# Noncommutative symmetric functions with many parameters

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Mots, Codes et Combinatoire Algébrique
Cetraro, July 1–5 2013

**Aim of the talk:**

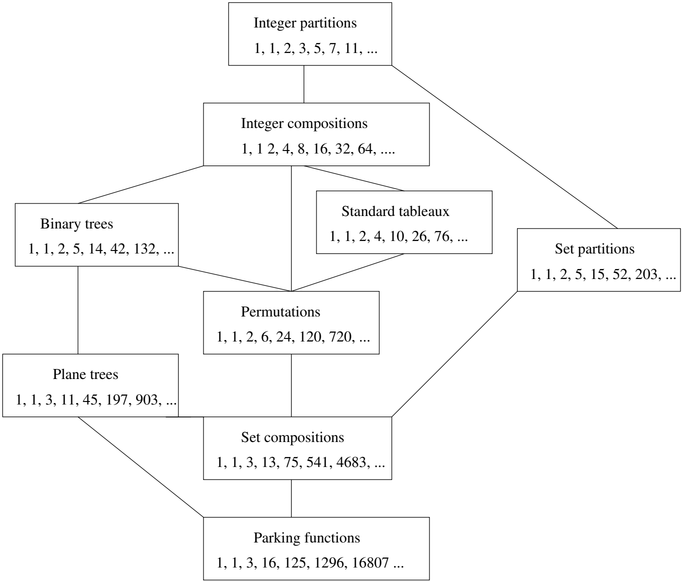
Explain the context of the paper in the IJAC special issue.

**General idea:**

Find interesting bases in combinatorial Hopf algebras

# Combinatorial Hopf algebras

- Algebras based on combinatorial objects (integer or set partitions, compositions, permutations, tableaux, trees, matroids or whatever)
- Product by summing over “compositions” of two structures, coproduct by summing over “decompositons”
- Heuristic notion (no formal definition)
- Integer partitions: *Sym*, symmetric functions. Nontrivial product and coproduct for Schur functions (Littlewood-Richardson)
- For us: CHA are generalizations of the algebra of symmetric functions.


Integer partitions
1, 1, 2, 3, 5, 7, 11, ...
Integer compositions
1, 1 2, 4, 8, 16, 32, 64, ....
Binary trees
1, 1, 2, 5, 14, 42, 132, ...
Standard tableaux
1, 1, 2, 4, 10, 26, 76, ...
Set partitions
1, 1, 2, 5, 15, 52, 203, ...
Permutations
1, 1, 2, 6, 24, 120, 720, ...
Plane trees
1, 1, 3, 11, 45, 197, 903, ...
Set compositions
1, 1, 3, 13, 75, 541, 4683, ...
Parking functions
1, 1, 3, 16, 125, 1296, 16807 ...

# Symmetric functions I

The algebra of symmetric functions is useful because it contains interesting elements: Schur, Hall-Littlewood, zonal, Jack, Macdonald ...

- Schur: character tables of symmetric groups, characters of $GL(n, \mathbb{C})$, zonal spherical functions of $(GL(n, \mathbb{C}), U(n))$, KP-hierarchy, Fock space, lots of combinatorial applications
- Hall-Littlewood (one parameter): Hall algebra, character tables of $GL(n, \mathbb{F}_q)$, geometry and toplogy of flag varieties, characters of affine Lie algebras, zonal spherical functions for $p$-adic groups, statistical mechanics
- Zonal polynomials: for orthogonal and symplectic groups
- Macdonald (two parameters): unification of the previous ones. Solutions of quantum relativistic models, diagonal harmonics, etc.

# Symmetric functions II

**Question:** Are there such things in combinatorial Hopf algebras? At least in *QSym* (pieces of symmetric functions) or **Sym** (projecting onto symmetric functions) ...

Actually, *two* different questions:

1. Find *analogs*, i.e., elements with similar definitions, properties, applications ...
2. Find *lifts* of *refinements*, e.g., noncommutative symmetric functions having Schur, HL or whatever classical symmetric functions as commutative image, or find bases of *QSym* on which the classical symmetric functions have a natural decomposition (sum over compositions with the same uderlying partition)

# Background on symmetric functions I

- "functions": polynomials in an infinite set of indeterminates

$$X = \{x_i | i \geq 1\}$$

$$\lambda_t(X) \text{ or } E(t;X) = \prod_{i\geq 1}(1+tx_i) = \sum_{n\geq 0} e_n(X)t^n$$

$$\sigma_t(X) \text{ or } H(t;X) = \prod_{i\geq 1}(1-tx_i)^{-1} = \sum_{n\geq 0} h_n(X)t^n$$

- $e_n$ = elementary symmetric functions
- $h_n$ = complete (homogeneous) symmetric functions
- Algebraically independent: $Sym(X) = K[h_1, h_2, \dots]$
- With $n$ variables: $K[e_1, e_2, \dots, e_n]$

- Bialgebra structure:

$$\Delta f = f(X + Y)$$

- $X + Y$: disjoint union; $u(X)v(Y) \simeq u \otimes v$
- Graded connected bialgebra: Hopf algebra
- Self-dual. Scalar product s.t.

$$\langle f \cdot g\,,\, h\rangle = \langle f \otimes g\,,\, \Delta h\rangle$$

- Linear bases: integer partitions

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0)$$

- Multiplicative bases:

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_r} \text{ and } h_\lambda$$

# Background on symmetric functions III

- Obvious basis: monomial symmetric functions

$$m_\lambda = \Sigma x^\lambda = \sum_{\text{distinct permutations}} x^\mu$$

- Hall's scalar product realizes self-duality

$$\langle h_\lambda \,,\, m_\mu \rangle = \delta_{\lambda\mu}$$

- *h* and *m* are adjoint bases, and

$$\sigma_1(XY) = \prod_{i,j\geq 1} (1 - x_i y_j)^{-1} = \sum_\lambda m_\lambda(X) h_\lambda(Y)$$

  (Cauchy type identity)
- Any pair of bases s.t. $\sigma_1(XY) = \sum_\lambda u_\lambda(X) v_\lambda(Y)$ are mutually adjoint

# Background on symmetric functions IV

- Original Cauchy identity for *Schur functions*

$$\sigma_1(XY) = \sum_\lambda s_\lambda(X)s_\lambda(Y)$$

where $s_\lambda = \det(h_{\lambda_i+j-i})$

- Schur functions encode irreducible characters of symmetric groups:

$$\chi^\lambda_\mu = \langle s_\lambda \, , \, p_\mu \rangle \quad \text{(Frobenius)}$$

- $p_n$: power-sums

$$p_n(X) = \sum_{i\geq 1} x_i^n \, , \quad \sigma_t(X) = \exp\left[\sum_{m\geq 1} p_m(X)\frac{t^m}{m}\right]$$

# Hecke algebra I

Permutations $\sigma \in \mathfrak{S}_n$ act on $\mathbb{K}[x_1, \ldots, x_n]$ by automorphisms: $\sigma(x_i) = x_{\sigma(i)}$. Let $s_i = (i, i+1)$ and

$$\pi_i(f) = \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}$$

(isobaric divided differences) and

$$T_i = (1-t)\pi_i + t s_i \quad (t = q^{-1})$$

Then,

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\ T_i T_j &= T_j T_i \ (|i-j| > 1) \\ T_i^2 &= (1-t) T_i + t \end{aligned}$$

Iwahori-Hecke algebra (of type $A_{n-1}$).

# Hecke algebra II

For a reduced decomposition $\sigma = s_{i_1} \cdots s_{i_r}$, let $T_\sigma = T_{i_1} \cdots T_{i_r}$ and set

$$\Omega_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\ell(\omega\sigma)} T_\sigma$$

Then, for $t = 1$, $H_n(1) = \mathbb{K}\,\mathfrak{S}_n$, and

$$m_\lambda = c_\lambda \Omega_n(1) x^\lambda \quad (c_\lambda \text{ a scalar})$$

while for $t = 0$,

$$s_\lambda = \Omega_n(0) x^\lambda$$

and (by definition), the Hall-Littlewood functions are

$$P_\lambda = c_\lambda(t) \Omega_n(t) x^\lambda$$

# Quasi-symmetric functions I

Represent a monomial $u = x_2^5 x_4^7 x_5 x_8^2$ by its *support*

$$A_u = \{x_2, x_4, x_5, x_8\}$$

and its *exponent sequence*

$$I_u = (5, 7, 1, 2)$$

(a composition of its degree $n = 15$).
The *quasi-symmetrizing action* of a permutation $\sigma$ is [Hivert]

$$\underline{\sigma}(u) = v \quad \text{with } A_v = \sigma(A_u) \text{ and } I_v = I_u$$

For example, $\underline{s}_4(u) = u$ and $\underline{s}_5(u) = x_2^5 x_4^7 x_6 x_8^2$

# Quasi-symmetric functions II

This is indeed an action of $\mathfrak{S}_n$ (not by automophisms) and its invariants is the algebra of *quasi-symmetric polynomials* [Gessel].
Precisely, one can still define $\underline{\pi}_i$ and $\underline{T}_i$ so as to get an action of $H_n(q)$, and with $\underline{\Omega}_n(t)$ as above, for a composition $I = (i_1, \ldots, i_r)$

$$M_I = c_I\, \underline{\Omega}_n(1) x^I \quad \text{(quasi-monomial functions)}$$

$$F_I = \underline{\Omega}_I(0) x^I \quad \text{(the fundamental basis)}$$

and so, Hivert defined naturally

$$P_I = c_I(t)\, \underline{\Omega}_I(t) x^I$$

This was the first example of a Hall-Littlewood-like basis in a combinatorial Hopf algebra.

Indeed, for infinite and totally ordered $X$, $QSym(X)$ becomes a Hopf algebra (coproduct by ordinal sum $X + Y$).
Its dual is **Sym** (noncommutative symmetric functions), as can be seen from the noncommutative Cauchy product

$$\mathcal{K}(X,A) := \prod_{i\geq 1}^{\rightarrow}\prod_{j\geq 1}^{\rightarrow}(1-x_i a_j)^{-1} = \sum_I M_I(X)S^I(A) = \sum_I F_I(X)R_I(A)$$

where $S^I = S_{i_1}\cdots S_{i_r}$,

$$S_n = \sum_{i_1\leq\ldots\leq i_n} a_{i_1}\cdots a_{i_n}$$

(complete functions), and $R_I$ are the *ribbon Schur functions* (sum of words with descent composition $I$).

The duality is [Malvenuto-Reutenauer]

$$\langle M_I, S^J\rangle = \delta_{IJ} = \langle F_I, R_J\rangle$$

and the dual basis $H_I$ of $P_I$ is a $t$-analogue of the product $S^I$, like the classical

$$Q'_\mu = \sum_\lambda K_{\lambda\mu}(t)s_\lambda$$

(Kostka-Foulkes polynomials, cf. [Lascoux-Schützenberger]). However, here, the coefficients $K_{IJ}(t)$ in

$$H_J = \sum_I K_{IJ}(t)R_I$$

are just powers of $t$ (KF-monomials!).

# Macdonald-like functions I

There is a simple closed formula for $K_{IJ}(t)$. Thus, we may be able to define simple Macdonald-like functions.
Precisely, we want noncommutative analogues of the

$$\tilde{H}_\mu(X;q,t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q,t)s_\lambda(X) = t^{n(\mu)}J_\mu\left(\frac{X}{1-t^{-1}};q,t^{-1}\right)$$

(bigraded Frobenius characteristics of certain realizations of the regular representations of the symmetric group [Haiman]).
Noncommutative analogues

$$\tilde{\mathrm{H}}_J(A;q,t) = \sum_I \tilde{k}_{IJ}(q,t)R_I(A) \qquad (1)$$

# Macdonald-like functions II

The $R_I$ are the characteristics of the indecomposable projective modules of the 0-Hecke algebra $H_n(0)$, each of them occuring with multiplicity one in the decomposition of the regular representation: the $\tilde{k}_{IJ}(q,t)$ have to be monomials $q^i t^j$.
The $\tilde{\mathrm{H}}_J(A;q,t)$ must reduce to HL functions for $q=0$, and we expect that the $(q,t)$-Kostka monomials should possess the symmetries

$$\tilde{k}_{I\bar{J}^{\sim}}(q,t) = \tilde{k}_{IJ}(t,q)\,, \quad (2)$$

$$\tilde{k}_{IJ}(q,t)\tilde{k}_{\bar{I}^{\sim}J}(q,t) = q^{\binom{n+1-l(J)}{2}}t^{\binom{l(I)}{2}}\,, \quad (3)$$

and that $\tilde{k}_{(n),J}(q,t)$ is always equal to 1.
These constraints determine the first matrices:

# Macdonald-like functions III

$$K_2 = \begin{pmatrix} 2 & 1 & q \\ 11 & 1 & t \end{pmatrix}$$

$$K_3 = \begin{pmatrix} 3 & 1 & q^2 & q & q^3 \\ 21 & 1 & t & q & tq \\ 12 & 1 & q & t & tq \\ 111 & 1 & t^2 & t & t^3 \end{pmatrix}$$

$$K_4 = \begin{pmatrix} 4 & 1 & q^3 & q^2 & q^5 & q & q^4 & q^3 & q^6 \\ 31 & 1 & t & q^2 & tq^2 & q & tq & q^3 & tq^3 \\ 22 & 1 & q^2 & t & tq^2 & q & q^3 & tq & tq^3 \\ 211 & 1 & t^2 & t & t^3 & q & t^2q & tq & t^3q \\ 13 & 1 & q^2 & q & q^3 & t & tq^2 & tq & tq^3 \\ 121 & 1 & t^2 & q & t^2q & t & t^3 & tq & t^3q \\ 112 & 1 & q & t^2 & t^2q & t & tq & t^3 & t^3q \\ 1111 & 1 & t^3 & t^2 & t^5 & t & t^4 & t^3 & t^6 \end{pmatrix}$$

# Macdonald-like functions IV

This is sufficient to guess the general formula, and an important property can be proved:

$$\det K_n(q,t) = \prod_{m=1}^{n-1} \prod_{k=1}^{m} \left(t^{m+1-k} - q^k\right)^{2^{n-1-m}\binom{m-1}{k-1}} .$$

There is such a factorization for the original Macdonald matrix, and there will be one for all our future generalizations.
We can in fact define *multiparameter noncommutative Macdonald-like functions* [Hivert-Lascoux-T. 2001]

$$\widetilde{\mathbf{H}}_J(A;Q,T) = \mathcal{K}_n(A;Z(J))$$

$$Z(J) = \{z_0 = 1, z_1 = \tilde{v}(J,1), z_2 = \tilde{v}(J,2), \ldots, z_{n-1} = \tilde{v}(J,n{-}1)\}$$

# Macdonald-like functions V

For $Z = \{z_0 = 1, z_1, z_2, \ldots\}$

$$\mathcal{K}_n(A; Z) = \sum_{|I|=n} \left( \prod_{d \in \mathrm{Des}(I)} z_d \right) R_I \,.$$

$$\tilde{v}(J,k) = \begin{cases} t_{1+d(J,k)} & \text{if } k \in \mathrm{Des}(J)\,, \\ q_{k-d(J,k)} & \text{if } k \notin \mathrm{Des}(J)\,. \end{cases}$$

and

$$d(I,k) = \#\{k' < k,\ k' \in \mathrm{Des}(I)\}$$

A few days after this paper was posted, another one by N. Bergeron and M. Zabrocki, defining a similar but different family of Macdonald-like functions appeared on the arXiv. It was not a specialization of our multiparameter family.

# $\mathbf{Sym}_n$ as a Grassmann algebra I

Both families can be unified by introducing more parameters [Lascoux-Novelli-T. 2012]. The construction is simplified by the following formalism.

For $n > 0$, $\mathbf{Sym}_n$ has dimension $2^{n-1}$, same as a Grassmann algebra on $n - 1$ generators $\eta_1, \ldots, \eta_{n-1}$

$$\eta_i \eta_j = -\eta_j \eta_i$$

If $I$ is a composition of $n$ with descent set $D = \{d_1, \ldots, d_k\}$,

$$R_I \longleftrightarrow \eta_D := \eta_{d_1} \eta_{d_2} \ldots \eta_{d_k} . \tag{4}$$

For example, $R_{213} \leftrightarrow \eta_2 \eta_3$. Then,

$$S^I \longleftrightarrow (1 + \eta_{d_1})(1 + \eta_{d_2}) \ldots (1 + \eta_{d_k})$$

# $\mathbf{Sym}_n$ as a Grassmann algebra II

Grassmann integral

$$\int d\eta\, f := f^{12\ldots n-1}, \qquad \text{where} \quad f = \sum_k \sum_{i_1<\cdots<i_k} f^{i_1\ldots i_k}\eta_{i_1}\ldots\eta_{i_k}.$$

Anti-involution $\eta_i^* = (-1)^i\eta_i$. Bilinear form on $\mathbf{Sym}_n$

$$(f,g) = \int d\eta\, f^* g$$

Then,

$$(R_I, R_J) = (-1)^{\ell(I)-1}\delta_{I,\bar{J}^{\sim}}$$

(Bergeron-Zabrocki “scalar product”).

# $\mathbf{Sym}_n$ as a Grassmann algebra III

For $Z = (z_1, \dots, z_{n-1})$, let

$$K_n(Z) = (1 + z_1\eta_1)(1 + z_2\eta_2)\dots(1 + z_{n-1}\eta_{n-1})\,. \qquad (5)$$

Then,

$$(K_n(X), K_n(Y)) = \prod_{i=1}^{n-1}(y_i - x_i)\,. \qquad (6)$$

We are interested in bases of $\mathbf{Sym}_n$ of the form

$$\tilde{\mathrm{H}}_I = K_n(Z_I) = \sum_J \tilde{\mathbf{k}}_{IJ} R_J$$

The HLT and BZ bases have this form.

# $\mathbf{Sym}_n$ as a Grassmann algebra IV

For both of them, the determinant of the Kostka matrix $\mathcal{K} = (\tilde{\mathbf{k}}_{IJ})$ is a product of linear factors. This is because these matrices have the form

$$\begin{pmatrix} A & xA \\ B & yB \end{pmatrix}$$

where $A$ and $B$ have a similar structure, and so on recursively:

$$\begin{vmatrix} A & xA \\ B & yB \end{vmatrix} = (y - x)^m \det A \cdot \det B \,.$$

We can now introduce many more parameters.

# $\mathbf{Sym}_n$ as a Grassmann algebra V

Let $\mathbf{y} = \{y_u\}$ for $u$ boolean word of length $\leq n-1$.
For $n = 3$: $y_0, y_1, y_{00}, y_{01}, y_{10}, y_{11}$.
Encode a composition $I$ with descent set $D$ by $u = (u_1, \ldots, u_{n-1})$ such that $u_i = 1$ if $i \in D$ and $u_i = 0$ otherwise.
Let $u_{m\ldots p}$ be the sequence $u_m u_{m+1} \ldots u_p$

$$P_I := (1 + y_{u_1}\eta_1)(1 + y_{u_{1\ldots 2}}\eta_2)\ldots(1 + y_{u_{1\ldots n-1}}\eta_{n-1})$$

or, equivalently,

$$P_I := K_n(Y_I) \qquad \text{with} \quad Y_I = [y_{u_1}, y_{u_{1\ldots 2}}, \ldots, y_u] =: (y_k(I))\,.$$

At this level of generality, the Kostka matrix, the product formula, and the dual basis can be computed explicitly.

# $\mathbf{Sym}_n$ as a Grassmann algebra VI

There are some interesting specializations. First, a family with two infinite matrix parameters $Q, T$:
Label the cells of $I$ with their matrix coordinates:

$$\mathrm{Diagr}\,(4,1,2,1) = \begin{array}{|c|c|c|c|c} \hline (1,1) & (1,2) & (1,3) & (1,4) & \multicolumn{1}{c}{} \\ \hline \multicolumn{3}{c|}{} & (2,4) & \multicolumn{1}{c}{} \\ \cline{4-5} \multicolumn{3}{c|}{} & (3,4) & \multicolumn{1}{c|}{(3,5)} \\ \cline{4-5} \multicolumn{4}{c|}{} & \multicolumn{1}{c|}{(4,5)} \\ \cline{5-5} \end{array}$$

Associate a variable $z_{ij}$ with each cell except $(1,1)$: $z_{ij} := q_{i,j-1}$ if $(i,j)$ has a cell on its left, and $z_{ij} := t_{i-1,j}$ if $(i,j)$ has a cell on its top. The alphabet $Z(I) = (z_j(I))$ is the sequence of the $z_{ij}$ in their natural order.

For $J \vDash n$

$$\tilde{\mathbf{k}}_{IJ}(Q,T) = \prod_{d \in \mathrm{Des}(J)} z_d(I)\,.$$

With $I = (4,1,2,1)$ and $J = (2,1,1,2,2)$, we have $\mathrm{Des}(J) = \{2,3,4,6\}$ and $\tilde{\mathbf{k}}_{IJ} = q_{12}q_{13}t_{14}q_{34}$.

Let $Q = (q_{ij})$ and $T = (t_{ij})$ $(i,j \geq 1)$ be two infinite matrices.

$\tilde{\mathrm{H}}_I(A;Q,T)$ is defined as

$$\tilde{\mathrm{H}}_I(A;Q,T) = K_n(A;Z(I)) = \sum_{J \vDash n} \tilde{\mathbf{k}}_{IJ}(Q,T) R_J(A)\,.$$

Note that $\tilde{\mathrm{H}}_I$ depends only on the $q_{ij}$ and $t_{ij}$ with $i + j \leq n$.

# $\mathbf{Sym}_n$ as a Grassmann algebra VIII

Let $(q_i)$, $(t_i)$, $i \geq 1$ be two sequences of indeterminates.
Let $\nu$ be the anti-involution of **Sym** defined by $\nu(S_n) = S_n$.

(i) For $q_{ij} = q_{i+j-1}$, $t_{ij} = t_{n+1-i-j}$, $\tilde{\mathrm{H}}_I(Q, T)$ becomes a multiparameter version of $\nu(\tilde{\mathrm{H}}_I^{BZ})$, to which it reduces under the further specialization $q_i = q^i$ and $t_i = t^i$.

(ii) For $q_{ij} = q_j$, $t_{ij} = t_i$, $\tilde{\mathrm{H}}_I(Q, T)$ reduces to $\tilde{\mathrm{H}}_I^{HLT}$.

The multivariate HL-BZ-polynomials have been recently interpreted by Jia Huang (arXiv:1306.1931) as graded Frobenius characteristics of the action of $H_n(0)$ on certain submodules of the Stanley-Reisner ring of the Boolean algebra.

In the Hopf algebra paradigm, monomial functions live on the quasi-symmetric side. But if one is willing to forget about the coproducts, noncommutative monomial functions can be defined [Tevlin]. Let

$$\Psi_n = \sum_{k=0}^{n-1} (-1)^k R_{1^k, n-k}$$

be the power-sums of the first kind (Dynkin elements) and

$$r\Psi_I \equiv r\Psi_{(i_1,\ldots,i_r)} = (-1)^{r-1} \begin{vmatrix} \Psi_{i_r} & 1 & 0 & \ldots & 0 & 0 \\ \Psi_{i_{n-1}+i_r} & \Psi_{i_{n-1}} & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\ldots+i_r} & \ldots & \ldots & \ldots & \Psi_{i_2} & n-1 \\ \boxed{\Psi_{i_1+\ldots+i_r}} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix}$$

(a quasi-determinant). In particular,

$$\Psi_{(n)} = \Psi_n, \text{ and } \Psi_{1^r} = \Lambda_r \,.$$

Equivalently,

$$r\Psi_{i_1,\ldots,i_r} = \Psi_{i_1}\Psi_{i_2,\ldots,i_r} - \Psi_{i_1+i_2}\Psi_{i_3,\ldots,i_r} + \ldots \\ + (-1)^{s-1}\Psi_{i_1+\cdots+i_s}\Psi_{i_{s+1},\ldots,i_r} + \cdots + (-1)^r\Psi_{i_1+\cdots+i_r}.$$

One can define an analog of Gessel's fundamental basis $F_I$ by

$$L_I = \sum_{J \succeq I} \Psi_J.$$

$$R_I = \sum_J G_{IJ} L_J = \sum_J K_{IJ} \Psi_J. \tag{7}$$

# Noncommutative monomial functions III

The $K_{IJ}$ and the $G_{IJ}$ are nonnegative integers with interesting combinatorial interpretations [Hivert-Novelli-Tevlin-T.]
Define the G-descent set of a permutation $\sigma \in \mathfrak{S}_n$ as

$$\mathrm{GDes}(\sigma) := \{i \in [2, n] | \sigma_j = i \Longrightarrow \sigma_{j+1} < \sigma_j\}.$$

The G-composition $\mathrm{GC}(\sigma)$ is the composition whose descent set is $\{d - 1 | d \in \mathrm{GDes}(\sigma)\}$.
Then,

$$R_I = \sum_{J \vDash n} G_{IJ} L_J,$$

where $G_{IJ}$ is the number of permutations $\sigma$ satisfying $C(\sigma^{-1}) = I$ and $\mathrm{GC}(\sigma) = J$.

The above $K_{IJ}$ and $G_{IJ}$ admit nontrivial $q$-analogues, which can be obtained from the combinatorial Hopf algebras **FQSym** (permutations) and **WQSym**) (packed words).
A packed word (over the integers) is a word $u$ whose support is an interval $[1, k]$.
An inversion $u_i = b > u_j = a$ (where $i < j$ and $a < b$) is *special* if $u_j$ is the *rightmost* occurence of $a$ in $u$. Let $\mathrm{sinv}(u)$ denote the number of special inversions in $u$.
The W-composition WC of $u$ is the composition whose descent set is given by the positions of the last occurrences of each letter in $u$.
Let $W(I, J)$ be the set of packed words $w$ such that

$$\mathrm{WC}(w) = I \qquad \text{and} \qquad C(w) \succeq J \tag{8}$$

and

$$C_I^J(q) = \sum_{w \in W(I,J)} q^{\mathrm{sinv}(w)}. \tag{9}$$

Then

$$S^J(q) := \sum_I C_I^J(q)\Psi_I$$

is a $q$-analogue of the product $S^J$ (like the classical $Q'_\mu$) defined in [Novelli-T.-Williams]. Its expansion on a simple $q$-analogue $L_I(q)$ of $L_I$ provides a $q$-enumeration of permutation tableaux. One can also define a basis $R_I(q)$ and $q$-analogues of the $G_{IJ}$. The $q$-deformed ribbons are given by

$$R_J(q) = \sum_I D_I^J(q)\Psi_I$$

where

$$D_I^J(q) = \sum_{w \in W'(I,J)} q^{\mathrm{sinv}(w)}. \qquad (10)$$

$W'(I,J)$ being the set of packed words $w$ such that

$$\mathrm{WC}(w) = I \qquad \text{and} \qquad C(w) = J \qquad (11)$$

Next, Tevlin defined noncommutative analogues of the $P$-HL functions by a $t$-deformation of the quasi-determinant for the $\Psi_I$, and defined Kostka-like polynomials by

$$R_J(A) = \sum_I K_{IJ}(t) P_I(t;A) \qquad (12)$$

Then,

$$K_{IJ}(t) = \tilde{D}_I^J(t) = t^{\mathrm{maj}(I)} D_I^J(t^{-1}) \qquad (13)$$

# Grand unification I

[HLT] and [BZ] have been unified in [LNT], but [NTW] and [T] seem to belong to different worlds.
Actually, [NTW] and [T] are related by the noncommutative version of the classical $(1-t)$-transform on symmetric functions

$$p_n((1-t)X) = (1-t^n)p_n(X)$$

It admits a multiparameter analogue, and the resulting multiparameter $P$-functions admit a simple description within the Grassmann formalism of [LNT].

# Grand unification II

The noncommutative $(1-t)$-transform acts on ribbons by

$$R_I((1-t)A)) = (-1)^{\ell(I)} \sum_{|J|=|I|, r=\ell(J)} (-1)^r (1-t^{j_r}) t^{\sum_{k\in\mathcal{A}(I,J)} j_k} S^J(A)$$

where

$$\mathcal{A}(I,J) = \{s < \ell(J) | j_1 + \cdots + j_s \notin \mathrm{Des}(I)\}.$$

Let $\mathbf{t} = (t_i)_{i\geq 1}$, and define

$$\mathcal{R}_I(\mathbf{t}; A) = (-1)^{\ell(I)} \sum_{|J|=|I|, r=\ell(J)} (-1)^r \left( (1-t_{j_r}) \prod_{k\in\mathcal{A}(I,J)} t_{j_k} \right) S^J(A)$$

$$\begin{aligned}
\mathcal{R}_3 &= (1-t_3)S^3 - (1-t_1)t_2 S^{21} - (1-t_2)t_1 S^{12} + (1-t_1)t_1^2 S^{111}, \\
\mathcal{R}_{21} &= -(1-t_3)S^3 + (1-t_1)S^{21} + (1-t_2)t_1 S^{12} - (1-t_1)t_1 S^{111}.
\end{aligned}$$

Define also

$$\mathcal{S}^I(\mathbf{t};A) = \sum_{J \le I} \mathcal{R}_J(\mathbf{t};A)$$

The $\mathcal{S}$-basis is multiplicative:

$$\mathcal{S}^I(\mathbf{t})\mathcal{S}^J(\mathbf{t}) = \mathcal{S}^{IJ}(\mathbf{t})\,.$$

Thus, $\mathcal{R}_I$ is the image of $R_I$ by the automorphism

$$\theta_{\mathbf{t}}:\ S_n(A) \longmapsto \mathcal{S}_n(\mathbf{t};A)\,.$$

The inverse of $\theta_{\mathbf{t}}$ is

$$\theta_{\mathbf{t}}^{-1}:\ S_n \mapsto \mathcal{K}_n(\mathbf{t};A) = \sum_{I \vDash n} \frac{\prod_{d\in \mathrm{Des}(I)} t_d}{(1-t_1)(1-t_2)\cdots(1-t_n)} R_I(A)$$

(the multiparameter Klyachko element).

# Grand unification IV

Recall the Grassmann algebra formalism. We need a small modification of the definition of $K_n$:
Let $U = (u_1, \ldots, u_{n-1})$ and $V = (v_1, \ldots, v_{n-1})$ be two sequences of parameters. Set

$$K_n(U,V) = (u_1 + v_1\eta_1)\cdots(u_{n-1} + v_{n-1}\eta_{n-1})$$
$$= \sum_{I \vDash n} \prod_{d \in \mathrm{Des}(I)} v_d \prod_{e \notin \mathrm{Des}(I)} u_e \, R_I$$

We build a pair of sequences $(U_I, V_I) = ((u_j^I), (v_j^I))_{j=1}^{n-1}$ from the diagram of $I$.

# Grand unification V

First, write $(1, q_1), \dots, (1, q_k)$ in this order, starting from the top left cell, in all cells which are non-descents of $I$. Then, write $(t_1, 1), \dots, (t_l, 1)$, in this order, in all cells which are descents of $I$, starting from the bottom right cell

$$
(U_{4121}, V_{4121}) = \begin{array}{|c|c|c|c|c|}
\hline
(1, q_1) & (1, q_2) & (1, q_3) & (t_3, 1) & \\
\hline
 & & & (t_2, 1) & \\
\hline
 & & & (1, q_4) & (t_1, 1) \\
\hline
 & & & & \times \\
\hline
\end{array}
\qquad (14)
$$

# Grand unification VI

Let $\mathcal{J}'_I(\mathbf{q}, \mathbf{t}, A) = K_n(U_I, V_I)$. Define Macdonald-like functions by

$$\mathcal{J}_I(\mathbf{q}, \mathbf{t}; A) = \theta_{\mathbf{t}}(\mathcal{J}'_I(\mathbf{q}, \mathbf{t}; A))\,. \tag{15}$$

If we regard the $\mathcal{J}$-functions as analogues of the Macdonald $J$-functions, we can define natural analogues of the classical $P$ and $Q$-functions by

$$\prod_{i=1}^{\ell(I)}(1 - t_i)\mathcal{P}_I(\mathbf{t}; A) = \mathcal{Q}_I(\mathbf{t}; A) = \mathcal{J}_I(0, \mathbf{t}; A)$$

Note that $\mathcal{Q}_n(\mathbf{t}; A) = \mathcal{R}_n(\mathbf{t}; A)$.

# Grand unification VII

The $\mathcal{P}$-functions satisfy the recurrence

$$\frac{1-t_r}{1-t_1}\mathcal{P}_I = \mathcal{P}_{i_1}\mathcal{P}_{i_2,\dots,i_r} - \mathcal{P}_{i_1+i_2}\mathcal{P}_{i_3,\dots,i_r} + \cdots + (-1)^{r-1}\mathcal{P}_{i_1+\cdots+i_r}\,.$$

Equivalently, we have the quasideterminantal expression

$$\mathcal{P}_I(\mathbf{t};A) = (-1)^{r-1}\frac{1-t_1}{1-t_r}\begin{vmatrix} \mathcal{P}_{i_r} & 1-t_1 & 0 & \dots & 0 & 0 \\ \mathcal{P}_{i_{r-1}+i_r} & P_{i_{r-1}} & 1-t_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{P}_{i_2+\dots+i_r} & \dots & \dots & \dots & \mathcal{P}_{i_2} & 1-t_{r-1} \\ \boxed{\mathcal{P}_{i_1+\dots+i_r}} & \dots & \dots & \dots & \mathcal{P}_{i_1+i_2} & P_{i_1} \end{vmatrix}$$

which reduces to Tevlin's definition for $t_i = t^i$. Their product formula and expansions on various bases can be computed explicitly.

# The real motivation

Multiparameter Macdonald polynomials? Up to $n = 5$ ... [HLT]
Heuristics: a conjecture on $R$-matrices, multiparameter HL for rectangular shapes, hook shapes, symmetries, determinant ...

| | (4) | (31) | (22) | (211) | (1111) |
|---|---|---|---|---|---|
| (4) | 1 | $q_1 + q_2 + q_3$ | $q_2 + q_1 q_3$ | $q_1 q_2 + q_2 q_3 + q_1 q_3$ | $q_1 q_2 q_3$ |
| (31) | 1 | $t_1 + q_1 + q_2$ | $q_2 + q_1 t_1$ | $q_1 t_1 + q_2 t_1 + q_1 q_2$ | $q_1 t_1 q_2$ |
| (22) | 1 | $q_1 + q_1 t_1 + t_1$ | $q_1^2 + t_1^2$ | $q_1 t_1^2 + q_1 t_1 + q_1^2 t_1$ | $q_1^2 t_1^2$ |
| (211) | 1 | $q_1 + t_1 + t_2$ | $q_1 t_1 + t_2$ | $t_1 t_2 + q_1 t_1 + q_1 t_2$ | $q_1 t_1 t_2$ |
| (1111) | 1 | $t_1 + t_2 + t_3$ | $t_2 + t_1 t_3$ | $t_1 t_2 + t_2 t_3 + t_1 t_3$ | $t_1 t_2 t_3$ |

$$\det = (t_2 - q_2)(t_1 - q_2)(t_2 - q_1)(t_1 - q_1)^3(t_3 - q_1)(t_1 - q_3)$$

| (5) | (41) | (32) | (311) |
|---|---|---|---|
| 1 | $q_1+q_2+q_3+q_4$ | $q_2+q_3+q_1q_3+q_1q_4+q_2q_4$ | $q_1q_2+q_1q_3+q_2q_3+q_1q_4+q_2q_4+q_3q_4$ |
| 1 | $t_1+q_1+q_2+q_3$ | $q_1t_1+q_2+q_2t_1+q_3+q_1q_3$ | $q_1t_1+q_2t_1+q_1q_2+t_1q_3+q_1q_3+q_2q_3$ |
| 1 | $t_1+q_1+q_1t_1+q_2$ | $q_1t_1+q_2+q_1^2t_1+t_1^2+q_1q_2$ | $q_1t_1+q_1t_1^2+q_1^2t_1+q_2t_1+q_1q_2+q_1t_1q_2$ |
| 1 | $q_1+q_2+t_1+t_2$ | $q_2+q_1t_1+q_2t_1+q_1t_2+t_2$ | $q_1q_2+q_1t_1+q_2t_1+q_1t_2+q_2t_2+t_1t_2$ |
| 1 | $q_1+t_1+q_1t_1+t_2$ | $q_1t_1+t_2+q_1t_1^2+q_1^2+t_1t_2$ | $q_1t_1+q_1^2t_1+q_1t_1^2+q_1t_2+t_1t_2+q_1t_1t_2$ |
| 1 | $q_1+t_1+t_2+t_3$ | $q_1t_1+t_2+q_1t_2+t_3+t_1t_3$ | $q_1t_1+q_1t_2+t_1t_2+t_3q_1+t_1t_3+t_2t_3$ |
| 1 | $t_1+t_2+t_3+t_4$ | $t_2+t_3+t_1t_3+t_1t_4+t_2t_4$ | $t_1t_2+t_1t_3+t_2t_3+t_1t_4+t_2t_4+t_3t_4$ |

| (221) | (2111) | (11111) |
|---|---|---|
| $q_1q_3+q_2q_3+q_2q_4+q_1q_2q_4+q_1q_3q_4$ | $q_1q_2q_3+q_1q_2q_4+q_1q_3q_4+q_2q_3q_4$ | $q_1q_2q_3q_4$ |
| $q_2t_1+q_1t_1q_2+q_1q_3+q_1q_3t_1+q_2q_3$ | $q_1q_2q_3+t_1q_2q_3+q_1q_3t_1+q_1t_1q_2$ | $t_1q_1q_2q_3$ |
| $q_1^2q_2+q_1t_1q_2+q_1t_1^2+q_2t_1+q_1^2t_1^2$ | $q_1^2q_2t_1+q_1q_2t_1^2+q_1t_1q_2+q_1^2t_1^2$ | $q_1^2t_1^2q_2$ |
| $q_1t_1t_2+q_2t_2+q_1t_2+q_2t_1+q_1t_1q_2$ | $t_1t_2q_2+q_1t_1t_2+q_1q_2t_2+q_1t_1q_2$ | $q_1q_2t_1t_2$ |
| $t_1^2t_2+q_1t_1t_2+q_1^2t_1+q_1t_2+q_1^2t_1^2$ | $q_1t_1^2t_2+t_2q_1^2t_1+q_1t_1t_2+q_1^2t_1^2$ | $q_1^2t_1^2t_2$ |
| $q_1t_2+q_1t_1t_2+t_1t_3+t_1t_3q_1+t_2t_3$ | $t_1t_2t_3+q_1t_2t_3+t_1t_3q_1+q_1t_1t_2$ | $q_1t_1t_2t_3$ |
| $t_1t_3+t_2t_3+t_2t_4+t_1t_2t_4+t_1t_3t_4$ | $t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4$ | $t_1t_2t_3t_4$ |

$$\det = (t_2-q_3)(t_1-q_3)(t_2-q_2)(t_1-q_2)^2(t_3-q_2)(t_2-q_1)^2(t_3-q_1)(t_1-q_1)^4(t_4-q_1)(t_1-q_4)$$

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