Combinatorial Hopf algebras

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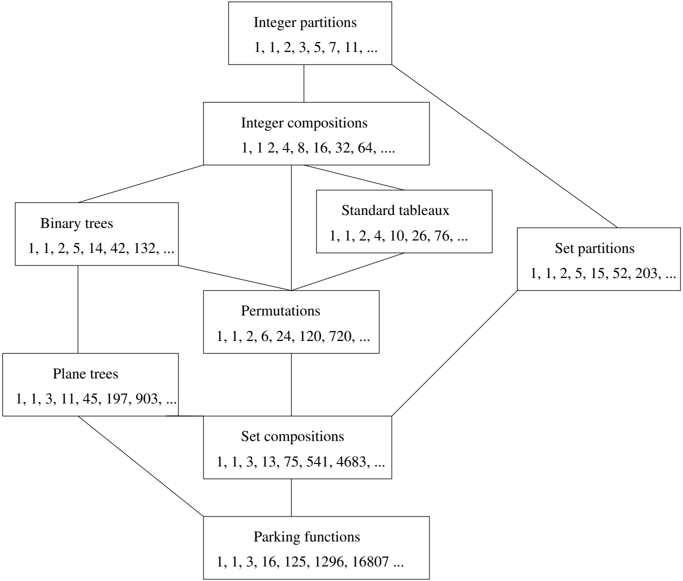
Combinatorial Hopf algebras

- Heuristic notion (no formal definition)
- Graded (bi-)Algebras based on combinatorial objects
- Arise in various contexts: combinatorics, representation theory, operads, renormalization, topology, singularities ...
- ... sometimes with very different definitions.
- Example: Integer partitions; Sym =symmetric functions. Nontrivial product and coproduct for Schur functions (Littlewood-Richardson)
- I like to see combinatorial Hopf algebras as generalizations of the algebra of symmetric functions.
- Jean-Louis had a different point of view. This was the basis of our interactions.


Integer partitions
1, 1, 2, 3, 5, 7, 11, ...
Integer compositions
1, 1 2, 4, 8, 16, 32, 64, ...
Binary trees
1, 1, 2, 5, 14, 42, 132, ...
Standard tableaux
1, 1, 2, 4, 10, 26, 76, ...
Set partitions
1, 1, 2, 5, 15, 52, 203, ...
Permutations
1, 1, 2, 6, 24, 120, 720, ...
Plane trees
1, 1, 3, 11, 45, 197, 903, ...
Set compositions
1, 1, 3, 13, 75, 541, 4683, ...
Parking functions
1, 1, 3, 16, 125, 1296, 16807 ...

Why symmetric functions? I

The algebra of symmetric functions contains interesting elements: Schur, Hall-Littlewood, zonal, Jack, Macdonald ... solving important problems:

- Schur: character tables of symmetric groups, characters of $GL(n, \mathbb{C})$, zonal spherical functions of $(GL(n, \mathbb{C}), U(n))$, KP-hierarchy, Fock space, lots of combinatorial applications
- Hall-Littlewood (one parameter): Hall algebra, character tables of $GL(n, \mathbb{F}_q)$, geometry and toplogy of flag varieties, characters of affine Lie algebras, zonal spherical functions for p-adic groups, statistical mechanics
- Zonal polynomials: for orthogonal and symplectic groups
- Macdonald (two parameters): unification of the previous ones. Solutions of quantum relativistic models, diagonal harmonics, etc.

Why symmetric functions? II

- One may ask whether there are such things in combinatorial Hopf algebras ...
- For our purposes, the example of Schur functions will be good enough
- Their product (LR-rule) solves a nontrivial problem (tensor products of representations of GL_n)
- This rule is now explained and generalized by the theory of crystal bases ...
- ... but it can also be interpreted in terms of a combinatorial Hopf algebra of Young tableaux, defined by means of the Robinson-Schensted correspondence
- and the Loday-Ronco Hopf algebra of binary trees admits a similar definition (LR-algebras?)

The Robinson-Schensted correspondence

Insertion algorithm: $w \in A^* \mapsto P(w)$ (semi-standard tableau) (A a totally ordered alphabet) Example: $P(132541)$

1	

1	3

3	
1	2

3		
1	2	5

3	5	
1	2	4

3		
2	5	
1	1	4

Bijection $w \mapsto (P(w), Q(w))$
$Q(w)$ standard tableau encoding the chain of shapes of $P(x_1), P(x_1x_2), \ldots, P(w)$.

$Q(132541) =$

6		
3	5	
1	2	4

Clearly, $Q(w)$ has the same shape as $P(w)$.

The plactic monoid

Equivalence relation $\sim$ on A^*

$$u \sim v \iff P(u) = P(v)$$

It is the congruence on A^* generated by the relations

$$\begin{aligned} xzy &\equiv zxy \quad (x \leq y < z) \\ yxz &\equiv yzx \quad (x < y \leq z) \end{aligned}$$

The *plactic monoid* on the alphabet A is the quotient $A^* / \equiv$, where $\equiv$ is the congruence generated by the *Knuth relations* above.

Free Schur functions

Tableau $T \mapsto$ monomial x^T

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 4 \\ \cline{1-2} 2 & 4 \\ \hline 1 & 2 & 2 & 5 \\ \hline \end{array} \longmapsto x^T = x_1 x_2^3 x_4^2 x_5$$

Shape of T: partition $sh(T) = \lambda = (4, 2, 1)$
Schur functions:

$$s_\lambda = \sum_{sh(T)=\lambda} x^T$$

Free Schur functions (labeled by standard tableaux)

$$\mathbf{S}_t = \sum_{Q(w)=t} w$$

Goes to s_λ (shape of t) by $a_i \mapsto x_i$.

The Hopf algebra **FSym**

t', t'' standard tableaux; k number of cells of t'.

$$\mathbf{S}_{t'}\mathbf{S}_{t''} = \sum_{t \in Sh(t', t'')} \mathbf{S}_t$$

$Sh(t', t'')$ set of standard tableaux in the shuffle of t' (row reading) with the plactic class of $t''[k]$.
Thus, the $\mathbf{S}_t$ span an algebra.
It is also a coalgebra for the coproduct $A \mapsto A' + A''$ (ordinal sum): Hopf algebra **FSym**.

[Littlewood-Richardson 1934; Robinson; Schensted; Knuth; Lascoux-Schützenberger; Poirier-Reutenauer; Lascoux-Leclerc-T.; Duchamp-Hivert-T. 2001]

Example

$$t' = t'' = \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}$$

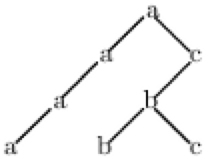
$$\begin{array}{|c|c|c|c|} \hline 3 & 6 & & \\ \hline 1 & 2 & 4 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & 4 & 6 \\ \hline 1 & 2 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 6 & & & \\ \hline 3 & & & \\ \hline 1 & 2 & 4 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 3 & 6 & \\ \hline 1 & 2 & 5 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 6 & & \\ \hline 3 & 4 & \\ \hline 1 & 2 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 4 & 6 \\ \hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 6 & & \\ \hline 4 & & \\ \hline 3 & & \\ \hline 1 & 2 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 6 & \\ \hline 4 & \\ \hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array}$$

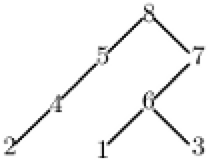
Binary search trees and the sylvester monoid

The sylvester correspondence $w \mapsto (\mathcal{P}(w), \mathcal{Q}(w))$
(binary search tree, decreasing tree) [Hivert-Novelli-T.]
For $w = bacaabca$,

$\mathcal{P}(w) =$



$\mathcal{Q}(w) =$



Equivalence relation $\sim$ on A^*

$$u \sim v \iff \mathcal{P}(u) = \mathcal{P}(v)$$

It coincides with the sylvester congruence, generated by

$$zxuy \equiv xzuy\,,\ x \leq y < z \in A\,,\ u \in A^*\,.$$

The cosylvester algebra

Flattening $\mathcal{P}(w)$ yields the nondecreasing rearrangement of w.
Thus, the only nontrivial information is its shape $\mathcal{T}(w)$.
Let

$$\mathbf{P}_T = \sum_{\mathcal{T}(w)=T} w$$

Then,

$$\mathbf{P}_{T'}\mathbf{P}_{T''} = \sum_{T \in Sh(T',T'')} \mathbf{P}_T \,,$$

where $Sh(T', T'')$ is the set of trees T in $u \sqcup\!\sqcup v$; ($u = w_{T'}$, $v = w_{T''}[k]$ are words read from the trees).
This is completely similar to the LRS rule.
The $\mathbf{P}_T$ span an algebra, and actually a bialgebra for the coproduct $A' + A''$ as above.
It is isomorphic to the Loday-Ronco algebra (free dendriform algebra on one generator): a *polynomial realization* of **PBT**.

Polynomial realizations

- Combinatorial objet $\longrightarrow$ "polynomial" in infinitely many variables (commuting or not)
- Combinatorial product $\longrightarrow$ ordinary product of polynomials
- Coproduct $\longrightarrow$ $A \mapsto A' + A''$

Can be found for most CHA.
In the special case of **PBT**:

- Dendriform structure implied by trivial operations on words

$$uv = u \prec v + u \succ v, \quad u \prec v = uv \text{ if } \max(u) > \max(v) \text{ or } 0$$

- Easy computation of the dual (via Cauchy type identity)
- Open problem: **FSym** free $\mathcal{P}$-algebra on one generator for some operad? (non-trivial implications: hook-length formulas)

Background on symmetric functions I

- "functions": polynomials in an infinite set of indeterminates

$$X = \{x_i | i \geq 1\}$$

$$\lambda_t(X) \text{ or } E(t; X) = \prod_{i \geq 1} (1 + tx_i) = \sum_{n \geq 0} e_n(X) t^n$$

$$\sigma_t(X) \text{ or } H(t; X) = \prod_{i \geq 1} (1 - tx_i)^{-1} = \sum_{n \geq 0} h_n(X) t^n$$

- e_n = elementary symmetric functions
- h_n = complete (homogeneous) symmetric functions
- Algebraically independent: $Sym(X) = K[h_1, h_2, \dots]$
- With n variables: $K[e_1, e_2, \dots, e_n]$

Background on symmetric functions II

- Bialgebra structure:

$$\Delta f = f(X+Y)$$

- $X+Y$: disjoint union; $u(X)v(Y) \simeq u \otimes v$
- Graded connected bialgebra: Hopf algebra
- Self-dual. Scalar product s.t.

$$\langle f \cdot g \, , \, h \rangle = \langle f \otimes g \, , \, \Delta h \rangle$$

- Linear bases: integer partitions

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0)$$

- Multiplicative bases:

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_r} \text{ and } h_\lambda$$

Background on symmetric functions III

- Obvious basis: monomial symmetric functions

$$m_\lambda = \Sigma x^\lambda = \sum_{\text{distinct permutations}} x^\mu$$

- Hall's scalar product realizes self-duality

$$\langle h_\lambda \, , \, m_\mu \rangle = \delta_{\lambda\mu}$$

- h and m are adjoint bases, and

$$\sigma_1(XY) = \prod_{i,j\geq 1} (1 - x_i y_j)^{-1} = \sum_\lambda m_\lambda(X) h_\lambda(Y)$$

  (Cauchy type identity)
- Any pair of bases s.t. $\sigma_1(XY) = \sum_\lambda u_\lambda(X) v_\lambda(Y)$ are mutually adjoint

Background on symmetric functions IV

- Original Cauchy identity for *Schur functions*

$$\sigma_1(XY) = \sum_{\lambda} s_\lambda(X) s_\lambda(Y)$$

where
$s_\lambda = \det(h_{\lambda_i+j-i}) = \sum_{\mathrm{shape}(T)=\lambda} x^T = \mathcal{A}(x^{\lambda+\rho})/\mathcal{A}(x^\rho)$

- Schur functions encode irreducible characters of symmetric groups:

$$\chi^\lambda_\mu = \langle s_\lambda \, , \, p_\mu \rangle \quad \text{(Frobenius)}$$

- p_n: power-sums

$$p_n(X) = \sum_{i \geq 1} x_i^n \, , \quad \sigma_t(X) = \exp\left[\sum_{m \geq 1} p_m(X) \frac{t^m}{m} \right]$$

Noncommutative Symmetric Functions I

- Very simple definition: replace the complete symmetric functions h_n by non-commuting indeterminates S_n, and keep the coproduct formula
- Realization: $A = \{a_i | i \geq 1\}$, totally ordered set of noncommuting variables

$$\sigma_t(A) = \prod_{i\geq 1}^{\rightarrow} (1 - ta_i)^{-1} = \sum_{n\geq 0} S_n(A) t^n \quad (\rightarrow h_n)$$

$$\lambda_t(A) = \prod_{1\leq i}^{\leftarrow} (1 + ta_i) = \sum_{n\geq 0} \Lambda_n(A) t^n \quad (\rightarrow e_n)$$

- Coproduct: $\Delta F = F(A + B)$ (ordinal sum, A commutes with B)

Noncommutative Symmetric Functions II

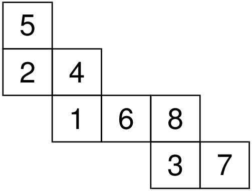
- **Sym** $= \bigoplus_{n\geq 0} \Sigma_n$, Σ_n descent algebra of $\mathfrak{S}_n$
  [Solomon,Malvenuto-Reutenauer, GKLLRT]
- **Sym** $= \bigoplus_{n\geq 0} K_0(H_n(0))$ (analogue of Frobenius) [Krob-T.]
- Topological interpretation: **Sym** $= H_*(\Omega\Sigma\,\mathbb{C}\,P^\infty))$ [Baker-Richter]
- Universal Leibniz Hopf algebra [Hazewinkel]
- Calling this algebra NCSF implies to look at it in a special way [GKLLRT]
- Find analogues of the classical families of symmetric functions ...
- ... and of the various interpretations of *Sym*

Noncommutative Symmetric Functions III

Analogues of Schur functions: the *ribbon basis*

- A *descent* of $w \in A^n$: an i s.t. $w_i > w_{i+1}$

$$w = \underline{5}2\underline{4}16\underline{8}37 \longrightarrow$$



- Descent set $\mathrm{Des}(w) = \{1, 3, 6\}$
- Descent composition $C(w) = I = (1, 2, 3, 2)$
- $\mathrm{Des}(I) = \{1, 3, 6\}$

Noncommutative Symmetric Functions IV

Analogue of the complete basis

$$S^I := S_{i_1} S_{i_2} \cdots S_{i_r} = \sum_{\mathrm{Des}(w) \subseteq \mathrm{Des}(I)} w$$

By inclusion exclusion

$$R_I := \sum_{C(w)=I} w = \sum_{Des(J) \subseteq \mathrm{Des}(I)} (-1)^{\ell(I)-\ell(J)} S^J$$

goes to a skew Schur function under $a_i \mapsto x_i$. It is a Schur like basis and the product rule is

$$R_I R_J = R_{I \triangleright J} + R_{IJ}$$

Free $As^{(2)}$-algebra on one generator.

Quasi-symmetric functions I

Sym is cocommutative:

$$\Delta S_n = S_n(A' + A'') = \sum_{i+j=n} S_i \otimes S_j$$

To find the dual, introduce an infinite set X of commuting indeterminates, and the Cauchy kernel

$$\mathcal{K}(X,A) := \overrightarrow{\prod_{i\geq 1}}\overrightarrow{\prod_{j\geq 1}}(1-x_i a_j)^{-1} = \sum_I M_I(X)S^I(A) = \sum_I F_I(X)R_I(A)$$

The M_I and F_I are bases of a commutative Hopf algebra: Quasi-symmetric functions [Gessel 1984].

$$M_I = \sum_{k_1<k_2<\ldots<k_r} x_{k_1}^{i_1} x_{k_2}^{i_2} \cdots x_{k_r}^{i_r}$$

(pieces of monomial symmetric functions).

Quasi-symmetric functions II

QSym is the free commutative tridendriform algebra on one generator.
The product rules for the M_I and the F_I have nontrivial multiplicities.

$$F_{11}F_{21} = F_{131} + 2F_{221} + F_{32} + F_{311} + F_{1121} + F_{122} + F_{1211} + F_{212} + F_{2111}$$

$$M_{11}M_{21} = M_{1121} + 2M_{1211} + M_{122} + M_{131} + 3M_{2111} + M_{212} + M_{221} + 2M_{311} + M_{32}$$

Their combinatorial understanding requires two larger Hopf noncommutative Hopf algebras, which can also be interpreted as operads:

- For the F_I: **FQSym**, based on permutations
- For the M_I: **WQSym**, based on packed words (surjections)

Standardization of a word: **FQSym**

The descent set of a word is compatible with a finer invariant: the *standardization*

$$\text{word of length } n \longmapsto \text{permutation of } \mathfrak{S}_n$$

$$w = a_1 a_2 \ldots a_n \longmapsto \sigma = \mathrm{std}(w)$$

for all $i < j$ set $\sigma(i) > \sigma(j)$ iff $a_i > a_j$.

Example: $\mathrm{std}(abcadbcaa) = 157296834$

a	b	c	a	d	b	c	a	a
a_1	b_5	c_7	a_2	d_9	b_6	c_8	a_3	a_4
1	5	7	2	9	6	8	3	4

Free Quasi-Symmetric Functions

Subspace of the free associative algebra $K\langle A\rangle$ spanned by

$$\mathbf{G}_\sigma(A) := \sum_{\mathrm{std}(w)=\sigma} w\,.$$

It is a subalgebra, with product rule for $\alpha \in \mathfrak{S}_m$, $\beta \in \mathfrak{S}_n$,

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\substack{\gamma = u\cdot v \\ \mathrm{std}(u)=\alpha,\ \mathrm{std}(v)=\beta}} \mathbf{G}_\gamma\,.$$

[Malvenuto-Reutenauer; Duchamp-Hivert-T.]

$$\mathbf{G}_{21}\mathbf{G}_{213} = \mathbf{G}_{54213} + \mathbf{G}_{53214} + \mathbf{G}_{43215} + \mathbf{G}_{52314} + \mathbf{G}_{42315} + \mathbf{G}_{32415} + \mathbf{G}_{51324} + \mathbf{G}_{41325} + \mathbf{G}_{31425} + \mathbf{G}_{21435}$$

Morphisms and duality

FQSym is a Hopf algebra for the coproduct

$$\Delta(\mathbf{G}_\sigma) = \mathbf{G}_\sigma(A' + A'')$$

The obvious embedding ι : **Sym** $\hookrightarrow$ **FQSym**

$$R_I = \sum_{C(\sigma)=I} \mathbf{G}_\sigma$$

is a morphism of Hopf algebras.
FQSym is self-dual, the dual basis of $\mathbf{G}_\sigma$ is

$$\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$$

Thus, ι^* : **FQSym** $\to$ *QSym* is an epimorphism of Hopf algebras. It is given by $a_i \mapsto x_i$ (commutative image). Then, $\mathbf{F}_\sigma \mapsto F_{C(\sigma)}$. The rule

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\gamma \in \alpha \sqcup\!\sqcup \beta[k]} \mathbf{F}_\gamma$$

projects to the product rule of the F_I.

FQSym as Zinbiel

FQSym is a dendriform (even bidendriform [Foissy]) algebra. It can also be interpreted as an operad.

A *rational mould* is a sequence $f = (f_n(u_1, \ldots, u_n))$ of rational functions. The mould product $*$ on single rational functions is

$$f_n * g_m = f(u_1, \ldots, u_n) g_m(u_{n+1}, \ldots, u_{m+n})$$

Chapoton has defined an operad structure on these rational functions.

The fractions

$$f_\sigma = \frac{1}{u_{\sigma(1)}(u_{\sigma(1)} + u_{\sigma(2)}) \cdots (u_{\sigma(1)} + u_{\sigma(2)} + \cdots + u_{\sigma(n)})}$$

satisfy the product rule of **FQSym**

$$f_\alpha * f_\beta = \sum_{\gamma \in \alpha \shuffle \beta[k]} f_\gamma$$

and their linear span is stable under the $\circ_i$: a suboperad which can be recognized as Zinbiel. [Chapoton-Hivert-Novelli-T.]

Packing of a word: **WQSym**

One can refine standardization by giving an identical numbering to all occurences of the same letter:
If $b_1 < b_2 < \ldots < b_r$ are the letters occuring in w, $u = \mathrm{pack}(w)$ is the image of w by the homomorphism $b_i \mapsto i$.
u is *packed* if $\mathrm{pack}(u) = u$. The, we set [Hivert; Novelli-T.]

$$\mathbf{M}_u := \sum_{\mathrm{pack}(w)=u} w\,.$$

For example,

$$\mathbf{M}_{13132} = 13132 + 14142 + 14143 + 24243 + 15152 + 15153 + 25253 + 15154 + 25254 + 35354 + \cdots$$

Under the abelianization $a_i \mapsto x_i$, the $\mathbf{M}_u \mapsto M_I$ where $I = (|u|_i)$.
The $\mathbf{M}_u$ span a subalgebra of $\mathbb{K}\langle A\rangle$, called **WQSym**

Structure of **WQSym**

Hopf algebra for $A \mapsto A' + A''$. It contains **FQSym**:

$$\mathbf{G}_\sigma = \sum_{\mathrm{std}(u)=\sigma} \mathbf{M}_u$$

It is a tridendriform algebra. Again, the tridendriform structure is induced by trivial operations on words

$$uv = u \prec v + u \succ v + u \circ v$$

(only one term is uv). The degree one element

$$a = \mathbf{G}_1 = \mathbf{F}_1 = \mathbf{M}_1 = \sum_{i \geq 1} a_i$$

generates a free tridendriform algebra (Schröder trees).
WQSym is also an operad of rational functions ($1 -$ **RatFct**)
$\mathbf{F}_\sigma$ of **FQSym** can be interpreted a the characteristic function of a simplex Δ_σ (product rule for iterated integrals).
Similarly, the $(-1)^{\max(u)}\mathbf{M}_u$ can be interpreted as characteristic functions of certain polyhedral cones [Menous-Novelli-T.].

Special words and equivalence relations I

A whole class of combinatorial Hopf algebras whose operations are usually described in terms of some elaborated surgery on combinatorial objets are in fact just subalgebras of $\mathbb{K}\langle A\rangle$

- **Sym**: $R_I(A)$ is the sum of all words with the same *descent set*
- **FQSym**: $\mathbf{G}_\sigma(A)$ is the sum of all words with the same *standardization*
- **PBT**: $\mathbf{P}_T(A)$ is the sum of all words with the same *binary search tree*
- **FSym**: $\mathbf{S}_t(A)$ is the sum of all words with the same *insertion tableau*
- **WQSym**: $\mathbf{M}_u(A)$ is the sum of all words with the same *packing*
- It contains the free tridendriform algebra one one generator, based on sum of words with the same *plane tree*

Special words and equivalence relations II

To these examples, one can add:

- **PQSym**: based on parking functions (sum of all words with the same *parkization*)

In all cases, the product is the ordinary product of polynomials, and the coproduct is $A' + A''$.

Parking functions I

- A parking function of length n is a word over w over $[1, n]$ such that in the *sorted word* $w^{\uparrow}$, the ith letter is $\leq i$.
- Example $w = 52321$ OK since $w^{\uparrow} = 12235$, but not 52521
- Parkization algorithm: sort w, shift the smallest letter if it is not 1, then if necessary, shift the second smallest letter of a minimal amount, and so on. Then put each letter back in its original place [Novelli-T.].
- Example: $w = (5, 7, 3, 3, 13, 1, 10, 10, 4)$,
  $w^{\uparrow} = (1, 3, 3, 4, 5, 7, 10, 10, 13)$,
  $p(w)^{\uparrow} = (1, 2, 2, 4, 5, 6, 7, 7, 9)$, and finally
  $p(w) = (4, 6, 2, 2, 9, 1, 7, 7, 3)$.

Parking functions II

- $\mathrm{PF}_n = (n+1)^{n-1}$
- Parking functions are related to the combinatorics of Lagrange inversion
- Also, noncommutative Lagrange inversion, antipode of noncommutative formal diffeomorphisms
- **PQSym***, Hopf algebra of (dual) Parking Quasi-Symmetric functions:

$$\mathbf{G_a} = \sum_{p(w)=\mathbf{a}} w$$

- Self dual in a nontrivial way. $\mathbf{G_a}^* =: \mathbf{F_a}$
- Many interesting quotients and subalgebras (**WQSym**, **FQSym**, Schröder, Catalan, 3^{n-1} ...)
- Tridendriform. Operadic interpretation is unknown

The Catalan subalgebra I

- Natural: group the parking functions $\mathbf{a}$ according the the sorted word $\pi = \mathbf{a}^{\uparrow}$ (occurs in the definition and in the noncommutative Lagrange inversion formula)
- Then, the sums
$$\mathbf{P}^{\pi} = \sum_{\mathbf{a}^{\uparrow}=\pi} \mathbf{F_a}$$
span a Hopf subalgebra **CQSym** of **PQSym**
- $\dim \mathbf{CQSym}_n = c_n$ (Catalan numbers 1,1,2,5,14)
- $\mathbf{P}^{\pi}$ is a multiplicative basis: $\mathbf{P}^{11}\mathbf{P}^{1233} = \mathbf{P}^{113455}$ (shifted concatenation)
- Free over a Catalan set $\{1, 11, 111, 112, \dots\}$ (start with 1)
- And it is cocommutative
- So it must be isomorphic to the Grossman-Larson algebra of ordered trees.
- However, this is a very different definition (no trees!)

The Catalan subalgebra II

- Duplicial algebras: two associative operations $\prec$ and $\succ$ such that
$$(x \succ y) \prec z = x \succ (y \prec z)$$
- The free duplicial algebra D on one generator has a basis labelled by binary trees. $\prec$ and $\succ$ are $\backslash$ (under) and $/$ (over)
- **CQSym** is the free duplicial algebra on one generator: Let $\mathbf{P}^{\alpha} \succ \mathbf{P}^{\beta} = \mathbf{P}^{\alpha}\mathbf{P}^{\beta}$ and
$$\mathbf{P}^{\alpha} \prec \mathbf{P}^{\beta} = \mathbf{P}^{\alpha \cdot \beta[\max(\alpha)-1]} =: \mathbf{P}^{\alpha \circ \beta}$$
For example, $\mathbf{P}^{12} \prec \mathbf{P}^{113} = \mathbf{P}^{12224}$.

Noncommutative Lagrange inversion I

Can be formulated as a functional equation in **Sym**

$$G = 1 + S_1 G + S_2 G_2 + S_3 G^3 + \cdots$$

[Garsia-Gessel; Gessel; Pak-Postnikov-Retakh; Novelli-T.]

Unique solution

$$\begin{aligned} G_0 &= 1, \qquad G_1 = S_1, \qquad G_2 = S_2 + S^{11}, \\ G_3 &= S^3 + 2S^{21} + S^{12} + S^{111}, \\ G_4 &= S^4 + 3S^{31} + 2S^{22} + S^{13} + 3S^{211} \\ &\quad + 2S^{121} + S^{112} + S^{1111}. \end{aligned}$$

(notice the sums of coefficients)

Noncommutative Lagrange inversion II

Sym embeds in **PQSym** by $S_n \mapsto \mathbf{F}_{1^n}$, and the sum of all parking functions

$$G = \sum_{\mathbf{a} \in \mathrm{PF}} \mathbf{F}_{\mathbf{a}}$$

solves the functional equation

$$G = 1 + S_1 G + S_2 G^2 + S_3 G^3 + \cdots$$

Actually, G belongs to **CQSym**, and solves the quadratic (duplicial) functional equation

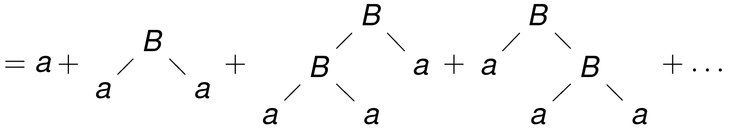
$$G = 1 + B(G, G) \quad (B(x, y) = x \succ \mathbf{P}^1 \prec y)$$

and each term $B_T(1)$ of the tree expansion of the solution is a single $\mathbf{P}^{\pi}$, thus forcing a bijection beween binary trees and nondecreasing parking functions.

Tree expansion for $x = a + B(x, x)$

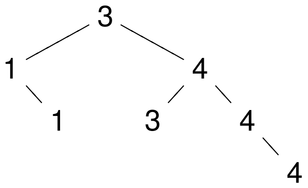
By iterated substitution

$$x = a + B(a, a) + B(B(a, a), a) + B(a, B(a, a)) + \cdots$$



$$x = \sum_{T:\ \text{Complete Binary Tree}} B_T(a)$$

For example,



$$\longrightarrow 1133444 = 11 \succ 1 \prec 1222\,.$$

Embedding of **Sym** in **PBT** I

- Sending S_n to the left (or right) comb with n (internal) nodes is a Hopf embedding of **Sym** in **PBT**
- Under the bijection forced by the quadratic equation, nondecreasing parking functions with the same packed evaluation I form an interval of the Tamari order, whose cardinality is the coefficient of S^I in G.
- The sum of the trees in this interval is the expansion of $R_{\bar{I}^{\sim}}$ in **PBT**.

Duality I

- Interesting property of the (commutative) dual: $\mathbf{CQSym}^*$ contains *QSym* in a natural way
- Recall $m_\lambda = \Sigma x^\lambda$ (monomial symmetric functions)

$$m_\lambda = \sum_{I\downarrow=\lambda} M_I \quad M_I(X) = \sum_{j_1<j_2<\dots<j_r} x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_r}^{i_r}$$

- Let $\mathcal{M}_\pi$ be the dual basis of $\mathbf{P}^\pi$. It can be realized by polynomials:

$$\mathcal{M}_\pi = \sum_{p(w)=\pi} \underline{w}$$

where $\underline{w}$ means commutative image ($a_i \to x_i$)

Duality II

- Example:

$$\mathcal{M}_{111} = \sum_i x_i^3$$

$$\mathcal{M}_{112} = \sum_i x_i^2 x_{i+1}$$

$$\mathcal{M}_{113} = \sum_{i,j;j \geq i+2} x_i^2 x_j$$

$$\mathcal{M}_{122} = \sum_{i,j;i<j} x_i x_j^2$$

$$\mathcal{M}_{123} = \sum_{i,j,k;i<j<k} x_i x_j x_k$$

Duality III

- Then,

$$M_I = \sum_{t(\pi)=I} \mathcal{M}_\pi.$$

  where $t(\pi)$ is the composition obtained by counting the occurences of the different letters of π. For example,

$$M_3 = \mathcal{M}_{111}, \quad M_{21} = \mathcal{M}_{112} + \mathcal{M}_{113}, \quad M_{12} = \mathcal{M}_{122}$$

- In most cases, one knows at least two CHA structures on a given family of combinatorial objects: a self-dual one, and a cocommutative one. Sometimes one can interpolate between them (in general, only by braided Hopf algebras).

Other aspects of CHA's not discussed in this talk I

- *Internal products*. Analogues of the tensor product of $\mathfrak{S}_n$ representations.
  - In **Sym**: given by the descent algebras. Application: Lie idempotents.
  - Subalgebras, e.g., peak algebras
  - In **FQSym**: just the product of $\mathfrak{S}_n$
  - In **WQSym**: the Solomon-Tits algebra
  - Also in **WSym** (invariants of $\mathfrak{S}(A)$ in $\mathbb{K}\langle A\rangle$)
  - In **PQSym** and **CQSym**: does exist, but mysterious ...
  - In $\mathbf{Sym}^{(r)}$ (colored multisymmetric functions, wreath products)
- *Categorification*
  - $Sym \simeq \bigoplus_n R(SG_n)$ (semisimple: $R = G_0 = K_0$)
  - $QSym \simeq \bigoplus_n G_0(H_n(0))$ and $\mathbf{Sym} \simeq \bigoplus_n K_0(H_n(0))$ (explains duality)
  - Colored version for 0-Ariki-Koike-Shoji algebras [Hivert-Novelli-T.]
  - Peak algebras for 0-Hecke-Clifford algebras [N. Bergeron-Hivert-T.]
  - Supercharacters of $U_n(q)$ for $\mathbf{WSym}_q$

Other aspects of CHA's not discussed in this talk II

- **PBT**: Tilting modules for $A_n^{(1)}$-quiver [Chapoton]
- *Other polynomial realizations*. With bi-indexed letters a_{ij} or x_{ij}
  - Commutative algebras on all objetcs from the diagram [Hivert-Novelli-T.]
  - **MQSym** [Duchamp-Hivert-T.]
  - Ordered forests, subalgebras and quotients, in particular Connes-Kreimer [Foissy-Novelli-T.]