Drawing heaps uniformly at random

Samy Abbes\textsuperscript{1}, Sébastien Gouëzel\textsuperscript{2,3}, Vincent Jugé\textsuperscript{2,4} & Jean Mairesse\textsuperscript{2,5}


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3 First convergence results

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Petri nets and dependency graphs

Consider your favorite one-bounded Petri net with . . .

- Set of transitions: $\Sigma = \{a, b, c, d\}$
Consider your favorite one-bounded Petri net with ...
Petri nets and dependency graphs

Consider your favorite one-bounded Petri net with . . .

- Set of transitions: $\Sigma = \{a, b, c, d\}$
- Set of infinite *sequential* executions: $S \subseteq \Sigma^\omega$, with
  $$S = (wx + y)^\omega + (wx + y)^* wz^\omega,$$$$
  w = ac + ca, \ x = bd + db, \ y = cdab, \ z = badc$$
Petri nets and dependency graphs

Consider your favorite one-bounded Petri net with . . .

- Set of transitions: $\Sigma = \{a, b, c, d\}$
- Set of infinite **sequential** executions: $S \subseteq \Sigma^\omega$, with
  \[
  S = (wx + y)^\omega + (wx + y)^* w z^\omega, \text{ and}
  \]
  
  \[w = ac + ca, \ x = bd + db, \ y = cdab, \ z = badc\]
- Set of infinite **concurrent** executions: $S \subseteq \Sigma^\omega / \equiv$, with
Consider your favorite one-bounded Petri net with . . .

- Set of transitions: $\Sigma = \{a, b, c, d\}$
- Set of infinite \textbf{sequential} executions: $S \subseteq \Sigma^\omega$, with
  
  $$S = (wx + y)^\omega + (wx + y)^* wz^\omega,$$
  and

  $$w = ac + ca, \ x = bd + db, \ y = cdab, \ z = badc$$

- Set of infinite \textbf{concurrent} executions: $S \subseteq \Sigma^\omega/\equiv$, with

  $$ac \equiv ca, \ ad \equiv da, \ bd \equiv db$$
Petri nets and dependency graphs

Consider your favorite one-bounded Petri net with . . .

- Set of transitions: \( \Sigma = \{ a, b, c, d \} \)
- Set of infinite **sequential** executions: \( S \subseteq \Sigma^\omega \), with
  \[
  S = (wx + y)^\omega + (wx + y)^*wz^\omega, \quad \text{and}
  \]
  \[
  w = ac + ca, \quad x = bd + db, \quad y = cdab, \quad z = badc
  \]
- Set of infinite **concurrent** executions: \( S \subseteq \Sigma^\omega/\equiv \), with
  \[
  uv \equiv vu \iff (\cdot u \cup u^*) \cap (\cdot v \cup v^*) = \emptyset
  \]
Petri nets and dependency graphs

Consider your favorite one-bounded Petri net with . . .

- Set of transitions: \( \Sigma = \{a, b, c, d\} \)
- Set of infinite \textbf{sequential} executions: \( S \subseteq \Sigma^\omega \), with\[ S = (wx + y)^\omega + (wx + y)^*wz^\omega, \text{ and} \]
  \[ w = ac + ca, \ x = bd + db, \ y = cdab, \ z = badc \]
- Set of infinite \textbf{concurrent} executions: \( S \subseteq \Sigma^\omega/\equiv \), with\[ uv \equiv vu \iff (\cdot u \cup u^\cdot) \cap (\cdot v \cup v^\cdot) = \emptyset \]

\textbf{Independence} relation: \( I = \{ (u, v) \mid (\cdot u \cup u^\cdot) \cap (\cdot v \cup v^\cdot) = \emptyset \} \)

\[ I = \{ (a, c), (c, a), (a, d), (d, a), (b, d), (d, b) \} \]
Consider your favorite one-bounded Petri net with 

Set of transitions: \( \Sigma = \{a, b, c, d\} \)

Set of infinite **sequential** executions: \( S \subseteq \Sigma^\omega \), with

\[
S = (wx + y)^\omega + (wx + y)^* wz^\omega, \text{ and}
\]

\[
w = ac + ca, x = bd + db, y = cdab, z = badc
\]

Set of infinite **concurrent** executions: \( S \subseteq \Sigma^\omega / \equiv, \) with

\[
uv \equiv vu \iff (\cdot u \cup u^\bullet) \cap (\cdot v \cup v^\bullet) = \emptyset
\]

Independence relation: \( l = \{(u, v) \mid (\cdot u \cup u^\bullet) \cap (\cdot v \cup v^\bullet) = \emptyset\} \)

\[
l = \{(a, c), (c, a), (a, d), (d, a), (b, d), (d, b)\}
\]
Petri nets and dependency graphs

Consider your favorite **full, symmetric** one-bounded Petri net with . . .

- Set of transitions: $\Sigma = \{a, b, c, d\}$

```
  a
    ↘
    ︾
  b
    ↘
    ︾
  c
    ↘
    ︾
  d
```
Consider your favorite **full, symmetric** one-bounded Petri net with . . .

- Set of transitions: $\Sigma = \{a, b, c, d\}$
- Set of infinite **sequential** executions: $S = \Sigma^\omega$
Petri nets and dependency graphs

Consider your favorite **full, symmetric** one-bounded Petri net with . . .

- Set of transitions: \( \Sigma = \{a, b, c, d\} \)
- Set of infinite **sequential** executions: \( S = \Sigma^\omega \)
- Set of infinite **concurrent** executions: \( S = \Sigma^\omega / \equiv \), with
  \[ uv \equiv vu \iff u^* \cap v^* = \emptyset \]
Consider your favorite full, symmetric one-bounded Petri net with ...

- Set of transitions: $\Sigma = \{a, b, c, d\}$
- Set of infinite **sequential** executions: $S = \Sigma^\omega$

- Set of infinite **concurrent** executions: $S = \Sigma^\omega/\equiv$, with

$$uv \equiv vu \iff u^\bullet \cap v^\bullet = \emptyset$$

**Independence** relation: $I = \{(u, v) \mid (\cdot u \cup u^\bullet) \cap (\cdot v \cup v^\bullet) = \emptyset\}$

$$I = \{(a, c), (c, a), (a, d), (d, a), (b, d), (d, b)\}$$
Petri nets and dependency graphs

Consider your favorite **full, symmetric** one-bounded Petri net . . .

Can you pick one of its infinite **concurrent** executions **uniformly at random**?
Consider your favorite **full, symmetric** one-bounded Petri net . . .

Can you pick one of its infinite **concurrent** executions **uniformly at random**?

Define your preferred notion of trace length
Consider your favorite full, symmetric one-bounded Petri net . . .

Can you pick one of its infinite concurrent executions uniformly at random?

1. Define your preferred notion of trace length
2. Study uniform distributions on traces of length $k$
Petri nets and dependency graphs

Consider your favorite **full, symmetric** one-bounded Petri net . . .

![Petri net diagram]

Can you pick one of its infinite **concurrent** executions **uniformly at random**?

1. Define your preferred notion of trace length
2. Study uniform distributions on traces of length $k$
3. Look for suitable **convergence properties** when $k \to +\infty$

---

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5  Going beyond...
Heaps of pieces and trace monoids

Heap of pieces

- Pieces:
  - a
  - b
  - c
  - d

Trace monoid

- Alphabet:
  - $\Sigma = \{a, b, c, d\}$

Independence relation:

$\text{I} \subseteq \Sigma \times \Sigma$

$\text{Trace monoid:}$

$M \subseteq \Sigma \times I \times \Sigma$

$ac \rightarrow ca$

$bd \rightarrow db$
Heaps of pieces and trace monoids

**Heap of pieces**
- Pieces:
  - a
  - b
  - c
  - d

- Purely vertical heaps:
  - d
  - a
  - c
  - a
  - b
  - d
  - c
  - d
  - a
  - c
  - d
  - a

**Trace monoid**
- Alphabet:
  - $\Sigma = \{a, b, c, d\}$
- Free monoid:
  - $\Sigma^* = \{1, a, b, c, d, a^2, ab, ac, ad, ba, \ldots\}$
Heaps of pieces and trace monoids

Heap of pieces

- Pieces:
  - a
  - b
  - c
  - d

- Purely vertical heaps:
  - d
    - a
      - c
        - a
          - b

- Horizontal layout:
  - a
  - b
    - c
      - c

Trace monoid

- Alphabet:
  \[ \Sigma = \{a, b, c, d\} \]

- Free monoid:
  \[ \Sigma^* = \{1, a, b, c, d, a^2, ab, ac, ad, ba, \ldots\} \]

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Drawing heaps uniformly at random
Heaps of pieces and trace monoids

Heap of pieces

- Pieces:
  - $a$
  - $b$
  - $c$
  - $d$

- Vertical heaps:
  - $a$
  - $c$
  - $d$
  - $a$
  - $c$
  - $d$
  - $a$
  - $b$
  - $c$
  - $d$

- Horizontal layout:
  - $a$
  - $b$
  - $c$

Trace monoid

- Alphabet:
  - $\Sigma = \{a,b,c,d\}$

- Free monoid:
  - $\Sigma^* = \{1,a,b,c,d,a^2,ab,ac,ad,ba,...\}$

- Independence relation:
  - $I = \{(a,c),(c,a),(a,d),(d,a),(b,d),(d,b)\}$

- Trace monoid:
  - $\mathcal{M}(\Sigma, I) = \langle a,b,c,d | ac=ca, ad=da, bd=db \rangle^+$

Dependency graph

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Drawing heaps uniformly at random
Heaps of pieces and trace monoids

**Heap of pieces**

- **Pieces:**
  - \(a\)
  - \(b\)
  - \(c\)
  - \(d\)

- **Vertical heaps:**
  - \(a\)
  - \(c\)
  - \(d\)
  - \(a\)
  - \(b\)
  - \(c\)
  - \(d\)
  - \(a\)
  - \(c\)
  - \(b\)
  - \(d\)
  - \(c\)
  - \(d\)

- **Horizontal layout:**
  - \(a\)
  - \(b\)
  - \(c\)

**Trace monoid**

- **Alphabet:**
  \(\Sigma = \{a, b, c, d\}\)

- **Free monoid:**
  \(\Sigma^* = \{1, a, b, c, d, a^2, ab, ac, ad, ba, \ldots\}\)

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- **Trace monoid:**
  \(\mathcal{M}(\Sigma, I) = \langle a, b, c, d | ac = ca, ad = da, bd = db \rangle^+\)

- **Dependency graph**
  - Drawing heaps uniformly at random
Heaps of pieces and trace monoids

Heap of pieces

- Pieces: a b c d
- Vertical heaps:
  - a c
  - a d
  - a c
  - a c
- Horizontal layout:
  - a b c d
  - a c b d
  - a c b d
  - a c b d

Dimer monoid

- Alphabet:
  \[ \Sigma = \{a, b, c, d\} \]
- Free monoid:
  \[ \Sigma^* = \{1, a, b, c, d, a^2, ab, ac, ad, ba, \ldots\} \]
- Independence relation:
  \[ I = \{(a, c), (c, a), (a, d), (d, a), (b, d), (d, b)\} \]
- Dimer monoid:
  \[ \mathcal{M}(\Sigma, I) = \langle a, b, c, d | ac=ca, ad=da, bd=db \rangle^+ \]

Dependency graph

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Heaps of pieces viewed from their places

Petri net

Heap of pieces

- Vertical heaps of pieces:

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Heaps of pieces viewed from their places

**Petri net**

```
1 a
 b
2
3 c
 d
```

**Heap of pieces**

- **Vertical heaps of pieces:**
  - 1. `a c` 2. `b` 3. `a d` 4. `a c`
  - 1. `c` 2. `d`

- **Place views:**
  - 1. `a c c`
  - 2. `a b d`
  - 3. `a c c`
  - 1. `c c`
  - 2. `a d`
  - 3. `b b c`
Heaps of pieces viewed from their places

Petri net

Heap of pieces

- Vertical heaps of pieces:

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- Place views:

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Heap of pieces ↔ Consistent place views
Heaps of pieces viewed from their places

Petri net

Heap of pieces

- Vertical heaps of pieces:

- Place views:

Heap of pieces $\Leftrightarrow \textbf{Consistent} \text{ place views}$
Heaps of pieces viewed from their places

Petri net

Heap of pieces

- Vertical heaps of disconnected pieces:

- Place views:

Heap of pieces $\iff$ Consistent place views
Heaps of pieces viewed from their places

Petri net

Heap of pieces

- Vertical heaps of disconnected pieces:

- Place views:

Heap of pieces $\Leftrightarrow$ **Consistent** place views
Heaps of pieces and Cartier-Foata normal forms

Heap of pieces

- Vertical heaps of pieces:

  - Vertical heap 1:
    - Diagram 1:
    - Diagram 2:

  - Vertical heap 2:
    - Diagram 3:
    - Diagram 4:

  - Vertical heap 3:
    - Diagram 5:
    - Diagram 6:

- Horizontal heaps:

  - Horizontal heap 1:
    - Diagram 7:
    - Diagram 8:

- Local conditions on consecutive cliques in heaps
Heaps of pieces and Cartier-Foata normal forms

Heap of pieces

- Vertical heaps of pieces:
  - \( \begin{array}{c}
  a \\
  b \\
  a \\
  a \\
  \end{array} \quad \begin{array}{c}
  c \\
  d \\
  b \\
  a \\
  \end{array} \)

- Cartier-Foata factorisations:
  - \( \begin{array}{c}
  ac \\
  b \\
  ad \\
  ac \\
  \end{array} \quad \begin{array}{c}
  c \\
  d \\
  ac \\
  b \\
  \end{array} \)
Heaps of pieces and Cartier-Foata normal forms

Heap of pieces

- Vertical heaps of pieces:

```
  a  c
  b
  a  d
  a  c
```

- Cartier-Foata factorisations:

```
  ac
  b
  ad
  ac
```

Cliques \((C)\)

- Horizontal heaps:

```
  a
  c
  a  c
  b  d
```

```
  b
  c
  d
```

```
  a  d
```

Local conditions on consecutive cliques in heaps

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Drawing heaps uniformly at random
Heaps of pieces and Cartier-Foata normal forms

Heap of pieces

- Vertical heaps of pieces:

```
   a   c
   b
  a   d
 a   c
```

- Cartier-Foata factorisations:

```
ac
b
ad
ac
```

Clique \((C)\)

- Horizontal heaps:

```
a
c
ac
d
```

- Local conditions on consecutive cliques in heaps:

```
b
d
```

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Heaps of pieces and left divisibility

Heap of pieces

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Drawing heaps uniformly at random
Heaps of pieces and left divisibility

Heap of pieces

Place views

Heaps of pieces and left divisibility

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Drawing heaps uniformly at random
Heaps of pieces and left divisibility

Heap of pieces

Place views

Cartier-Foata

+ upper commutativity

(bd ∈ C)

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Drawing heaps uniformly at random
Heaps of pieces and left divisibility

Heap of pieces

Place views

Cartier-Foata

Combinatorial properties

+ upper commutativity

(\(bd \in C\))

\(a \wedge b \) (and \(a \vee b\)) exist

\(h(a) \leq k \iff a \in C^k\)

maximality criterion:
Heaps of pieces and left divisibility

Heap of pieces

Place views

Cartier-Foata

Combinatorial properties

+ upper commutativity

(bd ∈ C)

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Drawing heaps uniformly at random
Heaps of pieces and left divisibility

Heap of pieces

Place views

Cartier-Foata

Combinatorial properties

- $a \wedge b$ (and $a \vee b$) exist
- $h(a) \leq k \Leftrightarrow a \in C^k$
- maximality criterion:

$C^k(a)$
Contents

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Probabilistic and topological setting

Probabilistic setting

Two notions of length:

1. # pieces: $|a|$
2. # floors: $h(a)$

$M_k = \{\text{heaps of size } k\}$

$\approx \text{regular language } \subseteq \Sigma^*$
Probabilistic and topological setting

**Probabilistic setting**

Two notions of length:

1. \(#\) pieces: \(|a|\)
2. \(#\) floors: \(h(a)\)

\(M_k = \{\text{heaps of size } k\}\)

\(\approx \text{regular language } \subseteq \Sigma^*\)

**Topological setting**

\(\mu_k \rightarrow \mu_\infty \iff \mathbb{P}_{\mu_k}[a \leq x] \rightarrow \mathbb{P}_{\mu_\infty}[a \leq x]\)
Probabilistic and topological setting

Probabilistic setting

Two notions of length:

1. \# pieces: \(|a|\)
2. \# floors: \(h(a)\)

\(M_k = \{\text{heaps of size } k\}\)

\(\approx \text{regular language } \subseteq \Sigma^*\)

Topological setting

\[\mu_k \rightarrow \mu_\infty \iff P_{\mu_k}[a \leq x] \rightarrow P_{\mu_\infty}[a \leq x]\]

\[\iff \mu_k(\uparrow a) \rightarrow \mu_\infty(\uparrow a)\]

\[\text{with } \uparrow a = \{x : a \leq x\}\]
Probabilistic and topological setting

**Probabilistic setting**

Two notions of length:

1. # pieces: $|a|$
2. # floors: $h(a)$

$M_k = \{\text{heaps of size } k\}$

$\approx$ regular language $\subseteq \Sigma^*$

**Topological setting**

$\mu_k \longrightarrow \mu_\infty \iff P_{\mu_k}[a \leq x] \rightarrow P_{\mu_\infty}[a \leq x]$

$\iff \mu_k(\uparrow a) \rightarrow \mu_\infty(\uparrow a)$

with $\uparrow a = \{x : a \leq x\}$

- Embed $M^+$ with the topology $\{\uparrow a\}$
- Make $M^+$ complete

**Theorem (S. Abbes & J. Mairesse 2015)**

The uniform distribution on $M_k$ converges weakly in $M^+$ when $k \to \infty$.
Probabilistic and topological setting

**Probabilistic setting**

Two notions of length:

1. \# pieces: \(|a|\)
2. \# floors: \(h(a)\)

\(M_k = \{\text{heaps of size } k\}\)

\(\approx\) regular language \(\subseteq \Sigma^*\)

**Topological setting**

\(\mu_k \xrightarrow{w} \mu_\infty \Leftrightarrow P_{\mu_k}[a \leq x] \rightarrow P_{\mu_\infty}[a \leq x]\)

\(\Leftrightarrow \mu_k(\uparrow a) \rightarrow \mu_\infty(\uparrow a)\)

with \(\uparrow a = \{x : a \leq x\}\)

- Embed \(M^+\) with the topology \(\{\uparrow a\}\)
- Make \(M^+\) complete

Theorem (S. Abbes & J. Mairesse 2015)

The uniform distribution on \(M_k\) converges weakly in \(M^+\) when \(k \rightarrow \infty\)

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Drawing heaps uniformly at random
Probabilistic and topological setting

**Probabilistic setting**

Two notions of length:

1. # pieces: $|a|$  
2. # floors: $h(a)$

$\mathcal{M}_k = \{\text{heaps of size } k\}$  
$\approx$ regular language $\subseteq \Sigma^*$

**Topological setting**

$\mu_k \xrightarrow{w} \mu_\infty \iff \mathbb{P}_{\mu_k}[a \leq x] \to \mathbb{P}_{\mu_\infty}[a \leq x]$  
$\iff \mu_k(\uparrow a) \to \mu_\infty(\uparrow a)$

with $\uparrow a = \{x : a \leq x\}$

- Embed $\mathcal{M}^+$ with the topology $\{\uparrow a\}$
- Make $\mathcal{M}^+$ complete

**Theorem (S. Abbes & J. Mairesse 2015)**

The uniform distribution on $\mathcal{M}_k$ converges weakly in $\overline{\mathcal{M}^+}$ when $k \to +\infty$
Weak convergence: \( \text{length}(\mathbf{a}) = |\mathbf{a}| \)

Generating series and Möbius polynomial

\[
G(z) = \sum_{\alpha \in \mathcal{M}^+} z^{|\alpha|} = \sum_{k \geq 0} \lambda_k z^k \quad \text{and} \quad H(z) = \sum_{\gamma \in \mathcal{C}} (-z)^{|\gamma|}
\]

Proposition (P. Cartier & D. Foata 1969)

\[
G(z)H(z) = 1
\]
Weak convergence: \( \text{length}(a) = |a| \)

Generating series and Möbius polynomial

\[ G(z) = \sum_{\alpha \in \mathcal{M}^+} z^{\alpha} = \sum_{k \geq 0} \lambda_k z^k \quad \text{and} \quad H(z) = \sum_{\gamma \in \mathcal{C}} (-z)^{|\gamma|} \]

**Proposition (P. Cartier & D. Foata 1969)**

\[ G(z)H(z) = 1 \]

**Proof**

\[ G(z)H(z) = \sum_{\alpha \in \mathcal{M}^+} z^{\alpha} \cdot \sum_{\gamma \in \mathcal{C}} (-z)^{|\gamma|} \]
Weak convergence: $\text{length}(a) = |a|$

Generating series and Möbius polynomial

$$G(z) = \sum_{\alpha \in \mathcal{M}^+} z^{\alpha} = \sum_{k \geq 0} \lambda_k z^k \quad \text{and} \quad H(z) = \sum_{\gamma \in \mathcal{C}} (-z)^{\gamma}$$

**Proposition (P. Cartier & D. Foata 1969)**

$$G(z)H(z) = 1$$

**Proof**

$$G(z)H(z) = \sum_{\alpha \in \mathcal{M}^+} \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} z^{\gamma} a^{\alpha}$$
Weak convergence: $\text{length}(a) = |a|$

Generating series and Möbius polynomial

$$G(z) = \sum_{\alpha \in \mathcal{M}^+} z^{|\alpha|} = \sum_{k \geq 0} \lambda_k z^k \quad \text{and} \quad H(z) = \sum_{\gamma \in \mathcal{C}} (-z)^{|\gamma|}$$

**Proposition (P. Cartier & D. Foata 1969)**

$$G(z)H(z) = 1$$

**Proof**

$$G(z)H(z) = \sum_{\alpha \in \mathcal{M}^+} \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} z^{|\gamma\alpha|}$$

$$= \sum_{\theta \in \mathcal{M}^+} \sum_{\gamma \in \mathcal{C}} 1_{\gamma \leq \theta} (-1)^{|\gamma|} z^{|\theta|}$$

where $\theta = \gamma\alpha$
Weak convergence: \( \text{length}(a) = |a| \)

Generating series and Möbius polynomial

\[
G(z) = \sum_{\alpha \in \mathcal{M}^+} z^{\left|\alpha\right|} = \sum_{k \geq 0} \lambda_k z^k \quad \text{and} \quad H(z) = \sum_{\gamma \in \mathcal{C}} (-z)^{|\gamma|}
\]

Proposition (P. Cartier & D. Foata 1969)

\[ G(z)H(z) = 1 \]

Proof

\[
G(z)H(z) = \sum_{\alpha \in \mathcal{M}^+} \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} z^{|\gamma\alpha|} \\
= \sum_{\theta \in \mathcal{M}^+} \sum_{\gamma \in \mathcal{C}} \mathbf{1}_{\gamma \leq \theta} (-1)^{|\gamma|} z^{|\theta|} \\
= \sum_{\theta \in \mathcal{M}^+} z^{|\theta|} \sum_{S \subseteq L(\theta)} (-1)^{|S|}
\]

where \( \theta = \gamma \alpha \), \( L(\theta) = \{ x \in \Sigma : x \leq \theta \} \) and \( \gamma = \bigvee S \)
Weak convergence: $\text{length}(a) = |a|$

Generating series and Möbius polynomial

$$G(z) = \sum_{\alpha \in \mathcal{M}^+} z^{\lvert \alpha \rvert} = \sum_{k \geq 0} \lambda_k z^k \text{ and } H(z) = \sum_{\gamma \in \mathcal{C}} (-z)^{\lvert \gamma \rvert}$$

Proposition (P. Cartier & D. Foata 1969)

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Proof

$$G(z)H(z) = \sum_{\alpha \in \mathcal{M}^+} \sum_{\gamma \in \mathcal{C}} (-1)^{\lvert \gamma \rvert} z^{\lvert \gamma \alpha \rvert}$$

$$= \sum_{\theta \in \mathcal{M}^+} \sum_{\gamma \in \mathcal{C}} 1_{\gamma \leq \theta} (-1)^{\lvert \gamma \rvert} z^{\lvert \theta \rvert}$$

$$= \sum_{\theta \in \mathcal{M}^+} z^{\lvert \theta \rvert} 1_{\mathcal{L}(\theta) = \emptyset}$$

where $\theta = \gamma \alpha$, $\mathcal{L}(\theta) = \{x \in \Sigma : x \leq \theta\}$ and $\gamma = \bigvee S$
Weak convergence: $\text{length}(a) = |a|$

Generating series and Möbius polynomial

$$G(z) = \sum_{\alpha \in \mathcal{M}} z^{\alpha} = \sum_{k \geq 0} \lambda_k z^k \quad \text{and} \quad H(z) = \sum_{\gamma \in \mathcal{C}} (-z)^{|\gamma|}$$

**Proposition (P. Cartier & D. Foata 1969)**

$$G(z)H(z) = 1$$

**Proof**

$$G(z)H(z) = \sum_{\alpha \in \mathcal{M}} \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} z^{\gamma \alpha}$$

$$= \sum_{\theta \in \mathcal{M}} \sum_{\gamma \in \mathcal{C}} 1_{\gamma \leq \theta} (-1)^{|\gamma|} z^{\theta} 1$$

$$= \sum_{\theta \in \mathcal{M}} z^{\theta} 1$$

where $\theta = \gamma \alpha$, $L(\theta) = \{x \in \Sigma : x \leq \theta\}$ and $\gamma = \bigvee S$
Weak convergence: $\text{length}(a) = |a|$

Generating series and Möbius polynomial

\[ G(z) = \sum_{\alpha \in \mathcal{M}^+} z^{\alpha|} = \sum_{k \geq 0} \lambda_k z^k \text{ and } H(z) = \sum_{\gamma \in \mathcal{C}} (-z)^{|\gamma|} \]

**Proposition (P. Cartier & D. Foata 1969)**

\[ G(z)H(z) = 1 \]

**Proof**

\[
G(z)H(z) = \sum_{\alpha \in \mathcal{M}^+} \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} z^{\gamma a} = \sum_{\theta \in \mathcal{M}^+} \sum_{\gamma \in \mathcal{C}} 1_{\gamma \leq \theta} (-1)^{|\gamma|} z^{\theta} = \sum_{\theta \in \mathcal{M}^+} z^{\theta} 1_{\theta = 1} = 1
\]

where $\theta = \gamma \alpha$, $L(\theta) = \{x \in \Sigma : x \leq \theta\}$ and $\gamma = \bigvee S$
Weak convergence: \( \text{length}(a) = |a| \)

Generating series and Möbius polynomial

\[
G(z) = \sum_{\alpha \in \mathcal{M}^+} z^{\alpha|} = \sum_{k \geq 0} \lambda_k z^k \quad \text{and} \quad H(z) = \sum_{\gamma \in \mathcal{C}} (-z)^{\gamma|}
\]

Proposition (P. Cartier & D. Foata 1969)

\[
G(z)H(z) = 1
\]

Corollary (D. Krob, J. Mairesse & I. Michos 2001)

\( H(z) \) has a smallest positive root \( p \) such that:

- \( (H(z) = 0 \land |z| \leq p) \iff z = p \)
- \( 0 < p \leq 1 \)

and there exists constants \( \Lambda > 0 \) and \( \ell \in \mathbb{N} \) such that \( \lambda_k \sim \Lambda p^{-k} k^\ell \)
Weak convergence: \( \text{length}(a) = |a| \)

**Proof of the theorem – length \( (a) = |a| \)**

1. \( \mu_k : S \rightarrow \frac{\#(S \cap M_k)}{\lambda_k} \)
Weak convergence: $\text{length}(a) = |a|$

Proof of the theorem – $\text{length}(a) = |a|$

1. $\mu_k : S \mapsto \frac{\#(S \cap M_k)}{\lambda_k}$
2. $x \mapsto ax$ maps $M_k$ to $(\uparrow a) \cap M_{k+|a|}$ bijectively
Weak convergence: \( \text{length}(a) = |a| \)

**Proof of the theorem – length(\(a\)) = |a|**

1. \( \mu_k : S \mapsto \frac{\#(S \cap M_k)}{\lambda_k} \)
2. \( x \mapsto ax \text{ maps } M_k \text{ to (} \uparrow a \text{)} \cap M_{k+|a|} \) bijectively
3. \( \mu_k(\uparrow a) = \frac{\lambda_k - |a|}{\lambda_k} \rightarrow p^{-|a|} \)
Weak convergence: $\text{length}(a) = |a|$

Proof of the theorem – $\text{length}(a) = |a|$

1. $\mu_k : S \mapsto \frac{\#(S \cap M_k)}{\lambda_k}$
2. $x \mapsto ax$ maps $M_k$ to $(\uparrow a) \cap M_{k+|a|}$ bijectively
3. $\mu_k(\uparrow a) = \frac{\lambda_k - |a|}{\lambda_k} \to p^{-|a|}$
4. $M^+$ is compact and $\emptyset \cup \{\uparrow a\}$ is closed under $\cap$
Weak convergence: \( \text{length}(a) = |a| \) and \( \text{length}(a) = h(a) \)

**Proof of the theorem – \( \text{length}(a) = |a| \)**

1. \( \mu_k : S \mapsto \frac{\#(S \cap M_k)}{\lambda_k} \)
2. \( x \mapsto ax \) maps \( M_k \) to \((\uparrow a) \cap M_{k+|a|} \) bijectively
3. \( \mu_k(\uparrow a) = \frac{\lambda_{k-|a|}}{\lambda_k} \to p^{-|a|} \)
4. \( M^+ \) is compact and \( \emptyset \cup \{ \uparrow a \} \) is closed under \( \cap \)

**Proof of the theorem – \( \text{length}(a) = h(a) \)**

5. Split \( \uparrow a \) into sets \( M^+(b) = \{ x : b = C^{h(a)}(x) \} \)
Weak convergence: \( \text{length}(a) = |a| \) and \( \text{length}(a) = h(a) \)

\[ \text{Proof of the theorem} - \text{length}(a) = |a| \]

1. \( \mu_k : S \mapsto \frac{\#(S \cap M_k)}{\lambda_k} \)
2. \( x \mapsto ax \) maps \( M_k \) to \( (\uparrow a) \cap M_{k+|a|} \) bijectively
3. \( \mu_k(\uparrow a) = \frac{\lambda_k - |a|}{\lambda_k} \rightarrow p^{-|a|} \)
4. \( M^+ \) is compact and \( \{\emptyset\} \cup \{\uparrow a\} \) is closed under \( \cap \)

\[ \text{Proof of the theorem} - \text{length}(a) = h(a) \]

5. Split \( \uparrow a \) into sets \( M^+(b) = \{x : b = C^{h(a)}(x)\} \)
6. Prove that \( \#(M^+(b) \cap M_k) \sim \Lambda_b q_b^k k^{\ell_b} \) for some \( \Lambda_b, q_b \) and \( \ell_b \)
Weak convergence: $\text{length}(a) = |a|$ and $\text{length}(a) = h(a)$

**Proof of the theorem – $\text{length}(a) = |a|$**

1. $\mu_k : S \mapsto \frac{\#(S \cap M_k)}{\lambda_k}$
2. $x \mapsto ax$ maps $M_k$ to $(\uparrow a) \cap M_{k+|a|}$ bijectively
3. $\mu_k(\uparrow a) = \frac{\lambda_k - |a|}{\lambda_k} \rightarrow p^{-|a|}$
4. $M^+$ is compact and $\{\emptyset\} \cup \{\uparrow a\}$ is closed under $\cap$

**Proof of the theorem – $\text{length}(a) = h(a)$**

5. Split $\uparrow a$ into sets $M^+(b) = \{x : b = C^{h(a)}(x)\}$
6. Prove that $\#(M^+(b) \cap M_k) \sim \Lambda_b q_b^k k^{\ell_b}$ for some $\Lambda_b$, $q_b$ and $\ell_b$
7. Complete the proof as above

Caution: $\lim \mu_k(\uparrow a)$ does not depend only on $h(a)$!
Contents

1 Introduction

2 Trace monoids and heaps

3 First convergence results

4 Bernoulli distributions

5 Going beyond...
Bernoulli distributions

A distribution $\mu$ on $M^+$ is . . .

Bernoulli if

- $\mu(\uparrow ab) = \mu(\uparrow a)\mu(\uparrow b)$
Bernoulli distributions

A distribution $\mu$ on $\mathcal{M}^+$ is ...

Bernoulli if

- $\mu(\uparrow ab) = \mu(\uparrow a)\mu(\uparrow b)$
- $\mu(\uparrow a_1a_2\ldots a_k) = \nu_{a_1}\nu_{a_2}\ldots\nu_{a_k}$
Bernoulli distributions

A distribution $\mu$ on $\mathcal{M}^+$ is . . .

**Bernoulli** if

- $\mu(\uparrow ab) = \mu(\uparrow a)\mu(\uparrow b)$
- $\mu(\uparrow a_1a_2\ldots a_k) = \nu_{a_1}\nu_{a_2}\ldots\nu_{a_k}$

**Uniform Bernoulli** if

- $\mu(\uparrow a) = \nu_{|a|}$
- $\nu_1 = \nu_2 = \ldots = \nu$
Bernoulli distributions

A distribution $\mu$ on $M^+$ is . . .

**Bernoulli** if

- $\mu(\uparrow ab) = \mu(\uparrow a) \mu(\uparrow b)$
- $\mu(\uparrow a_1a_2\ldots a_k) = \nu_{a_1}\nu_{a_2}\ldots \nu_{a_k}$

**Uniform Bernoulli** with parameter $\nu$ if

- $\mu(\uparrow a) = \nu^{|a|}$
- $\nu_1 = \nu_2 = \ldots = \nu$
Bernoulli distributions

A distribution $\mu$ on $\mathcal{M}^+$ is . . .

**Bernoulli** if

- $\mu(\uparrow ab) = \mu(\uparrow a)\mu(\uparrow b)$
- $\mu(\uparrow a_1 a_2 \ldots a_k) = \nu_1 \nu_2 \ldots \nu_k$

**Uniform Bernoulli** with parameter $\nu$ if

- $\mu(\uparrow a) = \nu^{|a|}$
- $\nu_1 = \nu_2 = \ldots = \nu$

**Finite uniform Bernoulli** if

- $\nu < p$
- $\mathcal{H}(z) > 0$ for all $z \in (0, p)$
Bernoulli distributions

A distribution $\mu$ on $\mathcal{M}^+$ is . . .

**Bernoulli if**

- $\mu(\uparrow ab) = \mu(\uparrow a)\mu(\uparrow b)$
- $\mu(\uparrow a_1a_2 \ldots a_k) = \nu_1\nu_2 \ldots \nu_k$

**Uniform Bernoulli** with parameter $\nu$ if

- $\mu(\uparrow a) = \nu^{|a|}$
- $\nu_1 = \nu_2 = \ldots = \nu$

**Finite uniform Bernoulli** if

- $\nu < p$
- $\mathcal{H}(z) > 0$ for all $z \in (0, p)$
- $\mu(\partial \mathcal{M}^+) = 0$
- $\mu(\{a\}) = \mathcal{H}(\nu)\nu^{|a|}$
Bernoulli distributions

Proving that finite uniform \( \Leftrightarrow \mu(\{x\}) = H(\nu)\nu^{|x|} \)
Bernoulli distributions

Proving that finite uniform $\iff \mu(\{x\}) = \mathcal{H}(\nu)\nu^{|x|}$

$\iff \mu(\uparrow x) = \mathcal{H}(\nu) \sum_\gamma \nu^{|x \gamma|} = \nu^{|x|} \mathcal{H}(\nu) \sum_\gamma \nu^{|\gamma|} = \nu^{|x|} \mathcal{H}(\nu) G(\nu)$
Bernoulli distributions

Proving that finite uniform $\iff \mu(\{x\}) = \mathcal{H}(\nu)\nu^{\lvert x \rvert}$

$\iff \mu(\uparrow x) = \mathcal{H}(\nu) \sum_{\gamma} \nu^{\lvert x \gamma \rvert} = \nu^{\lvert x \rvert} \mathcal{H}(\nu) \sum_{\gamma} \nu^{\lvert \gamma \rvert} = \nu^{\lvert x \rvert} \mathcal{H}(\nu) G(\nu)$

$\implies$ Proof #1: At most one measure works!
Bernoulli distributions

Proving that finite uniform $\Leftrightarrow \mu(\{x\}) = \mathcal{H}(\nu)\nu^{|x|}$

$\Leftarrow \mu(\uparrow x) = \mathcal{H}(\nu) \sum_{\gamma} \nu^{|x\gamma|} = \nu^{|x|} \mathcal{H}(\nu) \sum_{\gamma} \nu^{|\gamma|} = \nu^{|x|} \mathcal{H}(\nu) \mathcal{G}(\nu)$

$\Rightarrow$ Proof #1: At most one measure works!

$\Rightarrow$ Proof #2: Using inclusion-exclusion:

$\mu(\{x\}) = $
Bernoulli distributions

Proving that finite uniform $\iff \mu(\{x\}) = \mathcal{H}(\nu)\nu^{|x|}$

$\iff \mu(\uparrow x) = \mathcal{H}(\nu) \sum_\gamma \nu^{|x\gamma|} = \nu^{|x|} \mathcal{H}(\nu) \sum_\gamma \nu^{|\gamma|} = \nu^{|x|} \mathcal{H}(\nu) \mathcal{G}(\nu)$

$\implies$ Proof #1: At most one measure works!

$\implies$ Proof #2: Using inclusion-exclusion:

$\mu(\{x\}) = \nu(\uparrow x)$
Bernoulli distributions

Proving that finite uniform ⇔ \( \mu(\{x\}) = \mathcal{H}(\nu)\nu^{|x|} \)

\[ \mu(\uparrow x) = \mathcal{H}(\nu) \sum_\gamma \nu^{|x\gamma|} = \nu^{|x|} \mathcal{H}(\nu) \sum_\gamma \nu^{|\gamma|} = \nu^{|x|} \mathcal{H}(\nu) \mathcal{G}(\nu) \]

⇒ Proof #1: At most one measure works!
⇒ Proof #2: Using inclusion-exclusion:
\[ \mu(\{x\}) = \nu(\uparrow x) - \nu(\uparrow xa) - \nu(\uparrow xb) - \nu(\uparrow xc) - \nu(\uparrow xd) \]
Proving that finite uniform $\leftrightarrow \mu(\{x\}) = \mathcal{H}(\nu)\nu^{|x|}$

$\Leftrightarrow \mu(\uparrow x) = \mathcal{H}(\nu)\sum_\gamma \nu^{|x\gamma|} = \nu^{|x|}\mathcal{H}(\nu)\sum_\gamma \nu^{|\gamma|} = \nu^{|x|}\mathcal{H}(\nu)\mathcal{G}(\nu)$

$\implies$ Proof #1: At most one measure works!

$\implies$ Proof #2: Using inclusion-exclusion:

$$
\mu(\{x\}) = \nu(\uparrow x) - \nu(\uparrow xa) - \nu(\uparrow xb) - \nu(\uparrow xc) - \nu(\uparrow xd) + \\
\nu(\uparrow xac) + \nu(\uparrow xad) + \nu(\uparrow xbd)
$$
Bernoulli distributions

Proving that finite uniform $\iff \mu(\{x\}) = \mathcal{H}(\nu)\nu^{\|x\|}$

$\iff \mu(\uparrow x) = \mathcal{H}(\nu) \sum_\gamma \nu^{\|x\gamma\|} = \nu^{\|x\|} \mathcal{H}(\nu) \sum_\gamma \nu^{\|\gamma\|} = \nu^{\|x\|} \mathcal{H}(\nu) \mathcal{G}(\nu)$

$\implies$ Proof #1: At most one measure works!

$\implies$ Proof #2: Using inclusion-exclusion:

$$\mu(\{x\}) = \nu^{\|x\|} (1 - \nu^{|a|} - \nu^{|b|} - \nu^{|c|} - \nu^{|d|} + \nu^{|ac|} + \nu^{|ad|} + \nu^{|bd|})$$
Bernoulli distributions

Proving that finite uniform $\iff \mu(\{x\}) = \mathcal{H}(\nu)\nu^{|x|}$

$\iff \mu(\uparrow x) = \mathcal{H}(\nu) \sum_{\gamma} \nu^{|x\gamma|} = \nu^{|x|}\mathcal{H}(\nu) \sum_{\gamma} \nu^{|\gamma|} = \nu^{|x|}\mathcal{H}(\nu) G(\nu)$

$\Rightarrow$ Proof #1: At most one measure works!

$\Rightarrow$ Proof #2: Using inclusion-exclusion:

$\mu(\{x\}) = \nu^{|x|}(1 - \nu^{|a|} - \nu^{|b|} - \nu^{|c|} - \nu^{|d|} + \nu^{|ac|} + \nu^{|ad|} + \nu^{|bd|}) = \nu^{|x|}\mathcal{H}(\nu)$
Simulating finite, uniform Bernoulli distributions

Approach #1: Pick the length first

Pick a target length \( k \) with probability \( \lambda \) \( k \lambda^{k} \)

Pick a trace uniformly at random in \( t \) a \( PM`||a`k \)

Approach #2: Pick the ground floor first (Markov chain)

Order the generators from \( g_1 \) to \( g_n \) and choose whether \( g_i \in \mathcal{a} \) (based on \( t, g_j | 1 \leq j < i, g_j \in \mathcal{a} \) and on the previous floor)

Pick the upper floors recursively

Drawing heaps uniformly at random
Simulating finite, uniform Bernoulli distributions

Approach #1: Pick the length first

- Pick a **target length** $k$ with probability $\lambda_k \nu^k \mathcal{H}(\nu)$
- Pick a trace uniformly at random in $\{a \in \mathcal{M}^+ \mid |a| = k\}$
Simulating finite, uniform Bernoulli distributions

Approach #1: Pick the length first

- Pick a **target length** $k$ with probability $\lambda_k \nu^k \mathcal{H}(\nu)$
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Approach #2: Pick the ground floor first

- Order the generators from $g_1$ to $g_n$ and choose whether $g_i \leq a$
- Pick the upper floors recursively
Simulating finite, uniform Bernoulli distributions

Approach #1: Pick the length first

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- Order the generators from $g_1$ to $g_n$ and choose whether $g_i \leq a$
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Simulating Bernoulli distributions

Approach #1: Pick the length first

- Pick a **target length** \( k \) with probability \( \lambda_k \nu^k \mathcal{H}(\nu) \)
- Pick a trace uniformly at random in \( \{ a \in \mathcal{M}^+ | |a| = k \} \)

Approach #2: Pick the ground floor first (Markov chain)

- Order the generators from \( g_1 \) to \( g_n \) and choose whether \( g_i \leq a \) (based on \( \{ g_j | 1 \leq j < i, g_j \leq a \} \) and on the previous floor)
- Pick the upper floors recursively
Finite uniform Bernoulli distributions as Markov chains

Monoid cylinder

\[ \uparrow a = \{ b \in \mathcal{M}^+ \mid a \preceq b \} \]

Cartier-Foata cylinder

\[ \uparrow a = \{ b \in \mathcal{M}^+ \mid a = \mathcal{C}^{h(a)}(b) \} \]

Drawing heaps uniformly at random
Finite uniform Bernoulli distributions as Markov chains

Monoid cylinder

\[ \uparrow a = \{ b \in \mathcal{M}^+ \mid a \leq b \} \]

Cartier-Foata cylinder

\[ \uparrow a = \{ b \in \mathcal{M}^+ \mid a = C^{h(a)}(b) \} \]

Möbius inversion formula and Markov simulation

\[ \uparrow a = \bigcup_{a \leq b, h(a) = h(b)} \uparrow b \]
Finite uniform Bernoulli distributions as Markov chains

**Monoid cylinder**

\[ \uparrow a = \{ b \in \mathcal{M}^+ \mid a \leq b \} \]

**Cartier-Foata cylinder**

\[ \uparrow a = \{ b \in \mathcal{M}^+ \mid a = C^{h(a)}(b) \} \]

Möbius inversion formula and Markov simulation

\[ \uparrow a = \bigcup_{a \leq b, h(a) = h(b)} \uparrow b \]

\[ \nu|a| = \sum \mu(\uparrow b)1_{a \leq b, h(a) = h(b)} \]

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S. Abbes, S. Gouëzel, V. Jugé & J. Mairesse

Drawing heaps uniformly at random
Finite uniform Bernoulli distributions as Markov chains

Monoid cylinder

\[ \Uparrow a = \{ b \in \mathcal{M}^+ \mid a \leq b \} \]

Cartier-Foata cylinder

\[ \Uparrow a = \{ b \in \mathcal{M}^+ \mid a = C^h(a)(b) \} \]

Möbius inversion formula and Markov simulation

\[ \Uparrow a = \bigcup_{a \leq b, h(a) = h(b)} \Uparrow b \]

\[ \nu^{\mid a \mid} = \sum \mu(\Uparrow b) 1_{a \leq b, h(a) = h(b)} \]

\[ \mu(\Uparrow a) = \sum_{\gamma \in \mathcal{C}} (-1)^{\mid \gamma \mid} 1_{h(a) = h(a\gamma)} \nu^{\mid a\gamma \mid} \]
Finite uniform Bernoulli distributions as Markov chains

Monoid cylinder

\[ \uparrow a = \{ b \in \mathcal{M}^+ \mid a \leq b \} \]

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Möbius inversion formula and Markov simulation

\[ \uparrow a = \bigcup_{a \leq b, h(a) = h(b)} \uparrow b \]

\[ \nu^{|a|} = \sum \mu(\uparrow b) 1_{a \leq b, h(a) = h(b)} \]

\[ \mu(\uparrow a) = \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} 1_{h(a) = h(\gamma a)} \nu^{|a\gamma|} \]

\[ \mu(\uparrow a) = \nu^{|a|} \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} 1_{h(a) = h(\gamma a)} \nu^{|\gamma|} \]
Finite uniform Bernoulli distributions as Markov chains

**Monoid cylinder**

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**Möbius inversion formula and Markov simulation**

\[ \nu^{|a|} = \sum \mu(\uparrow b) 1_{a \leq b, h(a) = h(b)} \]

\[ \mu(\uparrow a) = \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} 1_{h(a) = h(a\gamma)} \nu^{|a\gamma|} \]

\[ = \nu^{|a|} \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} 1_{h(a) = h(a\gamma)} \nu^{|\gamma|} \]

\[ = \nu^{|a|} \mathcal{H}_a(\nu) \]
Finite uniform Bernoulli distributions as Markov chains

Monoid cylinder

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Möbius inversion formula and Markov simulation

\[ \uparrow a = \bigcup_{a \leq b, h(a) = h(b)} \uparrow b \]

\[ \nu^{|a|} = \sum \mu(\uparrow b) \mathbf{1}_{a \leq b, h(a) = h(b)} \]

\[ \mu(\uparrow a) = \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} \mathbf{1}_{h(a) = h(a\gamma)} \nu^{|a\gamma|} \]

\[ = \nu^{|a|} \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} \mathbf{1}_{h(a) = h(a\gamma)} \nu^{|\gamma|} \]

\[ = \nu^{|a|} H_{a_h(a)}(\nu) \]
Finite uniform Bernoulli distributions as Markov chains

Monoid cylinder
\[ \uparrow a = \{ b \in \mathcal{M}^+ \mid a \leq b \} \]

Cartier-Foata cylinder
\[ \uparrow a = \{ b \in \mathcal{M}^+ \mid a = C^h(a)(b) \} \]

Möbius inversion formula and Markov simulation
\[ \mu(\uparrow a) = \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} \mathbf{1}_{h(a) = h(a\gamma)} \nu^{a\gamma} \]
\[ = \nu^{a} \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} \mathbf{1}_{h(a) = h(a\gamma)} \nu^{\gamma} \]
\[ = \nu^{a} \mathcal{H}_{a\gamma}(\nu) \]
\[ = \mathbb{P}[\Theta^{\nu}_1 = a_1, \ldots, \Theta^{\nu}_{h(a)} = a_{h(a)}] \]

\[ \mathbb{P}[\Theta^{\nu}_1 = a] = \nu^{a} \mathcal{H}_a(\nu) \]
\[ \mathbb{P}[\Theta^{\nu}_{i+1} = b \mid \Theta^{\nu}_i = a] = \nu^{b} \frac{\mathcal{H}_b(\nu)}{\mathcal{H}_a(\nu)} \mathbf{1}_{a \rightarrow b} \]

\[ a \rightarrow b \iff \begin{array}{c} b \\ \hline a \end{array} \iff a = C^1(ab) \]
Infinite uniform Bernoulli distributions as Markov chains

Critical parameter: $\nu = p$
Critical parameter: $\nu = p$

Convergence of $(\Theta_i^\nu)$ when $\nu \to p$, with limit

\[
\mathbb{P}[\Theta_1^p = a] = p^{|a|} \mathcal{H}_a(p)
\]
\[
\mathbb{P}[\Theta_{i+1}^p = b \mid \Theta_i^p = a] = p^{|b|} \frac{\mathcal{H}_b(p)}{\mathcal{H}_a(p)} 1_{a \to b} 1_{\mathcal{H}_a(p) \neq 0}
\]
Infinite uniform Bernoulli distributions as Markov chains

Critical parameter: $\nu = p$

Convergence of $(\Theta_i^\nu)$ when $\nu \to p$, with limit

$$P[\Theta_1^p = a] = p|a|^\mathcal{H}_a(p)$$
$$P[\Theta_i^p = b \mid \Theta_i^p = a] = p|b|^\frac{\mathcal{H}_b(p)}{\mathcal{H}_a(p)}1_{a \to b}1_{\mathcal{H}_a(p) \neq 0}$$

Trivial supercritical parameter: $\nu = 1$

Possible only if $\mathcal{M}^+ = \mathbb{N}^n$ (i.e. $p = 1$)
Infinite uniform Bernoulli distributions as Markov chains

Critical parameter: $\nu = p$

Convergence of $(\Theta_i^{\nu})$ when $\nu \to p$, with limit

$$\mathbb{P}[\Theta_1^p = a] = p^{a|}\mathcal{H}_a(p)$$
$$\mathbb{P}[\Theta_{i+1}^p = b \mid \Theta_i^p = a] = p^{b|}\frac{\mathcal{H}_b(p)}{\mathcal{H}_a(p)}\mathbf{1}_{a\to b}\mathbb{1}_{\mathcal{H}_a(p) \neq 0}$$

Trivial supercritical parameter: $\nu = 1$

Possible only if $\mathcal{M}^+ = \mathbb{N}^n$ (i.e. $p = 1$)...

Non-trivial supercritical parameter: $p < \nu < 1$

No such distribution exists!
Infinite uniform Bernoulli distributions as Markov chains

**Critical parameter:** $\nu = p$

**Convergence** of $(\Theta_i^\nu)$ when $\nu \to p$, with limit

$$
\mathbb{P}[\Theta_1^p = a] = p^{|a|} \mathcal{H}_a(p) \\
\mathbb{P}[\Theta_{i+1}^p = b \mid \Theta_i^p = a] = p^{|b|} \frac{\mathcal{H}_b(p)}{\mathcal{H}_a(p)} 1_{a \to b} 1_{\mathcal{H}_a(p) \neq 0}
$$

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- Consider the **Garside matrix** $M^\nu$ with $M^\nu_{a,b} = 1_{a \to b} \nu^{|b|}$
Infinite uniform Bernoulli distributions as Markov chains

Critical parameter: $\nu = p$

Convergence of $(\Theta_i^\nu)$ when $\nu \to p$, with limit

\[
\mathbb{P}[\Theta_1^p = a] = p^{|a|} \mathcal{H}_a(p)
\]
\[
\mathbb{P}[\Theta_{i+1}^p = b \mid \Theta_i^p = a] = p^{|b|} \frac{\mathcal{H}_b(p)}{\mathcal{H}_a(p)} \mathbf{1}_{a \to b} \mathbf{1}_{\mathcal{H}_a(p) \neq 0}
\]

Trivial supercritical parameter: $\nu = 1$

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No such distribution exists!

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- $1 \geq \mu(\mathcal{M}^+) = \mathcal{H}(\nu) \mathcal{G}(\nu) \geq 0$, hence $\mathcal{H}(\nu) = 0$ and $\mu(\partial \mathcal{M}^+) = 1$
Infinite uniform Bernoulli distributions as Markov chains

Critical parameter: \( \nu = p \)

Convergence of \( (\Theta_i^\nu) \) when \( \nu \to p \), with limit

\[
\mathbb{P}[\Theta_1^p = a] = p^{|a|} \mathcal{H}_a(p) \\
\mathbb{P}[\Theta_{i+1}^p = b | \Theta_i^p = a] = p^{|b|} \frac{\mathcal{H}_b(p)}{\mathcal{H}_a(p)} 1_{a \to b} \mathbf{1}_{\mathcal{H}_a(p) \neq 0}
\]

Trivial supercritical parameter: \( \nu = 1 \)

Possible only if \( \mathcal{M}^+ = \mathbb{N}^n \) (i.e. \( p = 1 \))...

Non-trivial supercritical parameter: \( p < \nu < 1 \)

No such distribution exists!

- Consider the **Garside matrix** \( M^\nu \) with \( M^\nu_{a,b} = 1_{a \to b} \nu^{|b|} \)
- \( 1 \geq \mu(\mathcal{M}^+) = \mathcal{H}(\nu) \mathcal{G}(\nu) \geq 0 \), hence \( \mathcal{H}(\nu) = 0 \) and \( \mu(\partial \mathcal{M}^+) = 1 \)
- \( M^\nu \) and \( M^p \) are **stochastic Perron matrices** if \( \mathcal{M}^+ \) is irreducible
Contents

1 Introduction

2 Trace monoids and heaps

3 First convergence results

4 Bernoulli distributions

5 Going beyond...
From uniform to non-uniform Bernoulli measures

Generalisations from the uniform case

- Collection of parameters: \((\nu_a) \in (0, 1]^n\)
From uniform to non-uniform Bernoulli measures

Generalisations from the uniform case

- Collection of parameters: \((\nu_a) \in (0, 1]^n\)
- Multiplicative function: \(\overline{\nu} : a_1 \ldots a_k \mapsto \nu_{a_1} \ldots \nu_{a_k}\)
- Möbius polynomial: \(\mathcal{H}_a(\nu) = \sum_\gamma 1_{a \gamma \in \mathcal{C}}(-1)^{|\gamma|} \overline{\nu}(\gamma)\)
From uniform to non-uniform Bernoulli measures

Generalisations from the uniform case

- Collection of parameters: \((\nu_a) \in (0, 1]^n\)
- Multiplicative function: \(\nu : a_1 \ldots a_k \mapsto \nu_a \ldots \nu_a\)
- Möbius polynomial: \(H_a(\nu) = \sum_{\gamma} 1_{a_{\gamma} \in C} (-1)^{|\gamma|} \nu(\gamma)\)
- Subcritical domain: \(D = \{\nu \mid H_1(x\nu) > 0 \text{ when } 0 \leq x \leq 1\}\)
- Critical domain: \(\partial D \cap (0, 1]^n\)
From uniform to non-uniform Bernoulli measures

Generalisations from the uniform case

- Collection of parameters: \((\nu_a) \in (0, 1]^n\)
- Multiplicative function: \(\overline{\nu} : a_1 \ldots a_k \mapsto \nu_{a_1} \ldots \nu_{a_k}\)
- Möbius polynomial: \(\mathcal{H}_a(\nu) = \sum_{\gamma} 1_{a,\gamma \in \mathcal{C}} (-1)^{|\gamma|} \overline{\nu}(\gamma)\)
- Subcritical domain: \(\mathcal{D} = \{\nu \mid \mathcal{H}_1(x\nu) > 0 \text{ when } 0 \leq x \leq 1\}\)
- Critical domain: \(\partial \mathcal{D} \cap (0, 1]^n\)
- Markov chain: \(\mathbb{P}[\Theta_{1}^\nu = a] = \overline{\nu}(a) \mathcal{H}_a(\nu)\)
  \(\mathbb{P}[\Theta_{i+1}^\nu = b \mid \Theta_i^\nu = a] = \overline{\nu}(b) \frac{\mathcal{H}_b(\nu)}{\mathcal{H}_a(\nu)} 1_{a \rightarrow b} 1_{\mathcal{H}_a(\nu) \neq 0}\)

\[
\begin{align*}
\nu_a &\quad \text{and} \quad \nu_b \quad \text{in} \quad \mathbb{N} \times \mathbb{N} = \langle a, b \rangle^+ \\
\nu_a &\quad \text{and} \quad \nu_b \quad \text{in} \quad \mathbb{N}^2 = \langle a, b \mid ab = ba \rangle^+ \\
\nu_a &\quad \text{and} \quad \nu_c \quad \text{in} \quad \langle a, b, c \mid ac = ca \rangle^+
\end{align*}
\]
From uniform to non-uniform Bernoulli measures

Generalisations from the uniform case

- Collection of parameters: \((\nu_a) \in (0, 1]^n\)
- Multiplicative function: \(\nu : a_1 \ldots a_k \mapsto \nu_{a_1} \ldots \nu_{a_k}\)
- Möbius polynomial: \(\mathcal{H}_a(\nu) = \sum_{\gamma} \mathbf{1}_{a_{\gamma} \in C} (-1)^{|\gamma|} \nu(\gamma)\)
- Subcritical domain: \(\mathcal{D} = \{\nu \mid \mathcal{H}_1(x\nu) > 0 \text{ when } 0 \leq x \leq 1\}\)
- Critical domain: \(\partial \mathcal{D} \cap (0, 1]^n\)
- Markov chain: 
  \[
  \mathbb{P}[\Theta_1' = a] = \nu(a)\mathcal{H}_a(\nu)
  \]
  \[
  \mathbb{P}[\Theta_i' = b \mid \Theta_i' = a] = \nu(b) \frac{\mathcal{H}_b(\nu)}{\mathcal{H}_a(\nu)} \mathbf{1}_{a \rightarrow b} \mathbf{1}_{\mathcal{H}_a(\nu) \neq 0}
  \]
- No supercritical Bernoulli measures!

\[
\mathbb{N} \times \mathbb{N} = \langle a, b \rangle^+ \\
\mathbb{N}^2 = \langle a, b \mid ab = ba \rangle^+ \\
\langle a, b, c \mid ac = ca \rangle^+
\]
From weak convergence to central limit theorems

Some key ingredients:

- $\nu$: tuple $(\nu_1, \ldots, \nu_n) \in (0, +\infty)^n$
From weak convergence to central limit theorems

Some key ingredients:

- $\nu$: tuple $(\nu_1, \ldots, \nu_n) \in (0, +\infty)^n$
- $\mu_k$: $\nu$-uniform distribution on $\{x \in M^+ \mid |x| = k\}$
Some key ingredients:

- \( \nu \): tuple \((\nu_1, \ldots, \nu_n) \in (0, +\infty)^n\)
- \( \mu_k \): \(\nu\)-uniform distribution on \(\{x \in \mathcal{M}^+ \mid |x| = k\}\)
- \( \|x\|_a \): \# occurrences of \(a\) in the Cartier-Foata word of \(x\)
From weak convergence to central limit theorems

Some key ingredients:

- $\nu$: tuple $(\nu_1, \ldots, \nu_n) \in (0, +\infty)^n$
- $\mu_k$: $\nu$-uniform distribution on $\{x \in \mathcal{M}^+ \mid |x| = k\}$
- $\|x\|_a$: # occurrences of $a$ in the Cartier-Foata word of $x$
- $A_k$: law of $\|x\|_a$ when $x$ is distributed according to $\mu_k$
From weak convergence to central limit theorems

Some key ingredients:

- $\nu$: tuple $(\nu_1, \ldots, \nu_n) \in (0, +\infty)^n$
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- $\|x\|_a$: # occurrences of $a$ in the Cartier-Foata word of $x$
- $A_k$: law of $\|x\|_a$ when $x$ is distributed according to $\mu_k$

Central limit Theorem (S. A., S. G., V. J. & J. M. 2016$^+$)

There exists constants $\rho$ and $\sigma^2 > 0$ such that

1. $\frac{A_k}{k} \xrightarrow{\mathcal{L}} \rho$
2. $\sqrt{k} \left( \frac{A_k}{k} - \rho \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$ if $M^+$ is irreducible
From trace monoids to Artin–Tits and left-Garside monoids

- **Dimer** monoid: \( \langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i \rangle^+ \)
From trace monoids to Artin–Tits and left-Garside monoids

- **Dimer** monoid: $\langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i\sigma_j = \sigma_j\sigma_i \rangle^+$
- **Braid** monoid: $\langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i\sigma_j = \sigma_j\sigma_i, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle^+$
From trace monoids to Artin–Tits and left-Garside monoids

- **Dimer** monoid: $\langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i \rangle^+$
- **Braid** monoid: $\langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle^+$
- **Artin–Tits** monoid: $\langle \sigma_i \mid [\sigma_i \sigma_j]^{\ell(i,j)} = [\sigma_j \sigma_i]^{\ell(i,j)} \rangle^+$

$\ell(i,j) = 2 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i$
From trace monoids to Artin–Tits and left-Garside monoids

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- **Artin–Tits** monoid: $\langle \sigma_i \mid [\sigma_i \sigma_j]^{\ell(i,j)} = [\sigma_j \sigma_i]^{\ell(i,j)} \rangle^+$
  \[ \ell(i,j) = 3 \Rightarrow \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \]
From trace monoids to Artin–Tits and left-Garside monoids

**Dimer** monoid: $\langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i \rangle^+$

**Braid** monoid: $\langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle^+$

**Artin–Tits** monoid: $\langle \sigma_i \mid [\sigma_i \sigma_j]^{\ell(i,j)} = [\sigma_j \sigma_i]^{\ell(i,j)} \rangle^+$

\[ \ell(i,j) = 4 \Rightarrow \sigma_i \sigma_j \sigma_i \sigma_j = \sigma_j \sigma_i \sigma_j \sigma_i \]
From trace monoids to Artin–Tits and left-Garside monoids

- **Dimer** monoid: \( \langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i \rangle^+ \)
- **Braid** monoid: \( \langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle^+ \)
- **Artin–Tits** monoid: \( \langle \sigma_i \mid [\sigma_i \sigma_j]^{\ell(i,j)} = [\sigma_j \sigma_i]^{\ell(i,j)} \rangle^+ \)

\[ \ell(i, j) = + \infty \Rightarrow \text{no relation!} \]
From trace monoids to Artin–Tits and left-Garside monoids

- **Dimer** monoid: \( \langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i \rangle^+ \)
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- **Artin–Tits** monoid: \( \langle \sigma_i \mid [\sigma_i \sigma_j]^{\ell(i,j)} = [\sigma_j \sigma_i]^{\ell(i,j)} \rangle^+ \)
- **Trace** monoid: Artin–Tits with \( \ell(i,j) \in \{2, +\infty\} \)
From trace monoids to Artin–Tits and left-Garside monoids

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- **Left-Garside** monoid: cancellative, \( \leq \)-lower semi-lattice, finite generating family \( \Sigma \) closed under suffix and under \( \lor \)
From trace monoids to Artin–Tits and left-Garside monoids

**Theorem (— 2016⁺)**

1. Weak conv. in all A–T monoids
2. CLT 1 in all A–T monoids
3. CLT 2 in irreducible A–T monoids

+ some extensions to left-Garside monoids

- **Dimer** monoid: \( \langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i \rangle^+ \)
- **Braid** monoid: \( \langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle^+ \)
- **Artin–Tits** monoid: \( \langle \sigma_i \mid [\sigma_i \sigma_j]^{\ell(i,j)} = [\sigma_j \sigma_i]^{\ell(i,j)} \rangle^+ \)
- **Trace** monoid: Artin–Tits with \( \ell(i,j) \in \{2, +\infty\} \)
- **Left-Garside** monoid: cancellative, \( \leq \)-lower semi-lattice, finite generating family \( \Sigma \) closed under suffix and under \( \vee \)
And then?

Some directions of research

- Generalisation to all left-Garside monoids
- Generalisation to trace groups
- Sampling elements in regular languages \( L \cap M_k \) instead of \( M_k \)
- Identifying nice Markov chains when \( \text{length}(a) = h(ba) \)
- Your favorite one (tell me now!)