Vertex operators, Kronecker products, and Hilbert series

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Mathematical Foundations of Quantum Information
Sevilla, 2009

Summary of yesterday's lecture

1. Hopf:
   - $\Delta f = f(X+Y)$
   - $\langle f \cdot g, h\rangle = \langle f \otimes g, \Delta h\rangle$
   - $\mathrm{ch}(\chi^\lambda) = s_\lambda$
   - product = induction, coproduct = restriction
2. Vertex:

$$\sum_{n\in\mathbb{Z}} s_{(n,\nu)} = \Gamma_1 s_\nu = \sigma_1 D_{\lambda_{-1}} s_\nu$$

Summary of today's lecture

1. Kronecker
2. Kronecker + Hopf
3. Reduced notation = Kronecker + Hopf + Vertex
4. Application to Hilbert series of some invariant algebras

The internal product of *Sym*

- Can be defined without reference to characters
- Remember Cauchy's identity

$$\sigma_1(XY) = \sum_{\lambda} s_\lambda(X) s_\lambda(Y)$$

- $\delta: \; f \mapsto f(XY)$ is a coproduct
- Obviously, $\delta p_\mu = p_\mu \otimes p_\mu$
- The dual product is $p_\mu * p_\nu = z_\mu \delta_{\mu\nu} p_\mu$
- It corresponds under ch to the pointwise product of class functions
- $$s_\mu * s_\nu = \sum_{\lambda} g^\lambda_{\mu\nu} s_\lambda = \mathrm{ch}(\chi^\mu \chi^\nu)$$

The splitting (or Mackey) formula I

- There is a compatibility between $*$, $\cdot$ and Δ
- It reflects a general formula in group theory
- Again, it has a direct and elementary proof
- This is

$$(fg) * h = \mu[(f \otimes g) * \Delta h]$$

  where $\mu(u \otimes v) = uv$ and
  $(a \otimes b) * (a' \otimes b') = (a * a') \otimes (b * b')$
- Generalization $(f_1 f_2 \cdots f_r) * h$

The splitting (or Mackey) formula II

Proof (Hopf style):

$$\langle (fg) * h, u\rangle = \langle (fg)(X)h(Y), u(XY)\rangle$$

$$= \langle f(X')g(X'')h(Y), u(X'Y + X''Y)\rangle$$

$$= \sum_{(u)} \langle f(X')g(X'')h(Y), u_{(1)}(X'Y)u_{(2)}(X''Y)\rangle$$

(the right part is a Y product that we can dualize)

$$= \sum_{(u)} \langle f(X')g(X'')h(Y' + Y''), u_{(1)}(X'Y')u_{(2)}(X''Y'')\rangle$$

$$= \sum_{(h)} \langle f(X')g(X'')h_{(1)}(Y')h_{(2)}(Y''), u(X'Y' + X''Y'')\rangle$$

The splitting (or Mackey) formula III

$$= \sum_{(h)} \langle (f * h_{(1)})(X')(g * h_{(2)})(X''), u(X' + X'') \rangle$$

(now $X'Y' \to X'$ and $X''Y'' \to X''$)

$$= \langle \mu[(f \otimes g) * \Delta h, u \rangle \, .$$

Example:

$$h_\mu * h_\nu = \sum_{M \in \mathrm{M}(\mu,\nu)} h_M$$

The reduced notation I

- Murnaghan, Littlewood:

$$\langle\mu\rangle * \langle\nu\rangle = \sum_{\lambda} \bar{g}^{\lambda}_{\mu\nu} \langle\lambda\rangle$$

means

$$s_{\mu[n]} * s_{\nu[n]} = \sum_{\lambda} \bar{g}^{\lambda}_{\mu\nu} s_{\lambda[n]}$$

- But what is $\langle\lambda\rangle$, precisely ?
- Answer: image of s_λ by the vertex operator

$$\langle\lambda\rangle = \Gamma_1 s_\lambda = \sum_{m\in\mathbb{Z}} s_{(m,\lambda)}$$

That is, a generating series ...

The reduced notation II

- This follows from

$$(\sigma_1 f) * (\sigma_1 g) = \sigma_1 \sum_{\mu,\nu} (D_{v_\mu} f)(D_{v_\nu} g)(u_\mu * u_\nu)$$

where (u, v) is any pair of adjoint bases of *Sym*

Proof:

$$(\sigma_1 f) * (\sigma_1 g) = \mu[(\sigma_1 \otimes f) * \Delta\sigma_1 \Delta g]$$

$$= \mu[(\sigma_1 \otimes f) * \left(\sum_\gamma D_{v_\gamma} g \otimes u_\gamma\right) (\sigma_1 \otimes \sigma_1)]$$

$$\mu[(\sigma_1 \otimes f) * \left(\sum_\gamma \sigma_1 D_{v_\gamma} g\right) \otimes \sigma_1 u_\gamma]$$

$$= \sum_\gamma \mu[(\sigma_1 D_{v_\gamma} g) \otimes (f * \sigma_1 u_\gamma)]$$

The reduced notation III

$$= \sum_{\gamma} (\sigma_1 D_{v_\gamma} g)\mu[(\sigma_1 \otimes u_\gamma) * \sum_{\delta} D_{v_\delta} f \otimes u_\delta]$$

$$= \sum_{\gamma,\delta} (\sigma_1 D_{v_\gamma} g)(D_{v_\delta} f)(u_\gamma * u_\delta).$$

Applying this to $u = v = s$ and $f = s_\mu(X-1)$, $g = s_\nu(X-1)$, we get Littlewood's formula, which reads now

$$\Gamma_1 s_\mu * \Gamma_1 s_\nu = \sum_{\lambda} \bar{g}^{\lambda}_{\mu\nu} \Gamma_1 s_\lambda$$

or, more explicitly

$$\Gamma_1 s_\mu * \Gamma_1 s_\nu = \Gamma_1 \sum_{\alpha\beta\gamma} \left(D_{s_\gamma} D_{s_\alpha} s_\mu\right) \left(D_{s_\gamma} D_{s_\beta} s_\nu\right) \left(s_\alpha * s_\beta\right)$$

The reduced notation IV

Example: Two-row shapes, $s_{(n-k,k)} * s_{(n-l,l)}$

$$\Gamma_1 s_k * \Gamma_1 s_l = \Gamma_1 \sum_{p=0}^{\min(k,l)} \sum_{q=0}^{p} s_{k-p} s_{l-p} s_{p-q}$$

The triple product of one-part Schur functions is easily evaluated. With $k = 2, l = 3$, we get

$$\Gamma_1(s_2 s_3 s_0 + s_1 s_2(s_1 + s_0) + s_0 s_1(s_2 + s_1 + s_0))$$

$$= \Gamma_1(s_{32}+s_{41}+s_5+s_{211}+s_{22}+2s_{31}+s_4+2s_{21}+2s_3+s_{11}+s_2+s_1)$$

so that

$$s_{82} * s_{73} = s_{532}+s_{541}+s_{55}+s_{6211}+s_{622}+s_{631}+s_{64}+2s_{721}+2s_{73}+s_{811}+s_{82}+s_{91}$$

Schur functions and $GL(n,\mathbb{C})$

- I. Schur (1901) The irreducible polynomial representations of $GL(n,\mathbb{C})$ are parametrized by partitions in at most n parts
- if $V=\mathbb{C}^n$ the representations of degree k are those occuring in $V^{\otimes k}$

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k, l(\lambda)\leq n} V_\lambda^{\oplus f_\lambda}$$

  ($f_\lambda =$ nb of standard tableaux of shape λ)
- Character formula:

$$\mathrm{tr}\,\rho_\lambda(g) = s_\lambda(g)$$

  (as a symmetric function of the eigenvalues of g)
- Examples: symmetric tensors = h_k, alternating tensors = e_k, determinant = e_n
- Proof: Schur-Weyl duality

Schur functions and $SL(n, \mathbb{C})$

- V_λ remains irreducible, but now $e_n = 1$
- So $V_{\lambda+(1^n)} \simeq V_\lambda$
- In particular, $V_{(m^n)}$ is the trivial representation
- Invariants of $SL(n)$ come from rectangular shapes

Invariants of multilinear forms I

- Irreducible representations of a product group $G = \prod_{i=1}^{k} GL(n_i)$: the characters are $\prod_{i=1}^{k} s_{\lambda^{(i)}}(X_i)$
- We are interested in the relative invariants of G in $S^d(V_1 \otimes \cdots \otimes V_k)$, where $V_i = \mathbb{C}^{n_i}$, i.e., homogeneous polynomials F in the coordinates such that

$$g \cdot F = (\det g_1)^{l_1} (\det g_2)^{l_2} \cdots (\det g_k)^{l_k} F$$

  for any $g = (g_1, \dots, g_k) \in G$
- A *covariant* of degree $d = (d_0, d_1, \dots d_k)$ is a relative invariant of G in the representation space

$$S^{d_0}(V_1 \otimes \cdots \otimes V_k) \otimes S^{d_1}(V_1^*) \otimes \cdots \otimes S^{d_k}(V_k^*)$$

Invariants of multilinear forms II

If we write $A \in V_1^* \otimes \cdots \otimes V_k^*$ as

$$A(\mathbf{x}_1, \ldots \mathbf{x}_k) = A_{i_1 i_2 \ldots i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$$

The coordinate functions x_i^j, $j = 1, \ldots, n_i$, form a basis of V_i^* and the components $A_{i_1 \ldots i_k}$ are regarded as a basis of $(V_1^* \otimes \cdots \otimes V_k^*)^* = \mathbf{V}$. An invariant F is a homogeneous polynomial in the coefficients of the "groundform" A, such that $F = 0$ defines a G-invariant hypersurface of $\mathbb{P}(\mathcal{V})$. Similarly, a covariant is a multi-homogeneous polynomial in the original vector variables $\mathbf{x}_i$, whose coefficients are homogenous polynomials in the $A_{i_1 \ldots i_k}$, of which the simultaneous vanishing defines a G-invariant subvariety of $\mathbb{P}(\mathbf{V})$.

Invariants of multilinear forms III

A covariant is a G-equivariant map from $S^{d_0}(V_1 \otimes \cdots \otimes V_k)$ to the irreducible representation $S^{d_1}(V_1) \otimes \cdots \otimes S^{d_k}(V_k)$.

In general, a concomitant of degree d_0 and of type $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$, where the $\lambda^{(i)}$ are partitions, is an equivariant map from the same space to the irreducible representation $S_{\lambda^{(1)}}(V_1) \otimes \cdots \otimes S_{\lambda^{(k)}}(V_k)$ of G.

Invariants of multilinear forms IV

Characters of the irreducible polynomial representations of the product group G

$$S_\lambda = s_{\lambda^{(1)}}(X_1) \cdots s_{\lambda^{(k)}}(X_k)$$

$\lambda^{(i)}$ are partitions, $X_i = \{x_{i1}, \ldots, x_{in_i}\}$ is a set of n_i variables. The character of the one dimensional representation

$$\det{}^l(g) = (\det g_1)^{l_1} (\det g_2)^{l_2} \cdots (\det g_k)^{l_k}$$

is the product of rectangular Schur functions

$$s_{(l_1^{n_1})}(X_1) s_{(l_2^{n_2})}(X_2) \cdots s_{(l_k^{n_k})}(X_k)$$

The character of G in $S^d(\mathbf{V})$ is $h_d(X_1 X_2 \cdots X_k)$.

Invariants of multilinear forms V

Hence, the dimension of the space of invariants of degree d and weight l, which is also the multiplicity of the one dimensional character $\det^l$ in $S^d(\mathbf{V})$, is

$$\mathrm{dimInv}(d,l) = \langle h_d(X_1X_2\cdots X_k)\,,\, s_{(l_1^{n_1})}(X_1)s_{(l_2^{n_2})}(X_2)\cdots s_{(l_k^{n_k})}(X_k)\rangle_G$$

Replace the X_i by infinite sets of independent variables, and compute in $Sym^{\otimes k}$ is dual to the internal product $*$

$$\begin{aligned}\mathrm{dimInv}(d,l) &= \langle \delta^k(h_d), s_{(l_1^{n_1})}\otimes\cdots\otimes s_{(l_k^{n_k})})\rangle_{Sym^{\otimes k}} \\ &= \langle h_d, s_{(l_1^{n_1})} * \cdots * s_{(l_k^{n_k})})\rangle_{Sym}\,.\end{aligned}$$

Invariants of multilinear forms VI

The internal product of two homogenous symmetric functions being zero if these are not of the same degree, we see that $\mathrm{Inv}(d, l)$ can be nonzero only if the conditions

$$n_1 l_1 = n_2 l_2 = \cdots = n_k l_k = d$$

are satisfied. In particular, if all the n_i are equal, the l_i must also be all equal.

Invariants of multilinear forms VII

Let $c(d;l)$ be the dimension of the space of covariants of degree $d=(d_0,d_1,\dots,d_k)$ and weight $l=(l_1,\dots,l_k)$.

$$\begin{aligned} c(d;l) &= \langle h_{d_0}(X_1X_2\cdots X_k)\,,\,(s_{(l_1^{n_1})}h_{d_1})(X_1)\cdots(s_{(l_k^{n_k})}h_{d_k})(X_1)_G \\ &= \langle h_{d_0}\,,\,(s_{(l_1^{n_1})}h_{d_1})*\cdots*(s_{(l_k^{n_k})}h_{d_k})\rangle_{Sym}\,. \end{aligned}$$

For $SL(2)$, $s_{(l,l)}h_d = s_{(l+d,l)}$, so that the covariants are in bijection with highest weight vectors.

Multilinear binary forms (qubit systems) I

If all $V_i = \mathbb{C}^2$ we need only two-part partitions
For the size $(2,2,2)$, we have

$$\mathrm{dimInv}(2l;l,l,l) = \langle h_{2l}, s_{ll}^{*3}\rangle = \langle s_{ll} * s_{ll}, s_{ll}\rangle = \begin{cases} 0 & l \text{ odd} \\ 1 & l \text{ even} \end{cases}$$

using first the property $\langle f * g, h\rangle = \langle f, g * h\rangle$ and the formula for $s_{ll} * s_{ll}$. Hence,

$$\sum_{d\geq 0} \mathrm{dim} S^d(\mathbf{V})^G t^d = \frac{1}{1-t^4}\,.$$

The algebra of invariants is in this case $\mathbb{C}[\Delta]$, where $\Delta = \mathrm{Det}(A)$ is the hyperdeterminant.

Multilinear binary forms (qubit systems) II

The generating series for the covariants can be written in the form

$$C(t;\mathbf{u};\mathbf{v}) = \sum_{d,l} c(d,l) t^{d_0} u_1^{d_1} u_2^{d_2} u_3^{d_3} v_1^{l_1} v_2^{l_2} v_3^{l_3}$$

$$= \langle \sigma_1[tu_1 s_1 + t^2 v_1 s_{11}]\, ,\, \sigma_1[u_2 s_1 + v_2 s_{11}] * \sigma_1[u_3 s_1 + v_3 s_{11}] \rangle$$

since with two variables,

$$\sigma_1[v s_{11}] = \sum_{l\geq 0} v^l s_{ll}$$

and

$$\sigma_1[us_1 + vs_{11}] = \sum_{\ell(\lambda)\leq 2} u^{\lambda_1-\lambda_2} v^{\lambda_2} s_\lambda(X)\,.$$

Multilinear binary forms (qubit systems) III

The last sum can be obtained by combining vertex operators and MacMahon's linear operator $\Omega^{u}_{\geq}$, which maps any monomial containing a negative power of u to 0.

$$\sum_{\lambda_1\in\mathbb{Z}\,,\,\lambda_2\geq 0} t^{|\lambda|}u^{\lambda_1-\lambda_2}v^{\lambda_2}s_\lambda(X) = \Gamma_{tu}\Gamma_{tv/u}(1)$$

$$= \left(1-\frac{v}{u^2}\right)\sigma_t\left[\left(u+\frac{v}{u}\right)X\right].$$

Hence, if $\Omega^{\mathbf{u}}_{\geq}$ denotes the MacMahon operator annihilating any monomial containing a negative power of any of the u_i,

$$C(t;\mathbf{u};\mathbf{v}) = \Omega^{\mathbf{u}}_{\geq}\prod_{i=1}^{3}\left(1-\frac{v_i}{u_i^2}\right)\sigma_t\left[\prod_{i=1}^{3}\left(u_i+\frac{v_i}{u_i}\right)\right].$$

Multilinear binary forms (qubit systems) IV

Here, Ω is easily computed with the help of a computer algebra system by decomposing the right-hand side into partial fractions, and throwing away the terms leading to negative powers of the u_i in the Laurent expansion.
Setting the v_i equal to 1, one finds

$$\frac{1-t^6u_1^2u_2^2u_3^2}{(1-tu_1u_2u_3)(1-t^2u_1^2)(1-t^2u_2^2)(1-t^2u_3^2)(1-t^3u_1u_2u_3)(1-t^4)}$$

The structure of the generating series is the same for k qubits:

$$C(t;\mathbf{u};\mathbf{v})=\Omega^{\mathbf{u}}_{\geq}\prod_{i=1}^{k}\left(1-\frac{v_i}{u_i^2}\right)\sigma_t\left[\prod_{i=1}^{k}\left(u_i+\frac{v_i}{u_i}\right)\right].$$

Multilinear binary forms (qubit systems) V

For $k = 4$, the result is huge, and can be obtained only with more subtle algorithms (e.g., Xin's), but setting $u_i = 0$ after each $\Omega_{\geq}^{u_i}$ gives easily the Hilbert series of invariants.
For $k = 5$, this still works for the invariants.
Similar (but harder) calculations would give the Hilbert series of unitary or special unitary invariants.