NONCOMMUTATIVE UNICELLULAR LLT POLYNOMIALS

JEAN-CHRISTOPHE NOVELLI AND JEAN-YVES THIBON

Abstract. It is known that unicellular LLT polynomials are related to the quasi-symmetric chromatic polynomials of certain graphs by the $(t-1)$-transform of symmetric functions. We investigate the extension of this transformation to various combinatorial Hopf algebras and prove a noncommutative version of this property.

1. Introduction

In their proof of the shuffle conjecture [3], Carlsson and Mellit obtain a remarkable relation between unicellular LLT polynomials and the quasi-symmetric chromatic polynomials of certain graphs, namely

$$\mathrm{X}_G(t,X) = (t-1)^{-n}\,\mathrm{LLT}_G(t,(t-1)X). \tag{1}$$

The graphs G are simple graphs with vertices labelled $1,\dots,n$, characterized by the property that if there is an edge (i,j) with $i<j$, then all the (i',j') with $i \le i' < j' \le j$ are also edges of G. The number of such graphs is the Catalan number c_n. These are the incomparability graphs of certain posets P, known as unit interval orders [24].

A coloring of G is a map $c:\ V(G)\to\mathbb{N}^*$, which can be identified with a word $c_1c_2\cdots c_n$. A coloring is proper if $c_i \ne c_j$ whenever $(i,j)\in E(G)$. We denote by $C(G)$ the set of proper colorings of G. The chromatic quasi-symmetric function of G expands in the M basis of *QSym* as [24]

$$\mathrm{X}_G(t,X) = \sum_{c\in C(G)} t^{\mathrm{asc}_G(c)} x_{c_1}x_{c_2}\cdots x_{c_n} = \sum_{c\in \mathrm{PC}\,(\mathrm{G})} t^{\mathrm{asc}_G(c)} M_{\mathrm{ev}(c)}(X), \tag{2}$$

where $\mathrm{PC}\,(\mathrm{G})$ denotes the set of proper packed colorings, $\mathrm{asc}_G(c)$ is the number of edges $(i<j)$ such that $c_i<c_j$, and $\mathrm{ev}(c)$ is the evaluation of c. It can be shown that for the above graphs, $\mathrm{X}_G(t)$ is actually a symmetric function [24].

On another hand, LLT polynomials are t-analogues of products of skew Schur functions [19, 14]. By interpreting s_1 as $s_{\lambda/\mu}$ in various ways, one may obtain different t-analogues of the characteristic s_1^n of the regular representation of $\mathfrak{S}_n$. These can be parametrized by the same graphs as above, and their expression given by the dinv statistic of Haglund, Haiman and Loehr [14] can be rephrased as

$$\mathrm{LLT}_G(t,X) = \sum_{u\in PW_n} t^{\mathrm{asc}_G(u)} M_{\mathrm{ev}(u)}(X), \tag{3}$$

1991 *Mathematics Subject Classification.* 05E05, 20C30, 60C05.

Key words and phrases. Noncommutative symmetric functions, Quasi-symmetric functions, LLT polynomials.

where u runs now over *a*ll packed words of length n, regarded as colorings of G.

Therefore, Equation (1) tells us that the transformation $(t-1)X$ just eliminates the improper colorings, a fact which is far from obvious. A pedestrian proof can be found in the appendix of [13]. The aim of this note is to provide a conceptual explanation (and a generalization) in terms of combinatorial Hopf algebras.

2. Word quasi-symmetric functions

Let $A = \{a_1, a_2, \dots\}$ be a totally ordered alphabet. The *packed word* $u = \mathrm{pack}(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < b_2 < \dots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$.

A word u is said to be *packed* if $\mathrm{pack}(u) = u$. We denote by PW the set of packed words. With such a word, we associate the "polynomial"

$$\mathbf{M}_u := \sum_{\mathrm{pack}(w)=u} w\,. \tag{4}$$

For example, with $A = \{1,2,3,4,5\}$,

$$\begin{aligned}\mathbf{M}_{13132} = \quad & 13132 + 14142 + 14143 + 24243 \\ & + 15152 + 15153 + 25253 + 15154 + 25254 + 35354.\end{aligned} \tag{5}$$

The *evaluation* $\mathrm{ev}(w)$ of a word w is the sequence whose i-th term $|w|_{a_i}$ is the number of times the letter a_i occurs in w, regarded as a finite integer vector by removing the trailing zeros.

Let $\mathbb{K}$ be a field of characteristic 0, assumed to contain all formal series in the formal parameter t used in the sequel.

Under the abelianization $\chi:\ \mathbb{K}\langle A\rangle \to \mathbb{K}[X]$, the $\mathbf{M}_u$ are mapped to the monomial quasi-symmetric functions

$$M_I := \sum_{j_1<j_2\dots<j_r} x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_r}^{i_r}, \tag{6}$$

where $I = (|u|_a)_{a\in A} = (i_1, \dots, i_r)$ is the evaluation vector of u.

The $\mathbf{M}_u$ span a subalgebra of $\mathbb{K}\langle A\rangle$, called **WQSym** for Word Quasi-Symmetric functions, consisting in the invariants of the noncommutative version of Hivert's quasi-symmetrizing action [5], which is defined by $\sigma \cdot w = w'$ where w' is such that $\mathrm{std}(w') = \mathrm{std}(w)$ and $\chi(w') = \sigma\cdot\chi(w)$, where std stands for the usual standardization algorithm, namely the algorithm that sends any word to the permutation having the same inversions. Hence, two words are in the same $\mathfrak{S}(A)$-orbit iff they have the same packed word.

When A is infinite, $\mathbb{K}\langle A\rangle$ is interpreted as the algebra of formal series of bounded degree. Exactly as in the case of symmetric or quasi-symmetric functions, **WQSym** acquires then the structure of a Hopf algebra, with the natural coproduct given by the ordinal sum of mutually commuting alphabets.

The coproduct $A \hat{+} B$ is indeed well-defined on **WQSym** and allows to consider its graded dual **WQSym***. We shall denote by $\mathbf{N}_u \in \mathbf{WQSym}^*$ the dual basis of $\mathbf{M}_u$.

The algebra **FQSym** (Free Quasi-symmetric functions) may be defined as the subalgebra of $\mathbb{K}\langle A\rangle$ spanned by the

$$\mathbf{G}_\sigma(A) = \sum_{\mathrm{std}(w)=\sigma} w, \tag{7}$$

where σ runs over all permutations, and std is the classical standardisation map. It is also a Hopf algebra for the same coproduct. It is self-dual, and the dual basis $\mathbf{F}_\sigma = \mathbf{G}_\sigma^*$ can be identified with $\mathbf{G}_{\sigma^{-1}}$. It is isomorphic to the Malvenuto-Reutenauer Hopf algebra [21, 4].

There is therefore an inclusion of Hopf algebras $\iota:\ \mathbf{FQSym} \hookrightarrow \mathbf{WQSym}$ given by

$$\mathbf{G}_\sigma = \sum_{\mathrm{std}(u)=\sigma} \mathbf{M}_u, \tag{8}$$

whose dual is the projection $\iota^*:\ \mathbf{WQSym}^* \twoheadrightarrow \mathbf{FQSym}$

$$\mathbf{N}_u \mapsto \mathbf{F}_{\mathrm{std}(u)}. \tag{9}$$

Let AB be the alphabet $\{ab | a \in A,\ b \in B\}$ endowed with the lexicographic order on the pairs (a, b). It is easy to check that the coproduct $\delta:\ f \mapsto f(AB)$ is also well-defined: packing with respect to the lexicographic order makes sense, and

$$\mathbf{M}_u(AB) = \sum_{\mathrm{pack}\binom{v}{w}=u} \mathbf{M}_v(A)\mathbf{M}_w(B). \tag{10}$$

The dual of the coproduct AB is an internal product on each $\mathbf{WQSym}_n^*$ given by

$$\mathbf{N}_u * \mathbf{N}_v = \mathbf{N}_{\mathrm{pack}\binom{u}{v}}, \tag{11}$$

where $\binom{u}{v}$ denotes the word in *biletters* $\binom{u_i}{v_i}$, lexicographically ordered with priority to the top letter. In this way, the homogeneous component $\mathbf{WQSym}_n^*$ gets identified with the Solomon-Tits algebra of $\mathfrak{S}_n$ [22] for all n.

There is also a Hopf embedding of **Sym** (Noncommutative symmetric functions) into $\mathbf{WQSym}^*$ given by $S_n \mapsto \hat{S}_n := \mathbf{N}_{1^n}$:

$$S^I \mapsto \hat{S}^I = \sum_{\mathrm{ev}(u)=I} \mathbf{N}_u. \tag{12}$$

For example, $\hat{S}^{21} = \mathbf{N}_{112} + \mathbf{N}_{121} + \mathbf{N}_{211}$.

Under the projection ι^*,

$$\hat{S}^I \mapsto \iota^*(\mathbf{N}_{1^{i_1}} \cdots \mathbf{N}_{1^{i_r}}) = \mathbf{F}_{12\ldots i_1} \cdots \mathbf{F}_{12\ldots i_r} = S^I = \sum_{\mathrm{Des}(\sigma)\subseteq \mathrm{Des}(I)} \mathbf{G}_\sigma. \tag{13}$$

This projection is compatible with the internal products: on **Sym**, the internal product is defined as dual to the coproduct XY of $QSym$ [11, 21]. By definition, it maps $f(AB)$ to $f(XY)$. On **FQSym**, the internal product on the **F**-basis is ordinary composition: $\mathbf{F}_\sigma * \mathbf{F}_\tau = \mathbf{F}_{\sigma\circ\tau}$, so that on the **G**-basis, $\mathbf{G}_\sigma * \mathbf{G}_\tau = \mathbf{G}_{\tau\circ\sigma}$.

Now, although the $*$ product of $\mathbf{WQSym}^*$ does not coincide with composition on permutations, we have the following compatibility.

Lemma 2.1. *Define a right action of* $\mathfrak{S}_n$ *on* $\mathbf{WQSym}_n^*$ *by*

$$\mathbf{N}_u \cdot \sigma := \mathbf{N}_{u\sigma}, \text{ where } u\sigma = u_{\sigma(1)} u_{\sigma(2)} \cdots u_{\sigma(n)}. \tag{14}$$

Then, for any $I \vDash n$

$$\mathbf{N}_{u\sigma} * \hat{S}^I = (\mathbf{N}_u * \hat{S}^I) \cdot \sigma. \tag{15}$$

Proof – If $\hat{S}^I$ contains $\mathbf{N}_v$, it contains the $\mathbf{N}_{v\tau}$ for all permutations τ, and

$$\text{pack}\begin{pmatrix} u\tau \\ v \end{pmatrix} = \text{pack}\begin{pmatrix} u \\ v\tau^{-1} \end{pmatrix} \cdot \tau. \tag{16}$$

∎

For example, with $u = 111122$, $v = 212211$, $\tau = 451623$, we have $u\tau = 121211$, $v\tau^{-1} = 211212$, $\text{pack}\binom{121211}{212211} = 232411$, $\text{pack}\binom{111122}{211212} = 211234$, and $211234\tau = 232411$.

This implies that $(f \cdot \sigma) * g = (f * g) \cdot \sigma$ for all $f \in \mathbf{WQSym}_n^*, g \in \mathbf{Sym}_n$ and $\sigma \in \mathfrak{S}_n$.

A similar argument proves the existence of the descent algebra. If $\sigma = \text{std}(v)$,

$$\iota^*(\hat{S}^I * \mathbf{N}_v) = \sum_{\text{ev}(u)=I} \phi\left(\mathbf{N}_{\text{pack}\binom{u}{v}}\right) \tag{17}$$

$$= \sum_{\text{ev}(u)=I} \mathbf{F}_{\text{std}\binom{u}{v}} \tag{18}$$

$$= \sum_{\text{ev}(u)=I} \mathbf{F}_{\text{std}\binom{u\sigma^{-1}}{v\sigma^{-1}} \circ \sigma} \tag{19}$$

$$= \sum_{\text{ev}(u)=I} \mathbf{F}_{\text{std}(u\sigma^{-1}) \circ \sigma} \tag{20}$$

$$= S^I * \mathbf{F}_\sigma \tag{21}$$

$$= \iota^*(\hat{S}^I) * \iota^*(\mathbf{N}_v). \tag{22}$$

This implies in particular that in **FQSym**,

$$S^I * S^J = \iota^*(\hat{S}^I * \hat{S}^J) = \sum_{M \in \text{Mat}(I,J)} S^M, \tag{23}$$

where $\text{Mat}(I, J)$ denotes the set of nonnegative integer matrices with row-sums vector I and column-sums vector J (cf. [10]). Hence, the $S^I = \sum_{C(\sigma^{-1})=I} \mathbf{F}_\sigma$ span a sub $*$-algebra of **FQSym** isomorphic to $(\mathbf{Sym}, *)$, which is therefore anti-isomorphic to the Solomon descent algebra.

3. Transformations of alphabets

3.1. **Transformations in** $QSym$**.** Recall that the classical Cauchy identities for symmetric functions can be extended to the dual pair of Hopf algebras $(QSym, \mathbf{Sym})$ as follows. Let X be a totally ordered alphabet of commutative variables, and A be an alphabet of noncommuting variables, also totally ordered. The product alphabet XA

is the set of products xa endowed with the lexicographic order on the pairs (x,a). We can thus define the noncommutative symmetric functions of XA and we have

$$\sigma_1(XA) = \prod_{x\in X}^{\rightarrow}\prod_{a\in A}^{\rightarrow}(1-xa)^{-1} = \sum_I M_I(X)S^I(A) = \sum_I U_I(X)V_I(A) \tag{24}$$

for any pair (U,V) of mutually dual bases [10].

We can introduce a second commutative alphabet T, and compute in two ways

$$\sigma_1(XTA) = \sum_I M_I(XT)S^I(A) = \sum_I M_I(X)S^I(TA). \tag{25}$$

The alphabet T denoted by $\frac{1}{1-t}$ is $\{t^n|n\geq 0\}$, ordered by $t^i < t^j$ iff $i>j$ (which would be true for numerical values of t such that the geometric series converge). We introduce the notations

$$XT = \frac{X|}{|1-t} \quad \text{and} \quad TA = \frac{|A}{1-t|}. \tag{26}$$

The maps

$$M_I(X)\mapsto M_I\left(\frac{X|}{|1-t}\right) \quad \text{and} \quad S^I(A)\mapsto S^I\left(\frac{|A}{1-t|}\right) \tag{27}$$

are algebra automorphisms, and their inverses are consistently denoted by

$$M_I(X)\mapsto M_I(X(1-t)) \quad \text{and} \quad S^{(A)}\mapsto S^I\left((1-t)A\right) \tag{28}$$

Here, $(1-t)$ is an example of a virtual alphabet. More generally, a virtual alphabet T is defined as a morphism of algebras χ_T

$$M_I \mapsto \chi_T(M_I) =: M_I(T) \tag{29}$$

from $QSym$ to some commutative algebra. This defines the TA transform in **Sym** by

$$S_n(TA) := \sum_{I\vDash n} M_I(T)S^I(A) \tag{30}$$

and by duality, the XT transform on $QSym$

$$M_I(XT) = \sum_J \langle M_I(X), S^J(TA)\rangle M_J(X), \tag{31}$$

where

$$\langle M_I(X), S^J(TA)\rangle = \langle \Delta^s M_I, S^{j_1}(TA)\otimes\cdots\otimes S^{j_s}(TA)\rangle \tag{32}$$

$$= \begin{cases} M_{I_1}(T)\cdots M_{I_s}(T) & \text{if } I = I_1\cdots I_s \text{ with } I_k \vDash j_k \\ 0 & \text{otherwise.}\end{cases} \tag{33}$$

The $(t-1)$ transform is defined by writing $t-1 = t(1-t^{-1})$ so that $M_I(X(t-1)) = t^{|I|}M_I(X(1-t^{-1})$.

We note for further reference the specializations

$$M_I\left(\frac{1}{1-t}\right) = \frac{t^{\mathrm{maj}(I)}}{(1-t^{i_1})(1-t^{i_1+i_2})\cdots(1-t^{i_1+\cdots+i_r})}, \tag{34}$$

$$M_I\left(\frac{1}{t-1}\right) = \frac{1}{(1-t^{i_1})(1-t^{i_1+i_2})\cdots(1-t^{i_1+\cdots+i_r})}, \tag{35}$$

$$M_I(1-t) = (-1)^{\ell(I)-1}(t^{n-i_1}-t^n) \quad (I \vDash n), \tag{36}$$

$$M_I(t-1) = (-1)^{\ell(I)-1}(t^{i_1}-1). \tag{37}$$

3.2. **Transformations in WQSym.** The $1/(1-t)$ transform may be extended from *QSym* to **WQSym** by setting

$$\frac{A|}{|1-t} = \{a_i t^j | i \geq 1, j \geq 0\} \tag{38}$$

endowed with the total order $a_i t^j < a_k t^l \Leftrightarrow i < k$ or $i = k$ and $j > l$. Then, the commutative image of $\mathbf{M}_u\left(\frac{A|}{|1-t}\right)$ is $M_I\left(\frac{X|}{|1-t}\right)$, where $I = \mathrm{ev}(u)$.

The inverse transformations are consistently denoted by $\mathbf{M}_u \mapsto \mathbf{M}_u(A(1-t))$ on **WQSym** and $S^I \mapsto S^I((1-t)A)$ on **Sym**. These have been investigated in [23, 18].

The adjoint map of $\mathbf{M}_u \mapsto \mathbf{M}_u(A(1-t))$ is $\mathbf{N}_u \mapsto \mathbf{N}_u * \sigma_1((1-t)A)$, and similarly for the inverse maps.

Although there is no known polynomial realization of $\mathbf{WQSym}^*$, it will be convenient to define $\mathbf{N}_u(TA)$ as

$$\mathbf{N}_u(TA) := \mathbf{N}_u * \sigma_1(TA) = \sum_v \mathbf{M}_v(T)\mathbf{N}_u * \mathbf{N}_v \tag{39}$$

$$= \sum_w \left(\sum_{v;\mathrm{pack}\binom{u}{v}=w} \mathbf{M}_v(T) \right) \mathbf{N}_w. \tag{40}$$

Let

$$V(u,w) = \{v | \operatorname{pack}\binom{u}{v} = w\}. \tag{41}$$

Proposition 3.1. *Let u and w be two packed words of the same size. Let $w^{(i)} = \mathrm{pack}(w_{j_1}w_{j_2}\cdots w_{j_p})$, where $\{j_1,\ldots,j_p\} = \{j | u_j = i\}$. Then,*

$$\sum_{v\in V(u,w)} \mathbf{M}_v = \mathbf{M}_{w^{(1)}}\mathbf{M}_{w^{(2)}}\cdots\mathbf{M}_{w^{(\max(u))}}. \tag{42}$$

Proof – Since the packing process commutes with the right action of the symmetric group (see (16)), we can apply to u the smallest permutation σ such that $u\sigma$ is nondecreasing (i.e., $\sigma = \mathrm{std}(u)^{-1}$), so that pack $\binom{u\sigma}{v\sigma} = w\sigma$. We can therefore assume that u is nondecreasing. First note that no relation is required between the letters of v corresponding to different letters of u. The only order constraints are among places where u has identical letters, and these are the same as in the corresponding letters of w. This is precisely the definition of the convolution on packed words, describing the product of the $\mathbf{M}$ basis [22]. ■

Thus,

$$\mathbf{N}_u(TA) = \sum_w (\mathbf{M}_{w^{(1)}}\mathbf{M}_{w^{(2)}} \cdots \mathbf{M}_{w^{(\max(u))}})(T)\mathbf{N}_w, \tag{43}$$

and by duality,

$$\mathbf{M}_u(AT) = \sum_v \left(\sum_{w\in V(v,u)} \mathbf{M}_w(T) \right) \mathbf{M}_v(A) = \sum_v (\mathbf{M}_{u^{(1)}} \cdots \mathbf{M}_{u^{(r)}})(T)\mathbf{M}_v(A). \tag{44}$$

The morphism χ_T defining a virtual alphabet is naturally extended to **WQSym** by setting $\mathbf{M}_u(T) = M_I(T)$, where $I = \mathrm{ev}(u)$.

A packed word v is said to refine u if for all $i < j$, $v_i > v_j \iff u_i \geq u_j$ and $v_i = v_j \implies u_i = u_j$. In this case, we write $v \geqslant_{\mathrm{ref}} u$. This is the usual notion of refinement on set compositions: each block of u is a union of *consecutive* blocks of v.

For example, the packed words finer than 212 are 212, 213, and 312. The packed words finer than 2122 are

$$2122, 2123, 2132, 3122, 2133, 3123, 3132, 2134, 2143, 3124, 3142, 4123, 4132. \tag{45}$$

Lemma 3.2. *The coefficient of* $\mathbf{M}_v(A)$ *in* (44) *is 0 if* u *is not finer than* v*, and equal to the coefficient of* $M_{\mathrm{ev}(u)}(X)$ *in* $M_{\mathrm{ev}(v)}(XT)$ *otherwise.*

Proof – By definition, the words $u^{(i)}$ exist only when u is finer than v, and then, $\mathbf{M}_{w^{(1)}}\mathbf{M}_{w^{(2)}} \cdots \mathbf{M}_{w^{(\max(u))}})(T)$ is equal to $M_{I_1}(T) \cdots M_{I_s}(T)$, where $I = \mathrm{ev}(v)$, $J = \mathrm{ev}(u)$ and $I_k \vDash j_k$ for all k. ■

4. Dyck graphs

Definition 4.1. *A* Dyck graph is a simple undirected graph G with vertex set $V(G) = [n]$ and edge set $E(G)$ represented as pairs $(i < j)$ such that if $(i, j) \in E(G)$, then $(i', j') \in E(G)$ for all $i \leq i' < j' \leq j$.

Define for $\sigma \in \mathfrak{S}_n$

$$\mathrm{inv}_G(\sigma) = \#\{(i < j) \in E(G) | i \text{ is to the right of } j \text{ in } \sigma\} \tag{46}$$

$$\mathrm{Des}_G(\sigma) = \{i \,|\, \sigma_i > \sigma_{i+1} \text{ and } (\sigma_{i+1}, \sigma_i) \notin E(G)\} \tag{47}$$

$$\mathrm{maj}_G(\sigma) = \sum_{i\in \mathrm{Des}_G(\sigma)} i. \tag{48}$$

Set $\mathrm{st}_G(\sigma) = \mathrm{inv}_G(\sigma) + \mathrm{maj}_G(\sigma)$.

We shall make use of *descent bottoms* of a permutation associated with a graph, that are the values σ_{i+1} such that $\sigma_i > \sigma_{i+1}$ and $(\sigma_{i+1}, \sigma_i) \notin E(G)$.

Theorem 4.2. *Let G be a Dyck graph. For any $\sigma \in \mathfrak{S}_{n-1}$,*

$$\sum_{\tau \in \sigma \sqcup\!\sqcup n} t^{\mathrm{st}_G(\tau)} = [n]_t \, t^{\mathrm{st}_H(\sigma)}, \tag{49}$$

where H is the restriction of G to $[n-1]$ and $\sigma \sqcup\!\sqcup n$ means the set of all words τ such that the restriction of τ to $[1, n-1]$ gives back σ.

Corollary 4.3. *The map*

$$c(\sigma) = (\mathrm{st}_G(\sigma|_{[n-i]}))_{i=0\dots n-1} \tag{50}$$

is a bijection from $\mathfrak{S}_n$ to the set of integer vectors $v \in \mathbb{N}^n$ such that $v_i \le n-i$.

In particular, we recover a particular case of a result of Kasraoui [17]:

Corollary 4.4. *For any Dyck graph G, the statistic st_G is Mahonian:*

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{st}_G(\sigma)} = [n]_t! \tag{51}$$

The previous theorem is a direct consequence of the following lemma.

Lemma 4.5. *Consider the permutations τ obtained from σ by inserting n at each of the n possible positions.*

Then, $\mathrm{st}_G(\tau) - \mathrm{st}_H(\sigma)$ takes all the values from 0 to $n-1$ in this order if one visits the insertion positions in the following order:

start with the rightmost position, then, from right to left, insert n to the left of the values k such that $(k, n) \in E$ or that are descent bottoms of σ, then from left to right, run through the remaining ones.

Proof – First note that (50) implies that st_G is Mahonian by induction on n.

Let us now prove it. Let H be the restriction of G to $[n-1]$. There are four cases to be distinguished according to whether τ is obtained by inserting n:

(1) at the end of σ: then $\mathrm{st}_G(\tau) = \mathrm{st}_H(\sigma)$.
(2) to the left of a k such that $(k, n) \in E(G)$. Then, $(k, k') \in E(G)$ for all $k < k' < n$, so that k cannot be a descent bottom of σ. Thus,

$$\mathrm{st}_G(\tau) = \mathrm{st}_H(\sigma) + d_H(k) + e_G(k),$$

where $d_H(k)$ is the number of H-descent bottoms of σ to the right of n (since each descent is shifted by one position to the right), and $e_G(k)$ is the number of k' to the right of n such that $(k', n) \in E(G)$ (since all these values have an inversion with n).
(3) to the left of an H-descent bottom k. Then, the letter ℓ preceding k in σ is such that $(k, \ell) \notin E(G)$, so that $(l, k) \notin E(G)$. Therefore, inserting n between ℓ and k creates a descent (n, k) in τ which takes the place of the descent (k, i) in σ just one position to the right. Thus,

$$\mathrm{st}_G(\tau) = \mathrm{st}_H(\sigma) + d_H(k) + e_G(k),$$

as in the previous case. Here, the $+1$ due to moving the descent bottom k one place to the right is taken into account in $d_H(k)$.

(4) to the left of a k such that $(k,n) \notin E(G)$ and k is not an H-descent bottom. Then (n,k) is a new descent and

$$\mathrm{st}_G(\tau) = \mathrm{st}_H(\sigma) + \sigma_k^{-1} + d_H(k) + e_G(k).$$

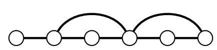
Let us now consider the sequence of insertions described in Lemma 4.5.

In the first part of the sequence going from right to left, one easily sees that the values of $d_H(k) + e_G(k)$ increase by one at each step since we stop at each element either creating an inversion with n or being a descent bottom. In the second part moving from left to right, the same property holds: at each step, the value of $\sigma_k^{-1} + d_H(k) + e_G(k)$ increases by one since, between two elements, σ_k^{-1} changes by one plus the number of values between these that are either descent bottoms or related with n in G, which is compensated by the fact that d_H and e_G decrease respectively on descent bottoms or values connected to n in G.

Finally, it is easily checked that both ways of evaluating the increment of st_H corrresponding to the leftmost insertion position do agree, whence the claim. ■

Note 4.6. This argument is similar to the one used for the maj-code (see, e.g., [16]). In particular, the definition of the sequence going backward then forward to visit each possible insertion position of n is essentially the same: one just has to add a special case when $(k,n) \in E(G)$.

Note 4.7. The statistic $s_G(\sigma)$ interpolates in a Catalan number of ways between the inversions number (G is the complete graph) and the major index (G is the graph with no edges).

Example 4.8. Consider the graph  on $V(G) = [6]$ with $E(G) = \{12, 23, 24, 34, 45, 46, 56\}$ and the permutation $\sigma = 52314$.

Then 4 and 5 belong to case (2), 1 and 2 belong to case (3), and 3 belongs to case (4). Then the order is ${}^4 5^3 2^5 3^2 1^1 4^0$, where the exponents encode the sequence.

Proposition 4.9. *For a Dyck graph,*

$$\mathrm{X}_G\left(t, \frac{1}{t-1}\right) = \frac{1}{(t-1)^n}. \tag{52}$$

Proof – According to [24, Theorem 9.3], the principal specialization of X_G satisfies

$$(q;q)_n \omega \mathrm{X}_G\left(t, \frac{1}{1-q}\right) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_G(\sigma)} q^{\mathrm{maj}_G(\sigma)}. \tag{53}$$

For $q = t$, this yields

$$(-1)^n (t;t)_n \mathrm{X}_G\left(t, \frac{1}{t-1}\right) = (t;t)_n \omega \mathrm{X}_G\left(t, \frac{1}{1-t}\right) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_G(\sigma)+\mathrm{maj}_G(\sigma)} = [n]_t!. \tag{54}$$

Dividing by $[n]_t!$, we are left with

$$(t-1)^n\, \mathrm{X}_G\left(t, \frac{1}{t-1}\right) = 1. \tag{55}$$ ■

5. The Guay-Paquet Hopf algebra

In his proof of the Shareshian-Wachs conjecture [12], Guay-Paquet introduces a Hopf algebra $\mathcal{G}$ based on ordered graphs, depending on a parameter t, and such that the map sending a graph to its quasi-symmetric chromatic function is a morphism of Hopf algebras $\mathcal{G} \to QSym$.

Its basis consists of finite simple undirected graphs with vertices labelled by the first integers. The product is the shifted concatenation: $G \cdot H = G \cup H[n]$ where $H[n]$ is H with labels shifted by the number n of vertices of G.

The parameter t arises in the coproduct. If G is a graph on n vertices and $w \in [r]^n$, regarded as a coloring of G, we denote by $G|_w$ the tensor product $G_1 \otimes \cdots \otimes G_r$ of the restrictions of G to vertices colored $1, 2, \dots, r$. The r-fold coproduct is then

$$\Delta^r G = \sum_{w \in [r]^n} t^{\operatorname{asc}_G(w)} G|_w. \tag{56}$$

At $t = 1$, $\mathcal{G}$ becomes cocommutative and is isomorphic to an algebra introduced in [25].

It is also proved in [12] that the subspace $\mathcal{D}$ of $\mathcal{G}$ spanned by Dyck graphs is a Hopf subalgebra. At $t = 1$, it is a free cocommutative graded connected Hopf algebra of graded dimension Catalan, and is therefore isomorphic to **CQSym** [15].

From these properties, we obtain a simple conceptual proof of (1):

Proposition 5.1.

$$(t-1)^n \mathrm{X}_G\left(t, \frac{X|}{|t-1}\right) = \sum_{u \in PW_n} t^{\operatorname{asc}_G(u)} M_{\operatorname{ev}(u)}(X) = \mathrm{LLT}_G(t, X). \tag{57}$$

Proof – If G is a Dyck graph and u a packed word, denote by $G_i(u)$ the restriction of G to the vertices j such that $u_j = i$. Then the coefficient of $M_I(X)$ in $\mathrm{X}_G\left(t, \frac{X|}{|t-1}\right)$ is

$$\left\langle \mathrm{X}_G\left(t, \frac{X|}{|t-1}\right), S^I \right\rangle = \left\langle \mathrm{X}_G(t, X), S^I\left(\frac{|A}{t-1|}\right) \right\rangle \tag{58}$$

$$= \left\langle \Delta^r \mathrm{X}_G(t, X), (S_{i_1} \otimes \cdots \otimes S_{i_r})\left(\frac{|A}{t-1|}\right) \right\rangle \tag{59}$$

$$= \sum_{u \in \mathrm{PW}_n} t^{\operatorname{asc}_G(u)} \prod_i \mathrm{X}_{G_i(u)}\left(t, \frac{1}{t-1}\right) \tag{60}$$

$$= (t-1)^{-n} \sum_{u \in \mathrm{PW}_n} t^{\operatorname{asc}_G(u)}. \tag{61}$$

Thanks to Lemma 3.2, this argument can be extended to the noncommutative case.

6. The noncommutative quasi-symmetric chromatic function

Given a Dyck graph G, define

$$\mathbf{X}_G(t, A) = \sum_{c \in \mathrm{PC}\,(\mathrm{G})} t^{\operatorname{asc}_G(c)} \mathbf{M}_c(A) \in \mathbf{WQSym}. \tag{62}$$

For example,

$$\mathbf{X}_{(\circ)} = \mathbf{M}_1. \tag{63}$$

$$\mathbf{X}_{(\circ\quad\circ)} = \mathbf{M}_{11} + \mathbf{M}_{12} + \mathbf{M}_{21}, \tag{64}$$

$$\mathbf{X}_{(\circ\!-\!\circ)} = t\,\mathbf{M}_{12} + \mathbf{M}_{21}. \tag{65}$$

$$\mathbf{X}_{(\circ\quad\circ\quad\circ)} = \sum_{w\in PW(3)} \mathbf{M}_w, \tag{66}$$

$$\begin{aligned}\mathbf{X}_{(\circ\!-\!\circ\quad\circ)} =& t\,\mathbf{M}_{121} + t\,\mathbf{M}_{122} + t\,\mathbf{M}_{123} + t\,\mathbf{M}_{132} + \mathbf{M}_{211}\\ &+ \mathbf{M}_{212} + \mathbf{M}_{213} + t\,\mathbf{M}_{231} + \mathbf{M}_{312} + \mathbf{M}_{321},\end{aligned} \tag{67}$$

$$\begin{aligned}\mathbf{X}_{(\circ\quad\circ\!-\!\circ)} =& t\,\mathbf{M}_{112} + \mathbf{M}_{121} + t\,\mathbf{M}_{123} + \mathbf{M}_{132} + t\,\mathbf{M}_{212}\\ &+ t\,\mathbf{M}_{213} + \mathbf{M}_{221} + \mathbf{M}_{231} + t\,\mathbf{M}_{312} + \mathbf{M}_{321},\end{aligned} \tag{68}$$

$$\mathbf{X}_{(\circ\!-\!\circ\!-\!\circ)} = t\,\mathbf{M}_{121}+t^2\,\mathbf{M}_{123}+t\,\mathbf{M}_{132}+t\,\mathbf{M}_{212}+t\,\mathbf{M}_{213}+t\,\mathbf{M}_{231}+t\,\mathbf{M}_{312}+\mathbf{M}_{321}, \tag{69}$$

$$\mathbf{X}_{\left(\overset{\frown}{\circ\!-\!\circ\!-\!\circ}\right)} = t^3\,\mathbf{M}_{123} + t^2\,\mathbf{M}_{132} + t^2\,\mathbf{M}_{213} + t\,\mathbf{M}_{231} + t\,\mathbf{M}_{312} + \mathbf{M}_{321}. \tag{70}$$

Proposition 6.1. *$G \mapsto \mathbf{X}_G(A)$ is a morphism of Hopf algebras from $\mathcal{G}$ to* **WQSym**.

Proof – The argument is essentially the same as for $QSym$. Multiplicativity is clear:

$$\mathbf{X}_{G_1}\mathbf{X}_{G_2} = \sum_{(u_1,u_2)\in \mathrm{PC}\,(\mathrm{G}_1)\times \mathrm{PC}\,(\mathrm{G}_2)} t^{\mathrm{asc}_{G_1}(u_1)+\mathrm{asc}_{G_2}(u_2)}\mathbf{M}_{u_1}\mathbf{M}_{u_2} \tag{71}$$

$$= \sum_{(u_1,u_2)\in \mathrm{PC}\,(\mathrm{G}_1)\times \mathrm{PC}\,(\mathrm{G}_2)} t^{\mathrm{asc}_{G_1}(u_1)+\mathrm{asc}_{G_2}(u_2)} \sum_{\substack{v=v_1v_2\\ \mathrm{pack}(v_1)=u_1\\ \mathrm{pack}(v_2)=u_2}} \mathbf{M}_v \tag{72}$$

$$= \sum_{v\in \mathrm{PC}\,(\mathrm{G}_1\mathrm{G}_2)} t^{\mathrm{asc}_{G_1G_2}(v)}\mathbf{M}_v. \tag{73}$$

Next, the coefficient of $\mathbf{M}_{u_1}\otimes\mathbf{M}_{u_2}$ in $\Delta\mathbf{X}_G$ is nonzero if and only if $u_1\in \mathrm{PC}\,(\mathrm{G}_1)$ and $u_2\in \mathrm{PC}\,(\mathrm{G}_2)$ for some splitting of the vertices of G into two complementary subsets, detemined by a word $w\in\{1,2\}^n$ as above.

Each such splitting determines a proper coloring u of G: color the vertices of G_1 with u_1 and those of G_2 with the shifted word $u_2[\max(u_1)]$ (recall that the shifted word $u[k]$ is obtained by adding k to all values of u). Thus, u_1 and u_2 are the restrictions of u to two consecutive intervals, so that $\mathbf{M}_{u_1}\otimes\mathbf{M}_{u_2}$ occurs in $\Delta\mathbf{M}_u$. In particular, note that u belongs to the shifted shuffle of u_1 and u_2.

Conversely, any $u\in \mathrm{PC}\,(\mathrm{G})$ and any term $\mathbf{M}_{u_1}\otimes\mathbf{M}_{u_2}$ occuring in $\Delta\mathbf{M}_u$ uniquely determines a splitting of $V(G)$ into two complementary subsets: the vertices of G_1 correspond to the positions of the letters of the subword u_1 of u. This proves that at $t=1$, $\mathbf{X}$ is indeed a morphism of coalgebras.

Now, in the above situation, we have

$$\mathrm{asc}_G(u) = \mathrm{asc}_{G_1}(u_1) + \mathrm{asc}_{G_2}(u_2) + r \tag{74}$$

where r is the number of ascending edges of G which are neither in G_1 nor in G_2. These correspond precisely to the G-ascents of the word $w \in \{1,2\}^n$ determining the splitting. ∎

From the product rule of the **M** basis of **WQSym**, one can easily check that

$$\mathbf{X}_{(\circ)}\mathbf{X}_{(\circ\ \ \circ)} = \mathbf{X}_{(\circ\ \ \circ\ \ \circ)}, \tag{75}$$

and that

$$\mathbf{X}_{(\circ)}\mathbf{X}_{(\circ\!-\!\circ)} = \mathbf{X}_{(\circ\ \ \circ\!-\!\circ)}. \tag{76}$$

For the coproduct, one can check the following example:

$$\begin{aligned}\Delta\mathbf{X}_{(\circ\!-\!\circ\ \ \circ)} &= \mathbf{X}_{(\circ\!-\!\circ\ \ \circ)} \otimes 1 + \mathbf{X}_{(\circ)} \otimes \big((1+t)\mathbf{X}_{(\circ\ \ \circ)} + \mathbf{X}_{(\circ\!-\!\circ)}\big)\\ &+ \big((1+t)\mathbf{X}_{(\circ\ \ \circ)} + \mathbf{X}_{(\circ\!-\!\circ)}\big) \otimes \mathbf{X}_{(\circ)} + 1 \otimes \mathbf{X}_{(\circ\!-\!\circ\ \ \circ)}.\end{aligned} \tag{77}$$

This example also allows to check that the restriction of the coproduct to the subalgebra of Dyck graphs is cocommutative.

Theorem 6.2.

$$(t-1)^n \mathbf{X}_G\left(t, \frac{A|}{|t-1}\right) = \sum_{u\in PW_n} t^{\mathrm{asc}_G(u)}\mathbf{M}_u(A). \tag{78}$$

The r.h.s. is therefore a noncommutative lift of the LLT polynomial LLT_G*.*

Definition 6.3. *Given a Dyck graph G, the* non-commutative LLT polynomial $\mathbf{LLT}_G$ *is*

$$\mathbf{LLT}_G := \sum_{u\in PW_n} t^{\mathrm{asc}_G(u)}\mathbf{M}_u(A). \tag{79}$$

Proof – The coefficient of $\mathbf{M}_v$ in $\mathbf{X}_G\left(t, \frac{A|}{|t-1}\right)$ is

$$c_v(t) = \left\langle S^{\mathrm{ev}(v)}, \sum_{\substack{u\in \mathrm{PC\,(G)}\\ u\geqslant_{\mathrm{ref}} v}} t^{\mathrm{asc}_G(u)} M_{\mathrm{ev}(u)}\left(\frac{X|}{|t-1}\right)\right\rangle. \tag{80}$$

Up to a power of t, the sum in the right-hand side of the bracket is the product of the quasi-symmetric chromatic polynomials of the graphs $G_i(v)$ evaluated at $\frac{X|}{|t-1}$. The power of t corresponds to the G-ascents of v on the deleted edges, that is

$$c_v(t) = t^{\mathrm{asc}_G(v)} \prod_i X_{G_i(v)}\left(\frac{1}{t-1}\right) = \frac{t^{\mathrm{asc}_G(v)}}{(t-1)^n}. \tag{81}$$

∎

Note 6.4. Alternatively, the r.h.s of (80) can be interpreted as a duality bracket for the pair $(\mathbf{WQSym}^*, \mathbf{WQSym})$:

$$c_v(t) = \left\langle S^{\mathrm{ev}(v)}, \sum_{\substack{u\in \mathrm{PC}\,(\mathrm{G})\\ u\geqslant_{\mathrm{ref}} v}} t^{\mathrm{asc}_G(u)}\mathbf{M}_u\left(\frac{A|}{|t-1}\right)\right\rangle \tag{82}$$

$$= \left\langle \mathbf{N}_{i_1}\cdots\mathbf{N}_{i_r}, \sum_{\substack{u\in \mathrm{PC}\,(\mathrm{G})\\ u\geqslant_{\mathrm{ref}} v}} t^{\mathrm{asc}_G(u)}\mathbf{M}_u\left(\frac{A|}{|t-1}\right)\right\rangle \tag{83}$$

$$= \left\langle \mathbf{N}_{i_1}\otimes\cdots\otimes\mathbf{N}_{i_r}, \sum_{\substack{u\in \mathrm{PC}\,(\mathrm{G})\\ u\geqslant_{\mathrm{ref}} v}} t^{\mathrm{asc}_G(u)}\Delta^r\mathbf{M}_u\left(\frac{A|}{|t-1}\right)\right\rangle \tag{84}$$

and evaluating the iterated coproducts $\Delta^r\mathbf{M}_u$ leads to the same conclusion.

6.1. **Special case: the chain graphs.** Let G_n be the graph on $[n]$ with edges $(i, i+1)$. Then,

$$\mathbf{LLT}_{G_n} = \sum_{u\in \mathrm{PW}_n} t^{\mathrm{asc}_{G_n}(u)}\mathbf{M}_u. \tag{85}$$

If we embed **Sym** in **WQSym** by sending S_n to the sum of nonincreasing words

$$S_n \mapsto \sum_{u\in\mathrm{PW}_n,\ u\downarrow} \mathbf{M}_u \quad \Leftrightarrow \quad \Lambda_n \mapsto \mathbf{M}_{12\cdots n}, \tag{86}$$

we can write

$$\mathbf{LLT}_{G_n} = \sum_{w\in A^n} t^{\mathrm{asc}(w)} w = \sum_{I\vDash n} (t-1)^{n-\ell(I)}\Lambda^I \tag{87}$$

so that

$$\mathbf{X}_{G_n} = \sum_{I\vDash n} \frac{\Lambda^I(A(t-1))}{(t-1)^{\ell(I)}} \tag{88}$$

which gives back by commutative image the generating series of [24]. The images of S_n and Λ_n by the $A\mapsto A(t-1)$ transform are given by

$$\sigma_1(A(t-1)) = \sum_{u\downarrow}\mathbf{M}_u(A(t-1)) = \sum_{u\downarrow} t^{|u|-\max(u)}(t-1)^{\max(u)}\mathbf{M}_u = \overleftarrow{\prod_{i\geq 1}} \frac{1-a_i}{1-ta_i} \tag{89}$$

and

$$\lambda_{-1}(A(t-1)) = \sum_{u=12\cdots n}\mathbf{M}_u(A(t-1)) = \sum_{u\uparrow}(1-t)^{\max(u)}\mathbf{M}_u = \overrightarrow{\prod_{i\geq 1}} \frac{1-ta_i}{1-a_i}. \tag{90}$$

Hence,

$$\left(\sum_{n\geq 0}\mathbf{X}_{G_n}\right)^{-1} = 1 + \sum_{n\geq 1}(-1)^n\sum_{u\uparrow}(1-t)^{\max(u)-1}\mathbf{M}_u. \tag{91}$$

At $t = 1$, this gives back the elementary fact that the sum of all Smirnov words is the inverse of the alternating sum of constant words.

7. The Dyck graphs subalgebra of **WQSym**

The goal of this section is to prove

Theorem 7.1. *The restriction of the morphism of Hopf algebras* $G \mapsto \mathbf{X}_G(t, A)$ *from* $\mathcal{G}$ *to* **WQSym** *to the subalgebra* $\mathcal{D}$ *of Dyck graphs is injective.*

We shall prove that the images of the Dyck graphs are already linearly independent for $t = 1$.

7.1. The Hopf algebra WSym.

The $\mathbf{X}_G(1, A)$ are the noncommutative chromatic polynomials defined by Gebhard [8, 9], and thus belong to the algebra of symmetric functions in noncommuting variables a_i, denoted here by **WSym**.

It consists of the invariants of $\mathfrak{S}(A)$ acting by automorphisms on the free algebra $\mathbf{K}\langle A\rangle$. Two words $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$ are in the same orbit whenever $u_i = u_j \Leftrightarrow v_i = v_j$. Thus, orbits are parametrized by set partitions into at most $|A|$ blocks. Assuming that A is infinite, we obtain an algebra based on all set partitions whose monomial basis is defined by

$$\mathbf{m}_\pi(A) = \sum_{w \in O_\pi} w \tag{92}$$

where O_π is the set of words such that $w_i = w_j$ iff i and j are in the same block of π.

As an example of expansion of a chromatic polynomial in terms of the $\mathbf{m}$, we have

$$\mathbf{X}_{(\circ\!-\!\circ\quad\circ)}(1) = \mathbf{m}_{121} + \mathbf{m}_{122} + \mathbf{m}_{123}. \tag{93}$$

The product of the $\mathbf{m}$ is given by the rule

$$\mathbf{m}_{\pi'}\mathbf{m}_{\pi''} = \sum_{\pi \in E(\pi', \pi'')} \mathbf{m}_\pi, \tag{94}$$

where $E(\pi', \pi'')$ consists of all set partitions whose parts are either a part of π', a part of π'', or a union of a part of π' and a part of π''.

Since set partitions are equivalence classes of set compositions which are in bijection with packed words, we shall often denote a set partition as the minimal packed word in its class.

Let us illustrate this notation on two examples of the product:

$$\mathbf{m}_1\mathbf{m}_{1123} = \mathbf{m}_{12234} + \mathbf{m}_{11123} + \mathbf{m}_{12213} + \mathbf{m}_{12231}, \tag{95}$$

and

$$\mathbf{m}_{1123}\mathbf{m}_1 = \mathbf{m}_{11234} + \mathbf{m}_{11233} + \mathbf{m}_{11232} + \mathbf{m}_{11231}. \tag{96}$$

7.2. **The chromatic polynomials.** To prove the linear independence of the images of the Dyck graphs, we shall show that they are triangular with respect to a basis of a subalgebra of **WSym** based on *nonnesting partitions.*

Define the *denesting* $\mathrm{dn}(\pi)$ of a set partition π as the nonnesting partition π' obtained by iterating the following process: for each sequence $i < j < k < l$ such that j and k are in a block B_1 of π and i and l in another block B_2 with no intermediate value, i.e., $B_2 = \{m_1 < \cdots < m_p = i < m_{p+1} = l < \cdots m_r\}$, split B_2 into $B_2' = \{m_1, \ldots, i\}$ and $B_2'' = \{l, \ldots, m_r\}$.

Up to $n = 3$, all set partitions are fixed by the denesting algorithm and there is only one set partition π of size 4 such that $\mathrm{dn}(\pi) \neq \pi$. In terms of set partitions, it is $\{\{1, 4\}, \{2, 3\}\}$ and $\mathrm{dn}(\pi) = \{\{1\}, \{2, 3\}, \{4\}\}$. In terms of packed words, it is 1221 and $\mathrm{dn}(1221) = 1223$.

If $\pi = 12341312$, then $\mathrm{dn}(\pi) = 12341356$.

Proposition 7.2. *For a nonnesting partition π, define*

$$\tilde{\mathbf{m}}_\pi = \sum_{\mathrm{dn}(\pi')=\pi} \mathbf{m}_{\pi'}. \tag{97}$$

Then, the $\tilde{\mathbf{m}}_\pi$ form the basis of a subalgebra of **WSym** *of homogeneous dimensions given by the Catalan numbers.*

For example,

$$\tilde{\mathbf{m}}_{1223} = \mathbf{m}_{1223} + \mathbf{m}_{1221}. \tag{98}$$

$$\tilde{\mathbf{m}}_{12334} = \mathbf{m}_{12334} + \mathbf{m}_{12331} + \mathbf{m}_{12332}. \tag{99}$$

$$\tilde{\mathbf{m}}_{12233} = \mathbf{m}_{12233} + \mathbf{m}_{12211}. \tag{100}$$

$$\tilde{\mathbf{m}}_{12324} = \mathbf{m}_{12324} + \tilde{\mathbf{m}}_{12321}. \tag{101}$$

Proof – Since the product of the $\mathbf{m}$-basis is multiplicity-free, we just have to check that for any partition π', $\mathbf{m}_{\pi'}$ occurs in $\tilde{\mathbf{m}}_{\pi_1}\tilde{\mathbf{m}}_{\pi_2}$ if and only if $\mathbf{m}_{\mathrm{dn}(\pi')}$ occurs in this product.

If $\pi_1 \vdash [n_1]$ and $\pi_2 \vdash [n_2]$, then $\mathbf{m}_{\pi'}$ occurs in $\tilde{\mathbf{m}}_{\pi_1}\tilde{\mathbf{m}}_{\pi_2}$ if and only if $\mathrm{dn}(\pi'|_{[1,n_1]}) = \pi_1$ and $\mathrm{dn}(\pi'|_{[n_1,n_1+n_2]}) = \pi_2$.

Since the denesting process is obviously compatible with restriction to intervals, this is equivalent to $\mathrm{dn}(\pi')|_{[1,n_1]} = \pi_1$ and $\mathrm{dn}(\pi')|_{[n_1,n_1+n_2]} = \pi_2$, which is the condition for $\mathbf{m}_{\mathrm{dn}(\pi')}$ to occur in $\tilde{\mathbf{m}}_{\pi_1}\tilde{\mathbf{m}}_{\pi_2}$. ∎

Lemma 7.3. *For a Dyck graph G,*

$$\mathbf{X}_G(1, A) = \sum_{\pi(i)\neq\pi(j) \text{ if } (i,j)\in E(G)} \tilde{\mathbf{m}}_\pi \tag{102}$$

where the sum runs over nonnesting partitions, and $\pi(i)$ denotes the block containing i.

Proof – If $\mathbf{m}_{\pi'}$ occurs in $\mathbf{X}_G$, then so does $\tilde{\mathbf{m}}_{\mathrm{dn}(\pi')}$, since $\mathrm{dn}(\pi')$ is finer than π', so is associated with proper colorings as well.

Conversely, if $\mathbf{m}_{\pi'}$ does not occur in $\mathbf{X}_G$, there exist i, j with $|j-i|$ minimal such that $(i,j) \in E(G)$ and i, j in the same block of π'. By minimality of $|j-i|$, i and j are consecutive in their block. Moreover, $(i,j) \in E(G)$ implies that $(i',j') \in E(G)$ for all $i < i' < j' < j$. Still by minimality of $|j-i|$, $i+1, \ldots, j-1$ are all in different blocks. Hence, i and j would not be separated by the denesting process, so that $\mathbf{m}_{\mathrm{dn}(\pi')}$ does not occur in $\mathbf{X}_G$ either. ∎

There is a simple bijection η between nonnesting partitions π (represented as diagrams of arcs) and Young diagrams λ contained in the staircase partition $(n-1, \ldots, 2, 1)$, represented as sets of cells above the diagonal in an $n \times n$ square: the arcs of π are the coordinates of the corners of λ. For example, the partition $\lambda = (221)$ corresponds to the nonnesting partition 13|24[5 which is read on the coordinates of the corners of the diagram. The edges of corresponding graph G are the coordinates of the empty cells above the diagonal, $(1,2), (2,3), (3,4), (3,5), (4,5)$.

×	×			5
×	×		4	
×		3		
	2			
1				

Thanks to that bijection, there is a natural partial order on nonnesting partitions: the Young lattice restricted to partitions contained in the staircase. We shall say that $\pi' \leq \pi$ if the image of π' is included in the image of π.

Lemma 7.4. *Given a Dyck graph G,*

$$\mathbf{X}_G(1, A) = \sum_{\pi' \leq \pi_G} \tilde{\mathbf{m}}_{\pi'}, \tag{103}$$

where the sum runs over nonnesting partitions smaller than the nonnesting partition π_G corresponding to the Young diagram encoding G.

Proof – Let $\lambda = \eta(\pi_G)$. Let π' be a nonnesting set partition. If $\eta(\pi') \subseteq \lambda$ then, thanks to the bijection between partitions and Dyck graphs, for all $(i,j) \in E(G)$, all pairs (i',j') such that $i \leq i' < j' \leq j$ are also edges of G, so that i' and j' can never be in the same part of π, hence of π'. So thanks to the previous lemma, $\tilde{\mathbf{m}}_{\pi'}$ appears in the expansion of $\mathbf{X}_G$.

Conversely, if $\eta(\pi') \not\subseteq \lambda$, then there exists a corner (i,j) of $\eta(\pi')$ that does not belong to λ. Then (i,j) is an edge of G since it is an empty cell in λ but i and j are in the same block of π' since they are consecutive by definition. So $\tilde{\mathbf{m}}_{\pi'}$ does not appear in the expansion of $\mathbf{X}_G$. ∎

For example,

$$\mathbf{X}_{(\bigcirc)}(1) = \tilde{\mathbf{m}}_1. \tag{104}$$

$$\mathbf{X}_{(\circ\ \ \circ)}(1) = \tilde{\mathbf{m}}_{11} + \tilde{\mathbf{m}}_{12}, \tag{105}$$

$$\mathbf{X}_{(\circ\!-\!\circ)}(1) = \tilde{\mathbf{m}}_{12}. \tag{106}$$

$$\mathbf{X}_{(\circ\ \ \circ\ \ \circ)}(1) = \tilde{\mathbf{m}}_{111} + \tilde{\mathbf{m}}_{112} + \tilde{\mathbf{m}}_{122} + \tilde{\mathbf{m}}_{121} + \tilde{\mathbf{m}}_{123}, \tag{107}$$

$$\mathbf{X}_{(\circ\!-\!\circ\ \ \circ)}(1) = \tilde{\mathbf{m}}_{122} + \tilde{\mathbf{m}}_{121} + \tilde{\mathbf{m}}_{123}, \tag{108}$$

$$\mathbf{X}_{(\circ\ \ \circ\!-\!\circ)}(1) = \tilde{\mathbf{m}}_{112} + \tilde{\mathbf{m}}_{121} + \tilde{\mathbf{m}}_{123} \tag{109}$$

$$\mathbf{X}_{(\circ\!-\!\circ\!-\!\circ)}(1) = \tilde{\mathbf{m}}_{121} + \tilde{\mathbf{m}}_{123} \tag{110}$$

$$\mathbf{X}_{\left(\circ\!-\!\circ\!-\!\circ \text{ (arc 1–3)}\right)}(1) = \tilde{\mathbf{m}}_{123} \tag{111}$$

8. A multiplicative basis

Recall that the *reverse refinement order*, denoted by $\leq$, on compositions is such that $I = (i_1, \ldots, i_k) \geq J = (j_1, \ldots, j_l)$ iff $\{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_k\}$ contains $\{j_1, j_1 + j_2, \ldots, j_1 + \cdots + j_l\}$. In this case, we say that I is finer than J. For example, $(2,1,2,3,1,2) \geq (3,2,6)$.

This order can be extended to packed words as follows. To avoid confusion with the order of packed words defined previously, we say that w is *strongly finer* than w', and write $w \geq w'$, iff w and w' have same standardized word and the evaluation of w is finer than the evaluation of w'. It also amounts to ask that w and w' have same standardized word and that $w \geqslant_{\text{ref}} w'$.

For example, the packed words strongly finer than 212 are 212 and 213. The packed words strongly finer than 2122 are

$$2122, 2123, 2133, 2134. \tag{112}$$

to be compared with the packed words finer that 2122 shown in (45).

Given a permutation, let us define the set $A(\sigma)$ of the *advances* of σ as the set of values i such that $i+1$ is to the right of i in σ. Note that it is the complementary set over $[1, n-1]$ of the usual recoils. Let $\mathrm{DST}(\sigma)$ be the set of packed words of standardized σ.

Lemma 8.1. *Let σ be a permutation. Then the elements of* $\mathrm{DST}(\sigma)$ *are in bijection with the subsets of $A(\sigma)$.*
In particular, this set of words has a natural structure of boolean lattice.

Proof – Let i be an element of $A(\sigma)$ and let $j < k$ be the respective positions of i and $i+1$ in σ. Then in any element w of $\mathrm{DST}(\sigma)$, either $u_j = u_k$ or $u_k = u_j + 1$. Since there are two independent choices for all elements of $A(\sigma)$, the result holds.

The bijection between the de-standardized of σ amounts to a convention: put i in its corresponding subset of $A(\sigma)$ if $u_k = u_j + 1$. ∎

Corollary 8.2. *Let w be a word. Then the elements strongly coarser than w are an interval of the boolean order of* $\mathrm{DST}\,(w)$ *described in Lemma 8.1, hence themselves a boolean order consisting of the subsets of* $A(\mathrm{std}(w))$ *that, following the notations of the previous lemma, necessarily contains the elements i such that $w_j = w_k$.*

The boolean order of the packed words of standardized 13425 is given in Figure 1.

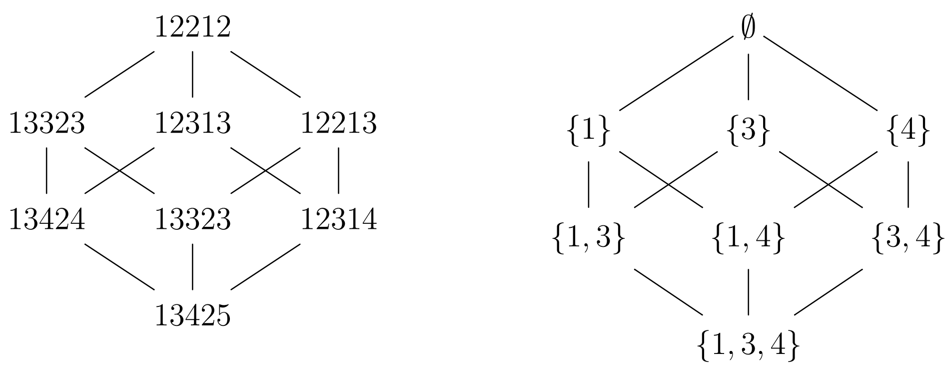


FIGURE 1. The boolean orders of the packed words of standardized 13425 and the corresponding subsets of $\{1, 3, 4\}$.

Let [22, 2]

$$\Phi_u := \sum_{v \geq u} \mathbf{M}_v. \tag{113}$$

For example,

$$\Phi_{111} = \mathbf{M}_{111} + \mathbf{M}_{112} + \mathbf{M}_{122} + \mathbf{M}_{123}; \quad \Phi_{212} = \mathbf{M}_{212} + \mathbf{M}_{213}. \tag{114}$$

$$\Phi_{133142} = \mathbf{M}_{133142} + \mathbf{M}_{134152} + \mathbf{M}_{144253} + \mathbf{M}_{145263}. \tag{115}$$

Since (Φ_u) is triangular over $(\mathbf{M}_u)$, it is a basis of **WQSym**. Note that the order used for summation is a restriction of the refinement order on compositions, so is a boolean lattice. Hence,

$$\mathbf{M}_u = \sum_{v \geq u} (-1)^{\max(v) - \max(u)} \Phi_v. \tag{116}$$

For example,

$$\mathbf{M}_{133142} = \Phi_{133142} - \Phi_{134152} - \Phi_{144253} + \Phi_{145263}. \tag{117}$$

By construction, the basis Φ satisfies a product formula similar to that of Gessel's basis F_I of *QSym* (whence the choice of notation). We shall not state it since we will not need it in the sequel but here follows an example illustrating the similarity with Gessel's basis.

$$\begin{aligned} \Phi_1 \Phi_{121} &= \Phi_{1121} + \Phi_{2121} + \Phi_{3121} + \Phi_{2132}, \\ F_1 F_{21} &= F_{31} + F_{22} + F_{211} + F_{121}. \end{aligned} \tag{118}$$

Proposition 8.3. *The noncommutative t-chromatic polynomial is Φ-positive:*

$$\mathbf{X}_G(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{asc}_G(\sigma)} \Phi_{\min_G(\sigma)}, \tag{119}$$

where $\min_G(\sigma)$ *is the packed word* u *defined as follows: let*

$$S = \{i | \sigma_{i-1}^{-1} < \sigma_i^{-1} \text{ and } (\sigma_{i-1}^{-1}, \sigma_i^{-1}) \notin E(G)\}. \tag{120}$$

Then,

$$u_i = \sigma_i - |S \cap [1,i]|. \tag{121}$$

All non trivial examples (excluding the case of the complete graph where S is always empty) of size 3 are given below.

First, here are all sets S and then all packed words $\min_G(\sigma)$.

	123	132	213	231	312	321
○ ○ ○	$\{2,3\}$	$\{2\}$	$\{3\}$	$\{3\}$	$\{2\}$	$\emptyset$
○—○ ○	$\{3\}$	$\{2\}$	$\{3\}$	$\emptyset$	$\{2\}$	$\emptyset$
○ ○—○	$\{2\}$	$\{2\}$	$\{3\}$	$\{3\}$	$\emptyset$	$\emptyset$
○—○—○	$\emptyset$	$\{2\}$	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$

	123	132	213	231	312	321
○ ○ ○	111	121	212	221	211	321
○—○ ○	122	121	212	231	211	321
○ ○—○	112	121	212	221	312	321
○—○—○	123	121	212	231	312	321

We then deduce

$$\mathbf{X}_{(\circ)} = \Phi_1. \tag{122}$$

$$\mathbf{X}_{(\circ\ \ \circ)} = \Phi_{11} + \Phi_{21}, \tag{123}$$

$$\mathbf{X}_{(\circ\!-\!\circ)} = t\,\Phi_{12} + \Phi_{21}. \tag{124}$$

$$\mathbf{X}_{(\circ\ \ \circ\ \ \circ)} = \Phi_{111} + \Phi_{121} + \Phi_{212} + \Phi_{221} + \Phi_{211} + \Phi_{321}. \tag{125}$$

$$\mathbf{X}_{(\circ\!-\!\circ\ \ \circ)} = t\,\Phi_{122} + t\,\Phi_{121} + \Phi_{212} + t\,\Phi_{231} + \Phi_{211} + \Phi_{321}, \tag{126}$$

$$\mathbf{X}_{(\circ\ \ \circ\!-\!\circ)} = t\,\Phi_{112} + \Phi_{121} + t\,\Phi_{212} + \Phi_{221} + t\,\Phi_{312} + \Phi_{321}, \tag{127}$$

$$\mathbf{X}_{(\circ\!-\!\circ\!-\!\circ)} = t^2\,\Phi_{123} + t\,\Phi_{121} + t\,\Phi_{212} + t\,\Phi_{231} + t\,\Phi_{312} + \Phi_{321}, \tag{128}$$

$$\mathbf{X}_{\left(\overset{\frown}{\circ\!-\!\circ\!-\!\circ}\right)} = t^3\,\Phi_{123} + t^2\,\Phi_{132} + t^2\,\Phi_{213} + t\,\Phi_{231} + t\,\Phi_{312} + \Phi_{321}. \tag{129}$$

Proof – Since Φ_w expanded in the $\mathbf{M}$ basis is a sum over elements of DST (std(w)), we have to show two things: first, the monomials in t that are coefficients of the $\mathbf{M}$ are constant among elements of DST (std(w)) and among the elements of DST (std(w)), the elements appearing in $\mathbf{X}_G$ expanded in the $\mathbf{M}$ basis are exactly the elements finer than $\min_G(\sigma)$.

Concerning the coefficients (monomials in t) of these elements, if w appears with a coefficient t^i then this is also the coefficient of std(w) and more generally of any element strongly finer than w. Indeed, the power of t counts the ascents among the pairs of edges of G and that does not change from w to $w' \geq w$ if w appears in $\mathbf{X}_G$. This comes from the fact that all ascents of w are ascents of w' and that the only positions (i,j) that could add an ascent from w to w' are those such that $w'_i > w'_j$ and $w_i = w_j$, but in that case (i,j) cannot be an edge of G since w is a proper coloring of G, and hence cannot count as an ascent of w'.

Let us now show that the packed words with standardized σ appearing in $\mathbf{X}_G$ are the elements finer than $\min_G(\sigma)$. Recall that thanks to Lemma 8.1, all words having a given standardized σ form a boolean lattice when equipped with the strong refinement order, and that any element corresponds to a subset of the set of values i such that $i-1$ is to the left of i in σ (or, equivalently $\sigma_{i-1}^{-1} < \sigma_i^{-1}$).

Since we are looking for the packed words that are proper colorings of G, it is pretty clear that the subsets containing an i such that $(\sigma_{i-1}^{-1}, \sigma_i^{-1}) \in E(G)$ cannot bring proper colorings since two connected vertices would have the same color. Conversely, excluding those values necessarily brings a proper coloring.

So all packed words appearing in the expansion of $\mathbf{X}_G$ in the M basis with a given standardized word σ are strongly finer than $\min_G(\sigma)$ and have all same coefficient, whence the statement. ∎

Corollary 8.4 ([24], Thm 3.1)**.** *The quasi-symmetric chromatic polynomial of a Dyck graph is F-positive and its expansion is*

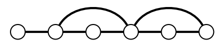
$$X_G(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_G(\sigma)} F_{\widetilde{DES_P(\sigma)}}, \tag{130}$$

where $\mathrm{inv}_G(\sigma)$ *is the pairs* $(i<j)$ *such that* $(\sigma_i, \sigma_j) \in E(G)$ *and* $\sigma_i > \sigma_j$, *and* $DES_P(\sigma)$ *is the composition encoding the set of* i *such that* $(\sigma_i, \sigma_{i+1}) \notin E(G)$ *and* $\sigma_i > \sigma_{i+1}$, *and where* ˜ *denotes the conjugate composition.*

Proof – Our formula in **WQSym** is projected to this expression by the morphism sending Φ_w to $F_{\mathrm{ev}(w)}$: the contribution of σ in our equation is the contribution of $\sigma' = \mathrm{inv}(r(\sigma))$ in their equation, where $r(\sigma)$ sends each value i to $n+1-i$ if $\sigma \in \mathfrak{S}_n$.

Indeed, our ascents of a permutation go to the inversions of [24] through $\mathrm{inv}\circ r$ since r changes ascents to inversions and inv exchanges values and positions. Moreover, two values i and $i+1$ of σ are equal in $\min_G(\sigma)$ (hence are not a descent of the composition $\mathrm{ev}(\min_G(\sigma))$) iff they are increasing and their positions do not correspond to an edge of G. Since σ' can also be described as $\sigma' = \overline{\mathrm{inv}(\sigma)}$, where $\bar{w}$ denotes the mirror-image of w, this exactly translates in σ' as the values in positions $n-i$, $n+1-i$

that decrease and do not form an edge of G, which is exactly the definition of the conjugate of $DES_P(\sigma')$. ■

For example, the graph $G =$  and the permutation 314652 contribute in our case to a term $t^3\mathbf{M}_{212321}$, whereas G and $(\overline{314652})^{-1} = (453162)^{-1} = 453162$ contribute in the case of [24] as $t^3 M_{231}$.

8.1. **Noncommutative LLT polynomials.** To formulate the noncommutative analogue of the F-expansion of unicellular LLT polynomials, we need another lift of the F-basis, defined as

$$\check{\Phi}_u = \sum_{v \geq \bar{u}} \mathbf{M}_{\bar{v}} \tag{131}$$

where the bar involution sends a word to its mirror image. Thus,

$$\check{\Phi}_u = \overline{(\Phi_{\bar{u}})} \tag{132}$$

and its commutative image is again $\check{\Phi}_u(X) = F_{\mathrm{ev}(\bar{u})}(X) = F_{\mathrm{ev}(u)}(X)$.

For a permutation σ and a Dyck graph G, define

$$\min{}'_G(\sigma) := \overline{\min_{\bar{G}}(\bar{\sigma})}, \tag{133}$$

where $\bar{G}$ is the mirror image of G (which amounts to relabeling the vertices by $i \mapsto n+1-i$).

Here are the non-trivial examples of $\min'_G$ for $n = 3$:

	123	132	213	231	312	321
○ ○ ○	123	122	112	121	212	111
○—○ ○	123	122	213	121	212	211
○ ○—○	123	132	112	121	212	221
○—○—○	123	132	213	121	212	321

With these definitions, Proposition 8.3 translates into:

Proposition 8.5. *The noncommutative t-chromatic polynomial is $\check{\Phi}$-positive:*

$$\mathbf{X}_G(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{asc}_G(\sigma)} \check{\Phi}_{\min'_G(\sigma)}. \tag{134}$$

For example,

$$\mathbf{X}_{(\circ\ \ \circ\ \ \circ)} = \check{\Phi}_{123} + \check{\Phi}_{122} + \check{\Phi}_{112} + \check{\Phi}_{121} + \check{\Phi}_{212} + \check{\Phi}_{111}. \tag{135}$$

$$\mathbf{X}_{(\circ\!-\!\circ\ \ \circ)} = t\,\check{\Phi}_{123} + t\,\check{\Phi}_{122} + \check{\Phi}_{213} + t\,\check{\Phi}_{121} + \check{\Phi}_{212} + \check{\Phi}_{211}, \tag{136}$$

$$\mathbf{X}_{(\circ\ \ \circ\!-\!\circ)} = t\,\check{\Phi}_{123} + \check{\Phi}_{132} + t\,\Phi_{112} + \check{\Phi}_{121} + t\,\check{\Phi}_{212} + \check{\Phi}_{221}, \tag{137}$$

$$\mathbf{X}_{(\circ\!-\!\circ\!-\!\circ)} = t^2\,\check{\Phi}_{123} + t\,\check{\Phi}_{132} + t\,\check{\Phi}_{212} + t\,\check{\Phi}_{121} + t\,\check{\Phi}_{212} + \check{\Phi}_{321}. \tag{138}$$

Theorem 8.6. *The noncommutative unicellular LLT polynomials are $\check{\Phi}$-positive:*

$$\mathbf{LLT}_G = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{asc}_G(\sigma)} \check{\Phi}_{\mathrm{min}'_{G_\emptyset}(\sigma)} \tag{139}$$

where $G_\emptyset$ is the graph with (n vertices, n omitted) and no edges.

Proof – If $\mathbf{M}_v$ occurs in $\check{\Phi}_u$, then $\mathrm{asc}_G(v) = \mathrm{asc}_G(u)$ for any graph G. Also, since $\mathrm{min}'_{G_\emptyset}(\sigma)$ is $\bar{v}$ where v is the minimal element of DST $(\bar{\sigma})$,

$$\sum_{\sigma\in\mathfrak{S}_n} \check{\Phi}_{\mathrm{min}'_{G_\emptyset}(\sigma)} = \sum_{u\in\mathrm{PW}_n} \mathbf{M}_u, \tag{140}$$

so that

$$\mathbf{LLT}_G = \sum_{u\in\mathrm{PW}_n} t^{\mathrm{asc}_G(u)}\mathbf{M}_u = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{asc}_G(\sigma)} \check{\Phi}_{\mathrm{min}'_{G_\emptyset}(\sigma)}. \tag{141}$$

■

For example,

$$\mathbf{LLT}_{(\circ\ \ \circ\ \ \circ)} = \check{\Phi}_{123} + \check{\Phi}_{122} + \check{\Phi}_{112} + \check{\Phi}_{121} + \check{\Phi}_{212} + \check{\Phi}_{111}, \tag{142}$$

$$\mathbf{LLT}_{(\circ\!-\!\circ\ \ \circ)} = t\,\check{\Phi}_{123} + t\,\check{\Phi}_{122} + \check{\Phi}_{112} + t\,\check{\Phi}_{121} + \check{\Phi}_{212} + \check{\Phi}_{111}, \tag{143}$$

$$\mathbf{LLT}_{(\circ\ \ \circ\!-\!\circ)} = t\,\check{\Phi}_{123} + \check{\Phi}_{122} + t\,\Phi_{112} + \check{\Phi}_{121} + t\,\check{\Phi}_{212} + \check{\Phi}_{111}, \tag{144}$$

$$\mathbf{LLT}_{(\circ\!-\!\circ\!-\!\circ)} = t^2\,\check{\Phi}_{123} + t\,\check{\Phi}_{122} + t\,\check{\Phi}_{112} + t\,\check{\Phi}_{121} + t\,\check{\Phi}_{212} + \check{\Phi}_{111}. \tag{145}$$

References

[1] P. Alexandersson, G. Panova, LLT polynomials, chromatic quasisymmetric functions and graphs with cycles, Discrete Mathematics 341 (2018), 3453–3482.

[2] N. Bergeron and M. Zabrocki, The Hopf algebras of symmetric functions and quasisymmetric functions in non-commutative variables are free and cofree, J. Algebra Appl. 8 (2009), no. 4, 581–600.

[3] E. Carlsson, A. Mellit, A proof of the shuffle conjecture, arXiv:1508.06239.

[4] G. Duchamp, F. Hivert, J.-Y. Thibon, Noncommutative symmetric functions VI: free quasi-symmetric functions and related algebras, Internat. J. Alg. Comput. 12 (2002), 671–717.

[5] G. Duchamp, F. Hivert, J.-C. Novelli, J.-Y. Thibon, Noncommutative symmetric functions VII: free quasi-symmetric functions revisited, Annals of Combinatorics 15 (2011), 655-673.

[6] K. Ebrahimi-Fard, F. Patras, N. Tapia, L. Zambotti, A Hopf-algebraic approach to cumulants-moments relations and Wick polynomials. arXiv:1710.00735.

[7] L. Foissy, Chromatic polynomials and bialgebras of graphs, arXiv:1611.04303

[8] D. Gebhard, Noncommutative symmetric functions and the chromatic polynomial, Proceedings of the 7th International Conference on Formal Power Series and Algebraic Combinatorics, Universit de Marne-la-Valle, 1995.

[9] D. Gebhard, B. Sagan, A Chromatic Symmetric Function in Noncommuting Variables, Journal of Algebraic Combinatorics 13 (2001) 227–255

[10] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, J.-Y. Thibon, Noncommutative symmetric functions, Adv. in Math. 112 (1995), 218–348.

[11] I. Gessel, Multipartite P-partitions and inner products of skew Schur functions, [in "Combinatorics and algebra", C. Greene, Ed.], Contemporary Mathematics, **34** (1984), 289–301.

[12] M. Guay-Paquet, A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra, arXiv:1601.05498
[13] J. Haglund, G. Xin, Lecture notes on the Carlsson-Mellit proof of the shuffle conjecture, arXiv:1705.11064.
[14] J. Haglund, M. Haiman, N. Loehr, A Combinatorial Formula for Macdonald Polynomials, J. Amer. Math. Soc. 18 (2005), 735–761.
[15] F. Hivert, J.-C. Novelli, J.-Y. THibon, Commutative combinatorial Hopf algebras, Journal of Algebraic Combinatorics, 28 n. 1 (2008), 65-95.
[16] F. Hivert, J.-C. Novelli, J.-Y. THibon, Multivariate generalizations of the Foata-Schtzenberger equidistribution. Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities, 289–299, Discrete Math. Theor. Comput. Sci. Proc., AG, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2006
[17] A. Kasraoui, A classification of Mahonian maj-inv statistics, Adv. in Apll. Math. 42 (2009), 569–595.
[18] D. Krob, B. Leclerc, J.-Y. Thibon, Noncommutative symmetric functions II : Transformations of alphabets, Internat. J. Alg. Comput. 7 (1997), 181–264.
[19] A. Lascoux, B. Leclerc, J.-Y. Thibon, Ribbon Tableaux, Hall-Littlewood Functions, Quantum Affine Algebras and Unipotent Varieties, J. Math. Phys. 38, (1997), 1041–1068.
[20] I.G. Macdonald, Symmetric functions and Hall polynomials, 2nd ed., Oxford University Press, Oxford, 1995.
[21] C. Malvenuto and C. Reutenauer, Duality between quasi-symmetric functions and the Solomon descent algebra, J. Algebra 177 (1995), 967–982.
[22] J.-C. Novelli, J.-Y. Thibon, Polynomial realizations of some trialgebras, FPSAC'06, San Diego.
[23] J.-C. Novelli, J.-Y. Thibon, Superization and (q,t)-specialization in combinatorial Hopf algebras, Electronic J. Combin. 16 (2) (2009), R21.
[24] J. Shareshian, M. Wachs, Chromatic quasisymmetric functions, Advances in Math. 295 (2016), 497–551.
[25] W. Schmitt, Incidence Hopf algebras, J. Pure Appl. Algebra 96 (1994), 299–330.

Laboratoire d'Informatique Gaspard Monge, UPEM (Université Paris-Est Marne-la-Vallée), 5 Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, FRANCE

E-mail address, Jean-Christophe Novelli: novelli@univ-mlv.fr
E-mail address, Jean-Yves Thibon: jyt@univ-mlv.fr