A polynomial realization of the Hopf algebra of uniform block permutations.

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Abstract. We investigate the combinatorial Hopf algebra based on uniform block permutations and we realize this algebra in terms of noncommutative polynomials in infinitely many bi-letters.

1 Introduction.

For several years, Hopf algebras based on combinatorial objects have been thoroughly investigated. Examples include the Malvenuto-Reutenauer Hopf algebra $\text{FQSym}$ whose bases are indexed by permutations $[7,4,2]$, the Loday-Ronco Hopf algebra $\text{PBT}$ whose bases are indexed by planar binary trees $[6]$, the Hopf algebra of free symmetric functions $\text{FSym}$ whose bases are indexed by standard Young tableaux $[11]$.

The product and the coproduct, which define the structure of a combinatorial Hopf algebra, are, in general, rules of composition and decomposition given by combinatorial algorithms such as the shuffle and the deconcatenation of permutations, as in $\text{FQSym}$ $[4]$, of packed words, as in $\text{WQSym}$ $[9]$ or of parking functions, as in $\text{PQSym}$ $[10]$. These can also be given by the disjoint union and the admissible cuts of rooted trees, as in the Connes-Kreimer Hopf algebra $[3]$, or by concatenation and subgraphs/contractions of graphs $[8]$.

Computing with these structures can be difficult, and polynomial realization can bring up important simplifications.

We can see a polynomial realization as the image by an injective Hopf algebra morphism from a combinatorial Hopf algebra to an algebra of polynomials (commutative or not).

The idea is to encode the combinatorial objects by polynomials, in such a way that the product of the combinatorial Hopf algebra become the ordinary product of polynomials, and that the coproduct be given by the disjoint union of alphabets, endowed with some extra structure such as an order relation. This can be done, e.g., for $\text{FQSym}$ $[4]$, $\text{WQSym}$ $[9]$, $\text{PQSym}$ $[10]$ and Connes-Kreimer $[5]$.

In this paper, we realize the Hopf algebra $\text{UBP}$ of uniform block permutations introduced by Aguiar and Orellana $[1]$. We begin by recalling the preliminary notions of uniform block permutations in Section 2. We recall, in Section 3 the Hopf algebra structure of uniform block permutations, we translate this...
structure via the decomposition and we describe the dual structure. Using polynomial realizations of Hopf algebras based on permutations and set partitions, we realize this algebra in Section 4. We finish by finding the Hopf algebra $WQSym$ as a quotient of $UBP$ and the dual as a subalgebra of $UBP^*$.

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2 Definitions and decompositions.

**Definition 1** A uniform block permutation of size $n$ is a bijection between two set partitions of $\{1, \cdots, n\}$ that maps each block to a block of the same cardinality.

**Notation 1** Let $A = \{A_1, A_2, \cdots, A_k\}$ and $A' = \{A'_1, A'_2, \cdots, A'_k\}$ be two set partitions of the same size. We denote the uniform block permutation $f : A \rightarrow A'$ for which the image of $A_i$ is $A'_i$ by $$(A_1 | A_2 | \cdots | A_k) \mapsto (A'_1 | A'_2 | \cdots | A'_k).$$

**Example 1** Here are all uniform block permutations of size 1 and 2:

$n = 1 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$n = 2 \quad \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$

In the case where the set partitions consist of singletons (i.e., $A = \{\{1\}, \cdots, \{n\}\}$), we find the permutations of size $n$.

**Definition 2** Let $f$ and $g$ be two uniform block permutations. The concatenation of $f$ and $g$, denoted by $f \times g$, is the uniform block permutation obtained by adding the size of $f$ to all entries of $g$ and concatenating the result to $f$.

**Example 2**

$$\begin{pmatrix} 13 & 2 \\ 23 & 1 \end{pmatrix} \times \begin{pmatrix} 13 & 2 & 46 & 5 \\ 46 & 2 & 15 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 2 & 46 & 5 & 79 & 48 & 6 \\ 23 & 1 & 79 & 5 & 9 \end{pmatrix}.$$ 

**Definition 3** Let $f : A \rightarrow A'$ and $g : B \rightarrow B'$ be two uniform block permutations of the same size $n$. The composition $g \circ f : C \rightarrow C'$ of $f$ and $g$ is defined by the following process. The blocks $C$ of the set partition $C$ are the subsets of $\{1, \cdots, n\}$ which are minimal for the two properties

1. $C$ is a union of blocks of $A$
2. $f(C)$ is a union of blocks $B_i$ of $B$.

The image of $C$ is the union of images $g(B_i)$.

Note that if $f$ is a permutation, then the set partition $C'$ is $B'$, and if $g$ is a permutation, then the set partition $C$ is $A$. 

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Uniform block permutations

Example 3

\[
\begin{pmatrix} 125 & 346 \\ 236 & 145 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 2 & 6 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 136 & 245 \\ 236 & 145 \end{pmatrix},
\]

\[
\begin{pmatrix} 15 & 2 & 3 & 4 & 6 \\ 46 & 2 & 1 & 5 & 3 \end{pmatrix} \circ \begin{pmatrix} 14 & 2 & 36 & 5 \\ 23 & 1 & 45 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 236 & 5 \\ 12 & 456 & 3 \end{pmatrix}.
\]

The data of a permutation and a set partition is enough to describe a uniform block permutation. Given a uniform block permutation \( f : A \rightarrow A' \) of size \( n \), there is a permutation \( \sigma \) in \( S_n \) such that

\[
A' = \bigcup_{H \in A} \left\{ \bigcup_{x \in H} \{ \sigma(x) \} \right\}.
\]

Conversely, a set partition of the set \( \{1, \ldots, n\} \) and a permutation of \( S_n \) determine a uniform block permutation of size \( n \). Any uniform block permutation can thus decompose (non-uniquely) as

\[
f = \sigma \circ \text{Id}_A = \text{Id}_{A'} \circ \sigma.
\]

The first equality expresses the fact that one can group the images \( \sigma_i \) whose indices are in the same block in \( A \), the second expresses the fact that one can group the images \( \sigma_i \) whose values are in the same block in \( A' \).

3 Algebraic structures.

In [1], Aguiar and Orellana introduced a Hopf algebra whose basis is the set of uniform block permutations. It will be denoted by \( \text{UBP} \), standing for Uniform Block Permutations. Let us recall the convolution of two permutations. Let \( \tau \) and \( \tau' \) be two permutations:

\[
\tau \ast \tau' = \sum_{\sigma = u \cdot v} \sigma
\]

where \( \text{std}(\sigma_1 \cdots \sigma_{m+n}) = 12 \cdots m + n \) and \( \text{std}(\sigma_{m+1} \cdots \sigma_{m+n}) = 12 \cdots n \)

\( \text{(i.e., } \sigma \text{ appears in } 12 \cdots m \ast 12 \cdots n\).
Let $I$ be a uniform block permutation of size $n$. Since $\sigma$ is a uniform block permutation, $\sigma^{-1}$ is also a uniform block permutation.

\begin{align*}
\Delta(f) := \sum_{i \in I(A')} f_i \otimes f'_{n-i}.
\end{align*}

Example 7

\begin{align*}
\Delta \left( \begin{array}{ccc}
13 & 256 & 4 \\
15 & 234 & 6
\end{array} \right) &= \left( \begin{array}{ccc}
13 & 256 & 4 \\
15 & 234 & 6
\end{array} \right) \otimes \left( \begin{array}{ccc}
13 & 256 & 4 \\
15 & 234 & 6
\end{array} \right) + \left( \begin{array}{ccc}
13 & 245 & 1 \\
15 & 234 & 6
\end{array} \right) \otimes \left( \begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3
\end{array} \right) + \left( \begin{array}{ccc}
13 & 256 & 4 \\
15 & 234 & 6
\end{array} \right) \otimes \left( \begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3
\end{array} \right).
\end{align*}
Uniform block permutations

One can rewrite this product and coproduct using the decomposition of a uniform block permutation. Given $f : A \rightarrow A'$ and $g : B \rightarrow B'$ two uniform block permutations, $f = \sigma \circ \text{Id}_A = \text{Id}_{A'} \circ \sigma$ and $g = \tau \circ \text{Id}_B = \text{Id}_{B'} \circ \tau$, we have

$$f \ast g = (\sigma \ast \tau) \circ \text{Id}_{A \times B}$$

$$(2)$$

$$\Delta(f) = \sum_{i \in I(A')} f_i^{(1, \ldots, i)} \otimes f_i^{(i+1, \ldots, n)}$$

$$(3)$$

where

$$f_i^{(1, \ldots, i)} = \text{Id}_{A'}^{(1, \ldots, i)} \circ \sigma^{(1, \ldots, i)}$$

$$f_i^{(i+1, \ldots, n)} = \text{Id}_{A'}^{(i+1, \ldots, n)} \circ \text{std}(\sigma^{(i+1, \ldots, n)}).$$

Example 8

$$\left( \begin{array}{c} 1 \\ 1 \\ 12 \\ 12 \end{array} \right) \ast \left( \begin{array}{c} 1 \\ 1 \\ 12 \\ 12 \end{array} \right) = 1 \circ \text{Id}_{f \{1\}} \ast 12 \circ \text{Id}_{f \{1,2\}}$$

$$= (1 \ast 12) \circ \text{Id}_{f \{1\}, \{2,3\}}$$

$$= 123 + 213 + 312 \circ \text{Id}_{f \{1\}, \{2,3\}}$$

$$= \left( \begin{array}{c} 1 \\ 1 \\ 23 \\ 23 \end{array} \right) + \left( \begin{array}{c} 1 \\ 2 \\ 13 \\ 13 \end{array} \right) + \left( \begin{array}{c} 1 \\ 3 \\ 12 \\ 12 \end{array} \right).$$

Example 9 We set $A = \{\{1, 5\}, \{2, 3, 4\}, \{6\}\}$. 

$$\Delta \left( \begin{array}{c} 13 \\ 15 \\ 256 \\ 234 \\ 4 \\ 6 \end{array} \right) = \Delta(\text{Id}_A \circ \text{Id}_{125634})$$

$$= \varepsilon \otimes \text{Id}_A \circ \text{Id}_{125634} + \text{Id}_{\{1,5\}, \{2,3,4\}} \circ \text{Id}_{f \{1\}} \circ 1 + \text{Id}_A \circ 125634 \otimes \varepsilon$$

$$= \left( \begin{array}{c} 13 \\ 15 \\ 256 \\ 234 \\ 4 \\ 6 \end{array} \right) \otimes \left( \begin{array}{c} 13 \\ 15 \\ 256 \\ 234 \\ 4 \\ 6 \end{array} \right) + \left( \begin{array}{c} 13 \\ 15 \\ 256 \\ 234 \\ 4 \\ 6 \end{array} \right) \otimes \left( \begin{array}{c} 13 \\ 15 \\ 256 \\ 234 \\ 4 \\ 6 \end{array} \right) \otimes \left( \begin{array}{c} 13 \\ 15 \\ 256 \\ 234 \\ 4 \\ 6 \end{array} \right).$$

It is proved in \cite{1} that UBP is a self-dual Hopf algebra. We can describe the dual structure. Let us first define another product and coproduct of uniform block permutations. Let $f : A \rightarrow A'$ and $g : B \rightarrow B'$ be two uniform block permutations, $f = \sigma \circ \text{Id}_A = \text{Id}_{A'} \circ \sigma$ and $g = \tau \circ \text{Id}_B = \text{Id}_{B'} \circ \tau$:

$$f \ast g = \text{Id}_{A' \times B'} \circ (\sigma \ast \tau)$$

$$(4)$$

$$\Delta_1(f) = \sum_{i \in I(A)} f_1 \cdots f_i \otimes f_{i+1} \cdots f_n$$

$$(5)$$

where

$$f_1 \cdots f_i = \text{std}(\sigma_1 \cdots \sigma_i) \circ \text{Id}_{A \{1, \ldots, i\}}$$

$$f_{i+1} \cdots f_n = \text{std}(\sigma_{i+1} \cdots \sigma_n) \circ \text{Id}_{A \{i+1, \ldots, n\}}.$$
Example 10

\[
\begin{pmatrix}
1 \\
1
\end{pmatrix} \sqcap \begin{pmatrix}
12 \\
12
\end{pmatrix} = \text{Id}_{\{1\},\{2,3\}} \circ (1 \sqcap 12)
\]

\[
= \text{Id}_{\{1\},\{2,3\}} \circ (123 + 213 + 231)
\]

\[
= \begin{pmatrix}
1 & 23 \\
1 & 23
\end{pmatrix} + \begin{pmatrix}
2 & 13 \\
1 & 23
\end{pmatrix} + \begin{pmatrix}
3 & 12 \\
1 & 23
\end{pmatrix}.
\]

\[
\Delta_1 \begin{pmatrix}
13 \\
15
\end{pmatrix} \begin{pmatrix}
256 \\
234
\end{pmatrix} \begin{pmatrix}
4 \\
6
\end{pmatrix} = \Delta_1(125634 \circ \text{Id}_{\{1,3\},\{2,5,6\},\{4\}})
\]

\[
= \varepsilon \otimes 125634 \circ \text{Id}_{\{1,3\},\{2,5,6\},\{4\}} + 125634 \circ \text{Id}_{\{1,3\},\{2,5,6\},\{4\}} \otimes \varepsilon
\]

\[
= \begin{pmatrix}
13 \\
15
\end{pmatrix} \begin{pmatrix}
256 \\
234
\end{pmatrix} \begin{pmatrix}
4 \\
6
\end{pmatrix} + \begin{pmatrix}
13 \\
15
\end{pmatrix} \begin{pmatrix}
256 \\
234
\end{pmatrix} \begin{pmatrix}
4 \\
6
\end{pmatrix} \otimes \begin{pmatrix}
13 \\
15
\end{pmatrix} \begin{pmatrix}
256 \\
234
\end{pmatrix} \begin{pmatrix}
4 \\
6
\end{pmatrix}.
\]

The two structures of Hopf algebra are dual to each other for the scalar product

\[
\langle f, g \rangle = \delta_{f,g^{-1}}.
\]  

(6)

In other words, we have:

\[
\langle \Delta f, g \otimes h \rangle = \langle f, g \sqcap h \rangle
\]  

(7)

\[
\langle \Delta_1 f, g \otimes h \rangle = \langle f, g * h \rangle.
\]  

(8)

4 Polynomial realization.

Given two alphabets \( B = \{b_i; 0 \leq i \} \) and \( C = \{c_i; 0 \leq i \} \), with \( C \) totally ordered, we consider an alphabet \( A = \{ (b_i c_j); 0 \leq i, j \} \) of bi-letters endowed with two relations:

- for all \( i \) and \( k \), \( (b_i c_j) \preceq (b_k c_l) \) if and only if \( c_j \leq c_l \),

- for all \( j \) and \( l \), \( (b_i c_j) \equiv (b_k c_l) \) if and only if \( b_i = b_k \).

The product of two biwords is the concatenation

\[
\begin{pmatrix}
u_1 \\
v_1
\end{pmatrix} \begin{pmatrix}
u_2 \\
v_2
\end{pmatrix} = \begin{pmatrix}
u_1u_2 \\
v_1v_2
\end{pmatrix}.
\]

We extend the definition of the standardization to biwords. Let \( w = \begin{pmatrix} u \\ v \end{pmatrix} \) be a biword. The standard-ized of \( w \) is the permutation \( \text{std}(v) \).
Example 11

\[
\text{std} \left( \begin{pmatrix} b_1 b_1 b_3 b_1 b_2 \\ c_2 c_3 c_2 c_1 c_6 \end{pmatrix} \right) = 24315.
\]

Let \( f : A \rightarrow A' \) be a uniform block permutation. Let \( \xi_f \) be the minimum permutation (for the lexicographic order) such that \( f = \xi_f \circ \text{Id}_A \). We say that a biword \( w \) is \( f \)-compatible if

- \( \text{std}(w) = \xi_f \)

and

- if \( i \) and \( j \) are in the same block in \( A \), then the bi-letters \( w_i \) and \( w_j \) satisfy \( w_i \equiv w_j \).

We set:

\[
G_f(A) = \sum_{w: w \text{ is } f \text{-compatible}} w.
\]

Example 12 Let \( f = \begin{pmatrix} 13 & 256 \\ 15 & 234 \end{pmatrix} \) be a uniform block permutation. The permutation \( \sigma = 125634 \) is the minimum permutation associated with \( f \) and we have

\[
G_f(A) = \sum_{i,j,k} \left( \begin{pmatrix} b_1 b_2 b_3 b_j \\ c_1 c_2 c_3 c_4 c_5 \end{pmatrix} \right).
\]

Theorem 1 The polynomials \( G_f(A) \) provide a realization of the Hopf algebra of uniform block permutations. That is to say

- \( G_f(A) G_g(A) = \sum_{h \in f \ast g} G_h(A) \)

- Let \( A' \) be an alphabet isomorphic to the alphabet \( A \) such that, for any bi-letter \( w' \) of \( A' \) and for any bi-letter \( w \) of \( A \), we have \( w \preceq w' \).

Define \( A \ast A' \) as the disjoint union of \( A \) and \( A' \) endowed with the relations \( \preceq \) and \( \equiv \). If we allow the bi-letters of \( A \) and \( A' \) to commute and identify \( P(A) \ast Q(A') \) with \( P \otimes Q \), we have:

\[
G_f(A \ast A') = \Delta(G_f).
\]

Example 13

\[
\begin{pmatrix} 1 & 1 \end{pmatrix} (A) \begin{pmatrix} 1 & 1 \end{pmatrix} (A) = \sum_{b \in B, c \in C} \begin{pmatrix} b & c \end{pmatrix}^2 \begin{pmatrix} b_1 b_2 \\ c_1 c_2 \end{pmatrix} + \sum_{b_1, b_2 \in B, c_1, c_2 \in C: c_1 \leq c_2} \begin{pmatrix} b_1 b_2 \\ c_1 c_2 \end{pmatrix} + \sum_{b_1, b_2 \in B, c_1, c_2 \in C: c_1 > c_2} \begin{pmatrix} b_1 b_2 \\ c_1 c_2 \end{pmatrix} = G \begin{pmatrix} 1 & 2 \end{pmatrix} (A) + G \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} (A).
\]
\[
G \left( \begin{array}{cc}
12 \\
12
\end{array} \right) (A) G \left( \begin{array}{cc}
1 \\
1
\end{array} \right) (A) = \sum_{b \in B \atop \text{std}(b)=12} \left( \begin{array}{c}
b \\
v
\end{array} \right) \sum_{c \in C} \left( \begin{array}{c}
c
\end{array} \right) \\
\sum_{i_1,i_2 \atop \text{std}(c_1)=12 \atop \text{std}(c_2)=1} \left( \begin{array}{c}
b_{i_1} b_{i_2}
\end{array} \right) v_1 v_2 \\
\sum_{c_1 \leq c_2 \leq c_3} \left( \begin{array}{c}
b_{i_1} b_{i_2} c_1 c_2 c_3
\end{array} \right) + \sum_{c_1 \leq c_2 \leq c_3} \left( \begin{array}{c}
b_{i_1} b_{i_2} c_1 c_2 c_3
\end{array} \right) + \sum_{c_1 \leq c_2 \leq c_3} \left( \begin{array}{c}
b_{i_1} b_{i_2} c_1 c_2 c_3
\end{array} \right)
\]

= \sum_{i_1,i_2 \atop \text{std}(c_1)=12} \left( \begin{array}{c}
b_{i_1} b_{i_2} c_1 c_2 c_3
\end{array} \right) v_1 v_2 \\
\sum_{c_1 \leq c_2 \leq c_3} \left( \begin{array}{c}
b_{i_1} b_{i_2} c_1 c_2 c_3
\end{array} \right) + \sum_{c_1 \leq c_2 \leq c_3} \left( \begin{array}{c}
b_{i_1} b_{i_2} c_1 c_2 c_3
\end{array} \right) + \sum_{c_1 \leq c_2 \leq c_3} \left( \begin{array}{c}
b_{i_1} b_{i_2} c_1 c_2 c_3
\end{array} \right)
\]

\[\sum_{12 \leq v} (A) = \sum_{12 \leq v} (A').\]

**Example 14** We set \( \tau = 2431. \)

\[
G \left( \begin{array}{ccc}
13 & 24 & 3 \\
13 & 24 & 2
\end{array} \right) (A + A') = \sum_{i,j \atop c_1 \leq c_2 \leq c_3 \leq c_4} \left( \begin{array}{c}
b_{i} b_{j} b_{k} c_1 c_2 c_3 c_4
\end{array} \right) + \sum_{i,j \atop c_1' \leq c_2' \leq c_3' \leq c_4'} \left( \begin{array}{c}
b_{i} b_{j} b_{k} c_1 c_2 c_3 c_4
\end{array} \right) + \sum_{i,j \atop c_1 \leq c_2 \leq c_3 \leq c_4} \left( \begin{array}{c}
b_{i} b_{j} b_{k} c_1 c_2 c_3 c_4
\end{array} \right)
\]

\[= G \left( \begin{array}{ccc}
13 & 24 & 3 \\
13 & 24 & 2
\end{array} \right) + G \left( \begin{array}{ccc}
13 & 24 & 3 \\
13 & 24 & 1
\end{array} \right).\]

\[
G \left( \begin{array}{ccc}
13 & 24 & 4 \\
13 & 24 & 1
\end{array} \right) (A + A') = \sum_{i,j,k \atop \text{std}(c_1 c_2 c_3 c_4) = \tau} \left( \begin{array}{c}
b_{i} b_{j} b_{k} c_1 c_2 c_3 c_4
\end{array} \right) + \sum_{i,j,k \atop \text{std}(c_1' c_2' c_3' c_4') = \tau} \left( \begin{array}{c}
b_{i} b_{j} b_{k} c_1 c_2 c_3 c_4
\end{array} \right) + \sum_{i,j,k \atop \text{std}(c_1 c_2 c_3 c_4) = \tau} \left( \begin{array}{c}
b_{i} b_{j} b_{k} c_1 c_2 c_3 c_4
\end{array} \right)
\]

\[= G \left( \begin{array}{ccc}
13 & 24 & 3 \\
13 & 24 & 1
\end{array} \right) + G \left( \begin{array}{ccc}
13 & 24 & 2 \\
13 & 24 & 1
\end{array} \right) + G \left( \begin{array}{ccc}
13 & 24 & 4 \\
13 & 24 & 1
\end{array} \right).\]

5 Packed words.

We recall the definition of a packed word and the structure of Hopf algebra given in [29]. The packed word \( \text{pack}(w) \) associated with a word \( w \in A^* \) is the word \( u \) obtained by the following process. If \( b_1 < b_2 < \cdots < b_r \) are the letters occurring in \( w \), \( u \) is the image of \( w \) by the homomorphism \( b_i \rightarrow a_i \). A word \( w \) is said to be packed if \( \text{pack}(w) = w \).
Uniform block permutations

5.1 WQSym*

The Hopf algebra $\text{WQSym}^*$ is the $K$–vector space generated by packed words endowed with the shifted shuffle product and the coproduct given by, using notations in [5]:

$$u \boxdot v = u \shuffle v[\max(u)],$$  \hspace{1cm} (11)

$$\Delta(w) = \sum_{u \cdot v = w} \text{pack}(u) \otimes \text{pack}(v).$$  \hspace{1cm} (12)

Let $\Psi$ be the map that associates the word $w$ of the length $n$ with a permutation $\sigma$ of size $n$ and a set partition $\mathcal{A}$ of $\{1, 2, \ldots, n\}$ constructed by the following process. The $i$th letter $u_i$ of $u$ is the minimum of the letters $\sigma_j$ whose values $\sigma_j$ are in the same block in $\mathcal{A}$ as $\sigma_i$. The word $w$ is obtained by packing. The map $\Psi$ induces a linear map from the vector space generated by uniform block permutations to that generated by packed words. We define it by writing uniform block permutations $f$ in the form $\text{Id}_{\mathcal{A}'} \circ \sigma$ and applying $\Psi$ to $\sigma$ and $\mathcal{A}'$. We also call this map $\Psi$.

**Example 15**

$$\Psi\left(\begin{array}{cc} 12 & 12 \\ 12 & 12 \end{array}\right) = 11 \quad \Psi\left(\begin{array}{cc} 12 & 3 \\ 23 & 1 \end{array}\right) = 221 \quad \Psi\left(\begin{array}{cc} 13 & 24 \\ 12 & 34 \end{array}\right) = 1212$$

If we endow the vector spaces with the shifted shuffle product, the map $\Psi$ is an algebra morphism.

**Proposition 1** Let $f$ and $g$ be two uniform block permutations.

$$\Psi(f \boxdot g) = \Psi(f) \boxdot \Psi(g).$$  \hspace{1cm} (13)

**Example 16**

$$\Psi\left(\begin{array}{cc} 12 & 3 \\ 12 & 13 \end{array}\right) \boxdot \left(\begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array}\right) = 11234 + 11324 + 11342 + 13124 + 13142 + 13412 + 31124 + 31142 + 31412 + 34112$$

$$= 112 \boxdot 12$$

$$= \Psi\left(\begin{array}{cc} 12 & 3 \\ 13 & 2 \end{array}\right) \boxdot \Psi\left(\begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array}\right).$$

The map $\Psi$ cannot be a coalgebra morphism as shown in the example

$$\Psi \otimes \Psi \circ \Delta\left(\begin{array}{cc} 12 & 3 \\ 23 & 1 \end{array}\right) = \Psi \otimes \Psi\left[\begin{array}{c} \left(\begin{array}{cc} 12 & 3 \\ 23 & 1 \end{array}\right) \otimes \left(\begin{array}{cc} 12 & 3 \\ 12 & 1 \end{array}\right) + \left(\begin{array}{cc} 12 & 3 \\ 12 & 1 \end{array}\right) \otimes \left(\begin{array}{cc} 12 & 3 \\ 12 & 1 \end{array}\right)\right]\right]$$

$$= \varepsilon \otimes 221 + 11 \otimes 1 + 221 \otimes \varepsilon$$

$$\neq \varepsilon \otimes 221 + 1 \otimes 21 + 11 \otimes 1 + 221 \otimes \varepsilon$$

$$\neq \Delta(221).$$
The idea is that the coproduct on packed words deconcatenates the word in all suffixes and prefixes, whereas the coproduct on uniform block permutations preserves the blocks. This suggests to define another coproduct:

\[ \Delta'(w) = \sum_{u,v \in \text{pack}(w) \cap \text{alph}(u) \cap \text{alph}(v) = \emptyset} \text{pack}(u) \otimes \text{pack}(v). \]

Endowed with the shifted shuffle product and with the coproduct \( \Delta' \), the vector space generated by packed words is a combinatorial Hopf algebra denoted by \( \text{WQSym}^* \).

**Proposition 2** The Hopf algebras \( \text{WQSym}^* \) and \( \text{WQSym}^{**} \) are isomorphic.

The morphism of Hopf algebras is given by the following process. We define the following relation on packed words of the same length. Let \( u \) and \( v \) be two packed words, the cover relation is \( u \preceq v \) if and only if

- \( \text{std}(u) = \text{std}(v) \)
- there exists a \( i \) such that
  \[
  u_k = \begin{cases} 
  v_k & \text{if } v_k \leq i \\
  v_k - 1 & \text{otherwise}.
  \end{cases}
  \]

The map \( \Gamma \) defined as

\[
\text{WQSym}^* \rightarrow \text{WQSym}^{**},
\]

\[
w \mapsto \sum_{\exists u \preceq w} u
\]

is an isomorphism of Hopf algebras. For example, the image of a permutation is a permutation, the image of the word of length \( n \) consisting only of 1 is the sum of all the non-decreasing packed words of length \( n \).

**Example 17**

\[
\Gamma(11 \sqcup 1) = \Gamma(112 + 121 + 211) \\
= 112 + 123 + 121 + 132 + 311 + 312 \\
= 11 \sqcup 1 + 12 \sqcup 1 \\
= \Gamma(11) \sqcup \Gamma(1)
\]

\[
\Gamma \otimes \Gamma(\Delta(121)) = 121 \otimes \varepsilon + 132 \otimes \varepsilon + 12 \otimes 1 + 1 \otimes 21 + \varepsilon \otimes 121 + \varepsilon \otimes 132 \\
= \Delta'(121) + \Delta'(132) \\
= \Delta'(\Gamma(121)).
\]

**Proposition 3** The map \( \Psi \) is a surjective Hopf algebra morphism from \( \text{UBP}^* \) to \( \text{WQSym}^{**} \).
5.2 WQSym.

We denote by $\text{WQSym}$ the $\mathbb{K}$–vector space generated by the packed words endowed with the convolution product and the coproduct given by

$$u *_W v = \sum_{w = u \cdot v' \in PW, \text{pack}(u') = u, \text{pack}(v') = v} w,$$

(14)

$$\Delta(u) = \sum_{0 \leq k \leq \max(u)} u_{\{1, \ldots, k\}} \otimes \text{pack}(u_{\{k+1, \ldots, \max(u)\}})$$

(15)

The dual $\text{WQSym}'$ of $\text{WQSym}^*$ is the Hopf algebra endowed with the product

$$u *_{W'} v = \sum_{w = u \cdot v' \in PW, \text{pack}(u') = u, \text{pack}(v') = v} w$$

(16)

for any packed word $u$ and $v$, and the coproduct $\Delta$.

**Proposition 4** The Hopf algebra $\text{WQSym}'$ and $\text{WQSym}$ are isomorphic.

In fact, the map $\Gamma$ induces a dual map. This is

$$\Gamma^* : WQSym' \rightarrow WQSym, \quad w \mapsto \sum_{u \preceq w} u.$$

For example:

$$\Gamma^*(1212334) = 1212334 + 1212223 + 1212333 + 1212222.$$

This map is an isomorphism of Hopf algebras.

**Example 18**

$$\Gamma^*(212 *_{W'} 11) = \Gamma^*(21233 + 31322 + 32311)$$

$$= (21233 + 21222) + (31322 + 21211) + 32311$$

$$= 212 *_W 11$$

$$= \Gamma^*(212) *_W \Gamma^*(11)$$

$$\Gamma^* \otimes \Gamma^* \circ \Delta(312341) = \Gamma^* \otimes \Gamma^*(\varepsilon \otimes 312341 + 11 \otimes 2123 + 121 \otimes 112 + 31231 \otimes 1 + 312341 \otimes \varepsilon)$$

$$= \varepsilon \otimes (312341 + 312331) + 11 \otimes (2123 + 2122) +$$

$$121 \otimes (112 + 111) + 31231 \otimes 1 + (312341 + 312331) \otimes \varepsilon$$

$$= \Delta(312341 + 312331)$$

$$= \Delta(\Gamma^*(312341))$$
Proposition 5 The map
\[ \varphi : \text{WQSym}' \rightarrow \text{UBP} \]
\[ w \mapsto \sum_{f : \Psi(f) = w} \hat{f} \]

is an injective morphism of Hopf algebras.

References


