Grandchildren weight-balanced binary search trees

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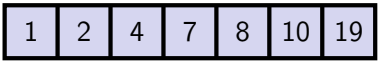

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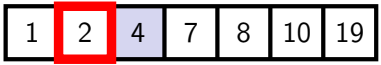

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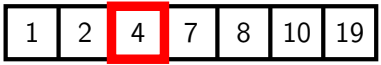
$3 \in S$? **No!**



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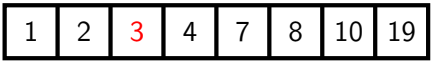

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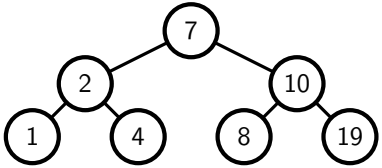

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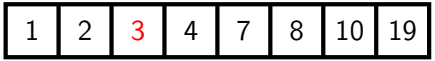

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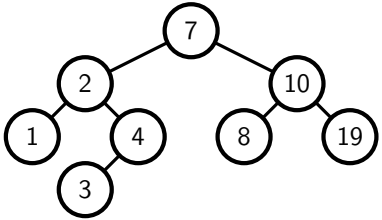

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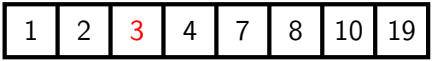

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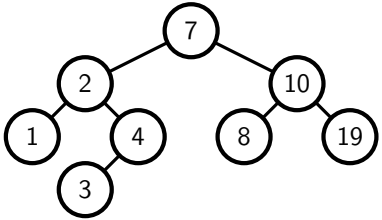

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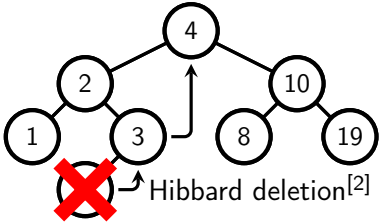
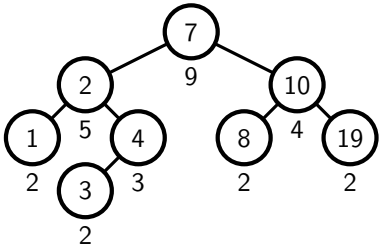


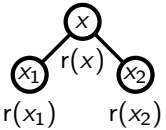
Remove 7!

Balanced binary search trees

Maintaining search trees of height $\mathcal{O}(\log(n))$ often requires some kind of **rank** and **balance**.

1. Weight-balanced trees[3]: $\mathsf{r}(x) = \#\mathcal{T}(x) + 1$ $\qquad (\mathsf{r}(x) = \mathsf{r}(x_1) + \mathsf{r}(x_2)$ and $\mathsf{r}(\bot) = 1)$
   $\mathsf{r}(x_i) \leqslant \alpha \mathsf{r}(x)$ $\qquad (1/\sqrt{2} \leqslant \alpha < 9/11)$





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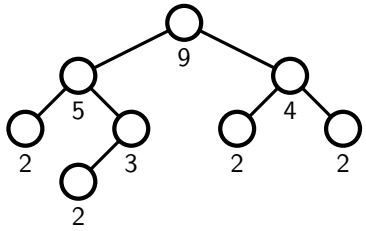

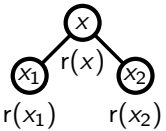

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2. AVL trees[1]: $\mathsf{r}(x) = \mathsf{h}(x)$ $\qquad (\mathsf{r}(x) = \max\{\mathsf{r}(x_1), \mathsf{r}(x_2)\} + 1$ and $\mathsf{r}(\bot) = -1)$
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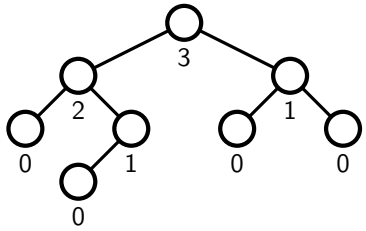

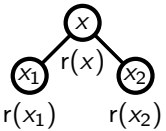

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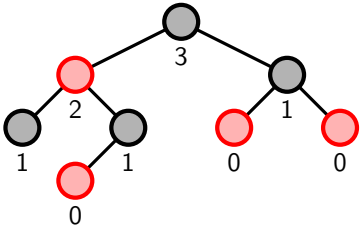
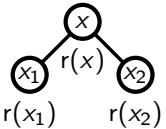
3. Red-black trees[5,9]: $\mathsf{r}(x) = ?$ $\qquad (\mathsf{r}(\bot) = -1)$
   $\mathsf{r}(x_i) + 1 \leqslant \mathsf{r}(x) \leqslant \mathsf{r}(x_i) + 2$ $\qquad (x_i$ red $\Leftrightarrow \lfloor \mathsf{r}(x_i)/2 \rfloor = \lfloor \mathsf{r}(x)/2 \rfloor)$





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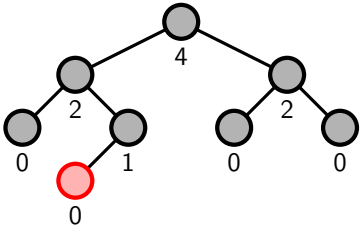
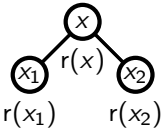
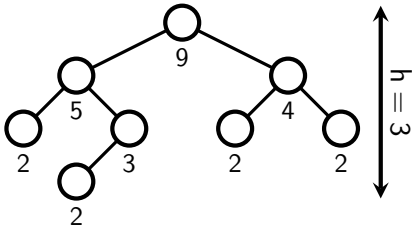


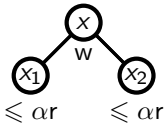


Height and internal/external path length

In weight-balanced trees with parameter $\alpha \in [1/\sqrt{2}, 9/11)$, in the worst case,

1. Height h $\approx \log_\alpha(n)$.





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1. Height $h \approx \log_{\alpha}(n)$.
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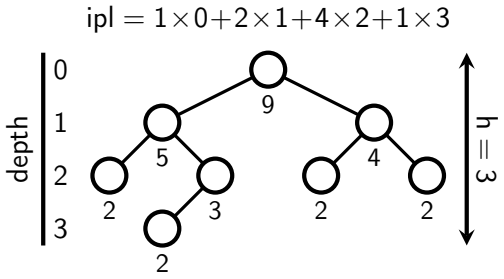
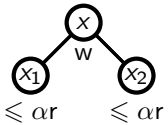
$$H_2(\alpha) = -\alpha \log_2(\alpha) - (1-\alpha)\log_2(1-\alpha).$$





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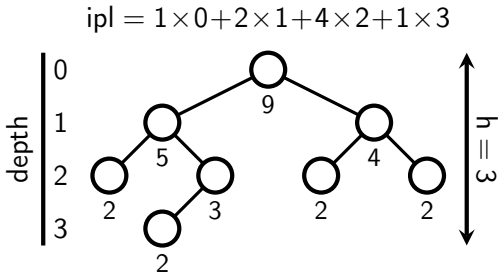

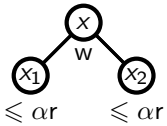

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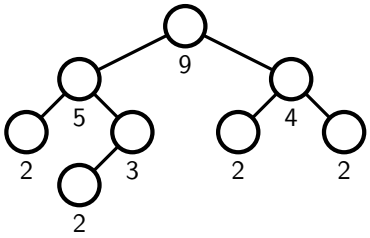

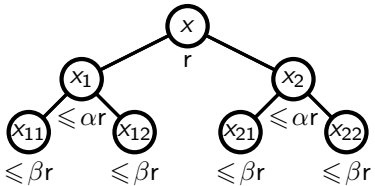

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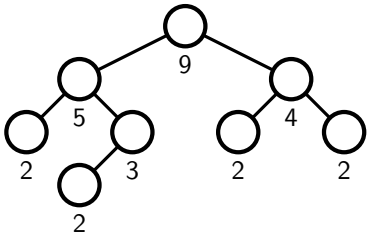
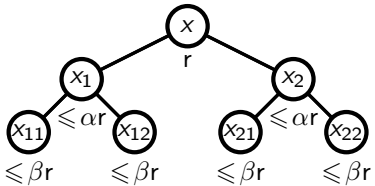
$$\text{where } \mathscr{B}(\alpha) = \frac{\sqrt{1+4\alpha}-1}{2}.$$

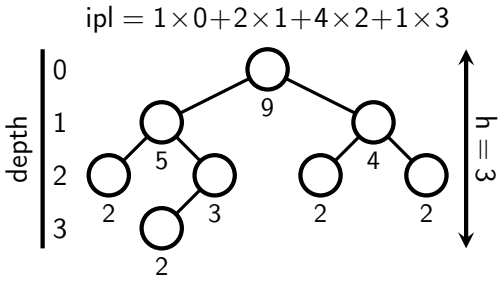


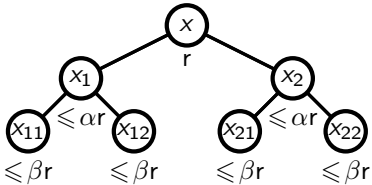


Improved height and internal/external path length

In trees with parameters $\alpha \in [1/\sqrt{2}, 3/4]$ and $\beta \in [\mathscr{B}(\alpha), \alpha^2]$, in the worst case,

1. Height $h \approx 2\log_\beta(n) \approx 1.880\log_2(n) < 2\log_2(n)$.





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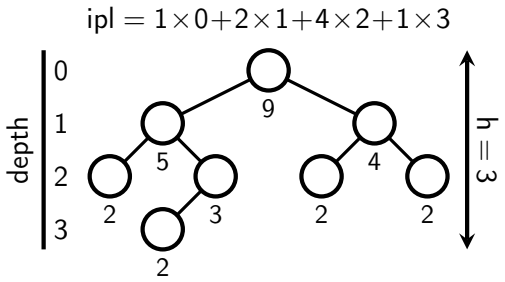
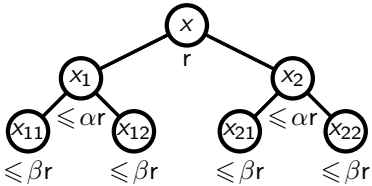
1. Height $h \approx 2\log_\beta(n) \approx 1.880\log_2(n) < 2\log_2(n)$.
2. Internal path length $ipl \approx n\log_2(n)/\Delta \approx 1.127 n\log_2(n) < 1.146 n\log_2(n)$, where

$$\Delta = \frac{H_2(\alpha) + \alpha H_2(\beta/\alpha)}{1+\alpha}.$$





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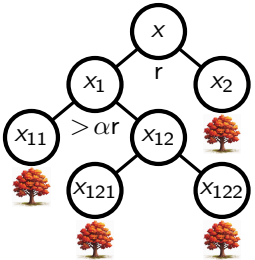
Routine **C**: Use rotations to keep your **C**hildren small[3]



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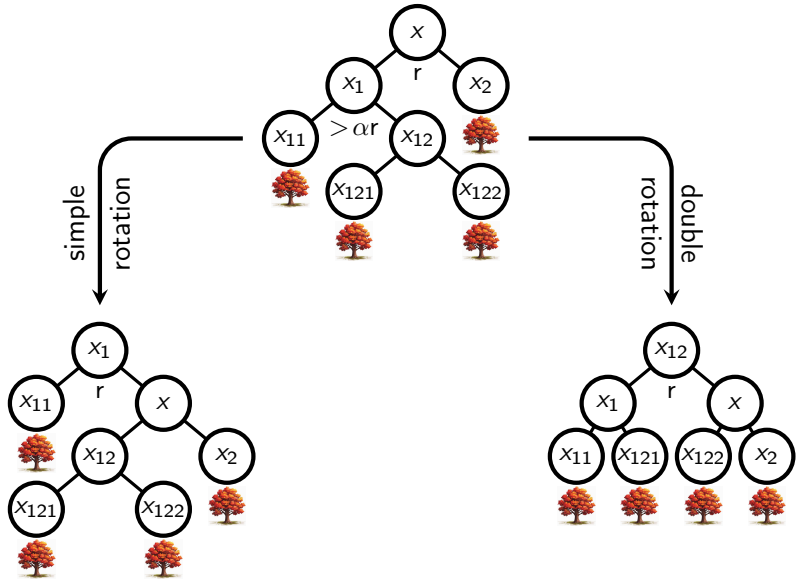

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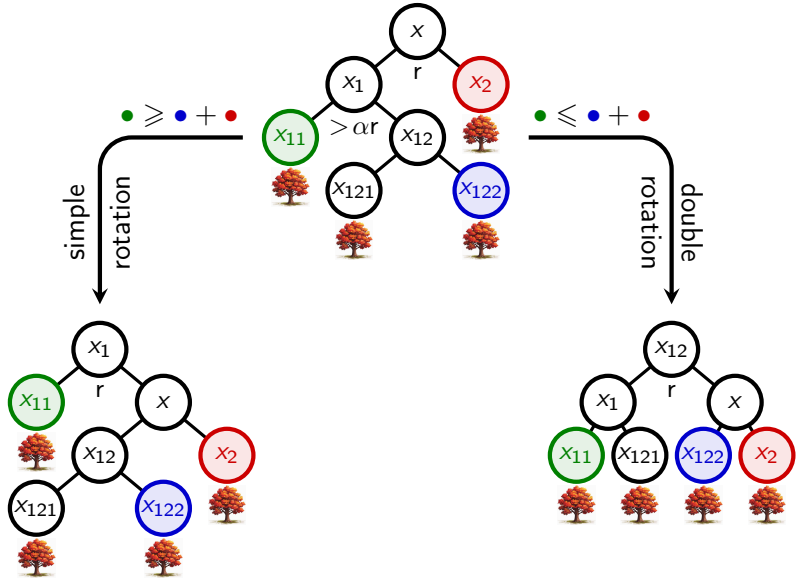
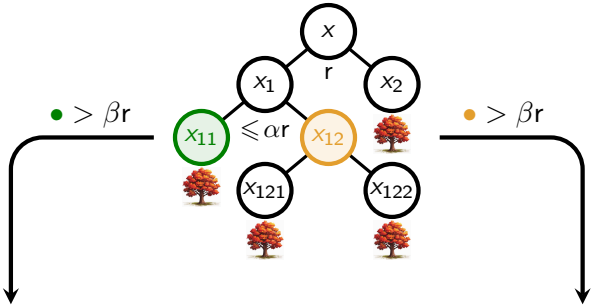


Routine **GC**: Use rotations to keep your **G**rand**C**hildren small



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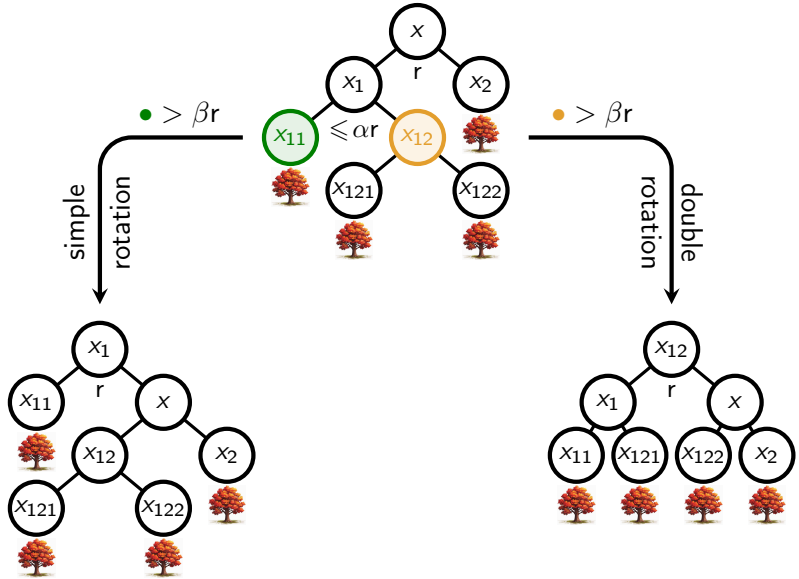
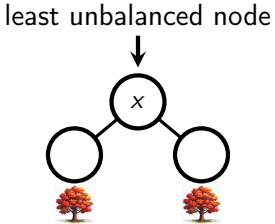


Combining both routines **C** and **GC**

Pros & cons of each routine:

- Routine **C** ensures everybody has small children once your descendants have small children. It **may** mess with the grandchildren balance of those nodes its displaces. ☹



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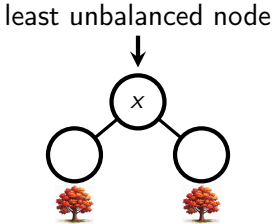

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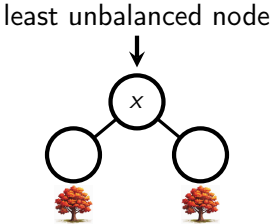

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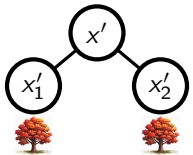

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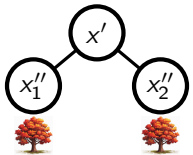

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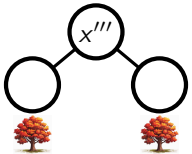
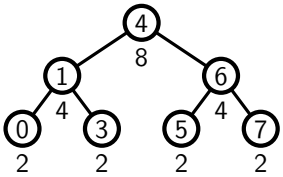


Contents

What is top-down maintenance?

Insert 2 **bottom-up**



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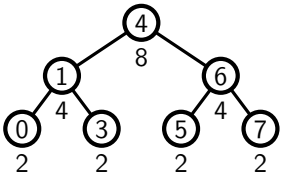

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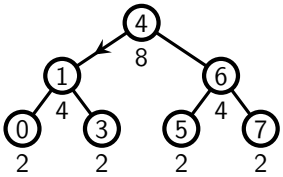

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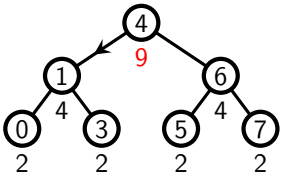

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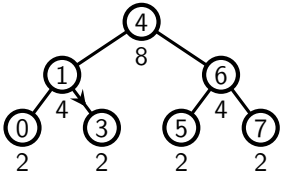

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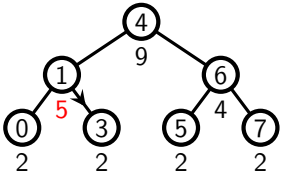

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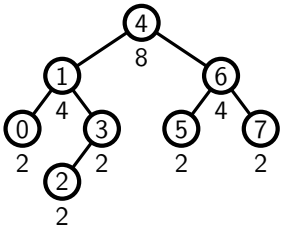

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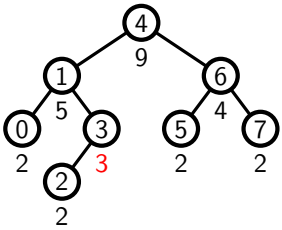

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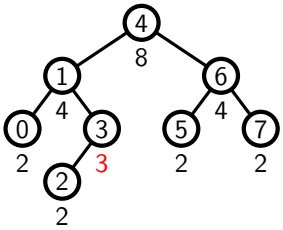

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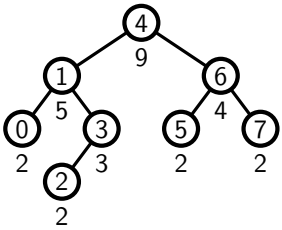

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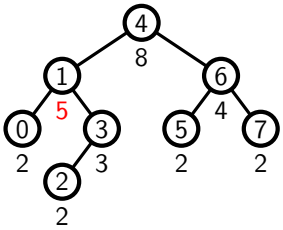

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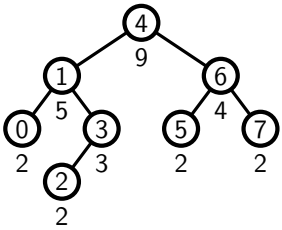

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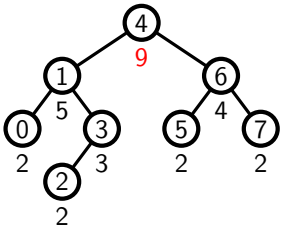

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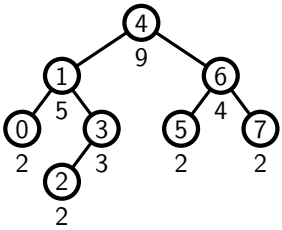

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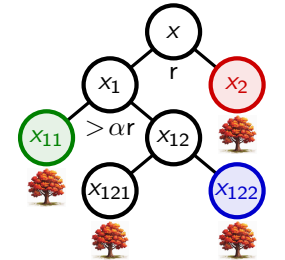
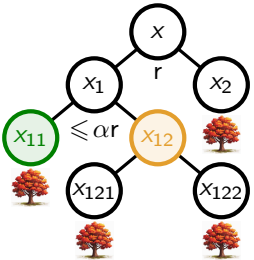
Make **C** and **GC** routines trigger rotations **as soon as** adding or deleting a leaf below x might make x unbalanced[7].



$\bullet \leqslant \bullet + \bullet$ or $\bullet \geqslant \bullet + \bullet$



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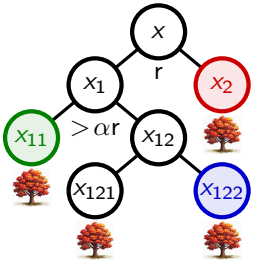
Redundant update = Trying to insert an already present key or delete an already absent key

At each node x, store $\mathrm{r}(x_i)$ for some child x_i **not** on the branch you modify[8]:
It just remains to restore the tree rank, instead of all node ranks on the modified branch.



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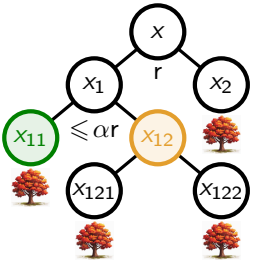

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Contents

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The sum of all counters

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- decreases by $\Omega(\mathsf{c})$ when a rotation takes place.

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