25 years of LLT polynomials

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Original motivation: plethysm

- Irreducible tensor representations of $GL(n, \mathbb{C})$:

$$\rho_\lambda : \ GL(n, \mathbb{C}) \longrightarrow GL(V_\lambda), \quad V_\lambda \subseteq (\mathbb{C}^n)^{\otimes k}$$

- λ partition of k with at most n parts
- Character: Schur function $s_\lambda = \mathrm{ch}(\rho_\lambda)$
- Composition of two representations ρ of character f and η of character g:

$\mathrm{ch}(\eta \circ \rho) =: g \circ f$ plethysm of f by g, also denoted by $g[f]$

- The problem: compute

$$s_\lambda[s_\mu] = \sum_\nu d^\nu_{\lambda\mu} s_\nu$$

- More precisely, find a *combinatorial* description
- if $\lambda \vdash d$, $s_\lambda[s_\mu]$ is a part of

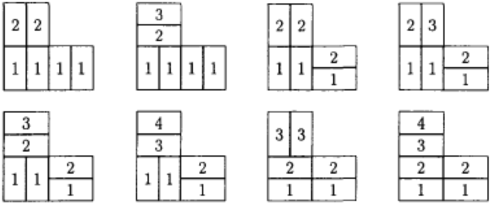
$$s_\mu^d = \sum_{\nu \vdash nd} c^\nu_{\mu\mu\cdots\mu} s_\nu = \sum_{\lambda \vdash d} f^\lambda s_\lambda[s_\mu]$$

  where $c^\nu_{\mu\mu\cdots\mu}$ are the Littlewood-Richardson coefficients, and f^λ the number of standard tableaux of shape λ.
- For $d = 2$, no multiplicities

$$V \otimes V = S^2(V) \oplus \Lambda^2(V) \Leftrightarrow s_\mu^2 = h_2[s_\mu] + e_2[s_\mu]$$

- First problem: split the Littlewood-Richardson tableaux into two sets, corresponding to the symmetric and antisymmetric parts of the square.
- Idea (B.L.) Formulate a version of the LR-rule with domino tableaux, and split according to the parity of half the number of horizontal dominos.

$$s_{21}^2 = s_{42} + s_{411} + s_{33} + 2s_{321} + s_{3111} + s_{222} + s_{2211}$$



$$\begin{cases} h_2[s_{21}] & = s_{42} + s_{321} + s_{3111} + s_{222} \\ e_2[s_{21}] & = s_{411} + s_{33} + s_{321} + s_{2211} \end{cases}$$

[C. Carré, B. Leclerc, Séminaire Lotharingien de Combinatoire, B31c (1993), 8 pp; J. Alg.Combin. **4** (1995), 201–231]

What about higher powers?

Next step suggested by previous LLT results on Hall-Littlewood functions at roots of unity

- Hall-Littlewood functions

$$P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu}(t) P_\lambda$$

  such that $g^\lambda_{\mu\nu}(q) = q^{n(\lambda)-n(\mu)-n(\nu)} f^\lambda_{\mu\nu}(q^{-1})$ (Hall algebra)

- Kostka numbers

$$s_\lambda = \sum_\mu K_{\lambda\mu}(t) P_\mu$$

- Kostka numbers are special LR coefficients

$$K_{\lambda\mu} = c^\lambda_{\mu_1,\mu_2,\dots,\mu_r}$$

- Dual HL functions

$$\langle Q'_\mu, P_\nu \rangle = \delta_{\mu\nu} \quad (\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu})$$

  are t-analogues of products h_μ

$$Q'_\mu = \sum_\lambda K_{\lambda\mu}(t) s_\lambda \longrightarrow h_\mu \ (t \to 1)$$

- The Kostka-Foulkes polynomials $K_{\lambda\mu}(t) \in \mathbb{N}[t]$
- The $\tilde{K}_{\lambda\mu}(q)$ are (parabolic) Kazhdan-Lusztig polynomials for the affine symmetric group

Roots of unity and plethysm formulae

- $t = 1$ is not the only interesting value
- For $t = \zeta$ a primitive rth root of unity

$$Q'_\lambda(X;\zeta) = Q'_\mu(X;\zeta) \prod_{i\geq 1} \left[Q'_{(i^r)}(X;\zeta)\right]^{q_i}$$

  where $\lambda = (1^{m_1} 2^{m_2} \dots n^{m_n})$, $m_i = rq_i + r_i$ with $0 \leq r_i < r$, and $\mu = (1^{r_1} 2^{r_2} \dots n^{r_n})$.
- and for rectangular partitions, we obtain plethysms with power-sums

$$Q'_{(n^r)}(X;\zeta) = (-1)^{(r-1)n} p_r[h_n(X)]$$

Interpretation

Consider the (reducible) $GL(n,\mathbb{C})$-module

$$V = \Lambda^{\nu_1}\mathbb{C}^n \otimes \Lambda^{\nu_2}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{\nu_r}\mathbb{C}^n$$

and the cyclic shift operator $\gamma : \ V^{\otimes d} \mapsto V^{\otimes d}$

$$\gamma(v_1 \otimes v_2 \otimes \cdots \otimes v_d) = v_d \otimes v_1 \otimes \cdots \otimes v_{d-1}$$

Its eigenspaces $W^{(k)}$ are representations of $GL(n,\mathbb{C})$.
The previous formulae imply a combinatorial description of their characters $\ell_d^{(k)}[e_\nu]$.

Can we do the same starting with $V = V_\lambda$ irreducible ?

Answer: ribbon tableaux

Ribbon tableaux and products of Schur functions

A Schur function $s_\lambda(X)$ is a sum over semi-standard Young tableaux t of shape λ

$$s_\lambda(X) = \sum_{t \in \mathrm{Tab}\,(\lambda)} X^t$$

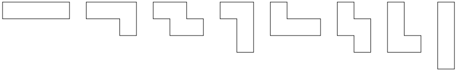
where $X^t = \prod_i x_i^{m_i}$, m_i number of occurences of i in t.
A product of r Schur functions $s_{\mu^{(i)}}$ is a sum over r-tuples of tableaux

$$s_{\mu^{(1)}} s_{\mu^{(2)}} \cdots s_{\mu^{(r)}} = \sum_{(t_1,\ldots,t_r)} X^{t_1} X^{t_2} \cdots X^{t_r}$$

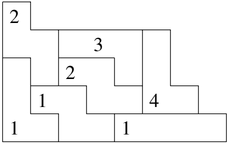
r-tuples of tableaux $\longleftrightarrow$ r-ribbon tableaux

Ribbons (rim-hooks) and ribbon tableaux

Here are the ($2^3 = 8$) 4-ribbons

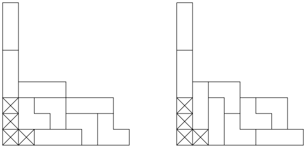


and a 4-ribbon tableau of shape (87661) and weight (3211)

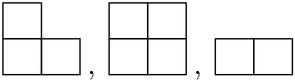


r-core and r-quotient

The partition $\lambda = (87^2 41^5)$ has as 3-core $\nu = (211)$



and as 3-quotient the triple $((21), (22), (2))$

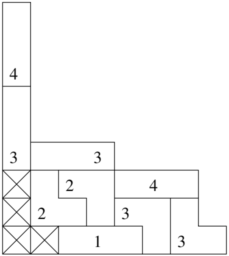


The Stanton-White bijection

Choosing as 3-core $\kappa = (211)$, the triple

$$\begin{array}{|c|cc}\hline 4 & \multicolumn{2}{c}{} \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\;,\quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}\;,\quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}$$

with weights $(0021), (1111), (0110)$ corresponds to the 3-ribbon tableau of shape $\lambda = (87^2 41^5)$ and weight $\mu = (1242)$.



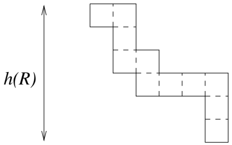
If μ is the partition with *r*-quotient $(\mu^{(0)}, \ldots, \mu^{(r-1)})$ and empty *r*-core

$$s_{\mu^{(0)}} s_{\mu^{(1)}} \cdots s_{\mu^{(r-1)}} = \sum_{T \in \mathrm{Tab}_r(\mu,\cdot)} X^T$$

where $\mathrm{Tab}_r(\mu, \cdot)$ is the set of *r*-ribbon tableaux of shape μ

A natural statistic on ribbon tableaux is the sum of the heights of the ribbons

Example: $r = 11$, $h(R) = 6$



Spin and cospin

The relevant statistic is rather $h(R)-1$, and for compatiblity with Hall-Littlewood functions, one introduces the *spin*

$$s(R)=\frac{1}{2}(h(R)-1),\quad s(T)=\sum_{R\in T} s(R)$$

(a half-integer in general) and the *cospin* (an integer)

$$\tilde{s}(T)=s_r^*(\mu)-s(T)\quad \text{for } T\in \mathrm{Tab}\,_r(\mu,\cdot)$$

The most general q-LR coefficients are defined by

$$\tilde{G}_\mu=\sum_{T\in \mathrm{Tab}\,_r(\mu,\cdot)} q^{\tilde{s}(T)}X^T=\sum_\lambda c^\lambda_{\mu^{(0)},\mu^{(1)},\ldots,\mu^{(r-1)}}(q)s_\lambda(X)$$

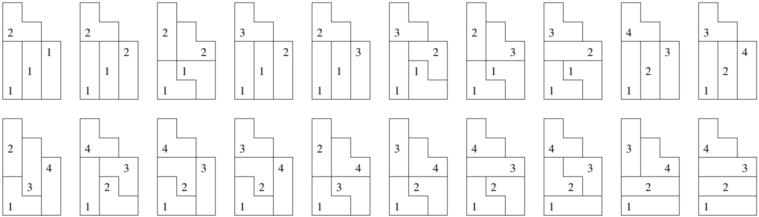
The 3-quotient of $\lambda = (33321)$ is $((1),(1,1),(1))$ and the q-analogue of $s_1 s_{11} s_1$ (in this order) is

$$m_{31} + (1+q)m_{22} + (2+2q+q^2)m_{211} + (3+5q+3q^2+q^3)m_{1111}$$

$$= (s_{31} - s_{22} - s_{211} + 2\,s_{1111}) + (1+q)(s_{22} - s_{211} + s_{1111})$$
$$+(2+2q+q^2)(s_{211} - 3\,s_{1111}) + (3+5q+3q^2+q^3)s_{1111}$$
$$= s_{31} + qs_{22} + (q+q^2)s_{211} + q^3 s_{1111}$$

The $c^{\lambda}_{\mu_1,\mu_2,\ldots,\mu_r}(q)$ are defined by an alternating sum but are in $\mathbb{N}[q]$.

The monomial expansion above is given by the 3-ribbon tableaux of shape (33321) and dominant weight



The H-functions

- Family of spin t-analogues related to HL functions.
- A partition of the form $\lambda = r\mu = (r\mu_1, \ldots, r\mu_s)$ has empty r-core
- Its r-quotient is obtained by grouping the parts of μ according to their class modulo r

$$\lambda(i) = \{\mu_j | j \equiv -i \mod r\}$$

- For any r, the symmetric functions

$$H_\mu^{(r)}(X;t) = \sum_{T \in \mathrm{Tab}_r(r\mu,\cdot)} t^{s(T)} X^T$$

  form a basis which is unitriangular on Schur functions
- It can be proved that for $r \geq \ell(\mu)$,

$$H_\mu^{(r)}(X;t) = Q'_\mu(X;t)$$

Some conjectures for H-functions

- **Monotonicity** $H_\mu^{(r+1)} - H_\mu^{(r)}$ is positive on the Schur basis, that is, the coefficients are in $\mathbb{N}[t]$.
- **Plethysm** When $\mu = \nu^r$, for ζ a primitive r-th root of unity,

$$H_{\nu^r}^{(r)}(\zeta) = (-1)^{(r-1)|\nu|}\, p_r[s_\nu]$$

and when $d|r$ and ζ is a primitive d-th root of unity,

$$H_{\nu^r}^{(r)}(\zeta) = (-1)^{(d-1)|\nu|r/d} p_d^{r/d}[s_\nu]\ .$$

- Equivalently,

$$H_{\nu^r}^{(r)}(t) \mod 1 - t^r = \sum_{i=0}^{r-1} t^i \ell_r^{(i)}[s_\nu]$$

- **Proved by Kazuto Iijima** [European J. Combin. **34** (2013) 968–986]

Examples

The H-functions associated with the partition $\lambda = (3211)$ are

$$\begin{aligned}
H^{(2)}_{3211} &= s_{3211} + t\, s_{322} + t\, s_{331} + t\, s_{4111} \\
&\quad + (t + t^2)\, s_{421} + t^2\, s_{43} + t^2\, s_{511} + t^3\, s_{52} \\
H^{(3)}_{3211} &= s_{3211} + t\, s_{322} + (t + t^2)\, s_{331} + t\, s_{4111} \\
&\quad + (t + 2t^2)\, s_{421} + (t^2 + t^3)\, s_{43} + (t^2 + t^3)\, s_{511} \\
&\quad + 2t^3\, s_{52} + t^4\, s_{61} \\
H^{(4)}_{3211} &= s_{3211} + t\, s_{322} + (t + t^2)\, s_{331} + t\, s_{4111} + (t + 2t^2 + t^3)\, s_{421} \\
&\quad + (t^2 + t^3 + t^4)\, s_{43} + (t^2 + t^3 + t^4)\, s_{511} \\
&\quad + (2t^3 + t^4 + t^5)\, s_{52} + (t^4 + t^5 + t^6)\, s_{61} + t^7\, s_7 \\
&= Q'_{3211}
\end{aligned}$$

The plethysms of s_{21} with the cyclic characters $\ell_3^{(i)}$ are given by the reduction modulo $1-t^3$ of $H_{222111}^{(3)}$

$$\begin{aligned}
H_{222111}^{(3)} &= t^9 s_{63} + (t+1)t^7 s_{621} + t^6 s_{6111} + (t+1)t^7 s_{54} \\
&+ (t^3+2t^2+2t+1)t^5 s_{531} + (t^2+2t+1)t^5 s_{522} \\
&+ (t^3+2t^2+2t+1)t^4 s_{5211} + (t+1)t^4 s_{51111} \\
&+ (t^2+2t+1)t^5 s_{441} + (t^3+2t^2+3t+2)t^4 s_{432} \\
&+ (2t^3+3t^2+3t+1)t^3 s_{4311} + (t^3+3t^2+3t+2)t^3 s_{4221} \\
&+ (t^3+2t^2+2t+1)t^2 s_{42111} + t^3 s_{411111} + (t^3+1)t^3 s_{333} \\
&+ (2t^3+3t^2+2t+1)t^2 s_{3321} + (t^2+2t+1)t^2 s_{33111} \\
&+ (t^2+2t+1)t^2 s_{3222} + (t^3+2t^2+2t+1)t s_{32211} \\
&+ (t+1)t s_{321111} + (t+1)t s_{22221} + s_{222111}
\end{aligned}$$

$$\ell_3^{(0)} = s_3 + s_{111}$$
$$\ell_3^{(1)} = s_{21}$$
$$\ell_3^{(2)} = s_{21}$$

In general,

$$\ell_n^{(k)} = \sum_{\substack{t \in \mathrm{STab}\,(n) \\ \mathrm{maj}(t) \equiv k \mod n}} s_{\mathrm{shape}(t)}$$

$$\begin{aligned}(h_3 + e_3)[s_{21}] &= s_{222111} + 2s_{33111} + 3s_{4311} + 2s_{32211} + 2s_{42111} \\ &+ 3s_{4221} + 2s_{3222} + 2s_{3321} + s_{411111} \\ &+ 2s_{333} + s_{6111} + 2s_{531} + 2s_{5211} + 2s_{432} \\ s_{21}[s_{21}] &= s_{3222} + 3s_{3321} + 2s_{32211} + 2s_{42111} + s_{22221} + s_{33111} \\ &+ s_{321111} + 3s_{4311} + 3s_{4221} + s_{441} + s_{522} \\ &+ 2s_{5211} + s_{51111} + 2s_{531} + 3s_{432} + s_{621} + s_{54}\end{aligned}$$

Ribbons tableaux and the Fock space

- The algebra of symmetric functions can be identified with the Fock space representation of $\widehat{\mathfrak{gl}}_\infty$.

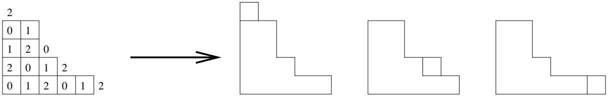
$$s_\lambda \leftrightarrow |\lambda\rangle = v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots \quad \text{where } i_k = \lambda_k - k + 1$$

- This induces actions of $\widehat{\mathfrak{gl}}_r = \widehat{\mathfrak{sl}}_r + \mathcal{H}_r$ where $\mathcal{H}_r$ is a Heisenberg algebra
- Bosonic Fock space $\mathcal{F} = \mathbb{C}[x_1, x_2, \dots] \simeq Sym\ (x_k = \frac{1}{k} p_k)$
- Action of $\widehat{\mathfrak{gl}}_r$ on $\mathcal{F}$:
  - the generator B_k of $\mathcal{H}_r$ acts by $rk\frac{\partial}{\partial p_{rk}}$ for $k > 0$ and as the multiplication by p_{-rk} for $k < 0$.
  - Action of the generators of $\widehat{\mathfrak{sl}}_r$ particularly simple in the basis of Schur functions s_λ.

For a node γ in ith row and jth column of λ let $r(\gamma) = j - i \mod r$.
Then,

$$e_i s_\lambda = \sum s_\nu , \qquad f_i s_\lambda = \sum s_\mu ,$$

where ν (resp. μ) runs through all partitions obtained from λ by removing (resp. adding) a node of residue i.
For example, f_2 of $\widehat{\mathfrak{s}l}_3$ acts on s_{5322} by



- $U(\mathcal{H}_r) = p_r \circ Sym$ is as well generated by the

$$V_k = \text{ `multiplication by } p_r \circ h_k\text{'}$$

$$V_k s_\lambda = \sum (-1)^{\mathbf{h}(\mu/\lambda)} s_\mu$$

sum over all partitions μ such that μ/λ is a horizontal r-ribbon strip of weight k, where

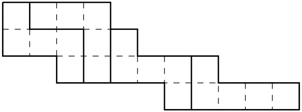
$$\mathbf{h}(\mu/\lambda) = \sum_R (h(R) - 1)$$

sum over all the r-ribbons R tiling μ/λ.

- and their adjoints U_k

$$U_k s_\mu = \sum (-1)^{\mathbf{h}(\mu/\lambda)} s_\lambda$$

sum over all partitions λ such that μ/λ is a horizontal r-ribbon strip of weight k.



A horizontal 5-ribbon strip of weight 4 and spin $\frac{7}{2}$

q-Fock space representation of $U_q(\widehat{\mathfrak{gl}}_r)$

- In the $\mathbb{Q}(q)$-vector space

$$\mathcal{F} = \bigoplus_{\lambda \in \mathbf{P}} \mathbb{Q}(q)\,|\lambda\rangle$$

  $\gamma = (a, b) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ is an indent i-node of λ if a box of residue $i = a - b \mod r$ can be added to λ at position (a, b)
- Similarly, a node of residue i which can be removed is called a removable i-node.
- $i \in \{0, 1, \ldots, r-1\}$
- λ, ν such that $\nu/\lambda = \gamma = \boxed{i}$

Defining some numbers associated with a partition

- $N_i(\lambda) = \sharp\{$ indent *i*-nodes of λ $\} - \sharp\{$ removable *i*-nodes of λ $\}$,
- $N_i^l(\lambda,\nu) = \sharp\{$ indent *i*-nodes of λ on the *left* of γ (not counting γ) $\}$ $-\sharp\{$ removable *i*-nodes of λ on the *left* of γ $\}$,
- $N_i^r(\lambda,\nu) = \sharp\{$ indent *i*-nodes of λ on the *right* of γ (not counting γ) $\} - \sharp\{$ removable *i*-nodes of λ on the *right* of γ $\}$,
- $N^0(\lambda) = \sharp\{$ 0-nodes of λ $\}$.

One can construct *q*-analogues of the previous representations

$$f_i|\lambda\rangle = \sum_\mu q^{N_i^r(\lambda,\mu)}|\mu\rangle\,, \qquad e_i|\mu\rangle = \sum_\lambda q^{N_i^l(\lambda,\mu)}|\lambda\rangle$$

$$q^{h_i}|\lambda\rangle = q^{N_i(\lambda)}|\lambda\rangle \quad \text{and} \quad q^D|\lambda\rangle = q^{-N^0(\lambda)}|\lambda\rangle$$

defines an action of $U_q(\widehat{\mathfrak{sl}}_r)$

- Can be extended to $U_q(\widehat{\mathfrak{gl}}_r)$ (q-wedges and q-bosons of [Kashiwara-Miwa-Stern 1996].)
- Key point: 'q-bosons' B_k can be replaced by q-analogues of U_k and V_k

$$V_k|\lambda\rangle = \sum(-q)^{-\mathbf{h}(\mu/\lambda)}|\mu\rangle \qquad U_k|\mu\rangle = \sum(-q)^{-\mathbf{h}(\mu/\lambda)}|\lambda\rangle$$

- The relations $[U_i, U_j] = [V_i, V_j] = 0$ prove that the H-functions are symmetric (more elementary proofs since then)

- Identify $\mathcal{F}_q \simeq \mathbb{Q}(q) \otimes Sym$ by $|\lambda\rangle = s_\lambda$
- Define a linear operator $\psi_q^r : \mathcal{F}_q \longrightarrow \mathcal{F}_q$ by

$$\psi_q^r(h_\lambda) = V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_r} |\emptyset\rangle$$

- Then,

$$\psi_q^n(h_\mu) = \sum_{T \in \mathrm{tab}_r(\cdot, \mu)} (-q)^{-2s(T)} s_{\mathrm{shape}(T)}$$

- The image $\{\psi_q^r(g_\lambda)\}$ of any basis $\{g_\lambda\}$ is a basis of the space of $U_q(\widehat{\mathfrak{sl}}_r)$-highest weight vectors in $\mathcal{F}_q$.
- Taking $g_\lambda = s_\lambda$, we have

$$\langle \psi_q^r(s_\lambda), s_\mu \rangle = (-q)^{2s_r^*(\mu)} c^\lambda_{\mu^{(0)},\ldots,\mu^{(r-1)}}(q^2)$$

$((\mu^{(0)} \ldots, \mu^{(r-1)})$ r-quotient of μ).

Canonical bases

- As an $U_q(\widehat{\mathfrak{gl}}_r)$-module, $\mathcal{F}_q$ is irreducible.
- But as $U_q(\widehat{\mathfrak{sl}}_r)$-module,

$$\mathcal{F}_q \simeq \bigoplus_{m\geq 0} L(\Lambda_0 - m\delta)^{\oplus p(m)}$$

- Each simple $U_q(\widehat{\mathfrak{sl}}_r)$-module $L(\Lambda_0 - m\delta)$ has a canonical basis but these cannot be pieced together to form a canonical basis of the whole $\mathcal{F}_q$ under $U_q(\widehat{\mathfrak{gl}}_r)$.
- Such a basis (G^-_λ) was defined in [Leclerc-T. 1996].
- All the q-plethysms $\psi^r_q(s_\nu)$ are members of this basis.
- The coefficients of the dual basis on Schur functions were conjectured to give the decomposition matrices of quantized Schur algebras at roots of unity.

- The proof of this conjecture [Varagnolo-Vasserot 1999] allows one to identify the q-LR coefficients with parabolic KL polynomials [Leclerc-T. 2000]
- Then, a result of [Kashiwara-Tanisaki 1999] shows that $c^{\lambda}_{\mu^{(0)},\dots,\mu^{(r-1)}}(q) \in \mathbb{N}[q]$
- A combinatorial proof is still wanted for general r.
- Combinatorial formula for $r = 3$ [J. Blasiak, Math. Z. **283** (2016), 601–628]
- LLT polynomials have been defined for other root systems by Lecouvey [European J. of Combin. **30** (2009) 157–191], and Grojnowski-Haiman (unpublished)
- In both versions, the coefficients are parabolic KL polynomials

Upper and lower canonical bases of $\mathcal{F}_q$

- There is a unique q-semi-linear endomorphism $x \mapsto \bar{x}$ of $\mathcal{F}_q$ such that $\overline{|\emptyset\rangle} = |\emptyset\rangle$, $\overline{f_i x} = f_i \bar{x}$ and $\overline{V_k x} = V_k \bar{x}$.
- In terms of q-wedges, reverse a prefix and normalize

$$|\lambda\rangle = u_I = u_{i_1} \wedge_q u_{i_2} \wedge_q \cdots u_{i_m} \wedge_q \cdots$$

$$\overline{u_I} = (-1)^{\binom{k}{2}} q^{\alpha_{n,k}(I)} u_{i_k} \wedge_q u_{i_{k-1}} \wedge_q \cdots \wedge_q u_{i_1} \wedge_q u_{i_{k+1}} \wedge_q u_{i_{k+2}} \wedge_q \cdots$$

- Let

$$\mathcal{L}^+ = \bigoplus_\lambda \mathbb{Z}[q]|\lambda\rangle \quad \text{and} \quad \mathcal{L}^- = \bigoplus_\lambda \mathbb{Z}[q^{-1}]|\lambda\rangle$$

- There exists bases G^+_λ and G^-_λ of $\mathcal{F}_q$ characterized by

  (i) $\overline{G^+_\lambda} = G^+_\lambda$, $\overline{G^-_\lambda} = G^-_\lambda$
  (ii) $G^+_\lambda \equiv |\lambda\rangle \mod q\mathcal{L}^+$, $G^-_\lambda \equiv |\lambda\rangle \mod q^{-1}\mathcal{L}^-$

- Let

$$G_\mu^+ = \sum_\lambda d_{\lambda\mu}(q)|\lambda\rangle$$

and

$$G_\lambda^- = \sum_\mu e_{\lambda\mu}(-q^{-1})|\mu\rangle$$

- Then,

$$e_{\lambda\mu}(q) = \sum_{x\in\widehat{\mathfrak{S}}(a)} (-q)^{\ell(x)} P_{w_v x, w_u}(q)$$

$$d_{\lambda\mu}(q) = \sum_{y\in\mathfrak{S}_m} (-q)^{\ell(y)} P_{y\hat{w}_u,\hat{w}_v}(q)$$

(parabolic KL polynomials of Deodhar).

Quantized Schur algebras at roots of 1

- $\mathcal{S}_n(\zeta)$ with ζ a primitive r-th root of 1
- $W(\lambda)$ Weyl modules. $L(\mu)$ simple modules
- **Conjecture** [LLT] let $\{W(\lambda)^i\}$ be the Jantzen filtration

$$d_{\lambda'\mu'}(q) = \sum_{i\geq 0} [W(\lambda)^i / W(\lambda)^{(i+1)} : L(\mu)] q^i$$

- Extends the LLT conjecture proved by Ariki.
- Proved by Varagnolo-Vasserot for $q = 1$.
- **Proved by P. Shan** [Represent. Theory 16 (2012), 212-269] for $\zeta = e^{2i\pi/k}$, $k \leq -3$
- One has $[d_{\lambda\mu}(q)] = [e_{\lambda'\mu'}(-q)]^{-1}$.

Back to Hall-Littlewood functions

- Why do we have $\tilde{K}_{\lambda\mu}(q) = c^{\lambda}_{\mu_1,\ldots,\mu_r}(q)$?
- One can now deduce it from an earlier result of Lusztig

$$e_{N\lambda,N\mu}(q) = \tilde{K}_{\lambda\mu}(q^2) \quad (N \geq m)$$

- Original proof [LLT97]: cell decompositions of unipotent varieties
- Open problem: similar interpretation for other LLT polynomials ?
- Cospin q-analogues $\tilde{G}_{\mu}(X; 1+q)$ of products of arbitrary vertical strips are e-positive [P. Alexandersson, arXiv:1903.03998; M. d'Adderio, JCTA 172 (2020)],
- Not true in general. Known for $\tilde{Q}'_{\mu}(X; 1+q)$, special case of a property of Hall polynomials

Unipotent varieties

- The coefficients $\tilde{g}_{\nu\mu}(q)$ of the monomial expansions

$$\tilde{Q}'_\mu(X;q) := \sum_\lambda \tilde{K}_{\lambda\mu}(q)s_\lambda = \sum_\nu \tilde{g}_{\nu\mu}(q)m_\nu$$

  are the Poincaré polynomials of certain algebraic varieties.
- Let $u \in GL(n, \mathbb{C})$ be a unipotent element of Jordan type μ, and let $\mathcal{F}_\nu$ be the variety of ν-flags in $V = \mathbb{C}^n$

$$V_{\nu_1} \subset V_{\nu_1+\nu_2} \subset \ldots \subset V_{\nu_1+\ldots\nu_r} = V$$

  where $\dim V_i = i$.
- The unipotent variety $\mathcal{F}^u_\nu$ is the set of fixed points of u in $\mathcal{F}_\nu$.

Cell decompositions

- Cell decomposition of $\mathcal{F}_\nu^u$ involving only cells of even real dimensions $\simeq \mathbb{C}^d$ [Shimomura 1980].
- Hence, the Poincaré polynomial has the form

$$\Pi_{\nu\mu}(t^2) = \sum_i t^{2i} \mathrm{dim} H_{2i}(\mathcal{F}_\nu^u, \mathbb{Z})$$

and $\Pi_{\nu\mu}(q) = |\mathcal{F}_\nu^u[\mathbb{F}_q]|$, which can be shown (by means of the Hall algebra) to be

$$|\mathcal{F}_\nu^u[\mathbb{F}_q]| = \tilde{g}_{\nu\mu}(q)$$

- Cells are parametrized by *tabloids*.

Tabloids

- For μ, ν arbitrary compositions of n, a μ-tabloid of shape ν is a filling of the diagram with row lengths $\nu_1, \nu_2, \ldots, \nu_r$ such that i occurs μ_i times, each row nondecreasing.
- For example,

3		
1	1	1
1	1	3
2	3	

is a $(5, 1, 3)$-tabloid of shape $(2, 3, 3, 1)$

Inversion statistic on tabloids

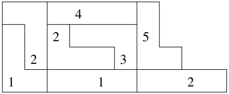
- Dimension $d(\mathbf{t})$ of the cell $c_{\mathbf{t}}$ explicitly given by Shimomura.
- A slightly modified version $e(\mathbf{t})$ (having the same distribution) can be interpreted as a kind of 'inversion number' on r-tuple of rows (e-inversions) [Terada 1993]
- Tabloid $\mathbf{t} = (w_1, \ldots, w_r) \simeq r$-tuple of row tableaux.
- y the k-th letter of w_i
- x the k-th letter of w_j
- For $x < y$ (y, x) is an e-inversion if either (a) $i < j$ or (b) $i > j$ and there is on the right of x in w_j a letter $u < y$
- $e(\mathbf{t})$ is equal to the number of inversions (y, x) in $\mathbf{t}$.

Inversions and cospin

Stanton-White correspondence maps $\mathbf{t}$ to T such that $\tilde{s}(T) = e(\mathbf{t})$ For example,

$$\mathbf{t} = \begin{pmatrix} \boxed{2}\boxed{3} & , & \boxed{1}\boxed{1}\boxed{2} & , & \boxed{4}\boxed{5} & , & \boxed{2} \\ 1\quad 1 & & 0\quad 0\quad 0 & & 3\quad 1 & & 1 \end{pmatrix}$$

has $e(T) = 7$ and is mapped to



of cospin 7.

Recent progress

- The generalized inversion number $e(\mathbf{t})$ has been extended to arbitrary r-tuples of tableaux [Schilling-Shimozono-White, Adv. Applied Math. **30** (2003) 258–272]
- Another version working with tuples of skew tableaux has been found by Haglund,Haiman, and Loehr [J. Amer. Math. Soc. **18** (2005), 735–761]
- It allowed these authors to prove the Schur positivity of Macdonald polynomials $\tilde{H}_\mu(x;q,t)$ by expressing them as $\mathbb{N}[q^{-1},t]$ linear combination of special LLT polynomials
- These special polynomials are q-analogues of products of ribbon Schur functions
- The proof uses quasi-symmetric functions
- This suggests connections with noncommutative symmetric functions and combinatorial Hopf algebras

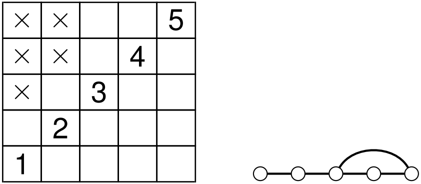
- Haglund and Wilson [arXiv:1701.05622]: Macdonald's $J_\mu(x;q,t)$ in terms of the quasi-symmetric chromatic polynomials [Shareshian-Wachs] of certain graphs
- Here, these chromatic polynomials are symmetric
- They are related to unicellular LLT-polynomials (t-analogues of s_1^n given by tuples of skew partitions with a single box) by

$$\mathrm{X}_G(t,X) = (t-1)^{-n}\,\mathrm{LLT}_G(t,(t-1)X)$$

[Carlsson and Mellit, J. Amer. Math. Soc. 31 (2018), 661–697]

Dyck graphs

- The graphs G are simple graphs with vertices labelled $1, \dots, n$, such that if there is an edge (i, j) with $i < j$, then all the (i', j') with $i \le i' < j' \le j$ are also edges of G.
- The number of such graphs is the Catalan number c_n.
- Encoding by partitions contained in a staircase



- A coloring is proper if $c_i \neq c_j$ whenever $(i, j) \in E(G)$. We denote by $C(G)$ the set of proper colorings of G.
- The chromatic quasi-symmetric function of G expands in the M basis of $QSym$

$$\mathrm{X}_G(t, X) = \sum_{c \in C(G)} t^{\mathrm{asc}_G(c)} x_{c_1} x_{c_2} \cdots x_{c_n} = \sum_{c \in \mathrm{PC}(G)} t^{\mathrm{asc}_G(c)} M_{\mathrm{Ev}(c)}(X),$$

where $\mathrm{PC}(G)$ denotes the set of proper packed colorings, $\mathrm{asc}_G(c)$ is the number of edges $(i < j)$ such that $c_i < c_j$, and $\mathrm{Ev}(c)$ is the evaluation of c.

Some combinatorial Hopf algebras

- $A = \{a_1 < a_2 < a_3 < \cdots\}$ totally ordered alphabet
- **WQSym**: "Word Quasi-Symmetric functions"

$$\mathbf{M}_u = \sum_{\mathrm{pack}(w)=u} w$$

$$\mathbf{M}_{121} = aba + aca + ada + bcb + bdb + cdc + \cdots$$

- Algebra:

$$\mathbf{M}_{u'}\mathbf{M}_{u''} = \sum_{\substack{u=vw \\ \mathrm{pack}(v)=u',\ \mathrm{pack}(w)=u''}} \mathbf{M}_u$$

- Hopf algebra $\Delta\mathbf{M}_u = \mathbf{M}_u(A \oplus B)$ (ordinal sum)
- Projection to *QSym*: $\mathbf{M}_u(X) = M_I(X)$

- The Guay-Paquet Hopf algebra $\mathcal{G}$: linear span of finite simple undirected graphs with vertices labelled by the first integers.
- Product: $G \cdot H = G \cup H[n]$ where $H[n]$ is H with labels shifted by the number n of vertices of G.
- Coproduct: G graph on n vertices, $w \in [r]^n$, coloring of G; $G|_w$ tensor product $G_1 \otimes \cdots \otimes G_r$ of the restrictions of G to vertices colored $1, 2, \ldots, r$.

$$\Delta^r G := \sum_{w \in [r]^n} t^{\mathrm{asc}_G(w)} G|_w. \tag{1}$$

- The subspace $\mathcal{D}$ of $\mathcal{G}$ spanned by Dyck graphs is a Hopf subalgebra.

- Given a Dyck graph G, define

$$\mathbf{X}_G(t,A) = \sum_{c\in \mathrm{PC}(G)} t^{\mathrm{asc}_G(c)} \mathbf{M}_c(A) \in \mathbf{WQSym}.$$

- Then, [Novelli, T., arXiv:1907.00077] $G \mapsto \mathbf{X}_G(A)$ is a morphism of Hopf algebras from $\mathcal{G}$ to **WQSym.**
- The $(1-t)$ transform and its inverse can be extended to **WQSym**
- Appliying it to $\mathbf{X}_G$, we find

$$(t-1)^n \mathbf{X}_G\left(t, \frac{A|}{|t-1}\right) = \sum_{u\in PW_n} t^{\mathrm{asc}_G(u)} \mathbf{M}_u(A).$$

The r.h.s. is therefore a noncommutative lift of the LLT polynomial LLT_G.

$$\mathbf{X}_{(\circ\quad\circ\quad\circ)} = \sum_{w\in PW(3)} \mathbf{M}_w$$

$$\begin{aligned}\mathbf{X}_{(\circ\!-\!\circ\quad\circ)} =& t\,\mathbf{M}_{121} + t\,\mathbf{M}_{122} + t\,\mathbf{M}_{123} + t\,\mathbf{M}_{132} + \mathbf{M}_{211}\\ &+ \mathbf{M}_{212} + \mathbf{M}_{213} + t\,\mathbf{M}_{231} + \mathbf{M}_{312} + \mathbf{M}_{321}\end{aligned}$$

$$\begin{aligned}\mathbf{X}_{(\circ\quad\circ\!-\!\circ)} =& t\,\mathbf{M}_{112} + \mathbf{M}_{121} + t\,\mathbf{M}_{123} + \mathbf{M}_{132} + t\,\mathbf{M}_{212}\\ &+ t\,\mathbf{M}_{213} + \mathbf{M}_{221} + \mathbf{M}_{231} + t\,\mathbf{M}_{312} + \mathbf{M}_{321}\end{aligned}$$

$$\begin{aligned}\mathbf{X}_{(\circ\!-\!\circ\!-\!\circ)} =& t\,\mathbf{M}_{121} + t^2\,\mathbf{M}_{123} + t\,\mathbf{M}_{132} + t\,\mathbf{M}_{212}\\ &+ t\,\mathbf{M}_{213} + t\,\mathbf{M}_{231} + t\,\mathbf{M}_{312} + \mathbf{M}_{321}\end{aligned}$$

$$\mathbf{X}_{\left(\overset{\frown}{\circ\!-\!\circ\!-\!\circ}\right)} = t^3\,\mathbf{M}_{123} + t^2\,\mathbf{M}_{132} + t^2\,\mathbf{M}_{213} + t\,\mathbf{M}_{231} + t\,\mathbf{M}_{312} + \mathbf{M}_{321}$$

Analogue of F-positivity

$$\check{\Phi}_u = \sum_{v \geq \bar{u}} \mathbf{M}_{\bar{v}} \ \mapsto F_I(X)$$

$$\mathbf{LLT}_G = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{asc}_G(\sigma)} \check{\Phi}_{\mathrm{min}'_{G_\emptyset}(\sigma)}$$

where $G_\emptyset$ is the graph with n vertices and no edges.

$$\mathbf{LLT}_{(\circ \quad \circ \quad \circ)} = \check{\Phi}_{123} + \check{\Phi}_{122} + \check{\Phi}_{112} + \check{\Phi}_{121} + \check{\Phi}_{212} + \check{\Phi}_{111},$$

$$\mathbf{LLT}_{(\circ\!-\!\circ \quad \circ)} = t\,\check{\Phi}_{123} + t\,\check{\Phi}_{122} + \check{\Phi}_{112} + t\,\check{\Phi}_{121} + \check{\Phi}_{212} + \check{\Phi}_{111},$$

$$\mathbf{LLT}_{(\circ \quad \circ\!-\!\circ)} = t\,\check{\Phi}_{123} + \check{\Phi}_{122} + t\,\Phi_{112} + \check{\Phi}_{121} + t\,\check{\Phi}_{212} + \check{\Phi}_{111},$$

$$\mathbf{LLT}_{(\circ\!-\!\circ\!-\!\circ)} = t^2\,\check{\Phi}_{123} + t\,\check{\Phi}_{122} + t\,\check{\Phi}_{112} + t\,\check{\Phi}_{121} + t\,\check{\Phi}_{212} + \check{\Phi}_{111}.$$

A conjecture

Let $\hat{Q}'(X;t) = (1-t)^{-\ell(\mu)} Q'(X;t)$.
Spin-unicellular LLT

$$X_G(t) = (1-t)^{-n} LLT_G((1-t)X;t)$$

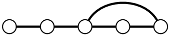
Define

$$(1-t)^{-n} LLT_G(X;t) = \sum_{\mu \vdash n} c_G^{\mu}(t) \hat{Q}'(X;t)$$

Conjecture (Novelli-T., in preparation)

The coefficient $c_G^{\mu}(t)$ is given by an explicit statistic $\mathrm{st}_G(\pi)$ on set partitions of type μ which are compatible with G, i.e. such that the extremities of an edge are not in the same block:

$$c_G^{\mu}(t) = \sum_{\pi \in \Pi_{\mu}} t^{\mathrm{st}_G(\pi)}$$

For the graph $G =$ 

$$LLT_G = \hat{Q}'_{11111} + (t^3 + 2t^2 + 2t)\hat{Q}'_{2111} + (t^3 + 2t^2 + t)\hat{Q}'_{221}$$

$\{\{1\},\{2,4\},\{3,5\}\}$	1
$\{\{1,4\},\{2\},\{3,5\}\}$	2
$\{\{1,4\},\{2,5\},\{3\}\}$	3
$\{\{1,5\},\{2,4\},\{3\}\}$	2
$\{\{1\},\{2\},\{3,5\},\{4\}\}$	1
$\{\{1\},\{2,5\},\{3\},\{4\}\}$	2
$\{\{1\},\{2,4\},\{3\},\{5\}\}$	1
$\{\{1,5\},\{2\},\{3\},\{4\}\}$	3
$\{\{1,4\},\{2\},\{3\},\{5\}\}$	2
$\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$	0

Thanks to the Haglund-Wilson formula, this would provide an explicit expression of Macdonald polynomials in terms of Hall-Littlewood functions.

Bon anniversaire Bernard !