CHAPTER 2

The Hopf algebra of symmetric functions

1. Symmetric polynomials

The theory of symmetric polynomials is as old as algebra itself, and had been actually its main topic for more than two centuries.

The story begins with the relations between the coefficients of a polynomial and its roots

$$P(x) = \prod_{i=1}^{n}(x - x_i) = \sum_{k=0}^{n}(-1)^k e_k(X)x^{n-k}, \quad (X = (x_1, \ldots, x_n)) \tag{287}$$

attributed to François Viète, which were known in the sixteenth century, and certainly to the ancient civilisations in the case of quadratic polynomials.

As is well known, the

$$e_k(x_1, \ldots, x_n) = \sum_{i_1<i_2<\ldots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k} \tag{288}$$

are the *elementary symmetric polynomials*, and the fundamental result of the theory states that every symmetric polynomial is the x_i is expressible in a unique way as a polynomial in the e_k.

The first developments, which aimed at the solution of algebraic equations, consisted essentially in expressing various families of symmetric polynomials in terms of each other. For example, the *power-sums*

$$p_m(X) = \sum_{k=1}^{n} x_k^m \tag{289}$$

or the *complete homogeneous symmetric polynomials*

$$h_m(X) = \sum_{k_1+\ldots+k_n=m} x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n} \tag{290}$$

(sum of all monomials of degree m), and the obvious linear basis

$$m_\lambda(X) = \Sigma x^\lambda := \sum_{\mu\in\mathfrak{S}_n(\lambda)} x_1^{\mu_1}x_2^{\mu_2}\cdots x_n^{\mu_n} \tag{291}$$

sum of all *distinct* permutations of the monomial

$$x^\lambda = x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n} \tag{292}$$

where we assume $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, and $\mathfrak{S}_n(\lambda)$ stands for the set of distinct permutations of λ.

We denote by $Sym(X)$ the algebra of symmetric polynomials (as usual, over some field $\mathbb{K}$ of characteristic 0). It is the algebra of invariants of the natural action of $\mathfrak{S}_n$ on $\mathbb{K}[X]$

$$(\sigma f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\,. \tag{293}$$

It is naturally graded

$$Sym(X) = \bigoplus_{k \geq 0} Sym_k(X) \tag{294}$$

and its Hilbert series is

$$H(t) = \sum_{k\geq 0} \dim\left(Sym_k(X)\right) t^k = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)} \tag{295}$$

A nonincreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq \ldots)$ of nonnegative integers, with finitely many nonzero terms, is called a *partition* of the integer $m = \sum_i \lambda_i$, also called the *weight* $|\lambda|$ of λ. We write then $\lambda \vdash m$. The number r of nonzero terms is called its *length* and is denoted by $\ell(\lambda)$. Thus, the dimension of $Sym_m(x_1, \ldots, x_n)$ is equal to the number of partitions of m of length at most n.

If we let n tend to infinity, we can get rid of the condition "length at most n". In this way, we arrive at the notion of symmetric "functions", which are defined as the symmetric "polynomials" in infinitely many variables (meaning, as usual, formal series of bounded degree). In this way, we can deal simultaneously with symmetric polynomials in any number of variables, and as we shall see, the resulting algebra acquires some extra structure.

2. Symmetric functions

Let now $X = \{x_1, x_2, \ldots\}$ be an infinite set of variables, which will be referred to as an *alphabet*. The elementary and complete symmetric functions of X e_n and h_n are best defined by their generating series

$$\lambda_t(X) := \prod_{i\geq 1}(1 + tx_i) = \sum_{n\geq 0} e_n(X) t^n, \tag{296}$$

$$\sigma_t(X) := \prod_{i\geq 1}(1 - tx_i)^{-1} = \sum_{n\geq 0} h_n(X) t^n = (\lambda_{-t}(X))^{-1}\,. \tag{297}$$

Alternatively, one may write $E(t;X)$ instead of $\lambda_t(X)$ and $H(t;X)$ instead of $\sigma_t(X)$.

Exercise 2.1. The h_n and the e_n determine each other via the relation

$$h_n - e_1 h_{n-1} + e_2 h_{n-2} - \cdots + (-1)^n e_n = 0 \text{ for } n \geq 1, \text{ and } h_0 = e_0 = 1. \tag{298}$$

We obtain the power-sums by taking a logarithm

$$\log \sigma_t(X) = \sum_{i\geq 1} \log\left(\frac{1}{1 - tx_i}\right) = \sum_{i\geq 1}\sum_{n\geq 1} \frac{x_i^n}{n} t^n = \sum_{n\geq 1} p_n(X) \frac{t^n}{n} \tag{299}$$

so that

$$\sigma_t(X) = \exp\left[\sum_{n\geq 1} p_n(X)\frac{t^n}{n}\right]. \tag{300}$$

Picking the coefficient of t^n, we find

$$h_n(X) = \sum_{\mu\vdash n} \frac{p_\mu}{z_\mu} \tag{301}$$

where $z_\mu = \prod_i i^{m_i} m_i!$ if m_i is the number of occurences of the part i in μ (this is often written $\mu = (1^{m_1}2^{m_2}\cdots n^{m_n})$).

Alternatively, taking the logarithmic derivatives of (296) and (297), we obtain the recurrrence relations

$$ne_n = p_1e_{n-1} - p_2e_{n-2} + p_3e_{n-3} - \cdots + (-1)^{n-1}p_n, \tag{302}$$

$$nh_n = p_1h_{n-1} + p_2h_{n-2} + p_3h_{n-3} + \cdots + p_n, \tag{303}$$

respectively attributed to Newton and Wronski.

Exercise 2.2. These relations provide a short way to prove that $Sym(X)$ is a polynomial algebra in either family e_k, h_k or p_k. Assume first that $X = \{x_1, \ldots, x_n\}$. Then, the Jacobian of $p_1, \ldots, p_n$ is a Vandermonde determinant, so that they are algebraically independent. Next, if λ is a partition of length r,

$$p_\lambda := p_{\lambda_1}\cdots p_{\lambda_r} = c_\lambda m_\lambda + \cdots \tag{304}$$

where c_λ is a nonzero integer and the dots stand for a linear combination of m_μ such that $\ell(\mu) < r$. This proves that the p_k generate $Sym(X)$ over $\mathbb{Q}$ (we need to invert the c_λ, which implies that the same is true of the e_k and of the h_k. We shall see that these last two families actually generate $Sym(X)$ over $\mathbb{Z}$.

Naturally, the *monomial symmetric function* $m_\lambda(X)$ is defined as above as the (now infinite) sum of all distinct permutations of the monomial x^λ.

At this point, we have at our disposal four bases of $Sym(X)$: three *multiplicative bases* e_λ, h_λ and p_λ, and the monomial basis m_λ. To understand the relations between them, let us take a second set of variables $Y = \{y_1, y_2, \ldots\}$, and form the *product alphabet*

$$XY := \{x_iy_j | i, j \geq 1\}. \tag{305}$$

We can evaluate our symmetric functions on it, and consider the following generating series

$$K(X,Y) = \sigma_1(XY) = \prod_{i,j\geq 1}(1 - x_iy_j)^{-1} \tag{306}$$

called the *Cauchy kernel.*

Writing it in the form

$$K(X,Y) = \prod_{i\geq 1}\sigma_{x_i}(Y) = \prod_{i\geq 1}\sum_{n_i\geq 0} x_i^{n_i}h_{n_i}(Y) \tag{307}$$

we obtain the expansion

$$K(X,Y) = \sum_\lambda m_\lambda(X) h_\lambda(Y) \tag{308}$$

Observing that

$$p_n(XY) = p_n(X)p_n(Y), \tag{309}$$

we have also

$$K(X,Y) = \sum_\lambda p_\lambda(X)\frac{p_\lambda(Y)}{z_\lambda}\,. \tag{310}$$

Exercise 2.3. Check this.

The multiplicative property (309) of the p_n can be nicely completed by a similar additive property if we define $X+Y$ as the (disjoint) union of X and Y. Then,

$$p_n(X+Y) = \sum_{z\in X\sqcup Y} z^n = p_n(X) + p_n(Y) \tag{311}$$

so that the power-sums appear as some kind of homomophisms for something which would look like an algebra of alphabets. We would then have as well

$$\lambda_t(X+Y) = \lambda_t(X)\lambda_t(Y) \quad \text{so that } e_n(X+Y) = \sum_{i=0}^n e_i(X)e_{n-i}(Y), \tag{312}$$

$$\sigma_t(X+Y) = \sigma_t(X)\sigma_t(Y) \quad \text{so that } h_n(X+Y) = \sum_{i=0}^n h_i(X)h_{n-i}(Y). \tag{313}$$

This can be made precise. Symmetric functions actually define functions on *multisets* of variables, or even of scalars if this does not lead to divergent series. Because of the symmetry, we can make sense of the substitution of, say

$$A = \{q, q, (-2), (-2), (-2), x^3y, x^3y, i\sqrt{2}\} \tag{314}$$

in a symmetric function: just specialize any two variables to q, three other ones to (-2), again two other ones to x^3y, a last one to $i\sqrt{2}$, and set all the remaining ones to zero. For example, with the above multiset,

$$p_n(A) = 2q^n + 3(-2)^n + 2(x^3y)^n + (i\sqrt{2})^n \tag{315}$$

and this defines all symmetric functions of A.

A multiset can be conveniently replaced by the *formal* sum of its elements, *e.g.*,

$$A = 2q + 3(-2) + 2x^3y + (i\sqrt{2}) \tag{316}$$

Note that the term $3(-2)$ should not be interpreted as -6.

Exercise 2.4. Check that $p_2(-2,-2,-2)$ is indeed different from $p_2(-6)$.

Exercise 2.5. Compute $h_2(A)$, $e_2(A)$, $e_8(A)$.

To avoid confusion, it is best to interpret symmetric functions as *operators* on polynomial rings, and to allow specialization of the variables only after application of the operators. The power sums are then ring homomorphisms defined by

$$p_n(z) = z^n \text{ if } z \text{ is a variable, and } p_n(1) = 1 \tag{317}$$

so that $p_n(\alpha) = p_n(\alpha \cdot 1) = \alpha$ for any scalar α.

Exercise 2.6. Compute $h_n(\alpha)$ and $e_n(\alpha)$.
Hint: $\sigma_t(\alpha) = (1-t)^{-\alpha}$.

A ring R endowed with such an action of Sym is called a ψ-ring, as the operators p_n are ofted denoted by ψ^n (and sometimes called Adams operations). This defines as well the action of the e_k, called exteriors powers and often denoted by Λ^k or λ^k, and of the h_k, called symmetric powers and often denoted by S^k or σ^k. A ring endowed with an action of the e_k is called a λ-ring[1].

The origin of these ideas can be traced back to the following facts from linear algebra. If $P(x)$ is the characteristic polynomial of a square matrix M, and the x_i are its eigenvalues, then $e_k(X) = \operatorname{tr} \Lambda^k(M)$, where $\Lambda^k(M)$ is the kth exterior power of M, i.e., the matrix whose entries are the minors of order k of M.

It is often more convenient to assume that the x_i are the reciprocals of the eigenvalues, so that

$$|I - tM| = \prod_{i=1}^{n}(1 - x_i t) = \sum_{k=0}^{n} e_k(X)(-t)^k \tag{318}$$

is invertible as a formal power series, and its inverse

$$|I - tM|^{-1} = \prod_{i=1}^{n}(1 - x_i t)^{-1} = \sum_{k\ge 0} h_k(X) t^k \tag{319}$$

has as coefficients the complete homogeneous symmetric functions $h_k(X)$, which can be interpreted as the traces of the symmetric powers $S^k(M)$. This last statement is essentially McMahon's "Master Theorem".

The power sums $p_k(X) = \sum x_i^k$ are obviously the traces of the powers M^k, and at a more advanced level, one knows that the traces of the images of M under the irreducible polynomial representation of GL_n, labelled by partitions λ, are the so-called Schur functions $s_\lambda(X)$, to be defined later.

Exercise 2.7. There are three basic kinds of polynomials in n variables. The algebra $\mathbb{K}[X]$ of ordinary polynomials is obtained by taking mutually commuting variables x_i. The algebra of non-commutative polynomials (or free associative algebra) $\mathbb{K}\langle A\rangle$ is built from noncommmuting letters a_i. The third kind, anticommutative polynomials, or the Grassmann algebra $\Lambda_{\mathbb{K}}(\eta)$, is built from anti-commuting variables η_i. That is, $\eta_i\eta_j = -\eta_j\eta_i$ (and in particular, $\eta_i^2 = 0$). If V is the n-dimensional vector space spanned by the variables, these algebras are respectively called the *symmetric algebra* $S(V)$, the *tensor algebra* $T(V)$ and the *exterior algebra* $\Lambda(V)$. All these algebras are graded. If f is an endomorphism of V, it induces endomorphisms $S^k(f)$, $T^k(f)$ and $\Lambda^k(f)$ of the homogeneous components of these three algebras. Check that $\Lambda^n(V)$ is one dimensional, so that $\Lambda^n(f)$ acts by a scalar, and prove that this scalar is the determinant of f. Then, prove that the matrix of $\Lambda^k(f)$

[1] Over a field of characteristic 0, there is no difference between ψ-rings and λ-rings.

is formed of the $k \times k$-minors of the matrix of f, and deduce that its trace is the k-th elementary symmetric function of the eigenvalues of f. [*Hint.* – Do it first for a diagonal matrix, and invoke a density argument for the general case.] Compute similarly the traces of $S^k(f)$ and $T^k(f)$.

Exercise 2.8. The following facts are obviously true for diagonal matrices D: (i) if $P(x) = |D - xI|$ is the characteristic polynomial of D, the $P(D) = 0$. (ii) $\det e^D = e^{\operatorname{tr} D}$. By a density argument, show that they are true for an arbitrary complex matrix.

3. Hopf algebras enter the scene

As we have seen, with a second alphabet Y, we can form symmetric functions of XY and $X + Y$, by the very simple rules (309) and (311). In each case, the result can be expanded as a sum of terms $u(X)v(Y)$. Such a term can be naturally identified with a tensor product $u \otimes v$, as the tensor product is nothing but a formal product having no other property than being linear in its arguments u and v. With this interpretation, the multiplication of symmetric functions is the linear map $u(X)v(Y) \mapsto u(X)v(X)$. These considerations allow us to reformulate the definition of an algebra, and to introduce the dual notion:

Definition 3.1. (i) *An algebra is a vector space V endowed with a product (or multiplication), that is, a linear map $\mu :\ V \otimes V \to V$.*
(ii) *A coalgebra is a vector space V endowed with a coproduct (or comultiplication), that is, a linear map $\Delta :\ V \to V \otimes V$.*

Clearly, these notions are dual to each other: the dual of a product is a coproduct and conversely.

Thus, our operations on alphabets provide us with two (very different) coproducts on *Sym*(X). Let us see what they are good for. We shall start with $X + Y$, which will be denoted by Δ. We have thus

(320) $$\Delta p_n = p_n \otimes 1 + 1 \otimes p_n,$$

(321) $$\Delta e_n = \sum_{i=0}^{n} e_i \otimes e_j,$$

(322) $$\Delta h_n = \sum_{i=0}^{n} h_i \otimes h_j\,.$$

On the last two equations, we can observe that the involutive algebra automorphism ω exchanging h_n and e_n is also an automophism of the coalgebra structure. The first equation says that the p_n are *primitive elements* (elements such that $\Delta f = f \otimes 1 + 1 \otimes f$).

Exercise 3.1. Compute $\omega(p_n)$.

It is also obvious that $(fg)(X+Y) = f(X+Y)g(X+Y)$, so that Δ is an algebra morphism from *Sym* to *Sym* $\otimes$ *Sym*.

Definition 3.2. *A bialgebra is an algebra V which is also a coalgebra, and such that the coproduct Δ is a morphism of algebras for the structure $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$ on $V \otimes V$ (called tensor product of algebras).*

Clearly, Δ endows *Sym* with the structure of a bialgebra. This is not all. Obviously, $(X+Y)+Z = X+(Y+Z)$ (recall that this is just the disjoint union of sets), so that

$$(\Delta \otimes I) \circ \Delta(f) = f((X+Y)+Z) = f(X+(Y+Z)) = (I \otimes \Delta) \circ \Delta(f)\,. \tag{323}$$

This property is called *coassociativity*. It is indeed dual to associativity, which can be expressed as

$$\mu \circ (\mu \otimes I) = \mu \circ (I \otimes \mu)\,. \tag{324}$$

Thus, *Sym* is an associative and coassociative bialgebra. It has of course a unit, the constant 1, which one may interpret as the linar map $u : \mathbb{K} \to Sym$ sending the scalar 1 to the unit element 1_{Sym} of *Sym*. Dually, one defines a *counit* on a coalgebra V as a linar map $\epsilon :\ V \to \mathbb{K}$ such that $\mu \circ (I \otimes \epsilon) \circ \Delta = I = I = \mu \circ (\epsilon \otimes I) \circ \Delta$.

Exercise 3.2. Check that this is indeed dual to the characteristic property of the unit u.

The counit of *Sym* is just the constant term map (projection onto the homogeneous component of degree 0).

There is more. If U is a coalgebra and V an algebra, one can define an operation $\star$ called *convolution* on the space $\mathcal{L}(U,V)$ of linear maps $U \to V$ by

$$(f \star g)(u) = \mu \circ (f \otimes g) \circ \Delta(u) \tag{325}$$

(this may not be well-defined in the infinite-dimensional case, but we shall never encounter this situation).

Exercise 3.3. If U is coassociative and V is associative, then $\star$ is associative.

Exercise 3.4. The polynomial algebra $\mathbb{K}[x]$ can be endowed with a bialgebra structure defined by the coproduct $\Delta(x) = x \otimes 1 + 1 \otimes x$ (so that one may write $\Delta P = P(x+y)$). Let $A = \mathbb{K}[x]^*$ be the dual space, endowed with pointwise multiplication of linear forms. Let $f, g \in A$ be defined by their values $f(x^n) = f_n$, $g(x^n) = g_n$, and let $h = f \star g$ be their convolution. Check that

$$\sum_{n\geq 0} h_n \frac{t^n}{n!} = \left(\sum_{n\geq 0} f_n \frac{t^n}{n!}\right)\left(\sum_{n\geq 0} g_n \frac{t^n}{n!}\right)\,. \tag{326}$$

Exercise 3.5. The free associative algebra $\mathbb{K}\langle A\rangle$ is a (coassociative) bialgebra for the coproduct defined on the letters by $\Delta(a_i) = a_i \otimes 1 + 1 \otimes a_i$. Each permutation $\sigma \in \mathfrak{S}_n$ defines a linear endomorphism of $\mathbb{K}\langle A\rangle$ acting on words by $g_\sigma(w) = 0$ is w is not of length n, and $g_\sigma(w) = w\sigma$ otherwise. Compute the convolution $g_\sigma \star g_\tau$, and make an interesting observation.

In particular, for endomorphisms of a bialgebra with unit and counit, $u \circ \epsilon$ is the neutral element of convolution. When the identity map is invertible for this operation, its inverse is called an *antipode*. That is, an antipode is an endomorphism S such that

$$(S \star I)(x) := \mu \circ (S \otimes I) \circ \Delta(x) = u \circ \epsilon(x) = (I \star S)(x) \tag{327}$$

For symmetric functions, it is clear that if we define $X - Y$ by

$$p_n(X-Y) = p_n(X) - p_n(Y) \tag{328}$$

then, we have the antipode

$$(Sf)(X) = f(-X) \tag{329}$$

Indeed, $(S \star I)(f) = f(X - X) = f(0) =$ constant term of $f = u \circ \epsilon(f)$.

Exercise 3.6. Compute $h_n(-X)$ and $e_n(-X)$. Interpreting $f(X-Y)$ as $(I \otimes S) \circ \Delta(f)$, compute $\sigma_t(X-Y)$.

DEFINITION 3.3. *A Hopf algebra is an associative, coassociative bialgebra with unit, counit and antipode.*

COROLLARY 3.4. *Sym is a Hopf algebra.*

As an algebra, *Sym* is commutative. It is also *cocommutative* which means that Δ commutes with the swap operator $P(u \otimes v) = v \otimes u$, or, less formally, that Δ is dual to a commutative product.

Exercise 3.7. The Hopf structure of *Sym* is a typical example of a Hopf algebra associated with a group. If we denote by G the multiplicative group of formal power series with constant term 1,

$$G = 1 + t\mathbb{K}[[t]] = \{a(t) = 1 + a_1 t + a_2 t^2 + \cdots\}, \tag{330}$$

we can interpret h_n as the coordinate function

$$h_n(a(t)) = a_n\,, \tag{331}$$

and *Sym* as the algebra of polynomial functions on G. The standard coproduct for functions on a group turns a function $f(x)$ into a function of two variables $\Delta f(x,y) = f(xy)$, and the antipode is $Sf(x) = f(x^{-1})$. Check that this defines the same Hopf algebra structure as described above.

4. Duality

The definition of a Hopf algebra H is self-dual in the sense that each defining property comes with its dual notion. If the dual space H' is well-defined, which will always be the case in this book, it is then automatically a Hopf algebra. This is the case when H is finite-dimensional. When it is graded, that is

$$H = \bigoplus_{n\geq 0} H_n \text{ with } \mu:\ H_m \otimes H_n \to H_{m+n} \text{ and } \Delta:\ H_n \to \bigoplus_{i+j=n} H_i \otimes H_j \tag{332}$$

with each H_n finite-dimensional, one can define the *graded dual*

$$H^* = \bigoplus_{n\geq 0} H'_n. \tag{333}$$

This is a Hopf algebra for the dual maps $\Delta^*, \mu^*, \epsilon^*, u^*$.

When H^* is isomorphic to H, one says that H is self-dual. In this case, there exists a scalar product on H such that

$$\langle fg, h\rangle = \langle f \otimes g, \Delta h\rangle, \quad \text{where } \langle u \otimes v, u' \otimes v'\rangle := \langle u, v\rangle\langle u', v'\rangle. \tag{334}$$

A closer look at the expansion (308) of the Cauchy kernel reveals that there is such a thing in *Sym*. Indeed, if, for two partitions λ, μ we denote by $\lambda \cup \mu$ the partition composed of the parts of λ and μ, on the one hand we have obviously

$$h_\lambda h_\mu = h_{\lambda\cup\mu} \tag{335}$$

and on the other hand

$$
\begin{aligned}
\sigma_1((X+Y)Z) &= \sum_\nu m_\nu(X+Y)h_\nu(Z) = \sigma_1(XZ)\sigma_1(YZ) \\
&= \sum_\nu \left(\sum_{\lambda\cup\mu=\nu} m_\lambda(X)m_\mu(Y) \right) h_\nu(Z)
\end{aligned} \tag{336}
$$

so that

$$\Delta m_\nu = \sum_{\lambda\cup\mu=\nu} m_\lambda \otimes m_\mu \,. \tag{337}$$

Hence, if we define the scalar product on *Sym* by

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} \text{ (Kronecker symbol)} \tag{338}$$

we have

$$\langle m_\nu, h_\lambda h_\mu \rangle = \langle \Delta m_\nu, h_\lambda \otimes h_\mu \rangle \tag{339}$$

which proves that *Sym* is self-dual.

The scalar product (338) is called Hall's scalar product. With it, we can now understand the significance of the Cauchy kernel. It is indeed a reproducing kernel, which means that for any $f \in Sym(X)$,

$$\langle K(X,Y), f(X) \rangle = f(Y) \,. \tag{340}$$

Or, interpreting $h_\mu(Y)$ as the dual basis of $m_\mu(X)$, this is the identity map of *Sym*, regarded as an element of $Sym \otimes Sym^*$.

This has the consequence that any pair of bases u_λ, v_λ satisfying

$$K(X,Y) = \sum_\lambda u_\lambda(X)v_\lambda(Y) \tag{341}$$

are dual to each other: $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$. In particular, the power-sum products form an orthogonal basis:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \,. \tag{342}$$

5. Schur functions

5.1. Antisymmetric polynomials. A polynomial $f(x_1,\ldots,x_n)$ is said to be *antisymmetric* if it changes sign when two variables are swapped. Equivalently,

$$f(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = (-1)^{\mathrm{inv}(\sigma)} f(x_1,\ldots,x_n) \,. \tag{343}$$

Hence, such a polynomial vanishes if one sets $x_i = x_j$ for some $j > i$. As the polynomials $x_i - x_j$ are pairwise relatively prime, f must be divisible by their product

$$\Delta(x_1,\ldots,x_n) = \prod_{j>i}(x_j - x_i) = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & & \ddots & \vdots & \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{vmatrix} \tag{344}$$

a Vandermonde determinant. The quotient is then a symmetric polynomial. This how Schur functions arise. Let $\mathcal{A}_n$ be the *antisymmetrization operator*

$$\mathcal{A}_n f(x_1, \dots, x_n) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \tag{345}$$

where $\varepsilon(\sigma) = (-1)^{\mathrm{inv}(\sigma)}$ is the *signature* of σ.

Then, for a monomial x^α, $\mathcal{A}_n(x^\alpha) = 0$ unless the exponents α_i are all distinct. In which case, since the result is antisymmetric, we can assume that $\alpha_1 > \alpha_2 \dots > \alpha_n \geq 0$, and write

$$\alpha = \lambda + \rho \tag{346}$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$ is a partition, and

$$\rho = (n-1, n-2, \dots, 1, 0) \tag{347}$$

Then, the *monomial antisymmetric functions*

$$A_\alpha(x_1, \dots, x_n) = \mathcal{A}_n(x^\alpha) \tag{348}$$

form a basis of the space of antisymmetric polynomials, and the ratios

$$s_\lambda = \frac{A_\alpha}{A_\rho} \tag{349}$$

are symmetric polynomials, called Schur functions. Up to a sign, A_ρ is the Vandermonde determinant, hence of degree $\frac{1}{2}n(n-1) = |\rho|$, so that s_λ is homogenous of degree $|\lambda|$. From the above discussion, it is clear that the s_λ form a basis of $Sym(x_1, \dots, x_n)$.

Note that $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$.

5.2. The Jacobi symmetrizer. The definition (349) can be rewritten as

$$s_\lambda = \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(\frac{x^{\lambda+\rho}}{A_\rho} \right) \tag{350}$$

since A_ρ is antisymmetric and takes care of the signature of σ. The linear operator on $\mathbb{K}[x_1, \dots, x_n]$

$$\Omega_n f = \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(\frac{f \cdot x^\rho}{A_\rho} \right) \tag{351}$$

is called the *Jacobi symmetrizer*. Its fundamental property is

Proposition 5.1. *If f is symmetric, then, for any g,*

$$\Omega_n(fg) = f\Omega_n(g)\,. \tag{352}$$

Proof – By linearity, it is sufficient to check this property with $f = p_k(X)$ and $g = x^\alpha$. In this case,

$$\begin{aligned}\Omega_n(p_k(X)x^\alpha) &= \frac{1}{A_\rho}\sum_{j=1}^{n}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)x_{\sigma(1)}^{\alpha_1+n-1}\cdots x_{\sigma(j)}^{\alpha_j+n-j+k}\cdots x_{\sigma(n)}^{\alpha_n}\\ &= \frac{1}{A_\rho}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\left(\sum_{j=1}^{n}x_{\sigma(j)}^{k}\right)x_{\sigma(1)}^{\alpha_1+n-1}\cdots x_{\sigma(j)}^{\alpha_j+n-j}\cdots x_{\sigma(n)}^{\alpha_n}\\ &= p_k(X)\Omega_n(x^\alpha)\,.\end{aligned} \tag{353}$$

5.2.1. *Muir's identity.* Remark that (350) can be used to define $s_\lambda(X)$ for an arbitrary $\lambda\in\mathbb{Z}^n$. The result is then either 0 (if $\lambda+\rho$ has repeated parts), or plus or minus a Schur function:

$$s_\lambda = \varepsilon(\sigma)s_\mu \quad \text{if } (\lambda+\rho)\cdot\sigma-\rho=\mu,\text{ a partition} \tag{354}$$

When μ defined as above has negative parts, the result is still a Schur function up to a negative power of $x_1x_2\cdots x_n$. When defining Schur functions of an infinite alphabet, this has to be changed, and as we shall see, the result is also zero in this case.

With these nonstandard Schur functions, we can now state Muir's rule for the product of a monomial function and a Schur function:

PROPOSITION 5.2. *Let λ, μ be two partitions in at most n parts, and $\alpha=\lambda+\rho$, $\beta=\mu+\rho$. Then,*

$$m_\mu s_\lambda = \sum_{\gamma\in\mathfrak{S}_n(\beta)} s_{\alpha+\gamma} \tag{355}$$

Proof – It suffices to write

$$\begin{aligned}m_\mu s_\lambda &= m_\lambda(X)\Omega_n(x^\alpha) = \Omega_n(m_\lambda(X)x^\alpha\\ &= \Omega_n\left(\sum_{\gamma\in\mathfrak{S}_n(\beta)}x^{\gamma+\alpha}\right) = \sum_{\gamma\in\mathfrak{S}_n(\beta)}s_{\alpha+\gamma}\,.\end{aligned} \tag{356}$$

■

Exercise 5.1. Show that

$$p_n = \sum_{k=0}^{n-1}(-1)^k s_{(n-k,1^k)}\,. \tag{357}$$

Hint – Apply Muir's formula to $p_n = m_{(n)}\cdot s_\emptyset$.

5.3. The first Pieri formula. Applying Muir's identity to $m_{(1^k)} = e_k = s_{(1^k)}$, we obtain

$$e_k s_\lambda = \sum_{\gamma\in\mathfrak{S}_n(0^{n-k}1^k)} s_{\alpha+\gamma} \tag{358}$$

and among the $\alpha+\gamma$ occuring in this expression, the only ones which are not weakly decreasing are those having two consecutive components of the form $\alpha_i, \alpha_{i+1}+1$ with $\alpha_i=\alpha_{i+1}$. But in this case, the Schur function $s_{\alpha+\gamma}$ is zero. Hence, PROPOSITION 5.3. *The product of a Schur function by an elementary function is a multiplicity free sum of Schur functions*

$$e_k s_\lambda = \sum_\mu s_\mu \tag{359}$$

where the sum is over all partitions μ whose diagram is obtained from λ by adding k boxes, no two ones in the same row.

5.4. The Cauchy kernel again.

LEMMA 5.4 (Cauchy). *Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. Then,*

$$D(X,Y) := \det\left(\frac{1}{x_i + y_j}\right) = \Delta(X)\Delta(Y)\prod_{i,j=1}^{n}\frac{1}{x_i+y_j}\,. \tag{360}$$

Proof – Subtract the first column to the other ones. Since

$$\frac{1}{x_i+y_j} - \frac{1}{x_i+y_1} = \frac{y_1 - y_j}{(xi_+y_1)(x_i+y_j)}\,, \tag{361}$$

we can extract a factor $1/(x_i + y_1)$ from the ith row and a factor $y_1 - y_j$ from the jth column ($j > 1$). Hence,

$$D(X,Y) = \frac{(y_1-y_2)(y_1-y_3)\cdots(y_1-y_n)}{(x_1+y_1)(x_2+y_1)\cdots(x_n+y_1)}\begin{vmatrix} 1 & \frac{1}{x_1+y_2} & \cdots \frac{1}{x_1+y_n} & \\ 1 & \frac{1}{x_1+y_2} & \cdots \frac{1}{x_1+y_n} & \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{x_n+y_2} & \cdots \frac{1}{x_n+y_n} & \end{vmatrix} \tag{362}$$

and subtracting the first row to the other ones, we extract a factor $x_1 - x_i$ from row i and a factor $1/(x_1 + y_j)$ from column j ($i, j > 1$). Thus,

$$D(X,Y) = \frac{\prod_{i=2}^n (y_1-y_i)(x_1-x_i)}{\prod_{i=1}^n (x_i+y_1))\prod_{j=2}^n (x_1+y_j)} D(x_2,\dots,x_n;y_2,\dots,y_n)\,. \tag{363}$$

∎

Replacing x_i by $1/x_i$ and y_i by $-y_i$, we obtain on the one hand

$$\det\left(\frac{1}{1-x_iy_j}\right) = \Delta(X)\Delta(Y)\prod_{i,j=1}^{n}\frac{1}{1-x_iy_j} = \Delta(X)\Delta(Y)\sigma_1(XY)\,. \tag{364}$$

On the other hand, the matrix $((1-x_iy_j)^{-1}$ is the product of the infinite Vandermonde matrix $V(X) = (x_i^{j-1})_{1\le i\le n,\ j\ge 1}$ by the transpose of $V(Y)$. Applying the Binet-Cauchy formula for minors of orders n to this product, we have

$$\det\left(\frac{1}{1-x_iy_j}\right) = \sum_\alpha |V(X)|_\alpha |V(Y)|_\alpha = \Delta(X)\Delta(Y)\sum_\lambda s_\lambda(X)s_\lambda(Y)\,. \tag{365}$$

Letting the number of variables tend to infinity, we obtain:

Corollary 5.5 (The Cauchy identity for Schur functions). *Let X, Y be any two alphabets. Then,*

$$K(X,Y) = \sum_\lambda s_\lambda(X)s_\lambda(Y)\,. \tag{366}$$

In particular, Schur functions form an orthonormal basis of Sym for the Hall scalar product.

5.5. The Jacobi-Trudi identity. As we have seen, the monomial antisymmetric functions are the minors of an infinite Vandermonde matrix. Since we know that they are divisible by the product of differences A_ρ, it is is natural to try to extract this factor by row and column manipulations. This is has been done by Jacobi, and independently by his student Trudi.

Let us first observe that for a single variable,

$$x_i^n = h_n(x_i) \tag{367}$$

and that if $i \neq j$,

$$h_n(X+x_i) - h_n(X+x_j) = (x_i - x_j)h_{n-1}(X+x_i+x_j)\,. \tag{368}$$

Let us now compute some Schur function, say $s_{522}(x_1,x_2,x_3)$. The numerator of (349) is the antisymmetrization of $x_1^{5+2}x_2^{2+1}x_3^{2+0}$, that is

$$A_{732} = \begin{vmatrix} x_1^7 & x_1^3 & x_1^2 \\ x_2^7 & x_2^3 & x_2^2 \\ x_3^7 & x_3^3 & x_3^2 \end{vmatrix} = \begin{vmatrix} h_6(x_1+x_2) & h_2(x_1+x_2) & h_1(x_1+x_2) \\ h_6(x_2+x_3) & h_2(x_2+x_3) & h_1(x_2+x_3) \\ h_7(x_3) & h_3(x_3) & h_2(x_3) \end{vmatrix} (x_1-x_3)(x_2-x_3) \tag{369}$$

the equality resulting from the subtraction of the last row to the first two ones and from (367) and (371). Subtracting now the second row to the first one, and appying again (371), we get

$$\begin{aligned} A_{732} &= \begin{vmatrix} h_5(x_1+x_2+x_3) & h_1(x_1+x_2+x_3) & h_0(x_1+x_2+x_3) \\ h_6(x_2+x_3) & h_2(x_2+x_3) & h_1(x_2+x_3) \\ h_7(x_3) & h_3(x_3) & h_2(x_3) \end{vmatrix} (x_1-x_3)(x_2-x_3)(x_1-x_2) \\ &= \begin{vmatrix} h_5(X_3) & h_1(X_3) & h_0(X_3) \\ h_6(X_3) & h_2(X_3) & h_1(X_3) \\ h_7(X_3) & h_3(X_3) & h_2(X_3) \end{vmatrix} (x_1-x_3)(x_2-x_3)(x_1-x_2) \end{aligned} \tag{370}$$

the final equality following from the expansion

$$h_n(X-x_i) = h_n(X) - x_i h_{n-1}(X) \tag{371}$$

which shows that inserting the extra variables does not change the value of the determinant:

$$h_n(x_3) = h_n((x_2+x_3)-x_2) = h_n(x_2+x_3) - x_2h_{n-1}(x_2+x_3) \tag{372}$$

so that x_3 can be replaced by x_2+x_3 in the last row, and so on.

We have thus proved:

THEOREM 5.6 (Jacobi-Trudi). *The Schur function s_λ is equal to the following determinant*

$$s_\lambda = \det\left(h_{\lambda_i+j-i}\right) . \tag{373}$$

In particular, adding null parts to λ does not change the value of the determinant, so that Schur functions are stable w.r.t. addition of new variables:

$$s_\lambda(x_1,\ldots,x_n,0) = s_\lambda(x_1,\ldots,x_n) . \tag{374}$$

This formula also makes sense of s_γ for an arbitrary $\gamma \in \mathbb{Z}^n$, if one makes the convention that $h_n = 0$ for $n < 0$.

5.6. The Kostka-Naegelsbach identity. The Jacobi-Trudi identity shows that Schur functions are minors of the infinite Toeplitz matrix $S = (h_{j-i})_{i,j\geq 1}$. The inverse of this matrix is $((-1)^{j-i}e_{j-i})$. There is another identity of Jacobi which relates the minors of order k of a matrix M to those of its inverse: if adj $M = \det M \cdot M^{-1}$ is the *adjugate* of M,

$$\Lambda^k M = (\det M)\cdot \mathrm{adj}^{\,(k)} M^{-1} \tag{375}$$

where the kth adjugate adj$^{(k)}A$ of a matrix A is defined as $\det A\cdot \Lambda^k A$.

Applying this to the Jacobi-Trudi determinant, we obtain

PROPOSITION 5.7 (Kostka-Naegelsbach). *Schur functions are determinants of elementary functions:*

$$s_\lambda = \det(e_{\lambda'_i+j-i}) \tag{376}$$

where λ' is the conjugate partition of λ. As a consequence,

$$\omega(s_\lambda) = s_{\lambda'} . \tag{377}$$

5.7. The second Pieri rule. Applying ω to the first Pieri rule, we obtain:

PROPOSITION 5.8. *The product of a Schur function by a complete function is a multiplicity free sum of Schur functions*

$$h_k s_\lambda = \sum_\mu s_\mu \tag{378}$$

where the sum is over all partitions μ whose diagram is obtained from λ by adding k boxes, no two ones in the same column.

5.8. Skew Schur functions. Schur functions are minors of the infinite Toeplitz matrix $S(X) = (h_{j-i}(X))$. This matrix satisfies $S(X+Y) = S(X)S(Y)$. Applying the Binet-Cauchy identity to this product, we obtain:

PROPOSITION 5.9. *The coproduct of a Schur function is given by*

$$s_\lambda(X+Y) = \sum_{\mu\subseteq\lambda} s_{\lambda/\mu}(X)s_\mu(Y) \tag{379}$$

where $\mu \subseteq \lambda$ means that $\mu_i \leq \lambda_i$ for all i, so that the diagram of μ is included in the one of λ, and the skew Schur functions *$s_{\lambda/\mu}$ are defined by*

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i-\mu_j+j-i}\right) . \tag{380}$$

The commutative images $r_I(X)$ of the noncommutative ribbon Schur functions $R_I(A)$ are particular skew Schur functions, precisely those such that the Ferrers diagram of λ/μ coincides with the ribbon diagram of the composition I.

5.9. Differential operators. Since **Sym** $= \mathbb{K}[p_1, p_2, \ldots]$, one may write a Taylor formula for $f(X+Y) = F[p_1(X)+p_1(Y), p_2(X)+p_2(Y), \ldots]$

$$f(X+Y) = \exp\left\{\sum_{n\geq 1} p_n(Y)\frac{\partial}{\partial p_n(X)}\right\} f(X) \tag{381}$$

Actually, the partial derivative with respect to p_n is almost the adjoint of multiplication by p_n. If, for any symmetric function g, we denote by D_g the adjoint of the map $f \mapsto gf$, we have

$$D_{p_n} f = n\frac{\partial}{\partial p_n} f\,. \tag{382}$$

Exercise 5.2. Check this by taking $f = p_\mu$.

Thus, the coproduct $f(X) \mapsto f(X+Y)$ can be rewritten as

$$f(X+Y) = \exp\left\{\sum_{n\geq 1} p_n(Y)\frac{D_{p_n(X)}}{n}\right\} f(X) = D^{(X)}_{\sigma_1(XY)} f(X)\,, \tag{383}$$

the notation $D^{(X)}$ meaning that we take the adjoint in $Sym(X)$, the y_i being regarded as parameters.

Expanding $\sigma_1(XY)$ by the Cauchy formula, we find

$$s_\lambda(X+Y) = \sum_\mu s_\mu(Y)\cdot\left(D_{s_\mu} s_\lambda\right)(X) \tag{384}$$

so that

$$s_{\lambda/\mu} = D_{s_\mu} s_\lambda\,, \tag{385}$$

a result due to Foulkes[2].

5.10. Dual Pieri rules. Writing $\langle h_k s_\mu, s_\nu\rangle = \langle s_\mu, s_{\nu/k}\rangle$ and $\langle e_k s_\mu, s_\nu\rangle = \langle s_\mu, s_{\nu/1^k}\rangle$, and applying the Pieri rules, we obtain

PROPOSITION 5.10. *The skew Schur function $s_{\nu/k}$ (resp. $s_{\nu/1^k}$)is the sum of all Schur functions s_μ such that the diagram of μ is obtained by removing k boxes from the diagram of λ, at most one from each column (resp. row).*

---

[2]The operators D_f are often called *Foulkes derivatives*. They are denote by $f^\perp$ in Macdonald [**66**].

5.11. Young tableaux. A semi-standard Young tableau of shape a partition λ (or a skew-partition λ/μ) is a filling of its Ferrers diagrams by postive integers, such that rows are nondecreasing from left to right and columns are strictly increasing from bottom to top.

For a tableau T, denote by X^T the monomial obtained by replacing each entry i of T by the variable x_i and taking the product. Let $m_i(T)$ be the number of occurences of i in T. The vector $(m_i(T))$ is called the *weight* of T.

We already know that the ribbon Schur function $r_I(X)$ is the sum of the monomials X^T for T of shape I. This includes in particular $r_n = s_n$ (single row tableaux) and $r_{1^n} = s_{1^n}$ (single column tableaux). We may suspect that a more general statement should be true. Let us first compute the coefficient of a monomial function m_μ in a Schur function s_λ. Writing

$$K_{\lambda\mu} := \langle s_\lambda, h_\mu \rangle = \langle s_{\lambda/\mu_s}, h_{\bar\mu} \rangle = \sum_\nu \langle s_\nu, h_{\bar\mu} \rangle \tag{386}$$

where $\mu = (\mu_1, \ldots, \mu_s)$ and $\bar\mu = (\mu_1, \ldots, \mu_{s-1})$, we prove by induction

Proposition 5.11. *The* Kostka number $K_{\lambda\mu}$ *is the number of semi-standard tableaux of shape λ and weight μ.*

■

This suggest that $s_\lambda(X)$ should actually be the sum of the X^T for T of shape λ. That this is true is easily seen by induction on the number of variables. Suppose that it is proved for $n-1$ variables. Then,

$$s_\lambda(X_{n-1} + x_n) = \sum_\mu s_\mu(X_{n-1}) s_{\lambda/\mu}(x_n) \tag{387}$$

Exercise 5.3. Show that a skew Schur function $s_{\lambda/\mu}(x)$ of a single variable x is nonzero iff the diagram of λ/μ is a disjoint union of rows.

Assuming the exercise, each $s_\mu(X_{n-1})$ in the sum above can be expanded as a sum of tableaux of shape μ, and the powers of x_n coming from the nonzero $s_{\lambda/\mu}(x_n)$ can now be added to these tableaux to build all the tableaux of shape λ over X_n.

Exercise 5.4. Extend this to skew Schur functions.

Theorem 5.12. *Schur functions are sums of tableaux:*

$$s_{\lambda/\mu}(X) = \sum_{T \in \mathrm{Tab}(\lambda/\mu)} X^T\,, \tag{388}$$

where $\mathrm{Tab}(\lambda/\mu)$ *denotes the set of semi-standard tableaux of shape λ/μ.*

5.12. The Littlewood-Richardson rule.

6. Group representations

A *linear representation* of a group G (over a field $\mathbb{K}$) is a homomorphism $R: G \to GL(V)$ from G to the group of automorphisms of some vector space V (over $\mathbb{K}$).

Exercise 6.1. Let $V = \mathbb{C}^n$. Then the map $R: \mathfrak{S}_n \to GL_n(\mathbb{C})$ defined by $R(\sigma)_{ij} = 1$ if $i = \sigma(j)$ and 0 otherwise is a linear representation (permutation matrices).

Identify now $\mathbb{C}^n$ with the linear span of variables $x_1, \ldots, x_n$. Then, $\mathfrak{S}_n$ acts on the space V_d homogeneous polynomials of degree d by $R_d(\sigma)(f) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. The representations R_d are the symmetric powers $R_d = S^d(R)$ of the representation $R = R_1$.

Exercise 6.2. Similarly, the group $G = GL_n(\mathbb{C})$ has a natural representation (the vector representation) $V = \mathbb{C}^n$ of dimension n, and representations in $T^k(V)$, $S^k(V)$, $\Lambda^k(V)$. Compute the dimensions of these representations. For $n = 2$, write down the matrices $S^2(g)$, $T^2(g)$ and $\Lambda^2(g)$ for a generic matrix g.

For a finite group, a linear representation is the same thing as a module over the group algebra $\mathbb{K}G$.

Two representations R, R' on V, V' are said to be isomorphic (or equivalent) if there is a linear isomorphism $f: V \to V'$ such that $f \circ R = R' \circ f$.

A representation is said to be *irreducible* if it has no nontrivial G-invariant subspace.

Exercise 6.3. Check that the representation of $\mathfrak{S}_n$ by permutation matrices is not irreducible. In the case $n = 3$, decompose $\mathbb{C}^3$ as a direct sum of irreducible representations.

Let V and V' be two *irreducible* representations of some group G, and suppose we have a map $f: V \to V'$ which commutes with G: $f \circ R = R' \circ f$. This is called an *intertwining operator.*

If f is not injective, its kernel is a G-invariant subspace, so we must have $f = 0$. Similarly, if f is not surjective, its image is G-invariant, so again, $f = 0$. Hence, if $f \neq 0$, it must be an isomorphism. We can then assume that $V = V'$ and $R = R'$. If we assume now that $\mathbb{K} = \mathbb{C}$ (which will always be the case in the sequel), then f has at least one eigenvalue λ. The map $f - \lambda I_V$ has now a nonzero kernel, and commutes with G. It must therefore be zero, and we have proved:

Lemma 6.1 (Schur's lemma). *There are no nonzero intertwining operators between two non-isomophic irreducible representations of a group. Moreover, over $\mathbb{C}$, the space of intertwining operators between two isomorphic irreducible representations is one dimensional. In particular, any endomorphism of an irreducible representation commuting with G must be a scalar multiple of the identity.*

6.1. Finite groups. We shall now concentrate on the case of representations of finite groups over $\mathbb{C}$. When there is no ambiguity, we shall drop the symbol R and write only gv for $R(g)v$.

So, let V, V' be two representations of a finite group G. Given any linear map $f: V \to V'$, we can build an intertwining operator $\hat{f}$ by averaging it as

$$\hat{f} = \frac{1}{|G|} \sum_{g \in G} g^{-1} f g\,. \tag{389}$$

Indeed, it is clear that $\hat{f}g = g\hat{f}$ for all $g \in G$.

If we assume now that V and V' are irreducible, we can consider two possibilities: (i) either they are not isomorphic, which implies $\hat{f} = 0$; (ii) $V = V'$ and $R = R'$, so that $\hat{f}$ is a scalar map, precisely $\hat{f} = \frac{1}{n}\operatorname{tr}(f) I_V$, where $n = \dim V$.

Let us choose bases in V, V' and let $a_{ij}(g)$, $a'_{ij}(g)$ be the matrix elements of $R(g)$ and $R'(g)$ in these bases. Note that we can also assume that V and V' are endowed with G-invariant hermitian scalar products so that all $R(g)$ and $R'(g)$ are unitary.

Exercise 6.4. Check this.

If we also introduce the natural hermitian scalar product

(390) $$(\phi, \psi) := \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi(g)$$

on the space of complex valued functions on G, we obtain in the case (i)

(391) $(a'_{ki}, a_{lj}) = 0$ for all i, j, k, l is V and V' are irreducible and non isomophic

and in the case (ii)

(392) $$(a_{ki}, a_{lj}) = \frac{1}{n} \delta_{kl} \delta_{ij} \text{ for all } i, j, k, l \text{ if } V \text{ is irreducible.}$$

These fundamental *orthogonality relations* have a number of interesting consequences.

Observe first that if two representations are isomorphic, $\operatorname{tr} R(g) = \operatorname{tr} R'(g)$ for all g. The map $\chi : \; g \mapsto \operatorname{tr} R(g)$ is called the *character* of the representation. The orthogonality relations imply

(393) $$(\chi, \chi') = 0 \text{ if } V,\ V' \text{ are irreducible and non-isomorphic}$$

and

(394) $$(\chi, \chi) = 1 \text{ if } V \text{ is irreducible}\,.$$

As a consequence, the characters of the different irreducible representations of G are linearly independent. Thus, a finite group has only a finite number of equivalence classes of irreducible representations.

Recall that we can always introduce a G-invariant scalar product on a finite dimensional representation V of G. If V is not irreducible, it has a nontrivial sub-representation U. Then $U^\perp$ is G-invariant, and $V = U \oplus U^\perp$. By induction, we see that we can find a maximal decomposition of V into a direct sum of irreducible representations $V = V_1 \oplus \cdots \oplus \mathbf{V}_m$.

This decomposition is in general not unique, but the irreducible representations occuring in it and their multiplicities are independent of the choice, and determined by the character χ of V.

Hence, two representations are isomorphic if and only if they have the same character.

Exercise 6.5. Suppose that G is abelian. What are its irreducible representations ?

Exercise 6.6. The *regular representation* of a finite group G is its group algebra $\mathbb{C}[G]$ regarded as a left module on itself. Show that its character is

$$\chi(g) = \begin{cases} 0 & g \neq e \\ |G| & g = e\,. \end{cases} \tag{395}$$

If (χ^α) are the irreducible characters of G, show that

$$(\rho, \chi^\alpha) = \chi^\alpha(e) \tag{396}$$

and deduce that the regular representation contains each irreducible representation with a multiplicity equal to its dimension.

A character in a *central function*, that is, is constant on conjugacy classes. We can now prove that this number is precisely the number of conjugacy classes of G.

Indeed, let ϕ be a central function which is orthogonal to all irreducible characters. For each representation $R:\ G \to GL(V)$, we can build an endomorphism of V

$$R^\phi = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} R(g) \tag{397}$$

obviously commuting with each $R(g)$. Thus, if R is irreducible, $R^\phi = \lambda I$ is a scalar, and

$$\lambda \cdot \dim V = \operatorname{tr} R^\phi = (\phi, \chi^R) = 0 \tag{398}$$

so that $R^\phi = 0$. On another hand, taking for R the regular representation,

$$R^\phi(e) = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} g = 0 \tag{399}$$

and therefore, $\phi = 0$.

Exercise 6.7. What is the number of irreducible representations of $\mathfrak{S}_n$?

Let $n_\alpha = \dim V_\alpha$ be the dimensions of the irreducible representations of G. From Exercise 6.6, we have on the one hand

$$|G| = \sum_\alpha n_\alpha^2\,. \tag{400}$$

On the other hand, the regular representation defines a linear map

$$f:\ \mathbb{C}[G] \longrightarrow \bigoplus_\alpha \operatorname{End}(V_\alpha) = \bigoplus_\alpha M_{n_\alpha}(\mathbb{C}) \tag{401}$$

since each $g \in G$ induces an endomorphism of each V_α. This map is clearly injective and is therefore an isomorphism.

Theorem 6.2. *The group algebra of a finite group G is isomorphic to a direct sum of complete matrix algebras*

$$\mathbb{C}[G] \simeq \bigoplus_\alpha M_{n_\alpha}(\mathbb{C}) \tag{402}$$

whose ranks are the dimensions of the irreducible representations. The group algebra has therefore a basis corresponding under this isomorphism to the matrix elements $E_{ij}^{(\alpha)}$. It contains in particular a complete system of orthogonal idempotents $E_{ii}^{(\alpha)}$.

The subalgebra spanned by the identity matrices I_α is the center of $C[G]$, and the I_α are central idempotents.

The central idempotents e_λ acting as I_λ can be constructed by the trick of (397), taking for ϕ an irreducible character χ^λ, and for R the irreducible representation R_λ. Again, by Schur's lemma, R^ϕ is a scalar, and its trace is the scalar product $(\chi^\lambda, \chi^\lambda) = 1$. Hence,

$$e_\lambda = \frac{\chi^\lambda(1)}{|G|} \sum_{g \in G} \overline{\chi^\lambda(g)} g \tag{403}$$

act as the identity on the irreducible representation V_λ. Moreover, if we choose for R a different irreducible R_μ, the trace of R^ϕ is now $(\chi^\lambda, \chi^\mu) = 0$, so R^ϕ acts by 0 on V_μ.

Thus, the e_λ form a complete set of orthogonal idempotents of the group algebra. They are obviously central, hence are indeed the central idempotents.

6.2. Lie groups and Lie algebras. For infinite groups, the matter is complicated by analytic and topological considerations. For compact topological groups, the theory is similar to that of finite groups, averages over the group being there integrals with respect to the Haar measure. This approach works, for example, for the unitary groups $U(n)$, the real orthogonal groups $O(n)$ and the compact symplectic groups $Sp(n)$. These are the compact forms of the so-called *classical groups*. Another approach is to separate the algebraic and the topological aspects of the theory, and to start with Lie algebras.

In the sequel, we shall need only basic information about the classical groups, mostly in their complex forms $GL(n, \mathbb{C})$, $O(n, \mathbb{C})$ and $Sp(2n, \mathbb{C})$, respectively the groups of all invertible $n \times n$ complex matrices, of those preseving a nondegenerate symmetric bilinear form, and a nondegenerate antisymmetric bilinear form (which exists only in even dimensions).

The idea is to start with the subgroup of exponentials. We know that

$$e^a e^b = e^{H(a,b)} \tag{404}$$

where the Hausdorff series, $H(a, b)$, is a Lie series. Since the conditions of convergence of this series are unclear at this point, let us associate with a matrix group G the set $\mathfrak{g}$ of matrices X such that $x(t) = e^{tX}$ is in G for t small enough (and actually for all t). This is a vector space, and it is stable under the commutator $[X, Y] = XY - YX$. It is the *Lie algebra* of G. A representation of G determines a representation of $\mathfrak{g}$ by taking derivatives at 0: $Xv := \frac{d}{dt}|_{t=0} e^{tX} v$.

To get rid of the scalar matrices in $GL(n, \mathbb{C})$, one considers its subgroup $SL(n, \mathbb{C})$ of matrices of determinant 1. Its Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ is then the space of complex matrices with trace 0.

A simple Lie group is a connected non-abelian Lie group G which does not have nontrivial connected normal subgroups. A simple Lie algebra is a non-abelian Lie algebra whose only ideals are 0 and itself. A direct sum of simple Lie algebras is called a semisimple Lie algebra.

It can be shown that SL_n and $\mathfrak{sl}_n$ are simple in the above sense. There is a classification of finite dimensional simple complex Lie algebras (E. Cartan): four infinite series A_n, B_n, C_n, D_n, and five exceptional types E_6, E_7, E_8, F_4, G_2. The $\mathfrak{sl}_n$ are of type A_{n-1}, $\mathfrak{so}_{2n+1}$ is of type B_n, $\mathfrak{so}_{2n}$ of type D_n and $\mathfrak{sp}_{2n}$ of type C_n (with some isomorphisms in small ranks, e.g., $A_1 \simeq B_2$.

To each type corresponds a *Weyl group*, a special case of a Coxeter group. For A_{n-1} this is $\mathfrak{S}_n$, for B_n and C_n, this is the hyperoctahedral group $\mathbb{Z}_2 \wr \mathfrak{S}_n$ (signed permutations), and for D_n, the subgroup of signed permutations with an even number of minus signs.

The Weyl group allows one to write down a character formula for irreducible representations, the *Weyl character formula*, of which the Jacobi expression of Schur functions as ratios of two alternants is a special case. Indeed, as we shall see later on, Schur has shown that Schur functions are the characters of the irreducible polynomial representations of $GL(n, \mathbb{C})$, and it is the reason why they are named after him.

Exercise 6.8.

7. Characters of symmetric groups

The character theory of finite groups is due to Frobenius. He initiated the following general strategy for constructing irreducible characters of a finite group G. Start from *permutational representations* constructed from subgroups of G, and to try to find linear combinations of the corresponding *permutational characters* satifying some irreducibility criterion.

If H is a subgroup of G, G permutes the left cosets gH, and this action of G on G/H defines a linear representation of G on the vector space $V_H := \mathbb{C}[G/H]$ spanned by G/H. Let ξ^H be the character of this representation, *i.e.,* $\xi^H(g) = |\{C \in G/H : gC = C\}|$. Then, one looks for linear combinations

$$\chi = \sum_H c_H \xi^H$$

with integer coefficients, such that

$$\begin{cases} (\chi\,,\,\chi) & = & 1 \\ \ \chi(1) & > & 0 \end{cases} \tag{405}$$

where

$$(\phi\,,\,\psi) = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi(g)$$

is the standard scalar product on the character ring of G. Then, Frobenius' result asserts that conditions (405) are satisfied if and only if χ is an irreducible character.

Using this method, Frobenius obtained the character table of the symmetric group $\mathfrak{S}_n$. He used the permutation representations over Young subgroups

$$\mathfrak{S}_I = \mathfrak{S}_{i_1} \times \mathfrak{S}_{i_2} \times \cdots \times \mathfrak{S}_{i_r}$$

associated with compositions $I = (i_1, \ldots, i_r)$ of n. Since two compositions differing only by the order of the parts give equivalent representations, one can consider only

Young subgroups indexed by partitions λ of n, which parametrize also the conjugacy classes of $\mathfrak{S}_n$. Frobenius has shown, by means of a computation of symmetric functions, that the linear combinations

$$\chi^\lambda = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\xi^{\lambda+\rho-\sigma(\rho)} \tag{406}$$

where $\rho = (n-1, n-2, \dots, 1, 0)$, satisfies to (405), so that the χ^λ, λ running over the partitions of n, are all the irreducible characters of $\mathfrak{S}_n$.

The encoding of characters by symmetric functions is given by the characteristic map, defined by $\mathrm{ch}\,(\xi^\lambda) = h_\lambda$. Then, equation (406) means that $\mathrm{ch}\,(\chi^\lambda) = s_\lambda$ (one recognizes the expansion of the Jacobi-Trudi determinant), and Frobenius computation is equivalent to the fact that Schur functions form an orthonormal basis of the scalar product defined by $\langle h_\lambda\,,\, m_\mu\rangle = \delta_{\lambda\mu}$.

Of course, the scalar products $(\chi^\lambda\,,\,\chi^\mu)$ are determined by the $(\xi^\lambda\,,\,\xi^\mu) = \langle h_\lambda\,,\, h_\mu\rangle$. This last number is easily seen to be equal to cardinal $|\mathrm{Mat}(\lambda,\mu)|$ of the set of matrices $M = (m_{ij})$ with coefficient in $\mathbb{N}$ such that $\sum_j a_{ij} = \lambda_i$ and $\sum_i a_{ij} = \mu_j$ for all i, j. But the general results for the scalar products of induced characters also imply that this is a number of double cosets

$$(\xi^\lambda\,,\,\xi^\mu) = (\mathrm{Ind}\,_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} 1\,,\, \mathrm{Ind}\,_{\mathfrak{S}_\mu}^{\mathfrak{S}_n} 1) = |\mathfrak{S}_\lambda\backslash\mathfrak{S}_n/\mathfrak{S}_\mu| \tag{407}$$

(in this particular case, this is simply the Cauchy-Frobenius lemma: the number of orbits of a finite group acting on a finite set is equal to the average number of fixed points).

Let us work this out in some detail. Let $A = \{a_1, \dots, a_n\}$ be an n letter alphabet. The symmetric group $\mathfrak{S}_n$ acts on the right on A^n by the usual rule $w\sigma = w_{\sigma(1)}\cdots w_{\sigma(n)}$, which can be turned into a left action by writing $\sigma w = w\sigma^{-1}$.

For a composition $I = (i_1, \dots, i_r)$ of n, let W_I be the vector space spanned by the orbit of the word $w_I := a_1^{i_1}\cdots a_r^{i_r}$. The action of $\mathfrak{S}_n$ on words induces a linear representation π_I on W_I. Its dimension is

$$\dim W_I = \binom{n}{i_1, i_2, \dots, i_r} = \frac{|\mathfrak{S}_n|}{|\mathrm{Stab}(w_I)|} \tag{408}$$

and its character $\xi^I(\sigma)$ is the number of fixed points of σ in the orbit of w_I. As this depends only on the conjugacy class of σ, if $\mu = (1^{m_1}2^{m_2}\cdots)$ is its cycle type, we can assume that σ is the canonical permutation

$$\sigma_\mu := (1)(2)\cdots(m_1)(m_1+1, m_1+2)\cdots\,. \tag{409}$$

The sum of all words fixed by σ_μ is clearly

$$P_\mu(A) := (a_1+\cdots+a_n)^{m_1}(a_1^2+\cdots+a_n^2)^{m_2}\cdots(a_1^n+\cdots+a_n^n)^{m_n} \tag{410}$$

and the number of words in this expression which are in the orbit of w_I is equal to the coefficient of the monomial $x^I = x_1^{i_1}\cdots x_r^{i_r}$ in the commutative image of $P_\mu(A)$, which is

$$p_\mu(X) := (x_1+\cdots+x_n)^{m_1}(x_1^2+\cdots+x_n^2)^{m_2}\cdots(x_1^n+\cdots+x_n^n)^{m_n}\,. \tag{411}$$

If λ is the partition obtained by reordering the parts of I, this is the coefficient of the monomial symmetric function m_λ in p_μ, which can be expressed as a scalar product

$$\xi^\lambda_\mu := \xi^\lambda(\sigma_\mu) = \xi^I(\sigma) = \langle h_\lambda, p_\mu\rangle \,. \tag{412}$$

We can therefore write

$$h_\lambda = \sum_{\mu\vdash n} \xi^\lambda_\mu p^*_\mu \tag{413}$$

where

$$p^*_\mu = \frac{p_\mu}{z_\mu} \tag{414}$$

and regard h_λ as a generating functions for the values of the character ξ^λ.

Definition 7.1. *The Frobenius characteristic of a central function χ on $\mathfrak{S}_n$ is the symmetric function*

$$\mathrm{ch}\,(\chi) = \sum_{\mu\vdash n} \chi(\mu) p^*_\mu \tag{415}$$

If we also set

$$Z(\sigma) = p_\mu \quad \text{for } \sigma \text{ of cycle type } \mu \tag{416}$$

the evaluation of a central function on a permutation reads

$$\chi(\sigma) = \langle \mathrm{ch}\,(\chi), Z(\sigma)\rangle \tag{417}$$

Let κ^μ be the characteristic function of the conjugacy class μ. That is,

$$\kappa^\mu_\nu = \delta_{\mu\nu} = \langle p_\mu, p^*_\nu\rangle \tag{418}$$

so that $\mathrm{ch}\,(\kappa^\mu) = p^*_\mu$, which proves that the characteristic map is an isometry:

$$(\kappa^\mu, \kappa^\nu)_{\mathfrak{S}_n} = \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \overline{\kappa^\mu(\sigma)}\kappa^\nu(\sigma) = \frac{1}{z_\mu}\delta_{\mu\nu} = \langle p^*_\mu, p^*_\nu\rangle \,. \tag{419}$$

Since h_λ is a basis of Sym, the characters ξ^λ are linearly independent. If we can find a $\mathbb{Z}$-linear combination of them satisfying (405), it will be an irreducible character. But we have already done this: the Schur functions are integral linear combinations of the h_λ. They form an orthonormal basis for the scalar product of Sym, so that the central functions

$$\chi^\lambda(\sigma) := \langle s_\lambda, Z(\sigma)\rangle \tag{420}$$

are an orthonormal basis of central functions, and finally, their values on the identity are

$$\chi^\lambda(\mathrm{id}_n) := \langle s_\lambda, p_1^n\rangle = f_\lambda, \tag{421}$$

the number of standard tableaux of shape λ, hence a positive number. We have thus established:

THEOREM 7.2. *The χ^λ ($\lambda \vdash n$) are the irreducible characters of $\mathfrak{S}_n$. Their characteristics are (by definition) the Schur functions s_λ. The dimension of the representation ρ_λ of character χ^λ is f_λ, the number of standard tableaux of shape λ* ∎

8. The internal product

Recall that

(422) $$\delta : \; f \longmapsto f(XY)$$

defines another (cocommutative and coassociative) coproduct on *Sym*, and in fact on each homogeneous component Sym_n. By duality, this defines a new (associative and commutative) product $*$, called the *internal product*, as it preserves each Sym_n:

(423) $$\langle f * g, h\rangle := \langle f \otimes g, \delta(h)\rangle, \text{ so that } f * g = \sum_\lambda \langle f \otimes g, \delta u_\lambda\rangle v_\lambda$$

for any pair of adjoint bases (u, v).

Recall also that $p_\lambda(XY) = p_\lambda(X)p_\lambda(Y)$, and note that this implies that $f * g = 0$ if f and g are homogeneous of different degrees.

Taking now $u_\lambda = p_\lambda$ and $v_\lambda = p^*_\lambda$, we see that

(424) $$p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda$$

so that if κ_λ is the characteristic function of the conjucacy class C_λ of $\mathfrak{S}_n$, we have

(425) $$\mathrm{ch}\,(\kappa_\lambda \kappa_\mu) = \mathrm{ch}\,(\kappa_\lambda) * \mathrm{ch}\,(\kappa_\mu)\,.$$

Hence, the Frobenius characteristic map is a homomorphism from the ring of central functions on $\mathfrak{S}_n$ (with pointwise multiplication) to Sym_n endowed with the internal product. In particular,

(426) $$\mathrm{ch}\,(\chi^\lambda \chi^\mu) = s_\lambda * s_\mu\,.$$

Exercise 8.1. Check that σ_1 is the neutral element of $*$, and that $(f * \lambda_{-1})(X) = f(-X)$.

Exercise 8.2. Show that for any $f, g, h \in Sym$,

(427) $$\langle f * g, h\rangle = \langle f, g * h\rangle\,.$$

PROPOSITION 8.1. *Let (u, v) be any pair of adjoint bases of Sym. For any symmetric function f,*

(428) $$f(XY) = \sum_\lambda (f * u_\lambda)(X) v_\lambda(Y)\,.$$

Proof – Let us first compute

$$
\begin{aligned}
s_\lambda(XY) &= \sum_\mu \left(\sum_\nu \langle s_\lambda, s_\mu * s_\nu \rangle s_\mu(X) \right) s_\nu(Y) \\
&= \sum_\mu \left(\sum_\nu \langle s_\lambda * s_\nu, s_\mu \rangle s_\mu(X) \right) s_\nu(Y) \qquad (429)\\
&= \sum_\nu (s_\lambda * s_\nu)(X) s_\nu(Y)
\end{aligned}
$$

Now, by linearity, we have for any f

$$
\begin{aligned}
f(XY) &= \sum_\nu (f * s_\nu)(X) s_\nu(Y) \\
&= f(X) *_X \sigma_1(XY) \text{ where } *_X \text{ is the internal product of } Sym(X) \qquad (430)\\
&= \sum_\lambda (f * u_\lambda(X)) v_\lambda(Y)\,.
\end{aligned}
$$

■

A fundamental problem of representation theory is to decompose tensor products of irreducible representations. Here, this amounts to expanding an internal product $s_\lambda * s_\mu$ as a linear combination of Schur functions (note that the coefficients are nonnegative integers, which is not at all obvious from the definition of $*$ as dual to δ). This problem is computationally difficult, and formulas are known only for very special cases. The most efficient tool for computing with internal products is the following *splitting formula*, a compatibility relation between the various operations.

THEOREM 8.2. *Let Δ^r denote the iterated coproduct*[3] *with values in $Sym^{\otimes r}$, μ_r be the r-fold multiplication and $*_r$ be the internal product on $Sym^{\otimes r}$. Then, for any $f_1, \dots, f_r, g \in Sym$,*

$$(f_1 f_2 \cdots f_r) * g = \mu_r \left[(f_1 \otimes f_2 \otimes \cdots \otimes f_r) *_r \Delta^r(g) \right] . \qquad (431)$$

Proof – Let again (u, v) be adjoint bases, and take $g = v_\mu$. On the one hand, we have by Prop. 8.1

(432)

$$(f_1 f_2 \cdots f_r)(XY) = \sum_{\lambda^{(1)},\dots,\lambda^{(r)}} (v_{\lambda_{(1)}} v_{\lambda_{(2)}} \cdots v_{\lambda_{(r)}})(Y)(f_1 * u_{\lambda^{(1)}})(X)(f_2 * u_{\lambda^{(2)}})(X) \cdots (f_r * u_{\lambda^{(r)}})(X)\,,$$

and on the other hand

$$(f_1 f_2 \cdots f_r)(XY) = \sum_\lambda v_\lambda(Y)[(f_1 f_2 \cdots f_r) * u_\lambda](X)\,, \qquad (433)$$

so that, writing

$$(v_{\lambda_{(1)}} v_{\lambda_{(2)}} \cdots v_{\lambda_{(r)}})(Y) = \sum_\lambda \langle v_{\lambda_{(1)}} v_{\lambda_{(2)}} \cdots v_{\lambda_{(r)}}, u_\lambda \rangle v_\lambda(Y) \qquad (434)$$

[3]This differs from the more usual convention denoting it by $\Delta^{(r-1)}$.

and identifying coefficients,we get the desired result. ∎

This formula can be interpreted as a special case of Mackey's theorem for a product of induced characters. Note we have established it without any reference to character theory, relying only upon the definition of $*$ as dual to the XY coproduct.

As an illustration, let us compute $h_\lambda * h_\mu$, which amounts to computing the product of permutational characters $\xi^\lambda \xi^\mu$, precisely those taken into account by Mackey's theorem.

PROPOSITION 8.3. *Let* $\lambda = (\lambda_1, \dots, \lambda_r)$ *and* $\mu = (\mu_1, \dots, \mu_s)$. *Then,*

$$h_\lambda * h_\mu = \sum_{M \in \mathrm{Mat}(\lambda,\mu)} h_M \tag{435}$$

where $\mathrm{Mat}(\lambda,\mu)$ *is the set of nonnegative integer* $r \times s$ *matrices with row sums* λ *and column sums* μ, *and* $h_M = \prod_{i,j} h_{m_{ij}}$.

Proof – We apply Prop. 8.2 with $f_i = h_{\lambda_i}$. Since h_m is neutral in Sym_m,

$$h_\lambda * h_\mu = \sum_{\lambda^{(1)} \vdash \lambda_1, \dots, \lambda^{(r)} \vdash \lambda_r} \langle m_{\lambda^{(1)}} \cdots m_{\lambda^{(r)}}, h_\mu \rangle h_{\lambda^{(1)}} \cdots h_{\lambda^{(r)}} \tag{436}$$

and since (m, h) are adjoint bases, the scalar product is equal to the coefficient of m_μ in the expansion of $m_{\lambda^{(1)}} \cdots m_{\lambda^{(r)}}$. This is the same as the coefficient of the monomial x^μ, which is the number of nonnegative integral matrices whose kth row is a permutation of $0^{s-r}\lambda^{(k)}$ (assuming $s \geq r$) and whose ith column has sum μ_i. Now, this is zero unless $\ell(\lambda^{(k)}) \leq s$ for all k, so that the product $h_{\lambda^{(1)}} \cdots h_{\lambda^{(r)}}$ can be written as $h_{M_1} \cdots h_{M_r}$ where $M_k = 0^{s-\ell(\lambda^{(k)})}\lambda^{(k)} =: (m_{kl})$, so that

$$\langle m_{\lambda^{(1)}} \cdots m_{\lambda^{(r)}}, h_\mu \rangle h_{\lambda^{(1)}} \cdots h_{\lambda^{(r)}} = \sum_M h_M \tag{437}$$

where M runs over matrices whose kth row is a permutation of M_k and with column sums μ. The sum of these expression is therefore $\sum_{M \in \mathrm{Mat}(\lambda,\mu)} h_M$, as claimed. ∎

9. Representation rings

9.1. The representation ring of $\mathfrak{S}_n$. For a finite group G, let $R(G)$ be the free abelian group generated by isomorphism classes of irreducible (complex) representations of G. It can be identified with the free $\mathbb{Z}$-module based on irreducible characters. Addition corresponds to direct sum of representations, and there is a multiplication induced by tensor products of representations. Thus, $R(G)$ is a commutative ring.

It is also a λ-ring: if G acts on a vector space V, it acts also on its exterior powers. The λ-operations are defined by $\lambda^k[V] = [\Lambda^k V]$. As we have seen, $\mathrm{tr}\, \Lambda^k M = e_k(X)$ if X is the alphabet of eigenvalues of the matrix M. The symmetric powers are denoted by σ^k and the Adams operations by ψ^k.

When $G = \mathfrak{S}_n$, both the irreducible representations and the λ-ring operations can be identified with symmetric functions. The kth exterior power of the irreducible representation $[\lambda]$ can then be regarded as the result of applying e_k to s_λ. This operation is called *inner plethysm*. We shall denote it by $\hat{e}_k[s_\lambda]$ (the hat is there to distinguish it from the more common operation of *outer plethysm*, to be introduced later.

Exercise 9.1. Let χ be a central function on G. Show that $\psi^k(\chi)(g) = \chi(g^k)$. Let ϕ_k be the adjoint (for the standard scalar product of central functions) of ψ^k, and let $\mathbf{1}$ be the trivial character. Show that $\phi_k(\mathbf{1})(g)$ is equal to the number of kth roots of g in G.

9.2. The Grothendieck ring of the tower of symmetric groups. There are natural embeddings $\mathfrak{S}_m \times \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{m+n}$. Hence, we can consider restriction of central functions of $\mathfrak{S}_{m+n}$ to $\mathfrak{S}_m \times SG_n$. If κ_μ is the characteristic function of a congugacy class C_μ of $\mathfrak{S}_{m+n}$, its restriction to $\mathbb{C}\mathfrak{S}_m \otimes \mathbb{C}\mathfrak{S}_n$ is

$$\kappa_\mu \downarrow_{m,n}^{m+n} = \sum_{\alpha\cup\beta=\mu\alpha\vdash m,\ \beta\vdash n} \kappa_\alpha \otimes \kappa_\beta \,, \tag{438}$$

and since the Frobenius characteristic of κ_λ is p_λ^*, we see that $\mathrm{ch}\,(\kappa_\mu \downarrow_{m,n}^{m+n})$ is the term of bidegree (m,n) in $p_\mu(X+Y) = \Delta p_\mu$. We have therefore the following representation theoretical interpretation of Δ:

Proposition 9.1. *If f is a central function of $\mathfrak{S}_n$,*

$$\Delta \mathrm{ch}\,(f) = \sum_{k+l=n} \mathrm{ch}\,\left(f \downarrow_{\mathfrak{S}_k\times\mathfrak{S}_l}^{\mathfrak{S}_n}\right) \,. \tag{439}$$

■

Since *Sym* is self-dual, this implies that the ordinary multiplication of symmetric functions encodes some operation on representations which is dual to restriction. This operation, called *induction*, has also been defined by Frobenius[4]:

Definition 9.2. *Let G be a finite group, H a subgroup of G and χ a central function on H. The induction of χ from H to G is defined by*

$$\chi \uparrow_H^G (g) = \frac{1}{|H|} \sum_{k\in G; kgk^{-1}\in H} \chi(kgk^{-1}) \tag{440}$$

Indeed, if ϕ is any character of G,

$$\langle \chi \uparrow_H^G, \phi\rangle_G = \frac{1}{|G|}\sum_{g\in G} \overline{\chi \uparrow_H^G (g)}\phi(g) \tag{441}$$

$$= \frac{1}{|G|}\sum_{g\in G}\frac{1}{|H|}\sum_{k\in G|kgk^{-1}\in H} \overline{\chi \uparrow_H^G (kgk^{-1})}\phi(g) \tag{442}$$

$$= \frac{1}{|G|}\sum_{g\in G}\frac{1}{|H|}\sum_{k\in G|kgk^{-1}\in H} \overline{\chi \uparrow_H^G (kgk^{-1})}\phi(kgk^{-1}) \tag{443}$$

$$= \frac{1}{|H|}\sum_{h\in H}\overline{\chi(h)}\phi(h) \times \frac{1}{|G|}|\{(k,g)\in G^2 | h = kgk^{-1}\}| \tag{444}$$

[4]In categorical terms, one says that induction and restriction are adjoint functors.

and

$$|\{(k,g)\in \mathbf{G}^2 | h = kgk^{-1}\}| = |\{(k,g)\in \mathbf{G}^2 | k^{-1}hk = g\}| = |C_h||Z_h| = |G| \tag{445}$$

where C_h is the conjugacy class of h, and Z_h its centralizer.

If χ is a character, so is $\chi\uparrow_H^G$. Let V be the representation of H with character χ. Then, $\chi\uparrow_H^G$ is the character of $\mathbb{C}G\otimes_{\mathbb{C}H} V$.

Exercise 9.2. Make sense of this statement, and check it.

Summarizing, we have proved:

THEOREM 9.3. *The direct sum*

$$R(\mathfrak{S}) = \bigoplus_{n\geq 0} R(\mathfrak{S}_n) \tag{446}$$

is a commutative, cocommutative and self-dual Hopf algebra. The coproduct of a representation of $\mathfrak{S}_n$ *is the sum of its restrictions to all subgroups* $\mathfrak{S}_k\times\mathfrak{S}_n$ *with* $k+l=n$*, and the product of two representations* $[\mu]$ *and* $[\nu]$ *of* $\mathfrak{S}_k$ *and* $\mathfrak{S}_l$ *is the induction of* $[\mu\times\nu]$ *to* $\mathfrak{S}_{k+l}$*.*

Note that this proves that the product of two Schur functions is a sum of Schur functions.

Thus, $R(\mathfrak{S})$ is an example of a commutative, cocommutative, *positive*, and self-dual Hopf algebras.Such algebras have been classified by Zelevinsky. They are all constructed on the same model from a few sequences of finite groups.

10. Schur-Weyl duality

Let $V=\mathbb{C}^N=\bigoplus_{i=1}^N \mathbb{C}a_i$, and identify the tensor algebra $T(V)$ with the free associative algebra $\mathbb{C}\langle A\rangle$. Having fixed a basis, we can identify $GL(V)$ with $GL(N,\mathbb{C})$. The natural action of $GL(V)$ on $T^n(V)$ and the right action of $\mathfrak{S}_n$ commute with each other: if $w=v_1\cdots v_n$,

$$g(w\sigma) = (gv_{\sigma(1)})\cdots(gv_{\sigma(n)}) = (gw)\sigma\,. \tag{447}$$

Recall that the *weight* (sometimes also called *evaluation*) of a word w is the vector $\mathrm{wt}(w) = (m_i(w))_{1\leq i\leq N}$, where $m_i(w)$ is the number of occurences of the letter a_i in w. For a weight $\alpha\in\mathbb{N}^N$ with $|\alpha|=n$,

$$V(\alpha) = \bigoplus_{\mathrm{wt}(w)=\alpha} \mathbb{C}w \tag{448}$$

is called a *weight space* of $V^{\otimes n}$. Weight spaces are not stable under $GL(V)$, but they are preserved by the subgroup T_N of diagonal matrices.

Clearly, any endomorphism f of $V^{\otimes n}$ commuting with $GL(V)$ must commute in particular with diagonal matrices, so it must preserve weight spaces. Thus, it must be a linear combination of permutations.

Conversely, one can check that any endomorphism of $T^n(V)$ commuting with all permutations must be of the form $T^n(u)$ for some endomorphism u of V.

Exercise 10.1. One way to do this is to compute the dimension of the commutant. An endomorphism f of $V^{\otimes n}$ is defined by a tensor F_I^J where $I, J \in [N]^n$. Such an f commutes with $\mathfrak{S}_n$ if and only if $F_{I\sigma}^{J\sigma} = F_I^J$ for every permutation σ. Hence, the dimension of the commutant is equal to the number of orbits, which is the same as the number of commutative monomials in indeterminates x_i^j. But this is also the number of independent matrix elements of an endomorphism of the form $T^n(u)$.

THEOREM 10.1 (Schur-Weyl duality). *In* End$(V^{\otimes n}$, $\mathbb{C}[GL(V)] = M_N(\mathbb{C})$ *and* $\mathbb{C}\mathfrak{S}_n$ *are the commutant of each other. Moreover, if* $N \geq n$*, the endomorphisms* $g_\sigma : \ w \mapsto w\sigma$ *are linearly independent.*

If (e_t) is a family of idempotents decomposing the regular representation of $\mathbb{C}\mathfrak{S}_n$ into irreducible components (t runs over standard tableaux of size n), then the subspaces $V^{\otimes n}e_t$ are irreducible representations of $GL(V)$, and we have a direct sum decomposition

$$V^{\otimes n} = \bigoplus_{t \in \mathrm{STab}_n} V_t \simeq \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus f_\lambda} \tag{449}$$

since $V_t \simeq V_{t'}$ if t and t' have the same shape λ. Thus, $\mathbf{V}^{\otimes n}$ is fully reducible, and its irreducible representations are labelled by partitions of n, in at most N parts if $n > N$.

Let us compute the character of V_λ. Since diagonalizable matrices are dense in $GL(N, \mathbb{C})$, it is sufficient to evaluate it on a diagonal matrix. So, let $g = \mathrm{diag}(z_1, \dots, z_N)$. On a weight space $V(\alpha)$, g acts by the scalar z^α. But $V(\alpha)$ is also stable under $\mathfrak{S}_n$, and its Frobenius characteristic is $h_\alpha = h_\mu$ if μ is the partition obtained by reordering α. Thus,

$$\sum_\alpha z^\alpha \cdot \mathrm{ch}\,(V(\alpha)) = \sum_\alpha z^\alpha h_\alpha = \sum_{\mu \vdash n} m_\mu(Z) h_\mu(X) = \sigma_1(ZX) = \sum_{\lambda \vdash n} s_\lambda(Z) s_\lambda(X)\,. \tag{450}$$

Hence, the character of V_λ is the Schur function s_λ, interpreted as a symmetric function of the eigenvalues of the matrix g:

THEOREM 10.2 (I. Schur, 1901). *The representation of* $GL(V)$ *on* $T^n(V)$ *is fully reducible. Its irreducible components* V_λ *are parametrized by partions of* n *in at most* $N = \dim V$ *parts, and if* $(z_1, \dots, z_N)$ *are the eigenvalues of* g,

$$\mathrm{tr}_{V_\lambda}(g) = s_\lambda(z_1, \dots, z_N)\,. \tag{451}$$

We had already the symmetric and exterior powers $S^n(V)$ and $\Lambda^n(V)$, which can be identified with symmetric and antisymmetric tensors respectively, and have as characters h_n and e_n. For $n = 2$, Schur's theorem reduces to the easy fact that every square matrix can be uniquely decomposed into the sum of a symmetric and an antisymmetric matrix. For $n = 3$, we have a new type of symmetry: the partition $(2, 1)$. The irreducible components of $V^{\otimes n}$ are the *symmetry classes* of tensors. The two classes of type $(2, 1)$ are composed of the tensors T_{ijk} wich are symmetric in i, j and antisymmetric in i, k, or symmetric in i, k and antisymmetric in i, j. These two classes correspond to the two standard tableaux of shape (2,1).

11. Plethysm

The λ-ring operators defined by symmetric functions can be composed. On symmetric functions, this operation is called *plethysm* or *outer plethysm* to distinguish it from *inner plethysm*, the action of Sym on Sym_n regarded as $R(\mathfrak{S}_n)$. It is denoted by $f \circ g$ or $f[g]$. Clearly,

$$p_n \circ p_m = p_{mn}\,. \tag{452}$$

Unfortunately, this simple rule does not give any practical way to compute plethysms of Schur functions, which is the really interesting case. Indeed, if $R: GL(n\mathbb{C}) \to GL(m,\mathbb{C})$ and $R': GL(m,\mathbb{C}) \to GL(p,\mathbb{C})$ are two representations of respective characters f and g, the character of the composittion $R' \circ R$ is the plethysm $g \circ f$. In particular, the characters of the exterior and symmetric powers $\Lambda^k(R)$ and $S^k(R)$ of R are respectively $e_k \circ f$ and $h_k \circ f$. For f a Schur function, this is explicitly known only for $k = 2$.

There are a few closed formulas, such as Schur's identity

$$\sigma_1[e_1 + e_2] = \prod_i \frac{1}{1-x_i} \prod_{j<k} \frac{1}{1-x_jx_k} = \sum_\lambda s_\lambda \tag{453}$$

(sum of all Schur functions).

12. Transformations of alphabets

The λ-ring formalism allows us to evaluate symmetric functions on various polynomials or even rational functions. If q is an indeterminate (element of rank 1), we can consider

$$p_n(1-q) = 1 - q^n\,,\ p_n\left(\frac{1}{1-q}\right) = \sum_{k\geq 0} p_n(q^k) = \frac{1}{1-q^n} \tag{454}$$

and we can now define symmetric functions of $(1-q)X$ or of $X/(1-q)$.

Exercise 12.1. Show that

$$h_n((1-q)X) = (1-q)\sum_{k=0}^{n-1} (-q)^k s_{n-k,1^k}(X)\,. \tag{455}$$

These transformations will play a fundamental role in the sequel. Let us define a *virtual alphabet* $\mathbb{Y}$ as any element of some λ-ring (so that symmetric functions of $\mathbb{Y}$ make sense). The map

$$f(X) \mapsto f(\mathbb{Y}X) \tag{456}$$

is called an *alphabet transformation.* Basic example, apart form $\mathbb{Y} = (1-q)^{\pm 1}$, include $\mathbb{Y} = \alpha$ (a scalar) and $\mathbb{Y} = \frac{1-t}{1-q}$ where t is of rank one.

Although there is no such thing as a noncommutative λ-ring, these transformations can in general be lifted to combinatorial Hopf algebras. Another useful expression involves the internal product:

Exercise 12.2. Show that

$$f(\mathbb{Y}X) = f(X) * \sigma_1(\mathbb{Y}X)\,. \tag{457}$$