Growth rates of braid monoids with many generators

Ramón Flores\textsuperscript{1}, Juan González-Meneses\textsuperscript{1} & Vincent Jugé\textsuperscript{2}

1: Universidad de Sevilla – 2: Université Gustave Eiffel (LIGM)

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Heaps of pieces vs Trace monoids

Heap of pieces\textsuperscript{[5]}

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Heaps of pieces vs Trace monoids

Heap of pieces

\[
\begin{array}{cccc}
S_1 & S_3 \\
S_2 &  \\
S_1 & S_4 \\
\end{array}
\]

Trace monoid

\[
T_4 = \left\langle S_1, S_2, S_3, S_4 \mid S_i S_j = S_j S_i \quad \text{if } i \neq j \pm 1 \right\rangle^+
\]

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Growth rates of wide braid monoids
Heaps of pieces vs Trace monoids

**Heap of pieces**

\[ S_1 \quad S_3 \]
\[ \quad S_2 \]
\[ S_1 \quad S_4 \]

- \(|h| = \#\text{pieces in } h\)

**Trace monoid**

\[ T_4 = \langle S_1, S_2, S_3, S_4 \mid S_i S_j = S_j S_i \text{ if } i \neq j \pm 1 \rangle^+ \]

- \(|\tau| = \#\text{generators needed to write } \tau\)

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Growth rates of wide braid monoids
Heaps of pieces vs Trace monoids

Heap of pieces\(^5\)

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- $|h| = \#\text{pieces in } h$
- $\lambda_{4,k} = \#\text{heaps of size } k$

Trace monoid

$T_4 = \langle S_1, S_2, S_3, S_4 \mid S_i S_j = S_j S_i \text{ if } i \neq j \pm 1 \rangle^+$

- $|\tau| = \#\text{generators needed to write } \tau$
- $\lambda_{4,k} = \#\text{traces of size } k$
Heaps of pieces vs Trace monoids

**Heap of pieces**\(^5\)

\[
\begin{array}{|c|c|}
\hline
S_1 & S_3 \\
\hline
S_2 \\
\hline
S_1 & S_4 \\
\hline
\end{array}
\]

- \(|h| = \#\text{pieces in } h\)
- \(\lambda_{4,k} = \#\text{heaps of size } k\)

**Trace monoid**

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T_4 = \left\langle S_1, S_2, S_3, S_4 \left| S_i S_j = S_j S_i \text{ if } i \neq j \pm 1 \right. \right\rangle^+
\]

- \(|\tau| = \#\text{generators needed to write } \tau\)
- \(\lambda_{4,k} = \#\text{traces of size } k\)

How does \(\lambda_{4,k}\) behave when \(k \to +\infty\)?
Growth rate of a finitely generated monoid

In a monoid $M$ generated by a finite family $F$,

- $|τ| = \#\text{generators (in } F\text{) needed to write } τ$
- $m_k = \#\text{elements of size } k$

Lemma:

$m_k \leq \ell \leq m_k (\log m_k \text{ is sub-additive})$

Corollary:

The sequence $m_1 \{k\}$ converges!

Proof:

If $m_k \neq 0$ for some $k ≥ 0$, then $m_\ell \neq 0$ for all $\ell ≥ k$.

Otherwise, set $x_k = p \log m_k q \{k \geq 0 \text{ and } X_k = \max t x_1, ..., x_k u$.

For all $\ell \leq k$ and $q ≥ 1$, we have $x_qk \leq \ell \leq p q k x_k \leq \ell x_qk \leq X_k \{q \leq q \leq x_k \text{ and thus } \lim sup_{k \to \infty} x_k \leq x_k$.
Growth rate of a finitely generated monoid

In a monoid $M$ generated by a finite family $\mathcal{F}$,
- $|\tau| = \# \text{generators (in } \mathcal{F}) \text{ needed to write } \tau$
- $m_k = \# \text{elements of size } k$

**Lemma:** $m_{k+\ell} \leq m_k m_\ell$ \hspace{1cm} (log $m_k$ is sub-additive)
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In a monoid $M$ generated by a finite family $F$,
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**Lemma:** $m_{k+\ell} \leq m_k m_\ell$ \hspace{1cm} (log $m_k$ is sub-additive)

**Corollary:** The sequence $m_k^{1/k}$ converges!
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**Lemma:** $m_{k+\ell} \leq m_k m_\ell$  
($\log m_k$ is sub-additive)

**Corollary:** The sequence $m_k^{1/k}$ converges!

**Proof:** If $m_k = 0$ for some $k \geq 0$, then $m_\ell = 0$ for all $\ell \geq k$.
Otherwise, set $x_k = (\log m_k)/k \geq 0$ and $X_k = \max\{x_1, \ldots, x_k\}$.
For all $\ell \leq k$ and $q \geq 1$, we have
\[
x_{qk+\ell} \leq (kq x_k + \ell x_\ell)/(qk + \ell) \leq x_k + X_k/q \xrightarrow{q\to\infty} x_k
\]
and thus $\limsup_{k\to\infty} x_k \leq x_k$.  

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Growth rates of wide braid monoids
Growth rate of a finitely generated monoid

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$$
and thus $\limsup_{k\to\infty} x_k \leq \liminf_{k\to\infty} x_k$. 

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Growth rate of a finitely generated monoid

In a monoid $M$ generated by a finite family $\mathcal{F}$,
- $|\tau| = \#\text{generators (in } \mathcal{F}) \text{ needed to write } \tau$
- $m_k = \#\text{elements of size } k$

**Lemma:** $m_{k+\ell} \leq m_k m_{\ell}$

$(\log m_k \text{ is sub-additive})$

**Corollary:** The sequence $m_k^{1/k}$ converges towards $M$’s growth rate!

**Proof:** If $m_k = 0$ for some $k \geq 0$, then $m_\ell = 0$ for all $\ell \geq k$.
Otherwise, set $x_k = (\log m_k)/k \geq 0$ and $X_k = \max\{x_1, \ldots, x_k\}$.
For all $\ell \leq k$ and $q \geq 1$, we have

$$x_{qk+\ell} \leq (k q x_k + \ell x_\ell)/(q k + \ell) \leq x_k + X_k/q \xrightarrow{q \to \infty} x_k$$

and thus $\lim_{k \to \infty} x_k \leq \lim_{k \to \infty} \inf x_k$. 
Growth rate of trace monoids

\[ T_4 = \left\langle S_1, S_2, S_3, S_4 \mid S_i S_j = S_j S_i \quad \text{if } i \neq j + 1 \right\rangle^+ \]

\[ \lambda_{4,k} = \#\{\tau \in T_4 : |\tau| = k\} \]

How does \( \lambda_{4,k} \) precisely behave when \( k \to +\infty \)?
Growth rate of trace monoids

\[ T_4 = \left\langle S_1, S_2, S_3, S_4 \middle| S_iS_j = S_jS_i \text{ if } i \neq j \pm 1 \right\rangle^+ \]

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How does \( \lambda_{4,k} \) precisely behave when \( k \to +\infty \)?

Generating function

\[ G_4(z) = \sum_{k \geq 0} \lambda_{4,k} z^k = \sum_{\tau \in T_4} z^{|\tau|} \]
Growth rate of trace monoids

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Möbius polynomial

\[ P_4(z) = G_4(z)^{-1} = 1 - 4z + 3z^2 \]

[Diagram showing the curve of \( P_4(z) \) from \( 0 \) to \( 1 \) with a peak at \( 1/3 \).]
Growth rate of trace monoids

\[ T_4 = \left\langle S_1, S_2, S_3, S_4 \bigg| S_i S_j = S_j S_i \right. \tag{5} \]

if \( i \neq j \pm 1 \) \( \implies \lambda_{4,k} = \#\{\tau \in T_4 : |\tau| = k\} \]

How does \( \lambda_{4,k} \) precisely behave when \( k \to +\infty? \)

Generating function

\[ G_4(z) = \sum_{k \geq 0} \lambda_{4,k} z^k = \sum_{\tau \in T_4} z^{|	au|} \]

\( \rho_4 = 1/3 \) and \( \lambda_{4,k} \sim -\frac{1}{\rho_4^{k+1} P_4' (\rho_4)} \)

Möbius polynomial

\[ P_4(z) = G_4(z)^{-1} = 1 - 4z + 3z^2 \]
Growth rate of trace monoids

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How does \( \lambda_{4,k} \) precisely behave when \( k \to +\infty \)?

**Generating function**

\[ G_4(z) = \sum_{k \geq 0} \lambda_{4,k} z^k = \sum_{\tau \in T_4} z^{|\tau|} \]

**Möbius polynomial**

\[ P_4(z) = G_4(z)^{-1} = 1 - 4z + 3z^2 \]

\[ \rho_4 = 1/3 \quad \text{and} \quad \lambda_{4,k} \sim \frac{-1}{\rho_4^{k+1} P_4'(\rho_4)} \]

**Corollary:** \( \lambda_{4,k}^{1/k} \to 1/\rho_4 = 3 \)
Growth rates of wide trace monoids

How does $\rho_n$ behave when $n \to +\infty$?

Recurrence equation

$P_{-1}(z) = P_0(z) = 1$

$P_n(z) = P_{n-1}(z) - zP_{n-2}(z)$

if $n \geq 1$
Growth rates of wide trace monoids

How does $\rho_n$ behave when $n \to +\infty$?

Recurrence equation

\[
P_{-1}(z) = P_0(z) = 1 \\
P_n(z) = P_{n-1}(z) - zP_{n-2}(z) \\
\text{if } n \geq 1
\]

$\rho_n \to \rho_\infty \geq 0$
Growth rates of wide trace monoids

How does $\rho_n$ behave when $n \to +\infty$?

Recurrence equation

$$
P_{-1}(z) = P_0(z) = 1$$

$$
P_n(z) = P_{n-1}(z) - zP_{n-2}(z)$$

if $n \geq 1$

$$
\rho_n = \frac{1}{4 \cos\left(\frac{\pi}{n+2}\right)^2}
$$

$$
\rho_n \to \rho_\infty = 1/4
$$
How did this proof work?

— Part #1: Introducing Möbius polynomials\textsuperscript{[3,7]} —

- Define the Möbius polynomial $P_n(z)$
- Prove that $P_n(z)G_n(z) = 1$
How did this proof work?

— Part #1: Introducing Möbius polynomials$^{[3, 7]}$ —

- Define the Möbius polynomial $P_n(z)$
- Prove that $P_n(z)G_n(z) = 1$

— Part #2: Computing Möbius polynomials —

- Find induction relation on polynomials $P_n(z)$
- Derive a closed-form expression of $P_n(z)$
How did this proof work?

— Part #1: Introducing Möbius polynomials\textsuperscript{[3,7]} —

- Define the Möbius polynomial $P_n(z)$
- Prove that $P_n(z)G_n(z) = 1$

— Part #2: Computing Möbius polynomials —

- Find induction relation on polynomials $P_n(z)$
- Derive a closed-form expression of $P_n(z)$

— Part #3: Conclusion —

- Compute the limit $\rho_\infty$ of roots $\rho_n$
Part #1: Introducing Möbius polynomials (1/2)

A few preliminary properties...

1. **length is additive**: $|\tau| + |\sigma| = |\tau \cdot \sigma|$
2. **$T_n$ is left-cancellative**: $\tau \cdot \sigma = \tau \cdot \sigma' \iff \sigma = \sigma'$
3. **($T_n, \leq$) is a lower-semilattice**: GCDs exist
   - **Corollary**: when a set $S$ has a common multiple, it has a LCM
4. **$F_n = \{S_1, \ldots, S_n\}$ is parabolic**: for all $\mathcal{F}' \subseteq \mathcal{F}_n$,
   - $T_{\mathcal{F}'} = \{\tau \in T_n : \text{all factors } S_j \text{ of } \tau \text{ belong to } \mathcal{F}'\}$ is a sub-monoid
   - for all $S \subseteq T_{\mathcal{F}'}$, if $S$ has a LCM, then $\text{LCM}(S) \in T_{\mathcal{F}'}$
Part #1: Introducing Möbius polynomials (1/2)

A few preliminary properties...

1. **length is additive:** $|\tau| + |\sigma| = |\tau \cdot \sigma|

2. **$T_n$ is left-cancellative:** $\tau \cdot \sigma = \tau \cdot \sigma' \iff \sigma = \sigma'$

3. **$(T_n, \leq)$ is a lower-semilattice:** GCDs exist
   - **Corollary:** when a set $S$ has a common multiple, it has a LCM

4. **$F_n = \{S_1, \ldots, S_n\}$ is parabolic:** for all $F' \subseteq F_n$, such that:
   - $T_{F'} = \{\tau \in T_n : \text{all factors } S_j \text{ of } \tau \text{ belong to } F'\}$ is a sub-monoid
   - for all $S \subseteq T_{F'}$, if $S$ has a LCM, then $\text{LCM}(S) \in T_{F'}$

and key objects:

5. **left set of a heap:** $L(\tau) = \{S_i : S_i \leq \tau\}

6. **left set of $T_n$:** $L_n = \{X \subseteq F_n : X \text{ has a LCM}\} = \{L(\tau) : \tau \in T_n\}$
Part #1: Introducing Möbius polynomials (1/2)

A few preliminary properties...

1. **length is additive**: $|\tau| + |\sigma| = |\tau \cdot \sigma|

2. $T_n$ is **left-cancellative**: $\tau \cdot \sigma = \tau \cdot \sigma' \iff \sigma = \sigma'$

3. $(T_n, \preceq)$ is a **lower-semilattice**: GCDs exist
   - **Corollary**: when a set $S$ has a common multiple, it has a LCM

4. $F_n = \{S_1, \ldots, S_n\}$ is **parabolic**: for all $F' \subseteq F_n$,
   - $T_{F'} = \{\tau \in T_n : \text{all factors } S_j \text{ of } \tau \text{ belong to } F'\}$ is a sub-monoid
   - for all $S \subseteq T_{F'}$, if $S$ has a LCM, then $\text{LCM}(S) \in T_{F'}$

and key objects:

5. **left set** of a heap: $L(\tau) = \{S_i : S_i \preceq \tau\}$

6. **left set** of $T_n$: $L_n = \{X \subseteq F_n : X \text{ has a LCM}\} = \{L(\tau) : \tau \in T_n\}$

This also holds in braid monoids!
Part #1: Introducing Möbius polynomials (1/2)

A few preliminary properties... 

1. **length is additive**: $|\tau| + |\sigma| = |\tau \cdot \sigma|$ 
2. **$T_n$ is left-cancellative**: $\tau \cdot \sigma = \tau \cdot \sigma' \Leftrightarrow \sigma = \sigma'$ 
3. **$(T_n, \leq)$ is a lower-semilattice**: GCDs exist 
   - **Corollary**: when a set $S$ has a common multiple, it has a LCM 
4. **$F_n = \{S_1, \ldots, S_n\}$ is parabolic**: for all $F' \subseteq F_n$, 
   - $T_{F'} = \{\tau \in T_n : \text{all factors } S_j \text{ of } \tau \text{ belong to } F'\}$ is a sub-monoid 
   - for all $S \subseteq T_{F'}$, if $S$ has a LCM, then $\text{LCM}(S) \in T_{F'}$

and key objects:

5. **left set of a heap**: $L(\tau) = \{S_i : S_i \leq \tau\}$ 
6. **left set of $T_n$**: $L_n = \{X \subseteq F_n : X \text{ has a LCM}\} = \{L(\tau) : \tau \in T_n\}$

This also holds in all Artin-Tits monoids!

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Growth rates of wide braid monoids
Part #1: Introducing Möbius polynomials (2/2)

**Theorem:** In the ring $\mathbb{Z}[T_n]$, \( \Delta_X = \text{LCM}(X) \)

\[
(S_1 + S_2 + 2S_3) \cdot (S_1 - S_3) = S_1^2 + S_2 \cdot S_1 + 2S_3 \cdot S_1 - S_1 \cdot S_3 - S_2 \cdot S_3 - 2S_3^2
\]
**Theorem:** In the ring $\mathbb{Z}[T_n]$, \(\Delta_X = \text{LCM}(X)\)

\[
(S_1 + S_2 + 2S_3) \cdot (S_1 - S_3) = S_1^2 + S_2 \cdot S_1 + 2S_3 \cdot S_1
- S_1 \cdot S_3 - S_2 \cdot S_3 - 2S_3^2
\]
Part #1: Introducing Möbius polynomials (2/2)

**Theorem:** In the ring $\mathbb{Z}[T_n]$, 

$$(\Delta_X = \text{LCM}(X))$$

$$(S_1 + S_2 + 2S_3) \cdot (S_1 - S_3) = S_1^2 + S_2 \cdot S_1 + S_3 \cdot S_1$$

$$- S_2 \cdot S_3 - 2S_3^2$$
Part #1: Introducing Möbius polynomials (2/2)

**Theorem:** In the ring $\mathbb{Z}[T_n]$,

$$\sum_{X \in L_n} (-1)^{|X|} \Delta_X \cdot \sum_{\tau \in T_n} \tau = 1$$

($\Delta_X = \text{LCM}(X)$)
Part #1: Introducing Möbius polynomials (2/2)

**Theorem:** In the ring $\mathbb{Z}[T_n]$, 

\[
\sum_{X \in L_n} (-1)^{|X|} \Delta_X \cdot \sum_{\tau \in T_n} \tau = \sum_{\tau \in T_n} \sum_{X \in L_n} (-1)^{|X|} (\Delta_X \cdot \tau)
\]

$(\Delta_X = \text{LCM}(X))$
Theorem: In the ring $\mathbb{Z}[T_n]$, 

\[
\sum_{X \in L_n} (-1)^{|X|} \Delta_X \cdot \sum_{\tau \in T_n} \tau = \sum_{\tau \in T_n} \sum_{X \in L_n} (-1)^{|X|} (\Delta_X \cdot \tau)
\]

\[
= \sum_{\theta \in T_n} \sum_{X \in L(\theta)} (-1)^{|X|} \theta
\]
Part #1: Introducing Möbius polynomials (2/2)

**Theorem:** In the ring $\mathbb{Z}[T_n]$, $(\Delta_X = \text{LCM}(X))$

\[ \sum_{X \in L_n} (-1)^{|X|} \Delta_X \cdot \sum_{\tau \in T_n} \tau = \sum_{\tau \in T_n} \sum_{X \in L_n} (-1)^{|X|} (\Delta_X \cdot \tau) \]

\[ = \sum_{\theta \in T_n} \sum_{X \subseteq L(\theta)} (-1)^{|X|} \theta \]

\[ = \sum_{\theta \in T_n} 1_{L(\theta)} = \emptyset \theta \]
Theorem: In the ring $\mathbb{Z} [ T_n ]$, 

$$\sum_{X \in L_n} (-1)^{|X|} \Delta_X \cdot \sum_{\tau \in T_n} \tau = \sum_{\tau \in T_n} \sum_{X \in L_n} (-1)^{|X|} (\Delta_X \cdot \tau)$$

$$= \sum_{\theta \in T_n} \sum_{X \subseteq L(\theta)} (-1)^{|X|} \theta$$

$$= \sum_{\theta \in T_n} 1_{L(\theta) = \emptyset} \theta = 1.$$
Theorem: In the ring $\mathbb{Z}[T_n]$, 
Corollary: In the ring $\mathbb{Z}[z]$, 

\[
\sum_{X \in L_n} (-1)^{|X|} z^{|\Delta X|} \cdot \sum_{\tau \in T_n} z^{|\tau|} = 1 = P_n(z) \cdot G_n(z)
\]
Parts #2 & #3: Computing Möbius polynomials and $\rho_\infty$

**Theorem:** $P_{-1}(z) = P_0(z) = 1$ and $P_n(z) = P_{n-1}(z) - zP_{n-2}(z)$ if $n \geq 1$
Theorem: \( P_{-1}(z) = P_0(z) = 1 \) and \( P_n(z) = P_{n-1}(z) - zP_{n-2}(z) \) if \( n \geq 1 \)

Proof: \( L_n = \{X \in L_n : S_n \notin X\} \sqcup \{X \in L_n : S_n \in X\} \)
\[
= L_{n-1} \sqcup \{X \cup \{S_n\} : X \in L_{n-2}\}
\]

and \( L_{-1} = L_0 = \emptyset \)

\( \Delta_{X \cup \{S_n\}} = \Delta_X \cdot S_n \) if \( X \subseteq L_{n-2} \)

and \( \Delta_{\emptyset} = 1 \)
Parts #2 & #3: Computing Möbius polynomials and $\rho_\infty$

**Theorem:** $P_{-1}(z) = P_0(z) = 1$ and $P_n(z) = P_{n-1}(z) - zP_{n-2}(z)$ if $n \geq 1$

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$$= L_{n-1} \sqcup \{X \cup \{S_n\} : X \in L_{n-2}\}$$

$\Delta_{X \cup \{S_n\}} = \Delta X \cdot S_n$ if $X \subseteq L_{n-2}$

and $L_{-1} = L_0 = \emptyset$ and $\Delta \emptyset = 1$

**Computing $P_n(z)$ and its roots:**

- For all $z \in \mathbb{C}$, the sequence $(P_n(z))_{n \geq -1}$ is recurrent linear of order 2
Parts #2 & #3: Computing Möbius polynomials and $\rho_\infty$

**Theorem:** $P_{-1}(z) = P_0(z) = 1$ and $P_n(z) = P_{n-1}(z) - zP_{n-2}(z)$ if $n \geq 1$

**Proof:**

$L_n = \{X \in L_n : S_n \notin X\} \uplus \{X \in L_n : S_n \in X\} \quad \text{and} \quad L_{-1} = L_0 = \emptyset$

$L_{n-1} \uplus \{X \cup \{S_n\} : X \in L_{n-2}\}$

$\Delta_X \cup \{S_n\} = \Delta_X \cdot S_n$ if $X \subseteq L_{n-2}$ \quad \text{and} \quad \Delta_{\emptyset} = 1$

**Computing $P_n(z)$ and its roots:**

- For all $z \in \mathbb{C}$, the sequence $(P_n(z))_{n \geq -1}$ is recurrent linear of order 2
- We find $P_n(z) = \frac{(1 + \delta)^{n+2} - (1 - \delta)^{n+2}}{2^{n+2}\delta}$, with $\delta = \sqrt{1 - 4z}$
  - and $P_n(z) = (3n + 4)/4^{n+1}$ if $z = 1/4$
Parts #2 & #3: Computing Möbius polynomials and $\rho_\infty$

**Theorem:** $P_{-1}(z) = P_0(z) = 1$ and $P_n(z) = P_{n-1}(z) - zP_{n-2}(z)$ if $n \geq 1$

**Proof:**

Let $L_n = \{ X \in L_n : S_n \notin X \} \cup \{ X \in L_n : S_n \in X \}$

$= L_{n-1} \cup \{ X \cup \{ S_n \} : X \in L_{n-2} \}$

and $L_{-1} = L_0 = \emptyset$

$\Delta_{X \cup \{ S_n \}} = \Delta_X \cdot S_n$ if $X \subseteq L_{n-2}$

and $\Delta_{\emptyset} = 1$

**Computing $P_n(z)$ and its roots:**

- For all $z \in \mathbb{C}$, the sequence $(P_n(z))_{n \geq -1}$ is recurrent linear of order 2
- We find $P_n(z) = \frac{(1 + \delta)^{n+2} - (1 - \delta)^{n+2}}{2^{n+2} \delta}$, with $\delta = \sqrt{1 - 4z}$
  and $P_n(z) = (3n + 4)/4^{n+1}$ if $z = 1/4$
- $P_n(z) = 0$ iff $z = \frac{1}{4 \cos(k\pi/(n + 2))^2}$ for some $k \in \mathbb{Z}$ (and $z \neq 1/4$)

**Conclusion:** $\rho_n = \frac{1}{4 \cos(\pi/(n + 2))^2} \rightarrow \rho_\infty = 1/4$
monoids vs monoids

\[ T_n = \langle S_1, \ldots, S_n \mid S_i S_j = S_j S_i \text{ if } i \neq j \pm 1, S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} \rangle^+ = A_n \]
monoids vs monoids

\[ T_n = \left\langle S_1, \ldots, S_n \right| \begin{array}{l}
S_iS_j = S_jS_i \text{ if } i \neq j \pm 1 \\
S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1}
\end{array} \right\rangle^+ = A_n \]
monoids vs monoids

\( T_n = \langle S_1, \ldots, S_n \mid S_i S_j = S_j S_i \text{ if } i \neq j \pm 1, S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} \rangle^+ = A_n \)

Artin braid diagram

Ramón Flores, Juan González-Meneses & Vincent Jugé
Growth rates of wide braid monoids
\[ T_n = \left< S_1, \ldots, S_n \mid S_i S_j = S_j S_i \text{ if } i \neq j \pm 1, S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} \right>^+ = A_n \]
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\[ T_n = \left\langle S_1, \ldots, S_n \right| S_i S_j = S_j S_i \text{ if } i \neq j \pm 1, S_i S_{i+1} S_i = S_{i+1} S_{i} S_{i+1} \right\rangle^+ = A_n \]
monoids vs monoids

\[ T_n = \left\langle S_1, \ldots, S_n \left| \begin{array}{c} S_i S_j = S_j S_i \text{ if } i \neq j \pm 1 \\ S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} \end{array} \right. \right\rangle^+ = A_n \]

- \( \mu_{n,k} = \# \{ \tau \in A_n : |\tau| = k \} \)
- \( H_n(z) = \sum_{k \geq 0} \mu_{n,k} z^k = Q_n(z)^{-1} \)
monoids vs monoids

\[ T_n = \left\langle S_1, \ldots, S_n \mid S_i S_j = S_j S_i \text{ if } i \neq j \pm 1, S_i S_{i+1} = S_{i+1} S_i S_{i+1} \right\rangle^+ = A_n \]

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\[ \mu_{n,k} \sim \frac{-1}{q_n^{k+1} Q'_n(q_n)} \]

Ramón Flores, Juan González-Meneses & Vincent Jugé
Growth rates of wide braid monoids
monoids vs monoids

\[ T_n = \langle S_1, \ldots, S_n \mid S_i S_j = S_j S_i \text{ if } i \neq j \pm 1, S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} \rangle^+ = A_n \]

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- \( H_n(z) = \sum_{k \geq 0} \mu_{n,k} z^k = Q_n(z)^{-1} \)

\( q_n \to q_\infty \geq 0: \text{What is } q_\infty? \)

\[ \mu_{n,k} \sim \frac{-1}{q_n^{k+1} Q'_n(q_n)} \]

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Growth rates of wide braid monoids
Möbius polynomials in braid monoids

**Theorem**: $Q_{-1}(z) = Q_0(z) = 1$ and

$$Q_n(z) = \sum_{j=0}^{n} (-1)^j z^{j(j+1)/2} Q_{n-1-j}(z) \text{ if } n \geq 1$$
Möbius polynomials in braid monoids

**Theorem**[6]: \( Q_{-1}(z) = Q_0(z) = 1 \) and

\[
Q_n(z) = \sum_{j=0}^{n} (-1)^j z^{j(j+1)/2} Q_{n-1-j}(z) \text{ if } n \geq 1
\]

**Proof:** \( L_n = 2^{F_n} = \bigsqcup_{i=1}^{n+1} \{ X \cup \{ S_i, \ldots, S_n \} : X \in L_{i-2} \} \) and \( L_{-1} = L_0 = \emptyset \)

\[
\Delta \{ S_i, \ldots, S_n \} = S_n \cdot S_{n-1} \cdots S_i \cdot \Delta \{ S_{i+1}, \ldots, S_n \} \quad \text{and} \quad \Delta \emptyset = 1
\]

\[
\Delta X \cup \{ S_i, \ldots, S_n \} = \Delta X \cdot \Delta \{ S_i, \ldots, S_n \} \text{ if } X \subseteq L_{i-2}
\]
Möbius polynomials in braid monoids

**Theorem**\cite{6}: \( Q_n(z) = 0 \) if \( n \leq -2 \), \( Q_{-1}(z) = 1 \) and

\[
Q_n(z) = \sum_{j \geq 0} (-1)^j z^{j(j+1)/2} Q_{n-1-j}(z) \quad \text{if} \quad n \geq 0
\]

**Proof:** \( L_n = 2^\mathcal{F}_n = \bigsqcup_{i=1}^{n+1} \{ X \cup \{ S_i, \ldots, S_n \} : X \in L_{i-2} \} \) and \( L_{-1} = L_0 = \emptyset \)

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\Delta \{ s_i, \ldots, s_n \} = S_n \cdot S_{n-1} \cdots S_i \cdot \Delta \{ s_{i+1}, \ldots, s_n \}
\]

and \( \Delta \emptyset = 1 \)

\[
\Delta X \cup \{ s_i, \ldots, s_n \} = \Delta X \cdot \Delta \{ s_i, \ldots, s_n \} \quad \text{if} \quad X \subseteq L_{i-2}
\]
The sequence \((Q_n(z))_{n \geq -1}\) is recurrent linear of infinite order, with \(Q_n(z) = 0\) if \(n \leq -2\) and \(Q_{-1}(z) = 0\).
Computing Möbius polynomials in braid monoids (1/2)

The sequence \((Q_n(z))_{n \geq -1}\) is recurrent linear of \textbf{infinite} order, with \(Q_n(z) = 0\) if \(n \leq -2\) and \(Q_{-1}(z) = 0\)

### Finite order

**Equation:**
\[
Q_n(z) = Q_{n-1}(z) - zQ_{n-2}(z)
\]

### Infinite order

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Q_n(z) = \sum_{j \geq 0}(-1)^j z^{j(j+1)/2} Q_{n-1-j}(z)
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The sequence \((Q_n(z))_{n \geq -1}\) is recurrent linear of infinite order, with \(Q_n(z) = 0\) if \(n \leq -2\) and \(Q_{-1}(z) = 0\).

**Finite order**

**Equation:**
\[
Q_n(z) = Q_{n-1}(z) - zQ_{n-2}(z)
\]

**Characteristic polynomial:**
\[
Q_z(X) = 1 - X + zX^2
\]

**Infinite order**

**Equation:**
\[
Q_n(z) = \sum_{j \geq 0} (-1)^j z^{j(j+1)/2} Q_{n-1-j}(z)
\]

**Characteristic function:**
\[
Q_z(X) = \sum_{j \geq 0} (-1)^j z^{j(j-1)/2} X^j
\]
The sequence \((Q_n(z))_{n \geq -1}\) is recurrent linear of infinite order, with \(Q_n(z) = 0\) if \(n \leq -2\) and \(Q_{-1}(z) = 0\)

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**Characteristic polynomial:**
\[ Q_z(X) = 1 - X + zX^2 \]

**General expression:**
\[ Q_z(r_1) = Q_z(r_2) = 0 \quad (r_1 \neq r_2): \]
\[ Q_n(z) = \alpha_1 r_1^{-n} + \alpha_2 r_2^{-n}, \text{ with} \]
\[ \alpha_i = -1/r_i Q_z'(r_i) \]

**Proof:** Cauchy residue formula

**Infinite order**

**Equation:**
\[ Q_n(z) = \sum_{j \geq 0} (-1)^j z^{j(j+1)/2} Q_{n-1-j}(z) \]

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Computing Möbius polynomials in braid monoids (1/2)

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### Finite order

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**Proof:** Cauchy residue formula

### Infinite order

**Equation:**
\[ Q_n(z) = \sum_{j \geq 0} (-1)^j z^{j(j+1)/2} Q_{n-1-j}(z) \]

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**General expression:**
\[ Q_z(r_i) = 0 \ (\text{for } i \geq 0): \]
\[ Q_n(z) = \sum_{i \geq 0} \alpha_i r_i^{-n}, \text{ with } \alpha_i = -1/r_i Q'_z(r_i) \]

(because \(r_i \approx z^{-i}\))
Computing Möbius polynomials in braid monoids (2/2)

and, some (ugly) computations later...

Theorem [10, 11]

$q_\infty \approx 0.30904 \ldots$ is the least real $z \geq 0$ such that $Q_z$ has a double root
Computing Möbius polynomials in braid monoids (2/2)

and, some (ugly) computations later...

Theorem [10, 11]

$q_\infty \approx 0.30904 \ldots$ is the least real $z \geq 0$ such that $Q_z$ has a double root

Theorem [11]

The same result holds in monoids of type $B$ and $D$

$A_n$: $S_1^3 S_2^3 S_3^3 S_4 \ldots S_n$

$B_n$: $S_1^4 S_2^3 S_3^3 S_4 \ldots S_n$

$D_n$: $S_2^3 S_3^3 S_4^3 S_5 \ldots S_n$

$S_1$
Pros and cons of our first approach

Pros:
- Similar techniques work in all cases
- No need for strong insights: careful computations are enough
- We get a nice, long sought result

Cons:
- Why does the proof work? Is this just random luck?
- Computations are not so nice
- What then? Let us find another proof!
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Let us find another proof!
Useful tools…

Direct limit of monoids: \( T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots \subseteq T_\infty \)

Embedding \( T_\infty \) into \( \mathbb{Z}[z, T_\infty] \): \( S \leftrightarrow \sum_{\tau \in S} \tau \leftrightarrow S(z) = \sum_{\tau \in S} z^{|\tau|} \)

Shift endomorphism: \( \text{sh}: S_i \mapsto S_{i+1} \)

Left-constrained traces: \( \mathcal{L}^n_i = \{ \tau \in T_n : L(\tau) \subseteq \{ S_1, \ldots, S_i \} \} \)

\( (T_n = \mathcal{L}^n_n) \)
Growth rates of trace monoids: an algebraic proof

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and associated results:

- \( \mathcal{L}^{n+1}_{i+1} = \operatorname{sh}(\mathcal{L}^n_i) \cdot \mathcal{L}^{n+1}_1 \)
- \( \mathcal{L}^n_1 = \mathcal{L}^n_0 + S_1 \cdot \mathcal{L}^n_2 \)

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Growth rates of wide braid monoids
Growth rates of trace monoids: an algebraic proof

Useful tools...

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and associated results:

- \( \mathcal{L}_{i+1}^{n+1} = \text{sh}(\mathcal{L}_i^n) \cdot \mathcal{L}_1^{n+1} \)
- \( \mathcal{L}_1^n = \mathcal{L}_0^n + S_1 \cdot \mathcal{L}_2^n \)
- \( \mathcal{L}_1^n(z) = 1 + z\mathcal{L}_1^{n-1}(z)\mathcal{L}_1^n(z) \):
  - \( \mathcal{L}_1^n(z) \leq 1/z \) when \( z < p_\infty \)
Growth rates of trace monoids: an algebraic proof

Useful tools...

Direct limit of monoids: \( T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots \subseteq T_\infty \)

Enbedding \( T_\infty \) into \( \mathbb{Z}[z, T_\infty] \): \( S \leftrightarrow \sum_{\tau \in S} \tau \leftrightarrow S(z) = \sum_{\tau \in S} z^{\lvert \tau \rvert} \)

Shift endomorphism: \( \text{sh}: S_i \mapsto S_{i+1} \)

Left-constrained traces: \( \mathcal{L}_i^n = \{ \tau \in T_n : \mathbf{L}(\tau) \subseteq \{S_1, \ldots, S_i\} \} \)

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  - \( \mathcal{L}_1^n(z) \leq 1/z \) when \( z < p_\infty \)
- \( p_n = \text{rad } \mathcal{L}_1^n \geq \text{rad } \mathcal{L}_1^{\infty} \)
Useful tools...

Direct limit of monoids: \[ T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots \subseteq T_\infty \]

Embedding \( T_\infty \) into \( \mathbb{Z}[z, T_\infty] \): \[ S \leftrightarrow \sum_{\tau \in S} \tau \leftrightarrow S(z) = \sum_{\tau \in S} z^{\mid \tau \mid} \]

Shift endomorphism: \[ \text{sh}: S_i \mapsto S_{i+1} \]

Left-constrained traces: \[ L_i^n = \{ \tau \in T_n : L(\tau) \subseteq \{S_1, \ldots, S_i\} \} \]
\[ (T_n = L_n^n) \]

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- \[ L_{i+1}^{n+1} = \text{sh}(L_i^n) \cdot L_{1}^{n+1} \]
- \[ L_1^n = L_0^n + S_1 \cdot L_2^n \]
- \[ L_1^n(z) = 1 + zL_1^{n-1}(z)L_1^n(z): \]
  - \( L_1^n(z) \leq 1/z \) when \( z < p_\infty \)
- \( p_n = \text{rad} L_1^n \geq \text{rad} L_1^\infty \)
- \( L_1^n \rightarrow L_1^\infty \) on \((0, p_\infty)\)
  - \( L_1^\infty(z) = (1 - \sqrt{1 - 4z})/2z \)
Growth rates of trace monoids: an algebraic proof

Useful tools...

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  \[ \mathcal{L}_1^n(z) \leq 1/z \text{ when } z < p_\infty \]
- \( p_n = \text{rad } \mathcal{L}_1^n \geq \text{rad } \mathcal{L}_1^\infty \)
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  \[ \mathcal{L}_1^\infty(z) = (1 - \sqrt{1 - 4z})/2z \]
- \( p_\infty = \text{rad } \mathcal{L}_1^\infty = 1/4 \)
Adapting previous tools...

**Direct limit of monoids:** $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_{\infty}$

**Left-constrained traces:** $\mathcal{L}_i^n = \{ \tau \in A_n : L(\tau) \subseteq \{S_1, \ldots, S_i\}\}$
Adapting previous tools...

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- \( \mathcal{L}_{i+1}^{n+1} = \text{sh}(\mathcal{L}_i^n) \cdot \mathcal{L}_1^{n+1} \)
Growth rates of braid monoids: an algebraic proof (1/2)

Adapting previous tools...

Direct limit of monoids: \( A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_\infty \)

Left-constrained traces: \( \mathcal{L}_i^n = \{ \tau \in A_n : L(\tau) \subseteq \{S_1, \ldots, S_i\} \} \)
\( \overline{\mathcal{L}}_i^n = \{ \tau \in A_n : L(\tau) = \{S_1, \ldots, S_i\} \} \)

and associated results:

- \( \mathcal{L}_{i+1}^{n+1} = sh(\mathcal{L}_i^n) \cdot \mathcal{L}_1^{n+1} \)
- \( \mathcal{L}_1^n = \mathcal{L}_0^n + \overline{\mathcal{L}}_1^n \)
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and associated results:

\[
\begin{align*}
\mathcal{L}_{i+1}^{n+1} &= \text{sh}(\mathcal{L}_i^n) \cdot \mathcal{L}_1^{n+1} \\
\mathcal{L}_1^n &= \mathcal{L}_0^n + \overline{\mathcal{L}}_1^n \\
\Delta \{S_1, \ldots, S_i\} \cdot \mathcal{L}_{i+1}^n &= \overline{\mathcal{L}}_i^n + \overline{\mathcal{L}}_{i+1}^n
\end{align*}
\]
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- \( \mathcal{L}_1^n = \mathcal{L}_0^n + \overline{\mathcal{L}}_1^n \)
- \( \Delta\{s_1, \ldots, s_i\} \cdot \mathcal{L}_{i+1}^n = \overline{\mathcal{L}}_i^n + \overline{\mathcal{L}}_{i+1}^n \)
- \( \sum_{i=0}^{n+1} (-1)^i \Delta\{s_1, \ldots, s_{i-1}\} \cdot \mathcal{L}_i^n = 0 \)
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- \( q_n = \text{rad} \mathcal{L}_1^n \geq \text{rad} \mathcal{L}_1^\infty \)
- \( Q_z(\mathcal{L}_1^\infty(z)) = 0 \) when \( z < \text{rad} \mathcal{L}_1^\infty \)
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- \( q_n = \text{rad} \mathcal{L}_1^n \geq \text{rad} \mathcal{L}_1^\infty \)
- \( Q_z(\mathcal{L}_1^\infty(z)) = 0 \) when \( z < \text{rad} \mathcal{L}_1^\infty \)
- No proof that \( q_\infty \leq \text{rad} \mathcal{L}_1^\infty \)!
Go into $\mathbb{Z}[z, \Theta, A_\infty]$ and study $Q_\infty = \sum_{n \geq 0} \sum_{T \in \mathcal{L}_{n-1}} \Theta^n (-1)^{|T|} \Delta_T$
set $\max T = \max \{ i : S_i \in T \}$ and $\max \emptyset = -1$
Growth rates of braid monoids: an algebraic proof (2/2)

Go into $\mathbb{Z}[z, \Theta, A_\infty]$ and study $Q_\infty = \sum_{n \geq 0} \sum_{T \in \mathcal{L}_{n-1}} \Theta^n (-1)^{|T|} \Delta_T$

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Lemma: Each $T \subseteq \mathcal{L}_\infty$ factors uniquely as $T = \{ S_1, \ldots, S_{k-1} \} \sqcup \text{sh}^k (\hat{T})$

and $\max T = \max \hat{T} + k - 1_{T=\emptyset}$
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Theorem: $Q_z(\Theta) \cdot Q_\infty(z) = Q_z(\Theta) \cdot \sum_{n \geq 0} \Theta^n Q_n(z) = 1$
Growth rates of braid monoids: an algebraic proof (2/2)

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Theorem: $Q_z(\Theta) \cdot Q_\infty(z) = Q_z(\Theta) \cdot \sum_{n \geq 0} \Theta^n Q_n(z) = 1$

Proof:

$$Q_\infty = \sum_{n \geq 0} \sum_{T \in L_{n-1}} \Theta^n (-1)^{|T|} \Delta_T$$
Go into $\mathbb{Z}[z, \Theta, A_{\infty}]$ and study $Q_\infty = \sum_{n \geq 0} \sum_{T \in L_{n-1}} \Theta^n(-1)^{|T|} \Delta_T$
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Lemma: Each $T \subseteq \mathbb{L}_\infty$ factors uniquely as $T = \{S_1, \ldots, S_{k-1}\} \sqcup \mathrm{sh}^k(\hat{T})$
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Proof:

$Q_\infty = \sum_{n \geq 0} \sum_{T \in L_{n-1}} \Theta^n(-1)^{|T|} \Delta_T = \sum_{T \in L_{\infty}} \sum_{n \geq \max T} \Theta^{n+1}(-1)^{|T|} \Delta_T$
Growth rates of braid monoids: an algebraic proof (2/2)

Go into $\mathbb{Z}[z, \Theta, A_\infty]$ and study $Q_\infty = \sum_{n \geq 0} \sum_{T \in L_{n-1}} \Theta^n (-1)^{|T|} \Delta_T$

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**Theorem:** $Q_z(\Theta) \cdot Q_\infty(z) = Q_z(\Theta) \cdot \sum_{n \geq 0} \Theta^n Q_n(z) = 1$

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\[
Q_\infty = \sum_{n \geq 0} \sum_{T \in L_{n-1}} \Theta^n (-1)^{|T|} \Delta_T = \sum_{T \in L_\infty} \sum_{n \geq \max T} \Theta^{n+1} (-1)^{|T|} \Delta_T
\]

\[
= 1 + \sum_{k \geq 1} \sum_{\hat{T} \in L_\infty} \sum_{n \geq \max \hat{T}} \Theta^{k+n+1} (-1)^{k-1+|\hat{T}|} \Delta\{S_1, \ldots, S_{k-1}\} \cdot \text{sh}^k(\Delta \hat{T})
\]
Go into $\mathbb{Z}[z, \Theta, A_\infty]$ and study $Q_\infty = \sum_{n \geq 0} \sum_{T \in L_{n-1}} \Theta^n (-1)^{|T|} \Delta_T$
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Lemma: Each $T \subseteq L_\infty$ factors uniquely as $T = \{ S_1, \ldots, S_{k-1} \} \sqcup sh^k(\hat{T})$
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Theorem: $Q_z(\Theta) \cdot Q_\infty(z) = Q_z(\Theta) \cdot \sum_{n \geq 0} \Theta^n Q_n(z) = 1$

Proof:

\[ Q_\infty = \sum_{n \geq 0} \sum_{T \in L_{n-1}} \Theta^n (-1)^{|T|} \Delta_T = \sum_{T \in L_\infty} \sum_{n \geq \max T} \Theta^{n+1} (-1)^{|T|} \Delta_T \]

\[ = 1 + \sum_{k \geq 1} \sum_{\hat{T} \in L_\infty} \sum_{n \geq \max \hat{T}} \Theta^{k+n+1} (-1)^{k-1+|\hat{T}|} \Delta_{\{S_1, \ldots, S_{k-1}\}} \cdot sh^k(\Delta_{\hat{T}}) \]

\[ = 1 - \sum_{k \geq 1} (-1)^k \Theta^k \Delta_{\{S_1, \ldots, S_{k-1}\}} \cdot sh^k(Q_\infty) \]
Growth rates of braid monoids: an algebraic proof (2/2)

Go into \( \mathbb{Z}[z, \Theta, A_{\infty}] \) and study
\[
Q_{\infty} = \sum_{n \geq 0} \sum_{T \in L_{n-1}} \Theta^n (-1)^{|T|} \Delta_T \text{ set max } T = \max\{i: S_i \in T\} \text{ and max } \emptyset = -1
\]

Lemma: Each \( T \subseteq I_{\infty} \) factors uniquely as \( T = \{S_1, \ldots, S_{k-1}\} \sqcup \text{sh}^k(\hat{T}) \) and \( \text{max } T = \text{max } \hat{T} + k - 1 \) \( T = \emptyset \)

Theorem: \( Q_z(\Theta) \cdot Q_{\infty}(z) = Q_z(\Theta) \cdot \sum_{n \geq 0} \Theta^n Q_n(z) = 1 \)

Proof:
\[
Q_{\infty} = \sum_{n \geq 0} \sum_{T \in L_{n-1}} \Theta^n (-1)^{|T|} \Delta_T = \sum_{T \in L_{\infty}} \sum_{n \geq \text{max } T} \Theta^{n+1} (-1)^{|T|} \Delta_T
\]
\[
= 1 + \sum_{k \geq 1} \sum_{\hat{T} \in L_{\infty}} \sum_{n \geq \text{max } \hat{T}} \Theta^{k+n+1} (-1)^{k-1+|\hat{T}|} \Delta\{S_1, \ldots, S_{k-1}\} \cdot \text{sh}^k(\Delta_{\hat{T}})
\]
\[
= 1 - \sum_{k \geq 1} \Theta^k (-1)^k \Delta\{S_1, \ldots, S_{k-1}\} \cdot \text{sh}^k(Q_{\infty})
\]
\[
Q_{\infty}(z) = 1 - \sum_{k \geq 1} \Theta^k (-1)^k z^{k(k-1)/2} Q_{\infty}(z)
\]
Go into $\mathbb{Z}[z, \Theta, A_\infty]$ and study $Q_\infty = \sum_{n \geq 0} \sum_{T \in L_{n-1}} \Theta^n (-1)^{|T|} \Delta_T$
set $\max T = \max \{i : S_i \in T\}$ and $\max \emptyset = -1$

**Theorem:** $Q_z(\Theta) \cdot Q_\infty(z) = Q_z(\Theta) \cdot \sum_{n \geq 0} \Theta^n Q_n(z) = 1$

**Corollary:** If $\operatorname{rad} L_1^\infty < z < q_\infty$, then $Q_z(\Theta) \cdot Q_\infty(z) = 1$ for all $\Theta \in \mathbb{C}$.

**Proof:** $P_n(z)/P_{n+1}(z) = L_{n+1}^n(z)/L_n^n(z) = L_{1}^{n+1}(z) \to L_1^\infty(z) = +\infty$, hence $\operatorname{rad} Q_z = \operatorname{rad}_\Theta Q_\infty(z) = +\infty$. 
Growth rates of braid monoids: an algebraic proof (2/2)

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Theorem [11]

In monoids of type $A$, $B$ and $D$, we have $q_\infty = \text{rad} L_1^\infty \approx 0.30904\ldots$
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4 Growth rates of trace and braid monoids: an algebraic proof

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Open problems & future research

- Generating random **infinitely wide & tall** heaps: \( \mathbb{P}[S_i \cdots] = 1/4 \)
Open problems & future research

• Generating random **infinitely wide & tall** heaps: \( \mathbb{P}[S; \cdots] = 1/4 \)

```
+ + + + - + - - + + + + +
- - + - - + + + - - + +
- + - + + - + - + - - +
+ + - - + - - - + + + - +
```

\( \cdots \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots \)
Open problems & future research

- Generating random **infinitely wide & tall** heaps: \( \mathbb{P}[S_i; \cdots] = 1/4 \)
Open problems & future research

- Generating random **infinitely wide & tall** heaps: $P[S_i; \ldots] = 1/4$

![Diagram of heaps](image-url)
Open problems & future research

- Generating random **infinitely wide & tall** heaps: \( \mathbb{P}[S; \cdots] = 1/4 \)

\[ ...
- + + - - + + + - + + + + \\
- - + + - + + - + - + + + \\
+ + - - + - - - - + - - + \\
...
\]

\[ \begin{array}{cccccc}
  & S_{-4} & S_{-1} & S_1 & S_4 & \cdots \\
-4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots
\end{array} \]
Generating random **infinitely wide & tall** heaps: \( \mathbb{P}[S_i; \cdots] = 1/4 \)
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- Generating random **infinitely wide & tall** heaps: $\mathbb{P}[S_i \cdots] = 1/4$

- What about **infinitely wide & tall** braids? $\mathbb{P}[S_i \cdots] = q_\infty \approx 0.30904$
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- Generating random **infinitely wide & tall** heaps: \( \mathbb{P}[S_i \cdots] = 1/4 \)

- What about **infinitely wide & tall** braids? \( \mathbb{P}[S_i \cdots] = q_\infty \approx 0.30904 \)

- Investigating other classes of **Artin-Tits** monoids:
  - Coincidence or not? \( Q_z(\Theta) \cdot Q_\infty(z) = 1 \) and \( Q_z(\mathcal{L}_{1}^\infty(z)) = 0 \)
  - Coincidence or not? \( q_\infty = \text{rad} \mathcal{L}_{1}^\infty \)
Bibliography

Thank you very much for your attention!
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Questions?