mathcal{G}$	$F(z) + G(z)$	$F'(z) + \mathcal{G}'(z)$
Product	$\mathcal{F} \times \mathcal{G}$	$F(z) \times G(z)$	$F'(z)G(z) + F(z)G'(z)$
Sequence	$\textsc{Seq}(\mathcal{G})$	$(1 - G(z))^{-1}$	$(1 - G(z))^{-2}$
Cycle	$\textsc{Cyc}(\mathcal{G})$	$\ln \dfrac{1}{1 - G(z)}$	$(1 - G(z))^{-1}$
Set	$\textsc{Set}(\mathcal{G})$	$e^{G(z)}$	$e^{G(z)}$

Any specification translates into a system of equations on generating series.

Jacobian matrix, dependency graph, and irreducible systems

$$\begin{cases} F(z) = e^{T_r(z)} \\ T_r(z) = z + \frac{R(z)}{1-T_b(z)} \\ T_b(z) = \frac{B(z)}{1-T_g(z)} \\ T_g(z) = \frac{G(z)}{1-T_r(z)} \\ R(z) = z^3 + \frac{z}{1-B(z)} \\ B(z) = \frac{z}{1-R(z)} \\ G(z) = z + G(z)^2 \end{cases}$$

Jacobian matrix, dependency graph, and irreducible systems

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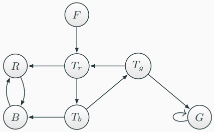
\qquad
\begin{pmatrix}
0 & e^{T_r(z)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{R(z)}{(1-T_b(z))^2} & 0 & \frac{1}{1-T_b(z)} & 0 & 0 \\
0 & 0 & 0 & \frac{B(z)}{(1-T_g(z))^2} & 0 & \frac{1}{1-T_g(z)} & 0 \\
0 & \frac{G(z)}{(1-T_r(z))^2} & 0 & 0 & 0 & 0 & \frac{1}{1-T_r(z)} \\
0 & 0 & 0 & 0 & 0 & \frac{z}{(1-B(z))^2} & 0 \\
0 & 0 & 0 & 0 & \frac{z}{(1-R(z))^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2G(z)
\end{pmatrix}$$

Jacobian matrix, dependency graph, and irreducible systems

$$\begin{cases} F(z) = e^{T_r(z)} \\ T_r(z) = z + \frac{R(z)}{1-T_b(z)} \\ T_b(z) = \frac{B(z)}{1-T_g(z)} \\ T_g(z) = \frac{G(z)}{1-T_r(z)} \\ R(z) = z^3 + \frac{z}{1-B(z)} \\ B(z) = \frac{z}{1-R(z)} \\ G(z) = z + G(z)^2 \end{cases} \qquad \begin{pmatrix} 0 & e^{T_r(z)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{R(z)}{(1-T_b(z))^2} & 0 & \frac{1}{1-T_b(z)} & 0 & 0 \\ 0 & 0 & 0 & \frac{B(z)}{(1-T_g(z))^2} & 0 & \frac{1}{1-T_g(z)} & 0 \\ 0 & \frac{G(z)}{(1-T_r(z))^2} & 0 & 0 & 0 & 0 & \frac{1}{1-T_r(z)} \\ 0 & 0 & 0 & 0 & 0 & \frac{z}{(1-B(z))^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{z}{(1-R(z))^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2G(z) \end{pmatrix}$$

The Jacobian matrix $\boldsymbol{\partial H}/\boldsymbol{\partial Y}$ *encodes* the dependency graph of the system.



Dependency graph

Jacobian matrix, dependency graph, and irreducible systems

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$$\begin{pmatrix} 0 & e^{T_r(z)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{R(z)}{(1-T_b(z))^2} & 0 & \frac{1}{1-T_b(z)} & 0 & 0 \\ 0 & 0 & 0 & \frac{B(z)}{(1-T_g(z))^2} & 0 & \frac{1}{1-T_g(z)} & 0 \\ 0 & \frac{G(z)}{(1-T_r(z))^2} & 0 & 0 & 0 & 0 & \frac{1}{1-T_r(z)} \\ 0 & 0 & 0 & 0 & 0 & \frac{z}{(1-B(z))^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{z}{(1-R(z))^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2G(z) \end{pmatrix}$$

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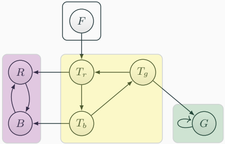


Strongly connected components

Definition (Irreducible)

A combinatorial system is *irreducible* when its dependency graph is strongly connected.

The analysis of general combinatorial systems often reduces to the special case of irreducible systems.

Well-founded systems of generating functions

Definition

Let $\boldsymbol{H}_{1:m} : \mathbb{C}^{m+1} \to \mathbb{C}^m$ be analytic in a neighborhood of $\mathbf{0}$.

The system $\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ is called *well founded* when

1. Each coordinate of $\boldsymbol{H}$ has nonnegative real Taylor coefficients at $\mathbf{0}$.
2. For $k = 0, \ldots, m$, the iteration
$$\boldsymbol{U}^{[0]} := \mathbf{0}, \qquad \boldsymbol{U}^{[k+1]} := \boldsymbol{H}(0, \boldsymbol{U}^{[k]})$$
that computes g.f. for the structures of size 0, is well defined.
3. The Jacobian matrix $\boldsymbol{\partial H}/\boldsymbol{\partial Y}$ is nilpotent at $(0, \boldsymbol{U}^{[m]})$.

This is by analogy with our extension of Joyal's implicit species theorem for combinatorial systems without the constraint $\boldsymbol{\mathcal{H}}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

Generating functions solutions of well-founded systems

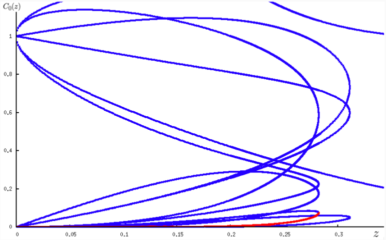
$$C_0(z) = zC_1(z)C_2(z)C_3(z)(C_1(z)+C_2(z))$$

$$C_1(z)=z+\frac{z}{1-C_1(z)^2C_3(z)^2}$$

$$C_2(z)=z+\frac{z^2}{(1-zC_2(z)^2/(1-z))(1-C_2(z))}$$

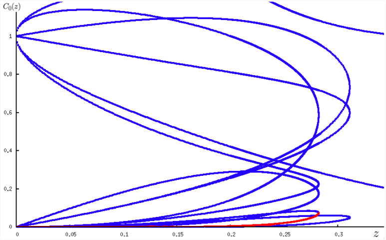
$$C_3(z)=z+\frac{z(3z+z^2+z^2C_1(z)C_3(z))}{1-C_1^2(z)}$$

A system of generating function equations may have many solutions...



Generating functions solutions of well-founded systems

$$C_0(z) = zC_1(z)C_2(z)C_3(z)(C_1(z)+C_2(z))$$
$$C_1(z)=z+\frac{z}{1-C_1(z)^2C_3(z)^2}$$
$$C_2(z)=z+\frac{z^2}{(1-zC_2(z)^2/(1-z))(1-C_2(z))}$$
$$C_3(z)=z+\frac{z(3z+z^2+z^2C_1(z)C_3(z))}{1-C_1^2(z)}$$



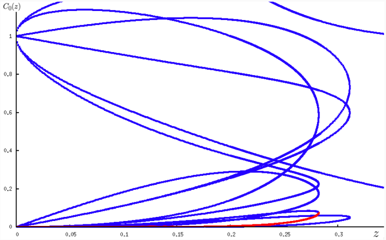
A system of generating function equations may have many solutions...

The only solution of interest is the one that corresponds to the value of the generating function of $\mathcal{C}$.

Generating functions solutions of well-founded systems

$$C_0(z) = zC_1(z)C_2(z)C_3(z)(C_1(z)+C_2(z))$$
$$C_1(z) = z + \frac{z}{1-C_1(z)^2 C_3(z)^2}$$
$$C_2(z) = z + \frac{z^2}{(1-zC_2(z)^2/(1-z))(1-C_2(z))}$$
$$C_3(z) = z + \frac{z(3z+z^2+z^2C_1(z)C_3(z))}{1-C_1^2(z)}$$

A system of generating function equations may have many solutions...



The only solution of interest is the one that corresponds to the value of the generating function of $\mathcal{C}$.

Definition (generating function solution)

If $\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ is *well-founded*, then it admits a **unique** solution $\boldsymbol{Y}(z)$ analytic in a neighborhood of 0 with $\boldsymbol{Y}(0) = \boldsymbol{U}^{[m]}$. This solution has nonnegative Taylor coefficients at 0.

$\boldsymbol{Y}(z)$ is called the *generating function solution* of the system.

Recall that $\boldsymbol{U}^{[m]}$ is the generating function for the structures of size 0.

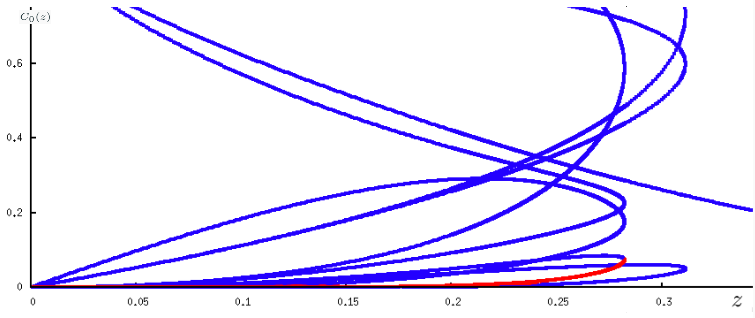
Second part: Analytic Functions

The centerpiece: computing the radius

Computation of the radius of convergence

First Principle of Coefficient Asymptotics: The location of a function's singularities dictates the exponential growth of its coefficients [Flajolet and Sedgewick, 2009].

Computation of the radius of convergence



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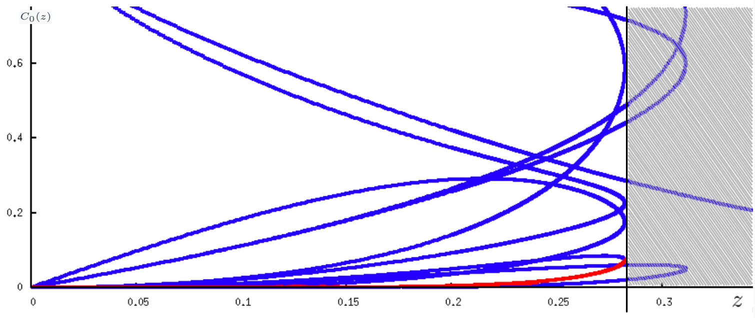
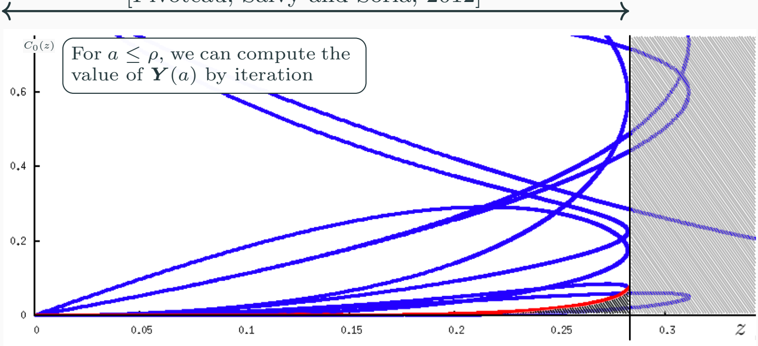


Computation of the radius of convergence

[Pivoteau, Salvy and Soria, 2012]



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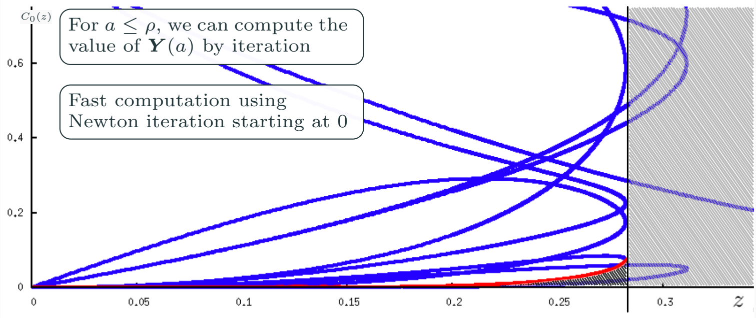
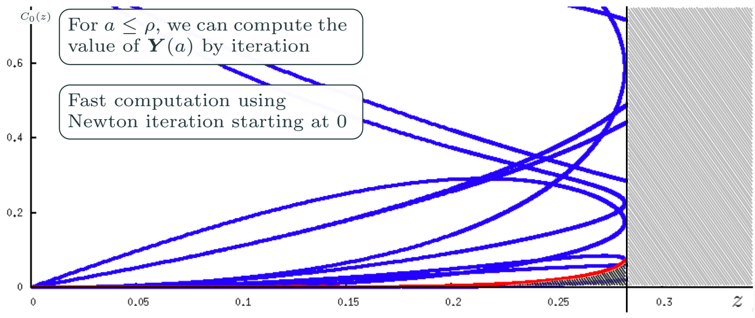


For $a \leq \rho$, we can compute the value of $\boldsymbol{Y}(a)$ by iteration

Fast computation using Newton iteration starting at 0

Computation of the radius of convergence

[Pivoteau, Salvy and Soria, 2012] [New]



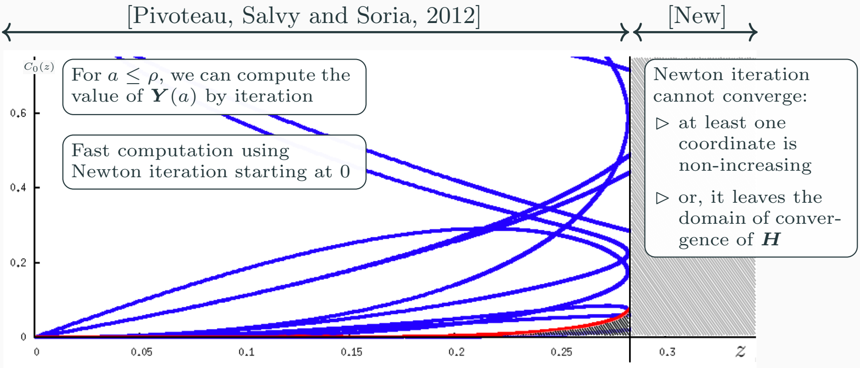
Theorem (Value of the generating function solution)

Let $\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ be a well-founded system and let ρ be the radius of convergence of its g.f. solution $\boldsymbol{Y}(z)$. Let $(a, \boldsymbol{B}) \in \mathbb{R}_{\geq 0}^{m+1}$ belong to the domain of convergence of $\boldsymbol{H}$ or its boundary and satisfy $\boldsymbol{B} = \boldsymbol{H}(a, \boldsymbol{B})$.

Then $\rho > 0$, $a \leq \rho$, $\boldsymbol{B} \geq \boldsymbol{Y}(a)$ and, moreover, $\Lambda_{\boldsymbol{H}}(a, \boldsymbol{B}) \leq 1 \Leftrightarrow \boldsymbol{B} = \boldsymbol{Y}(a)$.

$\Lambda_{\boldsymbol{H}}(x, \boldsymbol{y})$ is the *spectral radius* of the Jacobian matrix $\boldsymbol{\partial H}/\boldsymbol{\partial Y}$.

Computation of the radius of convergence



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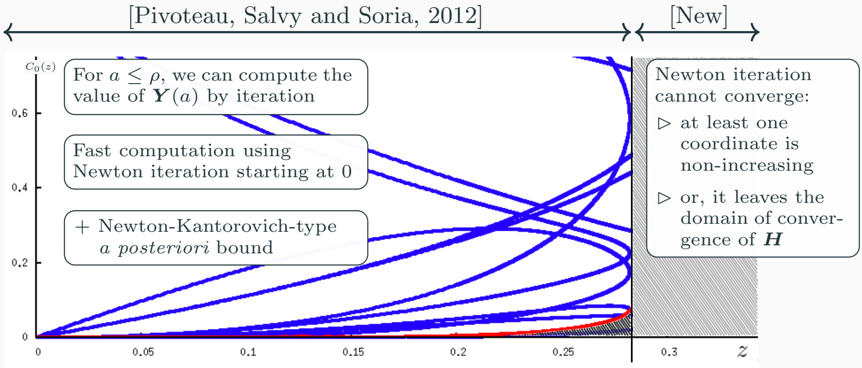

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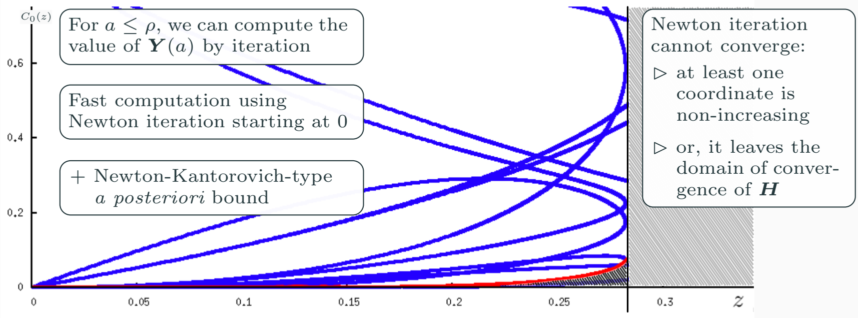

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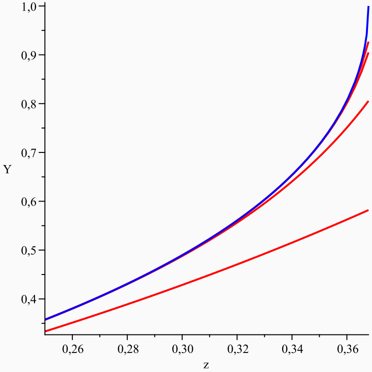
▷ We can compute the radius of convergence **by dichotomy**...

Another way to compute the radius

… by dichotomy, but Newton iteration is slow when a is close to ρ.



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Let $\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ *be an* irreducible system, *let* ρ *be the radius of convergence of its solution* $\boldsymbol{Y}(z)$ *and let* $\boldsymbol{\tau} = \lim_{u \to \rho-} \boldsymbol{Y}(u)$*. Then,*

1. *either* $(\rho, \boldsymbol{\tau})$ *is on the boundary of the domain of convergence of* $\boldsymbol{H}$ *or it is in the interior of it and* $\Lambda_{\boldsymbol{H}}(\rho, \boldsymbol{\tau}) = 1$*;*

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When $\Lambda_{\boldsymbol{H}}(\rho, \boldsymbol{\tau}) = 1$, the radius is a solution of the **characteristic system**

$$\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y}), \quad \det\left(\mathbf{Id} - \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{Y}}\right) = 0.$$

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But it may have several solutions... For instance, this characteristic system

$$\{y_1 - z(y_1^2 + y_2 + 1), y_2 - z(y_2^2 + y_1 + 1), 4y_1y_2z^2 - 2y_1z - 2y_2z - z^2 + 1\}$$

has 2 solutions for (z, y_1, y_2), $(1/3, 1, 1)$ and $((2\sqrt{2} - 1)/7, \sqrt{2} + 1, \sqrt{2} + 1)$.

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1. *either* $(\rho, \boldsymbol{\tau})$ *is on the boundary of the domain of convergence of* $\boldsymbol{H}$ *or it is in the interior of it and* $\Lambda_{\boldsymbol{H}}(\rho, \boldsymbol{\tau}) = 1$;
2. *if* $(a, \boldsymbol{B})$ *is s.t.* $\boldsymbol{B} = \boldsymbol{H}(a, \boldsymbol{B})$ *and* $\Lambda_{\boldsymbol{H}}(a, \boldsymbol{B}) = 1$, *then* $(a, \boldsymbol{B}) = (\rho, \boldsymbol{\tau})$.

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$\triangleright$ We can filter them with the criterion $\Lambda_{\boldsymbol{H}}(a, \boldsymbol{B}) = 1$.

Faster computation of the radius

- We can use Newton iteration on the characteristic system...
- ... but this is not a positive (combinatorial) system: there is no unconditional convergence.
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Algorithm

Decompose the system into **irreducible components**, and

- choose a "close enough" to the radius, by dichotomy, with Newton iteration for $\boldsymbol{Y} = \boldsymbol{H}(a, \boldsymbol{Y})$;
- use Newton iteration started at $(a, \boldsymbol{Y}(a))$ on the characteristic system;
- check that $\rho = a$ is found by verifying that $\Lambda_{\boldsymbol{H}}(a, \boldsymbol{Y}(a)) = 1$.

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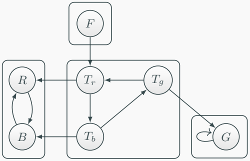
On the previous example, starting at $a = 0.05375$ and the values of $y_1(a)$ and $y_2(a)$, Newton's iteration on the characteristic system gives (for (y_1, z)):

$$(0.0569876, 0.05375), (1.98856, 1.76650), (-2.49792, 5.32162), (-1.25430, 5.31255), \ldots$$
$$(1.00003, 0.317617), (0.998510, 0.333309), (0.999996, 0.333333), \mathbf{(1.00000, 0.333333)}.$$

Computation of the radius for colored trees

- 1st: $S_G = \{G(z) = z + G(z)^2\}$
  - $G(z) = \frac{1-\sqrt{1-4z}}{2} \quad \to \quad \rho_G = 1/4.$



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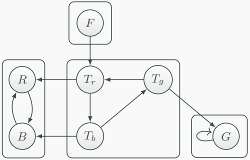

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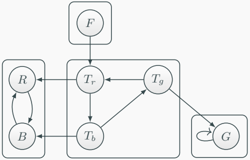
- 2nd: $S_{BR} = \left\{B(z) = \frac{z}{1-R(z)}, R(z) = z^3 + \frac{z}{1-B(z)}\right\}$
  - Find a candidate using Newton's iteration on S_{BR}: $z = 0.2$.
  - Starting with the value of (B, R) at $z = 0.2$, Newton's iteration on the characteristic system gives $\rho_{BR} \in [0.2462661, 0.2462662]$.

Computation of the radius for colored trees



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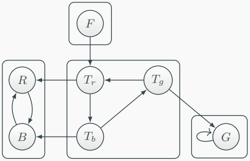
- 3rd: $S_T = \left\{T_r(z) = z + \frac{R(z)}{1-T_b(z)}, T_b(z) = \frac{B(z)}{1-T_g(z)}, T_g(z) = \frac{G(z)}{1-T_r(z)}\right\}$
  - the radius is either the minimum of ρ_G and ρ_{BR}, or smaller (where Λ should be 1).
  - Newton's iteration does not converge at 0.2462661 for S_T: $\rho < \rho_{BR}$
    - Find a candidate using Newton's iteration on S_T: $z = 0.16$.
    - Starting with the value of (T_r, T_b, T_g) at $z = 0.16$, Newton's iteration on the characteristic system gives $\rho_T \in [0.1703916, 0.1703917]$.

Computation of the radius for colored trees



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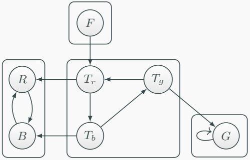
    - Starting with the value of (T_r, T_b, T_g) at $z = 0.16$, Newton's iteration on the characteristic system gives $\rho_T \in [0.1703916, 0.1703917]$.
- last: $S_F = \{F(z) = \exp(T_r(z))\}$
  - the radius of convergence is $\rho_T \in [0.1703916, 0.1703917]$.

Computation of the radius for colored trees



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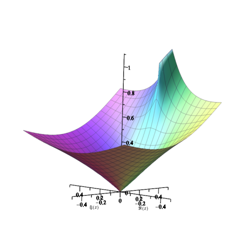
- 3rd: $S_T = \left\{T_r(z) = z + R(z)T_b(z)^2, T_b(z) = B(z)T_g(z)^2, T_g(z) = G(z)T_r(z)^2\right\}$
  - the radius is either the minimum of ρ_G and ρ_{BR}, or smaller (where Λ should be 1).
  - Newton's iteration converges at 0.2462661 for S_T
    - $\Lambda_{\boldsymbol{S}_T}(0.2462661) \approx 0.127 < 1$.
    - This implies that $\rho = \rho_{BR} \in [0.2462661, 0.2462662]$, from the component S_{BR}.
- last: $S_F = \{F(z) = \exp(T_r(z))\}$
  - the radius of convergence is $\rho_{BR} \in [0.2462661, 0.2462662]$.

The last pieces fall into place

Singularity Analysis [Flajolet and Odlyzko, 1990]

First Principle of Coefficient Asymptotics: The location of a function's singularities dictates the exponential growth of its coefficients.

Second Principle of Coefficient Asymptotics: The nature of a function's singularities determines the associated subexponential factor.



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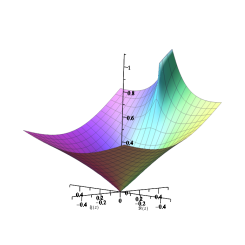


Next steps:

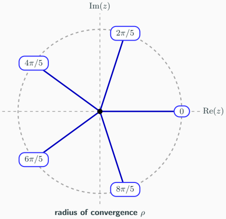
- locate all the **dominant singularities**;
- computation of the **local expansions** of the generating functions in the neighborhood of these singularities;
- **asymptotic behavior of the coefficients** using transfer theorems.

Dominant singularities

The radius of convergence ρ is a dominant singularity.

There might be others on the circle $|z| = \rho$.

Their arguments are **rational multiples of** π, induced by periodicity properties that can be computed from the combinatorial equations.



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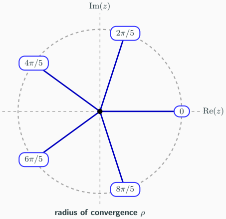
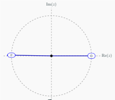


We extend scattered known results to all the well-founded specifications:

- An algorithm to **compute the periods** of the generating functions.
  ▷ Valuations + fixed-point iteration
- **Decompose the system** into strongly connected components and proceed bottom-up, restricting to the components having ρ as radius.
- Treating separately:
  - nonrecursive components
  - irreducible **linear/nonlinear** components

Dominant singularities of irreducible systems: examples

Nonlinear case:



$$\mathcal{A} = \mathcal{Z}^2 + \mathcal{Z}^2 \times \mathcal{B}^2, \quad \mathcal{B} = \mathcal{Z}^3 + \mathcal{Z}^3 \times \mathcal{A}^2.$$

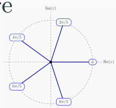
The valuations are $v_A = 2$ and $v_B = 3$ and the periods are solutions to the system $q_A = \gcd(6, q_B), q_B = \gcd(4, q_A)$.

Our algorithm gives $(0,0) \mapsto (6,4) \mapsto (2,2) \mapsto (2,2)$, which places dominant singularities at arguments at $e^{2ik\pi/2}$ for $k = 0$ and $k = 1$.

Linear case:

$$\mathcal{A} = \mathcal{Z}^2 + \mathcal{Z}^2 \times \mathcal{B}, \quad \mathcal{B} = \mathcal{Z}^3 + \mathcal{Z}^3 \times \mathcal{A}.$$

The equations for the periods are $q_A = \gcd(3, q_B)$ and $q_B = \gcd(2, q_A)$, which gives $q_A = q_B = 1$... but the generating functions solutions are



$$A(z) = z^2 \frac{1+z^3}{1-z^5}, \quad B(z) = z^3 \frac{1+z^2}{1-z^5}, \ldots$$

... which indicates dominant singularities at $e^{2ik\pi/5}$ for $k = 0, \ldots, 4$.

Dominant singularities of irreducible systems: algorithm

Algorithm Dominant singularities of the generating function:

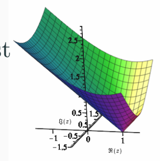
Input: $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\mathcal{Z}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Y}})$ an irreducible constructive component with $\partial \boldsymbol{\mathcal{H}} / \partial \mathcal{U} \neq 0$;
R radius of convergence of its generating function solution $\boldsymbol{Y}$;
$S_{\boldsymbol{\mathcal{U}}} := \boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{G}}(\mathcal{Z}, \boldsymbol{\mathcal{U}})$
$\Delta_{\boldsymbol{\mathcal{U}}} := \bigcup_{\mathcal{U} \in \boldsymbol{\mathcal{U}}} \Delta_{\mathcal{U}}(R)$ with $\Delta_{\mathcal{U}}(R) \supseteq \{\alpha \mid Re^{i\alpha}$ singularity of $U\}$

Output: $\mathcal{S} \supseteq \{\alpha \mid Re^{i\alpha}$ singularity of $\boldsymbol{Y}\}$

if $\partial^2 \boldsymbol{\mathcal{H}} / \partial \boldsymbol{\mathcal{Y}}^2 \neq 0$ **then** ◁ nonlinear case
  $\boldsymbol{q} :=$ Periods$(\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\mathcal{Z}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Y}}), S_{\boldsymbol{\mathcal{U}}})$
  return $\{2k\pi/q_1 \mid k = 0, \dots, q_1 - 1\}$ ◁ Theorems
else ◁ linear case
  if $\Lambda_{\boldsymbol{H}}(R) < 1$ **then** ◁ inequality
    return $\Delta_{\boldsymbol{\mathcal{U}}}$
  else
    Define $\boldsymbol{\mathcal{A}}(\mathcal{Z}, \boldsymbol{\mathcal{U}}), \boldsymbol{\mathcal{B}}(\mathcal{Z}, \boldsymbol{\mathcal{U}})$ by $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{Y}}$ ◁ $\boldsymbol{\mathcal{Y}} = (\mathbf{Id} - \boldsymbol{\mathcal{B}})^{-1}\boldsymbol{\mathcal{A}}$
    $q := \gcd($Periods$(\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{B}}(\mathcal{Z}, \boldsymbol{\mathcal{U}}),\ S_{\boldsymbol{\mathcal{U}}}))$
    if $q = 0$ **then**
      $\boldsymbol{Q} :=$ Periods$(\boldsymbol{\mathcal{Y}} = \mathbf{Id} + \boldsymbol{\mathcal{B}}(\mathcal{Z}, \boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{Y}},\ S_{\boldsymbol{\mathcal{U}}})$
      $q := \gcd(\text{diagonal}(\boldsymbol{Q}))$
    return $\Delta_{\boldsymbol{\mathcal{U}}} \cup \{2k\pi/q \mid k = 0, \dots, q - 1\}$ ◁ Theorems

Singular behavior (1/2)

We **classify the singular behaviors** for all constructive systems and prove a **gap property**: the generating function...



- ... is either of **superpolynomial growth**: it grows at least as fast as $\exp(c \ln^2(1 - z/\rho))$ for some $c > 0$.
- or it has an **algebraic-logarithmic** behavior.
- No other behavior is possible.

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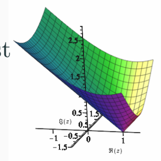

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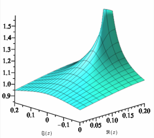
In the algebraic-logarithmic case:



- Either it behaves like:

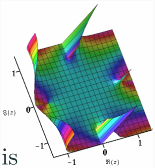
$$c_0 + c_1(1 - z/\rho)^{1/r} + c_2(1 - z/\rho)^{2/r} + \cdots ,$$

  where r is a power of 2 [Banderier and Drmota, 2015].
- Otherwise, it has an infinite limit and is equivalent to

$$C(1 - z/\rho)^{\alpha} \ln^k\left(\frac{1}{1 - z/\rho}\right), \qquad C > 0, \alpha \le 0, k \in \mathbb{N}, (\alpha, k) \neq (0, 0).$$

- More complex exponents may occur at other singularities.

Singular behavior (2/2)



Example: the generating function of $\mathcal{Y} = \textsc{Set}(\mathcal{Z} \times \textsc{Cyc}(\mathcal{Z}^4))$ is

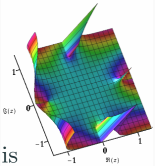
$$Y(z) = \exp\left(z \ln\left(\frac{1}{1-z^4}\right)\right)$$

and has 4 dominant singularities at $\{\pm 1, \pm i\}$. Its local behavior is

$$\begin{aligned}
f(z) &= \tfrac{1/4}{1-z} - \tfrac{1}{4}\ln\left(\tfrac{1}{1-z}\right) + \tfrac{3}{8} + \tfrac{\ln 2}{2} + O\left((1-z)\ln^2\left(\tfrac{1}{1-z}\right)\right), && z \to 1,\\
f(z) &= 4(1+z) + O\left((1+z)^2 \ln\left(\tfrac{1}{1+z}\right)\right), && z \to -1,\\
f(z) &= 4^{-i}(1-z/i)^{-i} + O\left((1-z/i)^{-i+1}\ln\left(\tfrac{1}{1-z/i}\right)\right), && z \to i
\end{aligned}$$

and the conjugate of the last one as $z \to -i$.

Singular behavior (2/2)



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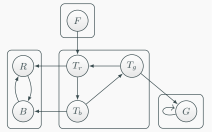
Theorem (Singular behavior)

Consider a constructive system and $\sigma \in \mathbb{C}$ that lies inside or on the boundary of the domain of convergence of its generating function solution.

Our **algorithm** *computes truncations of the expansions of all its coordinates that are analytic or have an algebraic-logarithmic singularity at σ. It returns* Exp *for the other coordinates.*

Asymptotic behavior



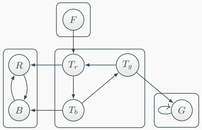
The expansion of $F(z)$ at its dominant singularity ρ is of the form

$$c_0 + c_1 u^{1/2} + c_2 u + c_3 u^{3/2} + \cdots ,$$

with $u = 1 - z/\rho$. Numerical values can be refined to arbitrary precision:

$$F = 1.801525 - 0.892560\, u^{1/2} + 0.044708\, u + 0.044641\, u^{3/2} + \cdots .$$

Asymptotic behavior



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By singularity analysis, the asymptotic behavior of the coefficients of $F(z)$ is deduced directly from this expansion. The nth coefficient behaves like:

$$\frac{\rho^{-n}}{n!}\left(c_1 \frac{\Gamma(n-1/2)}{\Gamma(-1/2)} + c_3 \frac{\Gamma(n-3/2)}{\Gamma(-3/2)} + c_5 \frac{\Gamma(n-5/2)}{\Gamma(-5/2)} + O(\Gamma(n-7/2))\right), \quad n \to \infty.$$

With the numerical values and $\rho \in [0.1703916, 0.1703917]$:

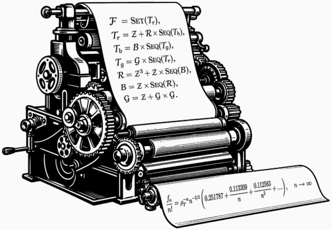
$$\frac{f_n}{n!} = \rho^{-n} n^{-3/2}\left(0.251787 + \frac{0.113309}{n} + \frac{0.112583}{n^2} + \cdots\right), \quad n \to \infty.$$

Theorem (Asymptotic behavior)

Let $\boldsymbol{Y}$ be a constructible generating function. An **asymptotic expansion** *can be computed for all Y_i that have an algebraic-logarithmic dominant singularity.*

Conclusion

We give an almost **complete algorithmic chain** going from combinatorial systems to **asymptotic expansions** for all the exponential generating functions produced by the symbolic method of Flajolet and Sedgewick.

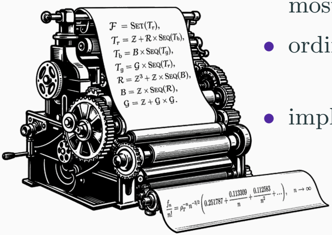


Conclusion

We give an almost **complete algorithmic chain** going from combinatorial systems to **asymptotic expansions** for all the exponential generating functions produced by the symbolic method of Flajolet and Sedgewick.

There is still work to be done:

- superpolynomial growth;
- cardinality constraints for $\textsc{Seq}, \textsc{Set}, \textsc{Cyc}$; mostly technical;
- ordinary generating functions: for some simple cases;
- implementation: Maple worksheets, but no library.



The End

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Decidability issues

Our algorithms need to decide equality between real values. This is a difficult question in general. We define an oracle tailored for our needs:

Constant Oracle

Given a constructive system, the *Constant Oracle* can

- decide whether $\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ has a solution, for a fixed $z \geq 0$;
- if so, decide whether it has a solution with $\Lambda(z, \boldsymbol{Y}) = 1$;
- test equality or inequality between solutions of such systems.

Proposition

If the radius of convergence of a constructible generating function is finite, then it can be computed with the help of the Constant Oracle.