Noncommutative symmetric functions and combinatorial Hopf algebras

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Asymptotics in dynamics, geometry and PDEs, generalized Borel summation

- Aim of this talk: describe a class of algebras which are increasingly popular in Combinatorics, and tend to permeate other fields as well.
- In particular (some of) these algebras have at least superficial connections with some topics of this conference.
- They can be approached in many different ways.
- Here, they will be regarded as generalizations of the algebra of symmetric functions.
- Plan:
  1. Reminder about symmetric functions as a Hopf algebra
  2. Noncommutative symmetric functions (with some details)
  3. Random walk through more complicated examples

Symmetric functions I

- “functions”: polynomials in an infinite set of indeterminates

$$X = \{x_i | i \geq 1\}$$

$$\lambda_t(X) \text{ or } E(t;X) = \prod_{i\geq 1}(1+tx_i) = \sum_{n\geq 0} e_n(X)t^n$$

- e_n = elementary symmetric functions
- Algebraically independent: $Sym(X) = \mathbb{K}[e_1, e_2, \ldots]$
- With n variables: stop at e_n

Symmetric functions II

- Bialgebra structure:

$$\Delta f = f(X+Y)$$

- $X+Y$: disjoint union; $u(X)v(Y) \simeq u \otimes v$
- One interpretation: e_n as a function on the multiplicative group

$$G = 1 + t\,\mathbb{K}[[t]] = \{a(t) = 1 + a_1 t + a_2 t^2 + \cdots\}$$

$$e_n(a(t)) = a_n$$

- Then, $\Delta e_n(a(t) \otimes b(t)) = e_n(a(t)b(t))$

Symmetric functions III

- Graded connected bialgebra: Hopf algebra
- Self-dual. Scalar product s.t.

$$\langle f \cdot g \,,\, h \rangle = \langle f \otimes g \,,\, \Delta h \rangle$$

- To define it, we need more interesting elements
- Complete homogeneous functions: h_n sum of all monomials of degree n

$$\sigma_t(X) \text{ or } H(t;X) = \prod_{i\geq 1}(1 - tx_i)^{-1} = \sum_{n\geq 0} h_n(X)t^n$$

Symmetric functions IV

- Linear bases: labeled by unordered sequences of positive integers (integer partions), usually displayed as nonincreasing sequences

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0)$$

- Multiplicative bases:

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_r} \text{ and } h_\lambda$$

- Obvious basis: monomial symmetric functions

$$m_\lambda = \Sigma x^\lambda = \sum_{\text{distinct permutations}} x^\mu$$

Symmetric functions V

- Hall's scalar product realizes self-duality

$$\langle h_\lambda \, , \, m_\mu \rangle = \delta_{\lambda\mu}$$

- h and m are adjoint bases, and

$$\sigma_1(XY) = \prod_{i,j\geq 1} (1 - x_i y_j)^{-1} = \sum_\lambda m_\lambda(X) h_\lambda(Y)$$

(Cauchy type identity)

- Any pair of bases s.t. $\sigma_1(XY) = \sum_\lambda u_\lambda(X) v_\lambda(Y)$ are mutually adjoint

Symmetric functions VI

- Original Cauchy identity for *Schur functions*

$$\sigma_1(XY) = \sum_{\lambda} s_\lambda(X) s_\lambda(Y)$$

where $s_\lambda = \det(h_{\lambda_i+j-i})$

- Schur functions encode irreducible characters of symmetric groups:

$$\chi^\lambda_\mu = \langle s_\lambda \, , \, p_\mu \rangle \quad \text{(Frobenius)}$$

- p_n: power-sums

$$p_n(X) = \sum_{i\geq 1} x_i^n \, , \quad \sigma_t(X) = \exp\left[\sum_{m\geq 1} p_m(X) \frac{t^m}{m}\right]$$

Symmetric functions VII

- $\delta f = f(XY)$ is another coproduct
- its dual is the *internal product* $*$
- it corresponds to the pointwise product of characters (tensor product of $\mathfrak{S}_n$ representations
- Other interpretations of Schur functions: characters of $U(n)$, zonal spherical functions for the Gelfand pair $(GL(n, \mathbb{C}), U(n)$, basis vectors of Fock space representations of some affine Lie algebras
- q and (q, t) deformations related to finite linear groups, Hecke algebras, quantum groups ...
- coproduct from composition of series: Faa di Bruno algebra

Combinatorial Hopf algebras I

- *Sym* is the prototype of a rather vast family of Hopf algebras
- based on “combinatorial objects” (for *Sym*: integer partitions)
- Schur-like bases with structure constants in $\mathbb{N}$
- coproduct $A + B$
- internal product $*$
- lots of morphisms between them
- connections with representation theory
- and with operads

Combinatorial Hopf algebras II

- Examples of combinatorial objects: integer compositions, set partitions, set compositions, permutations, Young tableaux, parking functions, various kinds of trees ...
- Motivations:
  - better understanding classical symmetric functions,
  - combinatorial description of solutions of functional equations, renormalization
  - operads
- The simplest one: Noncommutative Symmetric Functions

Noncommutative Symmetric Functions I

- Very simple definition: replace the complete symmetric functions h_n by non-commuting indeterminates S_n, and keep the coproduct formula
- Realization: $A = \{a_i | i \geq 1\}$, totally ordered set of noncommuting variables

$$\sigma_t(A) = \overrightarrow{\prod_{i\geq 1}} (1 - ta_i)^{-1} = \sum_{n\geq 0} S_n(A) t^n \quad (\rightarrow h_n)$$

$$\lambda_t(A) = \overleftarrow{\prod_{1\leq i}} (1 + ta_i) = \sum_{n\geq 0} \Lambda_n(A) t^n \quad (\rightarrow e_n)$$

Noncommutative Symmetric Functions II

- Coproduct: $\Delta F = F(A+B)$ (ordinal sum, A commutes with B)
- Obvious interpretation: multiplicative group of formal power series over a noncommutative algebra
- More exotic interpretations: **Sym** $= H_*(\Omega\Sigma\,\mathbb{C}\,P^\infty))$ ...
- Calling this algebra NCSF implies to look at it in a special way
- Find analogues of the classical families of symmetric functions ...
- ... and of the various interpretations of *Sym*

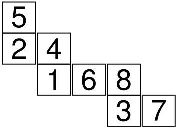
Some connections with the topics of the conference

- Illustration of Mould calculus (moulds over positive integers)
- Alien derivations $\leftrightarrow$ Lie idempotents in $\mathbb{C}\,\mathfrak{S}_n$
- Noncommutative formal diffeomorphisms (Noncommutative Lagrange inversion)
- Combinatorial Dyson-Schwinger equations
- **Sym**$^* = QSym$: Multiple Zeta Values are $M_I(1, \frac{1}{2}, \frac{1}{3} \ldots)$

Descent algebras I

- A *descent* of $\sigma \in \mathfrak{S}_n$: an i s.t. $\sigma(i) > \sigma(i+1)$

$\sigma = \underline{5}2\underline{4}16\underline{8}37 \longrightarrow$



- Descent set $\mathrm{Des}(\sigma) = \{1, 3, 6\}$
- Descent composition $C(\sigma) = I = (1, 2, 3, 2)$
- $\mathrm{Des}(I) = \{1, 3, 6\}$

Descent algebras II

- *Descent algebras* (L. Solomon, 1976): the sums

$$D_I = \sum_{C(\sigma)=I} \sigma$$

  span a subalgebra Σ_n of $\mathbb{Z}\,\mathfrak{S}_n$
- $\bigoplus_{n\geq 0} \Sigma_n \simeq \mathbf{Sym}$
- Linear basis of **Sym**: $S^I = S_{i_1} \cdots S_{i_r}$ (*compositions I*)
- Linear map $\alpha :\ \mathbf{Sym}_n \to \Sigma_n$

$$\alpha(S^I) = \sum_{\mathrm{Des}(\sigma)\subseteq \mathrm{Des}(I)} \sigma$$

- Internal product $*$ on $\mathbf{Sym}_n$: α *anti*isomorphism
- goes to the internal product of *Sym* under commutative image

Compatibility between structures

- The Mackey formula for a product of induced characters, applied to parabolic subgroups of $\mathfrak{S}_n$ translates into an identity for symmetric functions
- Solomon's motivation for the descent algebra was to lift this Mackey formula to the group algebra
- This implies an identity on noncommutative symmetric functions

$$(f_1 \dots f_r) * g = \mu_r[(f_1 \otimes \cdots \otimes f_r) *_r \Delta^r g]$$

μ_r is r-fold multiplication, Δ^r is the iterated coproduct with values in $\mathbf{Sym}^{\otimes r}$

Noncommutative power sums I

- Commutative case: power-sums are the primitive elements, $\sigma_t(X) = \exp\left\{\sum_{k\geq 1} p_k(X)\frac{t^k}{k}\right\}$ equivalent to Newton's recursion
  $nh_n = h_{n-1}p_1 + h_{n-2}p_2 + \cdots + h_1 p_{n-1} + p_n$
- Both make sense in the noncommutative case but define different "power sums":
  1. $\sigma_t(A) = \exp\left\{\sum_{k\geq 1} \Phi_k(A)\frac{t^k}{k}\right\}$
  2. $nS_n = S_{n-1}\Psi_1 + S_{n-2}\Psi_2 + \cdots + S_1\Psi_{n-1} + \Psi_n,$
- $\Phi(t) = \log \sigma_t$ where σ_t is the solution of $\frac{d}{dt}\sigma_t = \sigma_t\psi(t)$ satisfying $\sigma_0 = 1$, and $\psi(t) = \sum_{k\geq 1} t^{k-1}\Psi_k$

Noncommutative power sums II

- Now we have some elements to play with ...
- The relation between S and Ψ is given by a mould

$$m_I = \frac{1}{i_1(i_1+i_2)\cdots(i_1+i_2+\cdots i_r)} \qquad S_n = \sum_{|I|=n} m_I \Psi^I$$

easily obtained by solving $\frac{d}{dt}\sigma_t = \sigma_t \psi(t)$ with iterated integrals:

$$\sigma(t) = 1 + \int_0^t dt_1\, \psi(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2\, \psi(t_2)\psi(t_1) + \dots$$

Noncommutative power sums III

- Replacing A by $A+B$ in the differential equation

$$\frac{d}{dt}\sigma_t(A+B) = \sigma_t(A+B)\psi(t;A+B)$$

  shows immediately

$$\psi(t;A+B) = \psi(t;A) + \psi(t;B)$$

  i.e., the Ψ_n are primitive (or, the mould m_I is symmetral).
- This mould occurs in the formal linearization of vector fields (here in dimension 1)
- OK, but this is very basic. So what ?

Lie idempotents I

- The point is: our elements Φ_n, Ψ_n are interpretable as elements of $\Sigma_n \subset \mathbb{C}\,\mathfrak{S}_n$, that is, as symmetrizers ...
- ... and quite famous ones:
- $\Psi_n = n\theta_n$ where θ_n is Dynkin's idempotent (1947)

$$\theta_n = \frac{1}{n}[\dots[1,2],3],\dots],n] = \frac{1}{n}\sum_{k=0}^{n-1} D_{(1^k,n-k)}$$

- $\Phi_n = n\phi_n$ where ϕ_n is Solomon's idempotent (1968):

$$\phi_n = \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} \frac{(-1)^{d(\sigma)}}{\binom{n-1}{d(\sigma)}}\sigma \qquad (d(\sigma) = |\operatorname{Des}(\sigma)|)$$

Lie idempotents II

- And we shall also encounter Klyachko's idempotent (1974):

$$\kappa_n = \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} \omega^{\mathrm{maj}(\sigma)} \sigma$$

$$\omega = e^{2\mathrm{i}\pi/n}, \qquad \mathrm{maj}(\sigma) = \sum_{j \in \mathrm{Des}(\sigma)} j\,.$$

- $\pi \in \mathbb{K}\,\mathfrak{S}_n$ is a *Lie idempotent* if it acts as a projector from the free associative algebra $\mathbb{K}_n\langle A\rangle$ onto the free Lie algebra $L_n(A)$ generated by A

Lie idempotents III

- It seems that many important moulds have canonical representatives (in **Sym** for the 1-dimensional case, in other CHA's in general)
- Another example: the analog of a classical transformation of symmetric functions (related to Hall algebras, finite fields, Hecke algebras) is

$$\sigma_t\left(\frac{A}{1-q}\right) := \overleftarrow{\prod_{k\geq 1}} \sigma_{tq^k}(A)$$

Lie idempotents IV

- It is given by the mould

$$S_n\left(\frac{A}{1-q}\right)=\sum_{|I|=n}\frac{q^{\mathrm{maj}(I)}}{(1-q^{i_1})(1-q^{i_1+i_2})\dots(1-q^{i_1+\cdots+i_r})}S^I(A)$$

which occurs in the formal linearization of diffeomorphisms

- Expanding on the R-basis yields

$$(q)_n S_n\left(\frac{A}{1-q}\right)=\sum_{|I|=n}q^{\mathrm{maj}(I)}R_I(A)$$

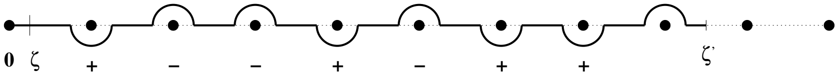
- This has at least two interpretations:
  1. Commutative image is a Hall-Littlewood function (q-character of the symmetric group in coinvariants)
  2. $q=\omega=e^{2i\pi/n}$ gives back Klyachko's idempotent

Lie idempotents V

- One may wonder whether other examples in mould calculus correspond to interesting noncommutative symmetric functions
- The answer is yes, but the deepest connections appear to come from Alien Calculus

Alien operators on $\mathrm{RESUR}(\mathbb{R}^+//\mathbb{N},\mathrm{int.})$ I

- A sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-1}) \in \{\pm\}^{n-1}$ defines an operator $D_{\varepsilon\bullet}$ on $\mathrm{RESUR}(\mathbb{R}^+//\mathbb{N},\mathrm{int.})$
- $\mathrm{RESUR}(\mathbb{R}^+//\mathbb{N},\mathrm{int.})$ is a convolution algebra of functions holomorphic on $]0,1[$ and analytically continuable along paths like this one:



$$D_{\varepsilon\bullet}\hat{\varphi} = \hat{\varphi}^{\varepsilon+}(\zeta + I(\varepsilon\bullet)) - \hat{\varphi}^{\varepsilon-}(\zeta + I(\varepsilon\bullet))$$

Alien operators on $\mathrm{RESUR}(\mathbb{R}^+//\mathbb{N}, \mathrm{int.})$ II

- The composition of such operators is given by:

$$D_{\mathbf{a}\bullet} D_{\mathbf{b}\bullet} = D_{\mathbf{b}+\mathbf{a}\bullet} - D_{\mathbf{b}-\mathbf{a}\bullet}$$

- This is, up to a sign, the product formula for noncommutative ribbon Schur functions

$$R_I \cdot R_J = R_{I.J} + R_{I \triangleright J}$$

- The sign can be taken into account, and there is a natural isomorphism of Hopf algebras

$$\mathrm{ALIEN} \longrightarrow \mathbf{Sym}$$

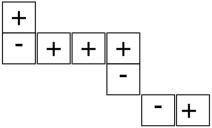
Alien operators on $\mathrm{RESUR}(\mathbb{R}^+//\mathbb{N}, \mathrm{int.})$ III

- It is given by

$$D_{\varepsilon\bullet} \quad \leftrightarrow \quad \varepsilon_1 \dots \varepsilon_{n-1} R_\varepsilon$$

- The ribbon Schur function R_ε is obtained by reading backwards the sequence $\varepsilon+$:

$$\varepsilon+ \;=\; + - - + + + - + \quad \rightarrow \quad R_\varepsilon =$$



Alien operators on $\mathrm{RESUR}(\mathbb{R}^+//\mathbb{N}, \mathrm{int.})$ IV

- Under this isomorphism,

$$\begin{aligned}\Delta_n^+ &= D_{+\ldots+\bullet} \leftrightarrow S_n\\ \Delta_n^- &= -D_{-\ldots-\bullet} \leftrightarrow (-1)^n\Lambda_n\\ \Delta_n &= \sum_{\varepsilon\in\mathcal{E}_{n-1}} \frac{p!q!}{(p+q+1)!} D_{\varepsilon\bullet} \leftrightarrow \frac{1}{n}\Phi_n\end{aligned}$$

- Given these identifications, is is not so surprising that ALIEN can be given Hopf algebra structure, for which Δ^+ and Δ^- are grouplike, and Δ primitive
- However, the analytical definition (Ecalle 1981) is not at all trivial. Grouplike elements are the alien automorphisms, and primitives are the alien derivations.

Alien operators on $\mathrm{RESUR}(\mathbb{R}^+//\mathbb{N}, \mathrm{int.})$ V

- Thus, alien derivations correspond to Lie idempotents in descent algebras
- Nontrivial examples are known on both sides
- For example, alien derivations from the Catalan family:

$$\mathrm{ca}_n = \frac{(2n)!}{n!(n+1)!}$$

$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) = (\pm)^{n_1}(\mp)^{n_2}(\pm)^{n_3} \ldots (\varepsilon_n)^{n_s} \quad (n_1 + \ldots + n_s = n)$$

$$\mathrm{ca}^\varepsilon = \mathrm{ca}_{n_1}\mathrm{ca}_{n_2} \ldots \mathrm{ca}_{n_s}$$

$$\mathrm{Dam}_n = \sum_{l(\varepsilon\bullet)=n} \mathrm{ca}^\varepsilon D_{\varepsilon\bullet}$$

Alien operators on $\mathrm{RESUR}(\mathbb{R}^+//\mathbb{N}, \mathrm{int.})$ VI

- The corresponding Lie idempotents were not known

$$\mathrm{Dam}_4 = 5R_4 - 5R_{1111} - 2R_{13} + 2R_{211} - 2R_{31} + 2R_{112} - R_{22} + R_{121}$$

- and up to now, no natural way to prove their primitivity in **Sym**
- On another hand, is there any application of the q-Solomon idempotent

$$\varphi_n(q) = \frac{1}{n} \sum_{|I|=n} \frac{(-1)^{d(\sigma)}}{\begin{bmatrix} n-1 \\ d(\sigma) \end{bmatrix}_q} q^{\mathrm{maj}(\sigma) - \binom{d(\sigma)+1}{2}} \sigma$$

in alien calculus ?

Permutations and Free Quasi-Symmetric Functions

- To go further, we need larger algebras
- The simplest one is based on permutations
- It is large enough to contain algebras based on binary trees and on Young tableaux
- To accomodate other kinds of trees, one can imitate its construction, starting from special words generalizing permutations

Standardization of a Word

word of length n	$\longmapsto$	permutation of $\mathfrak{S}_n$
$w = l_1 l_2 \ldots l_n$	$\longmapsto$	$\sigma = \mathrm{std}(w)$

for all $i < j$ set $\sigma(i) > \sigma(j)$ iff $a_i > a_j$.

Example: $\mathrm{std}(abcadbcaa) = 157296834$

a	b	c	a	d	b	c	a	a
a_1	b_5	c_7	a_2	d_9	b_6	c_8	a_3	a_4
1	5	7	2	9	6	8	3	4

Free Quasi-Symmetric Functions

Subspace of the free associative algebra $K\langle A\rangle$ spanned by

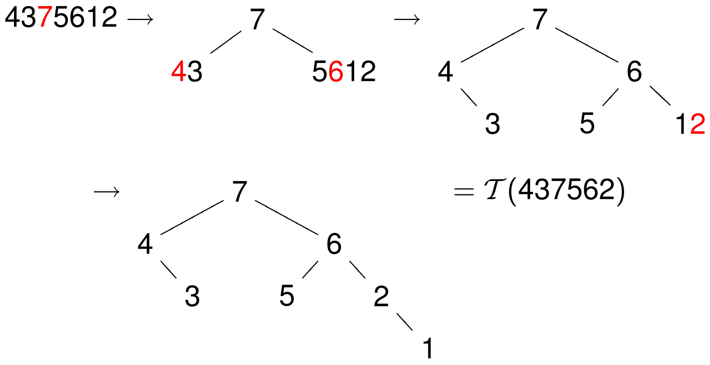
$$\mathbf{G}_\sigma(A) := \sum_{\mathrm{std}(w)=\sigma} w\,.$$

It is a subalgebra, with product rule for $\alpha \in \mathfrak{S}_m$, $\beta \in \mathfrak{S}_n$,

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\substack{\gamma=u\cdot v \\ \mathrm{Std}(u)=\alpha,\ \mathrm{Std}(v)=\beta}} \mathbf{G}_\gamma\,.$$

$$\mathbf{G}_{21}\mathbf{G}_{213} = \mathbf{G}_{54213} + \mathbf{G}_{53214} + \mathbf{G}_{43215} + \mathbf{G}_{52314} + \mathbf{G}_{42315} + \mathbf{G}_{32415} + \mathbf{G}_{51324} + \mathbf{G}_{41325} + \mathbf{G}_{31425} + \mathbf{G}_{21435}$$

Decreasing tree of a permutation



$= \mathcal{T}(437562)$

The Hopf algebra of planar binary trees

- Loday-Ronco algebra:

$$\mathbf{PBT} = \bigoplus \mathbb{K}\,\mathbf{P}_T$$

where

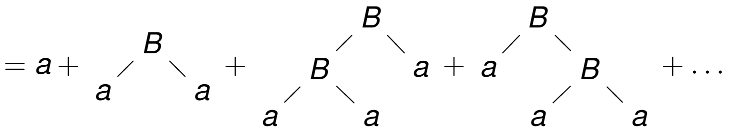
$$\mathbf{P}_T = \sum_{\mathcal{T}(\sigma)=T} \mathbf{G}_\sigma$$

- Several motivations can lead to this algebra. Originally: dendriform structure
- It also arises from a formal Dyson-Schwinger equation

Tree expansion for $x = a + B(x, x)$ I

For suitable bilinear maps B on an associative algebra, its is solved by iterated substitution

$$x = a + B(a, a) + B(B(a, a), a) + B(a, B(a, a)) + \cdots$$



$$x = \sum_{T\text{: Complete Binary Tree}} B_T(a)$$

Tree expansion for $x = a + B(x, x)$ II

For example, $x(t) = \frac{1}{1-t}$ is the unique solution of

$$\frac{dx}{dt} = x^2 , \quad x(0) = 1$$

This is equivalent to the fixed point problem

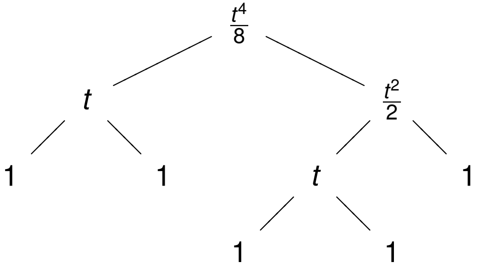
$$x = 1 + \int_0^t x^2(s)ds = 1 + B(x, x)$$

where

$$B(x, y) := \int_0^t x(s)y(s)ds$$

Tree expansion for $x = a + B(x, x)$ III

The terms in the tree expansion look like



$$B_T(1) = \frac{t^4}{8}$$

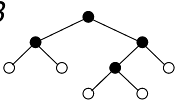
Tree expansion for $x = a + B(x, x)$ IV

The general expression is:

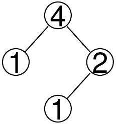
$$B_T(1) = t^{\#(T')} \prod_{\bullet \in T'} \frac{1}{HL(\bullet)}$$

- T' : incomplete tree associated to T;
- $HL(\bullet)$: size of the subtree rooted at $\bullet$.

B



$(1) \quad = \quad t^4 \prod \dfrac{1}{}$



$= \quad \dfrac{t^4}{8}$

Tree expansion for $x = a + B(x, x)$ V

- The number of permutations whose decreasing tree has shape T is $n!B_T(1)$ [Knuth - AOCP 3]
- In **FQSym**,

$$\mathbf{G}_1^n = \sum_{\sigma \in \mathfrak{S}_n} \mathbf{G}_\sigma$$

- $\phi : \mathbf{G}_\sigma \longmapsto \frac{t^n}{n!}$ is a homomorphism. Hence,

$$x(t) = \frac{1}{1-t} = \phi\left((1 - \mathbf{G}_1)^{-1}\right)$$

- There is a derivation ∂ of **FQSym** such that $\mathbf{X} = (1 - \mathbf{G}_1)^{-1}$ satisfies $\partial \mathbf{X} = \mathbf{X}^2$
- Moreover, there is a bilinear map B such that $\partial B(f, g) = fg$

Tree expansion for $x = a + B(x, x)$ VI

- $\mathbf{X}$ is the unique solution of $\mathbf{X} = 1 + B(\mathbf{X}, \mathbf{X})$
- $B_T(1) = \mathbf{P}_T$ (Loday-Ronco basis)
- This approach motivates the introduction of $\mathbf{P}_T$ ...
- ... and leads to new combinatorial results by using more sophisticated specializations of the $\mathbf{G}_\sigma$
- In particular, one recovers the Björner-Wachs q-analogs from $x = 1 + B_q(x, x)$, with

$$B_q(x, y) = \int_0^t x(s) \cdot y(qs)\, d_q s$$

Special words and equivalence relations I

A whole class of combinatorial Hopf algebras whose operations are usually described in terms of some elaborated surgery on combinatorial objets are in fact just subalgebras of $\mathbb{K}\langle A\rangle$

- **Sym**: $R_I(A)$ is the sum of all words with the same *descent set*
- **FQSym**: $\mathbf{G}_\sigma(A)$ is the sum of all words with the same *standardization*
- **PBT**: $\mathbf{P}_T(A)$ is the sum of all words with the same *binary search tree*

Special words and equivalence relations II

To these examples, one can add:

- **WQSym**: $\mathbf{M}_u(A)$ is the sum of all words with the same *packing*
- It contains the free tridendriform algebra one one generator, based on sum of words with the same *plane tree*
- **PQSym**: based on parking functions (sum of all words with the same *parkization*)

In all cases, the product is the ordinary product of polynomials, and the coproduct is $A + B$.

Parking functions I

- A parking function of length n is a word over w over $[1, n]$ such that in the *sorted word* $w^{\uparrow}$, the ith letter is $\leq i$.
- Example $w = 52321$ OK since $w^{\uparrow} = 12235$, but not 52521
- Parkization algorithm: sort w, shift the smallest letter if it is not 1, then if necessary, shift the second smallest letter of a minimal amount, and so on. Then put each letter back in its original place
- Example: $w = (5, 7, 3, 3, 13, 1, 10, 10, 4)$,
  $w^{\uparrow} = (1, 3, 3, 4, 5, 7, 10, 10, 13)$,
  $p(w)^{\uparrow} = (1, 2, 2, 4, 5, 6, 7, 7, 9)$, and finally
  $p(w) = (4, 6, 2, 2, 9, 1, 7, 7, 3)$.

Parking functions II

- $\mathrm{PF}_n = (n+1)^{n-1}$
- Parking functions are related to the combinatorics of Lagrange inversion
- Also, noncommutative Lagrange inversion, antipode of noncommutative formal diffeomorphisms
- **PQSym**, Hopf algebra of Parking Quasi-Symmetric functions:

$$\mathbf{G_a} = \sum_{p(w)=\mathbf{a}} w$$

- Many interesting quotients and subalgebras (**WQSym**, **FQSym**, Schröder, Catalan, 3^{n-1} ...)

The Catalan subalgebra I

- Natural: group the parking functions **a** according the the sorted word $\pi = \mathbf{a}^{\uparrow}$ (occurs in the definition and in the noncommutative Lagrange inversion formula)
- Then, the sums

$$\mathbf{P}^{\pi} = \sum_{\mathbf{a}^{\uparrow}=\pi} \mathbf{G}_{\mathbf{a}}$$

  span a Hopf subalgebra **CQSym** of **PQSym**
- $\dim \mathbf{CQSym}_n = c_n$ (Catalan numbers 1,1,2,5,14)
- $\mathbf{P}^{\pi}$ is a multiplicative basis: $\mathbf{P}^{11}\mathbf{P}^{1233} = \mathbf{P}^{113455}$ (shifted concatenation)
- Free over a Catalan set $\{1, 11, 111, 112, \dots\}$ (start with 1)
- And it is cocommutative

The Catalan subalgebra II

- So it must be isomorphic to the Grossman-Larson algebra of ordered trees.
- However, this is a very different definition (no trees!)
- It reveals an interesting property of the (commutative) dual: $\mathbf{CQSym}^*$ contains *QSym* in a natural way

The Catalan subalgebra III

- Recall $m_\lambda = \Sigma x^\lambda$ (monomial symmetric functions)

$$m_\lambda = \sum_{I^\downarrow=\lambda} M_I \quad M_I(X) = \sum_{j_1<j_2<\ldots<j_r} x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_r}^{i_r}$$

- Let $\mathcal{M}_\pi$ be the dual basis of $\mathbf{P}^\pi$. It can be realized by polynomials:

$$\mathcal{M}_\pi = \sum_{p(w)=\pi} \underline{w}$$

where $\underline{w}$ means commutative image $(a_i \to x_i)$

The Catalan subalgebra IV

- Example:

$$\mathcal{M}_{111} = \sum_{i} x_i^3$$

$$\mathcal{M}_{112} = \sum_{i} x_i^2 x_{i+1}$$

$$\mathcal{M}_{113} = \sum_{i,j;j\geq i+2} x_i^2 x_j$$

$$\mathcal{M}_{122} = \sum_{i,j;i<j} x_i x_j^2$$

$$\mathcal{M}_{123} = \sum_{i,j,k;i<j<k} x_i x_j x_k$$

The Catalan subalgebra V

- Then,

$$M_I = \sum_{t(\pi)=I} \mathcal{M}_\pi .$$

where $t(\pi)$ is the composition obtained by counting the occurences of the different letters of π. For example,

$$M_3 = \mathcal{M}_{111}, \quad M_{21} = \mathcal{M}_{112} + \mathcal{M}_{113}, \quad M_{12} = \mathcal{M}_{122}$$

- In most cases, one knows at least two CHA structures on a given family of combinatorial objects: a self-dual one, and a cocommutative one. Sometimes one can interpolate between them.

Conclusion

- Many combinatorial Hopf algebras can be realized with just ordinary polynomials (commutative or not)
- If necessary, with double variables a_{ij} or x_{ij}
- No need for general Hopf algebra theory (just $A + B$)
- Morphisms are conveniently described by specializations of the variables (e.g., $a_i \to x_i \to q^i$)