Efficient top-down updates in AVL trees

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08/05/2025

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Add 3!

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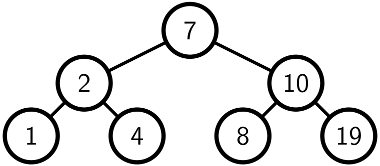
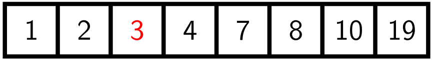
Use a low-height binary search tree instead.



Add 3!



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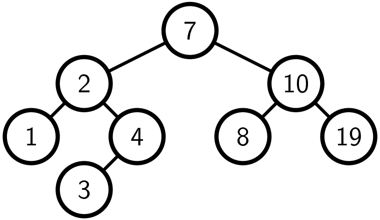

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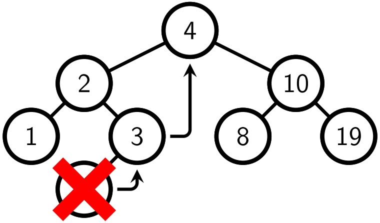

Remove 7!

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Balanced binary search trees

Maintaining search trees of height $\mathcal{O}(\log(n))$ often requires some kind of **rank** and **balance**.

1. Weight-bounded trees: $\mathrm{r}(x) = \#\mathcal{T}(x) + 1$ $\quad(\mathrm{r}(x) = \mathrm{r}(x_1) + \mathrm{r}(x_2)$ and $\mathrm{r}(\perp) = 1)$
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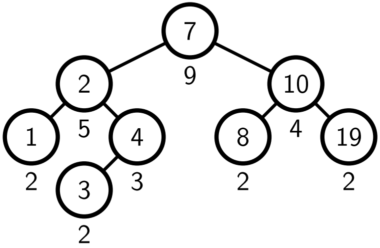

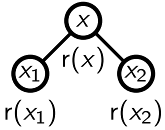

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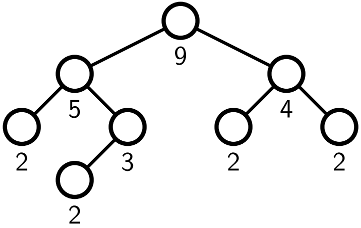

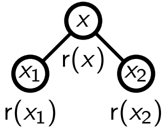

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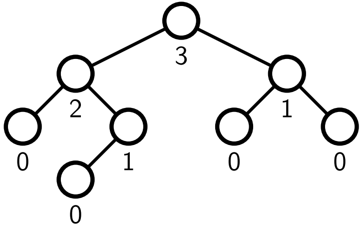

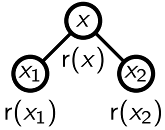

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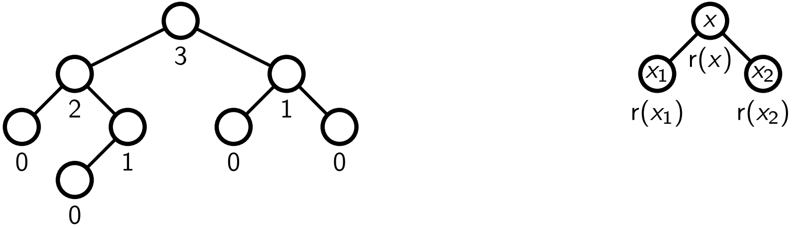

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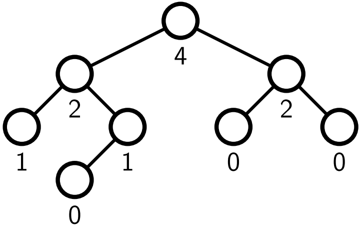

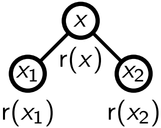

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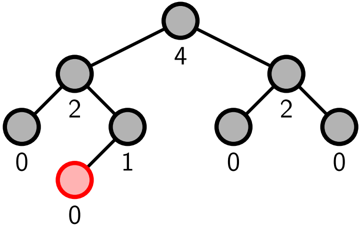

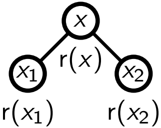

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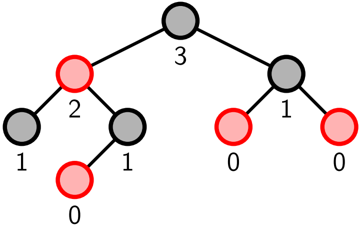

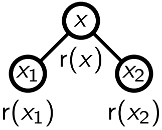

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Tree	Invented	Height	Am. writes/update	Top-down
Weight-balanced	1972	$2\log_2(n)$	$\Theta(1)$	yes
AVL	1962	$1.44\log_2(n)$	$\Theta(\log(n))$	no
Weak AVL	2015	$2\log_2(n)$	$\Theta(1)$	yes
Red-black	1978	$2\log_2(n)$	$\Theta(1)$	yes

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Write operations are quite more expensive than **read** operations.

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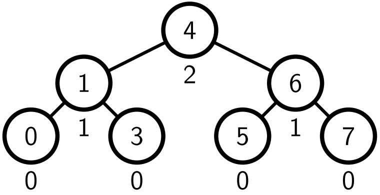

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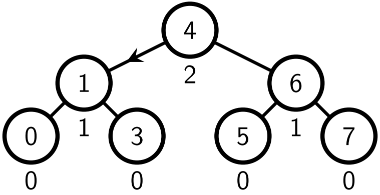

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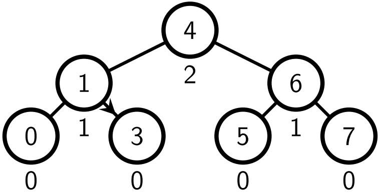

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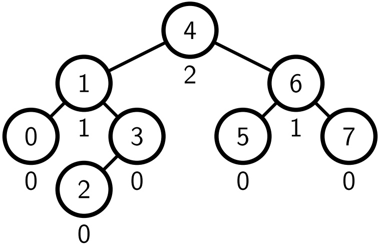

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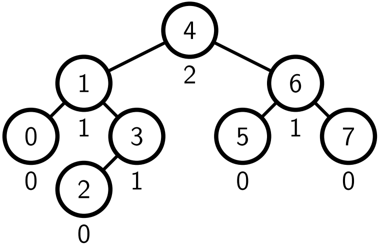

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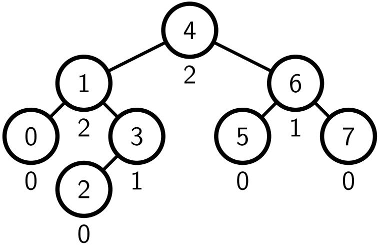

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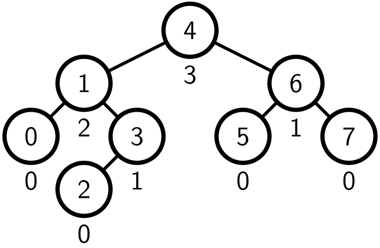

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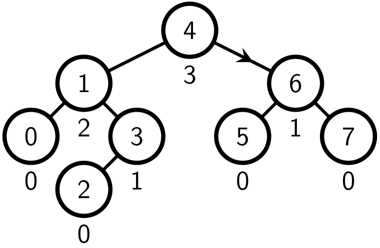

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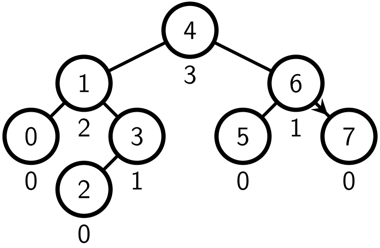

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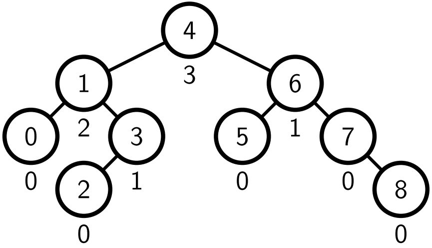

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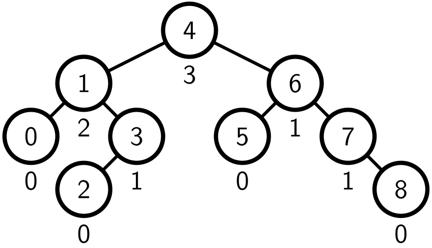

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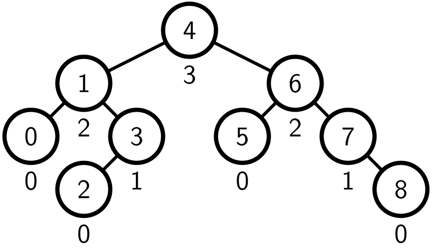

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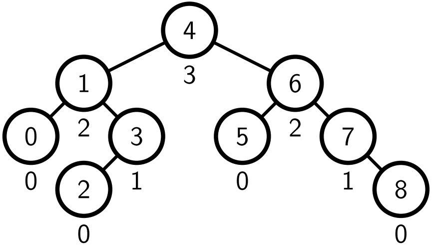

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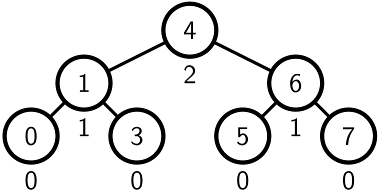

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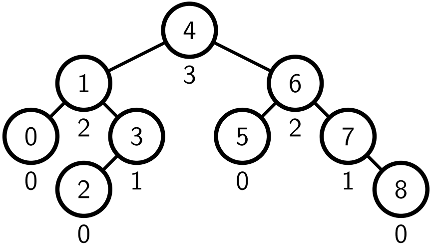

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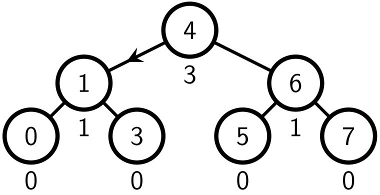

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Write operations are quite more expensive than **read** operations.

Amortised complexity: Worst-case **average** complexity over **arbitrary** sequences of operations

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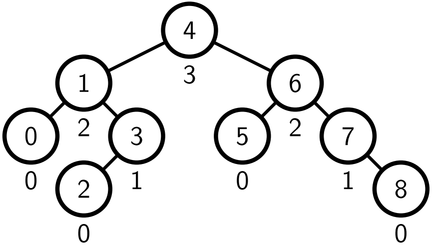

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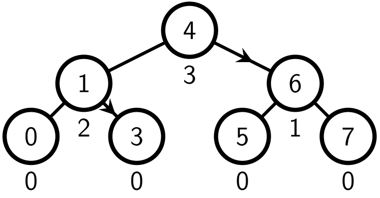

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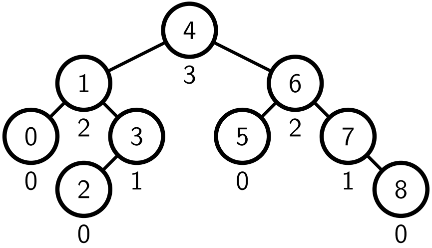

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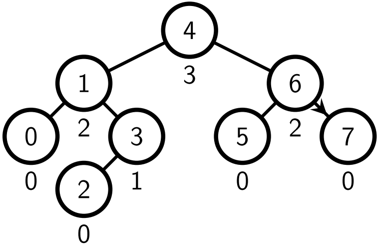

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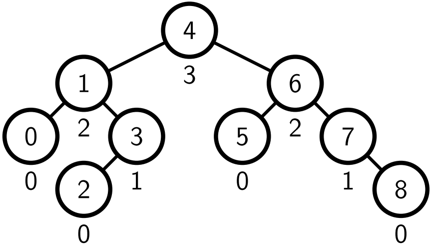

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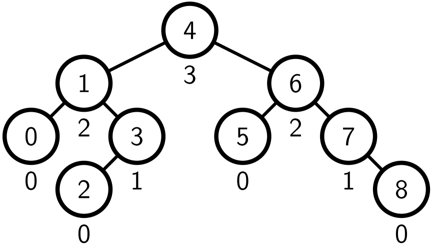

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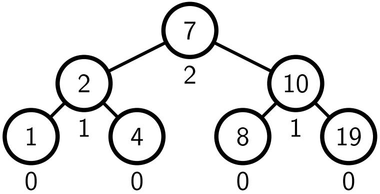

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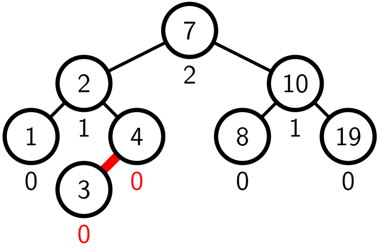

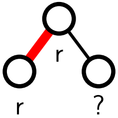

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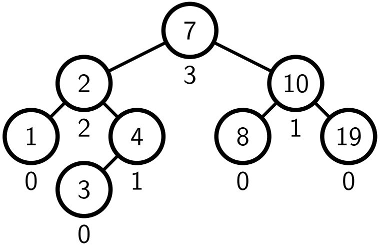

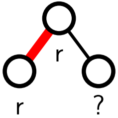

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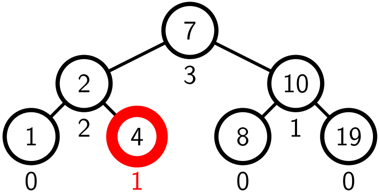

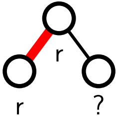

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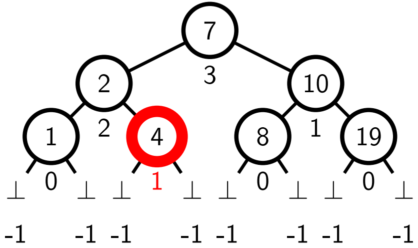

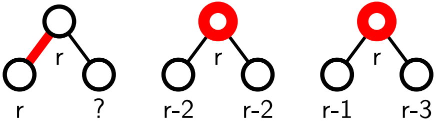

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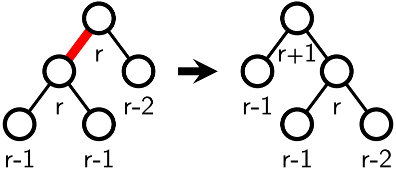

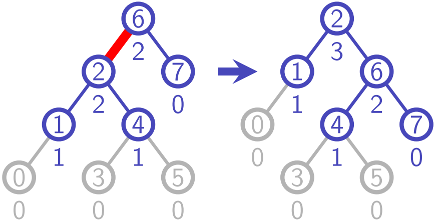

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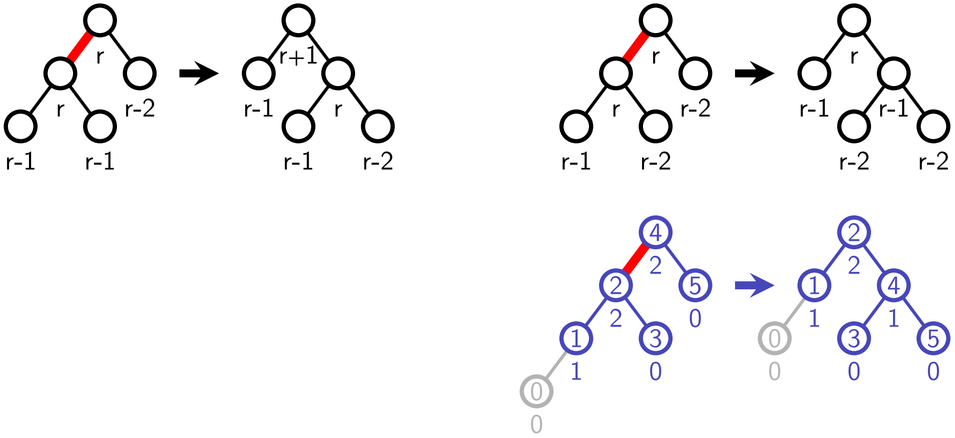

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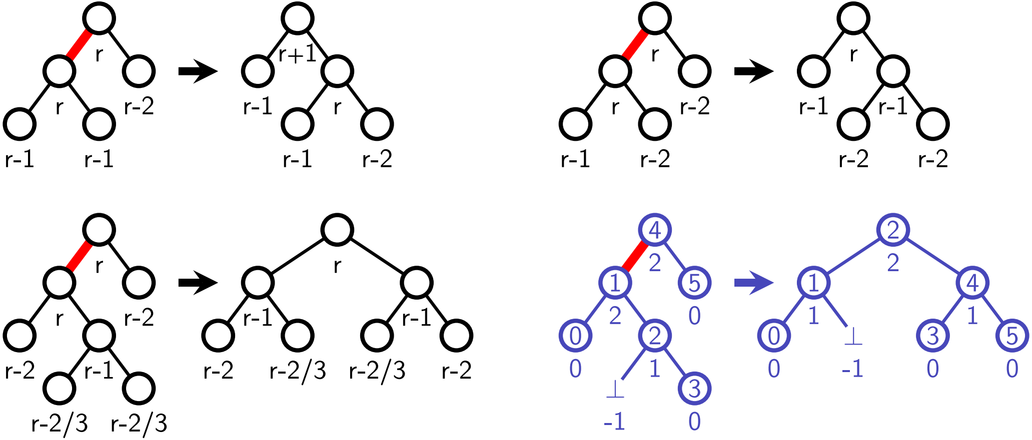

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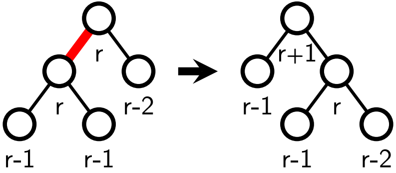

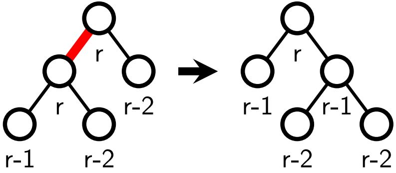

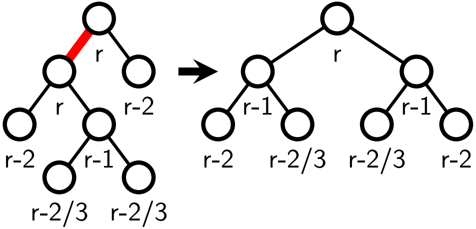

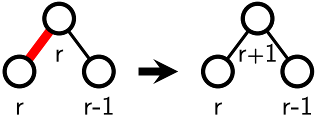

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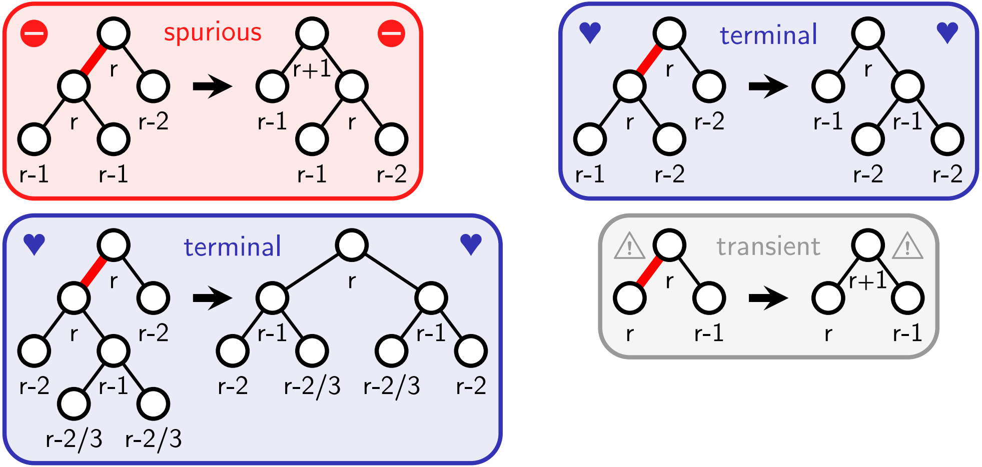

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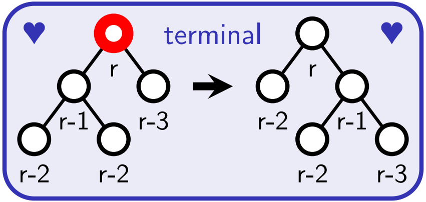

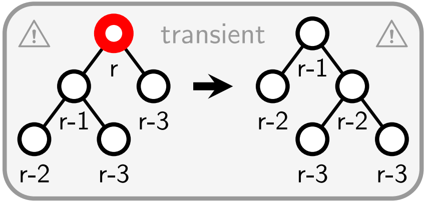

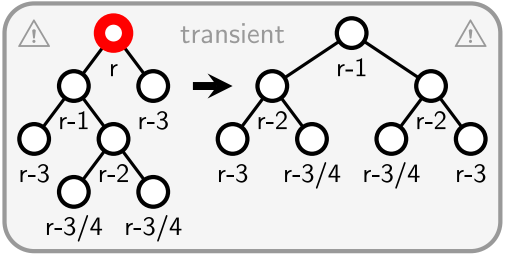

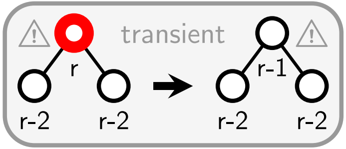

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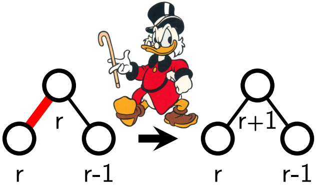

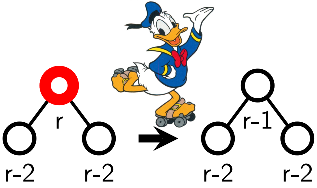

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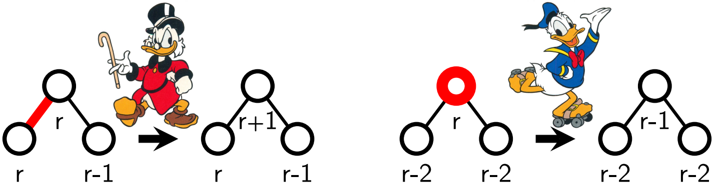

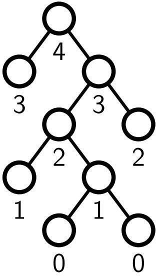

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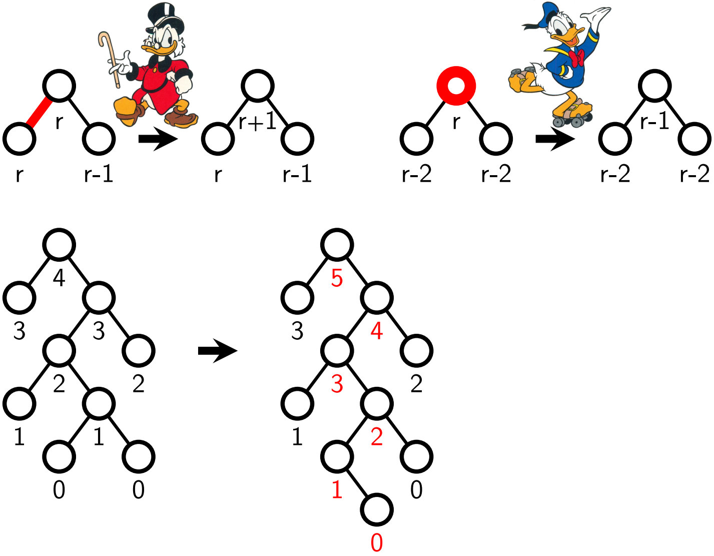

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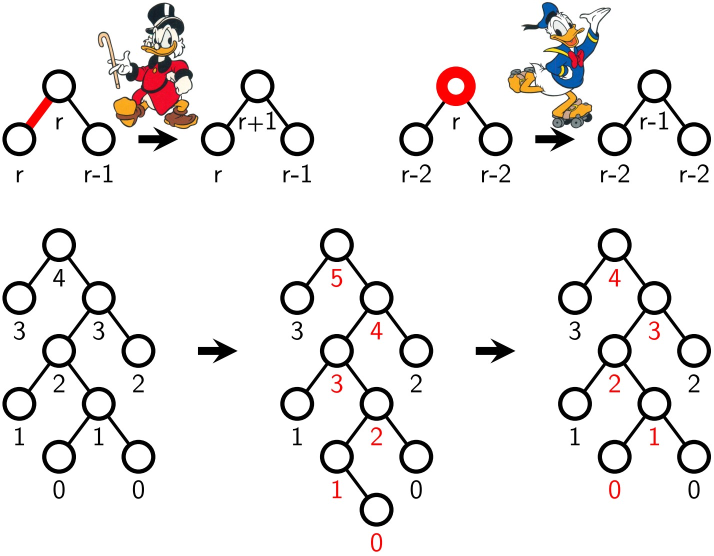

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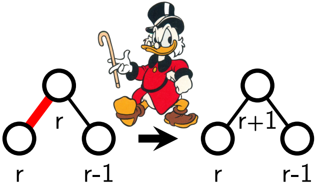

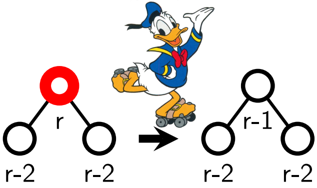

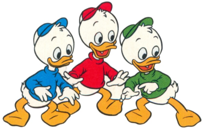

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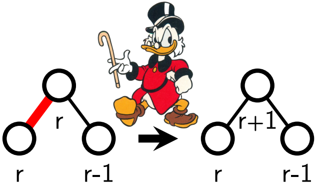

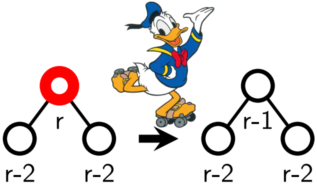

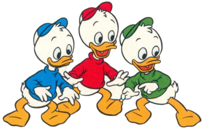

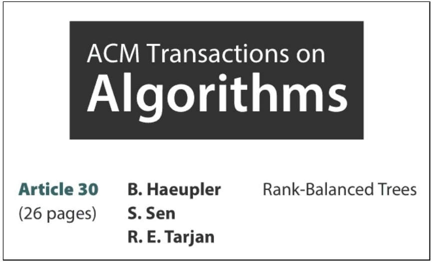
ACM Transactions on
Algorithms

Article 30 (26 pages) — B. Haeupler, S. Sen, R. E. Tarjan — Rank-Balanced Trees

Updating AVL trees: Stopping anomalies faster

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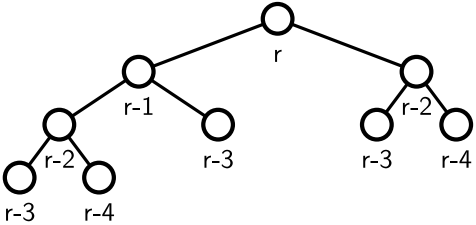

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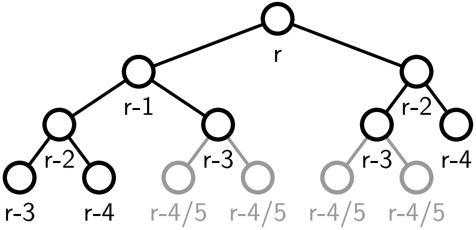

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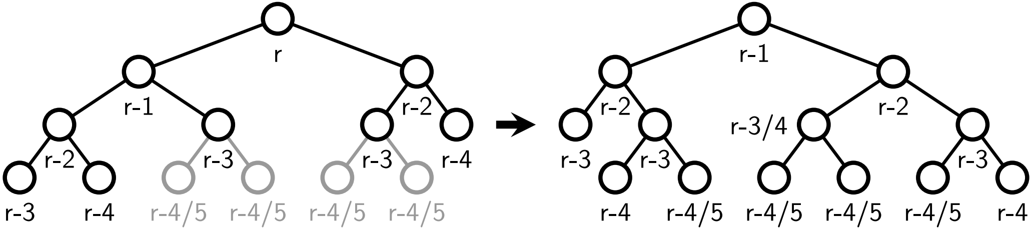

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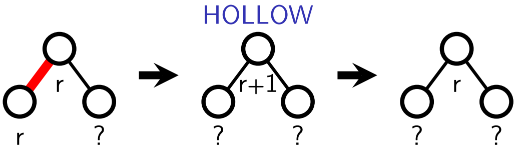

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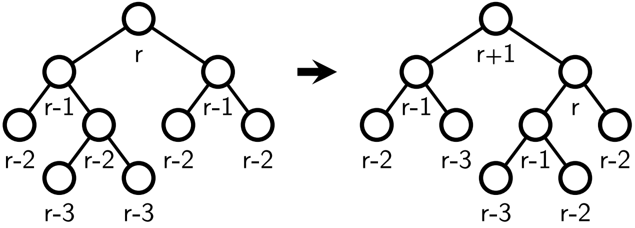

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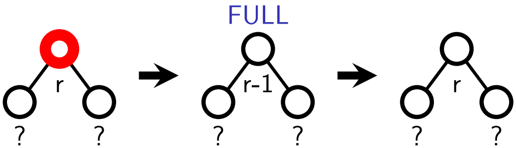

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With q queries + $\mathbf{T}$ transient operations:

- tree potential increases by $\mathcal{O}(q) - \mathbf{T}$ or less, and
- tree potential remains non-negative, hence
- $\mathbf{T} = \mathcal{O}(q)$.

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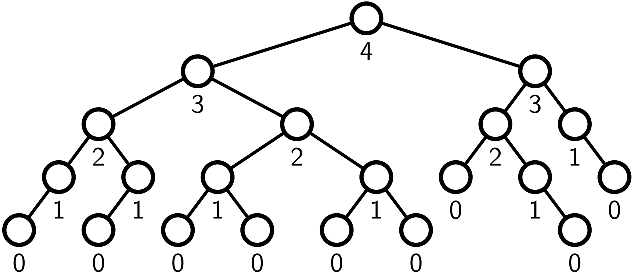

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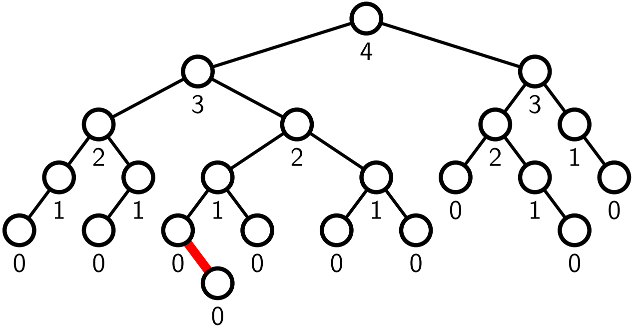

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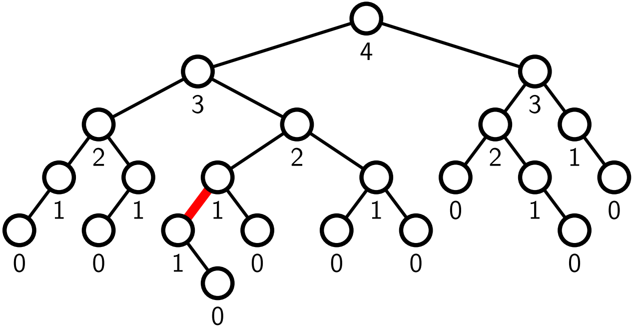

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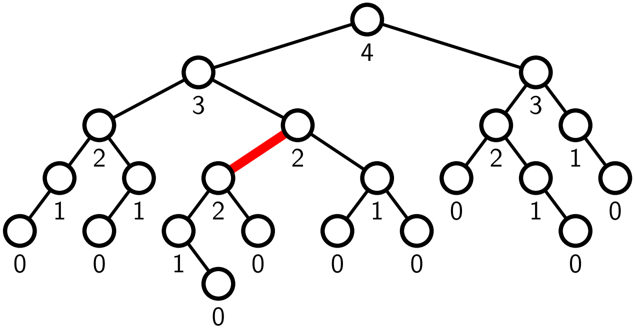

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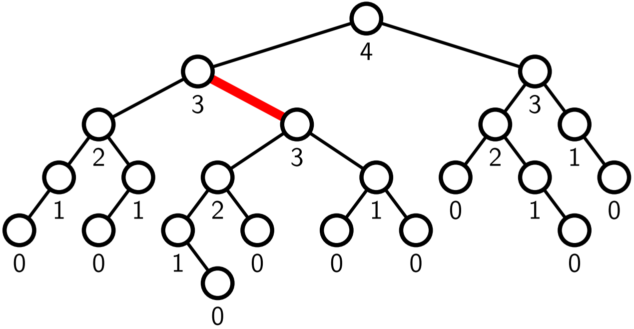

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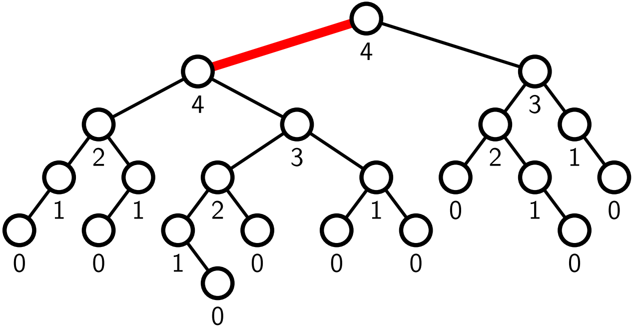

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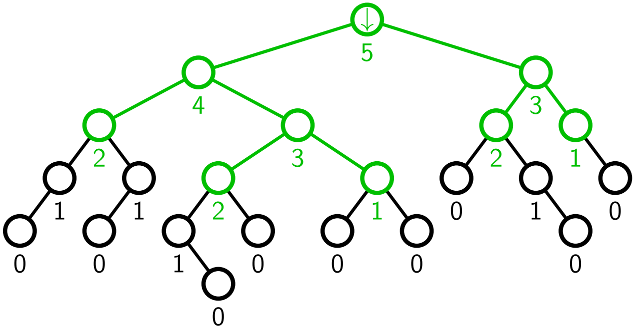

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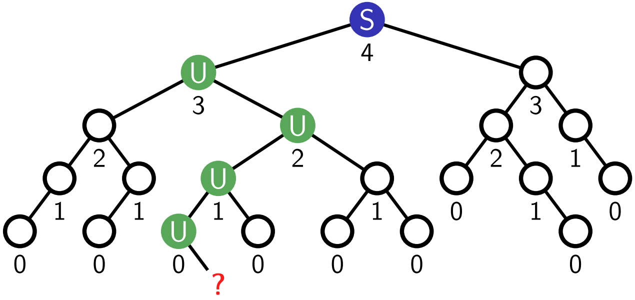

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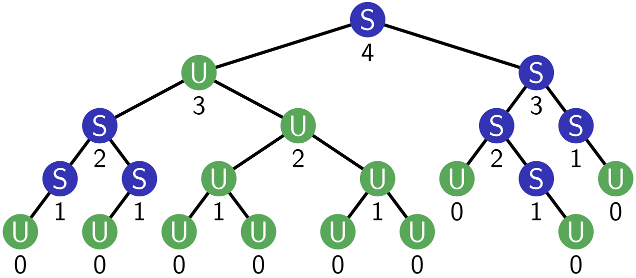
Nodes of rank $r \geqslant 2$ are insertion-**safe** when they are not **full**.

(Other insertion-safe nodes are those with rank $r = 1$ and one 2-child.)



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(Other insertion-safe nodes are those with rank $r = 1$ and one 2-child.)

A node is deletion-**safe** if it stops propagating **four-nodes**.
Whether a node is deletion-**safe** depends on a finite neighbourhood **and**
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Safe nodes: When do we stop anomalies?

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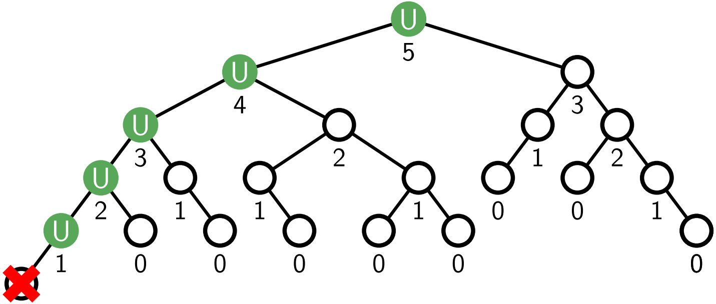

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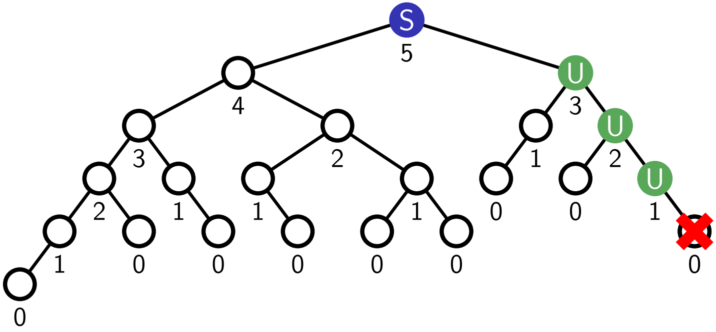
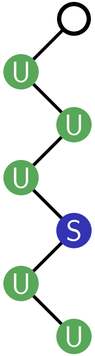


Finding or creating safe nodes (1/2)

Top-down insertion algorithm:

1. Look for a **safe** node on your insertion branch.



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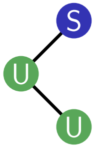

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1. Look for a **safe** node on your insertion branch.
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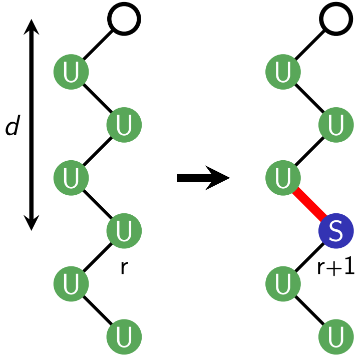

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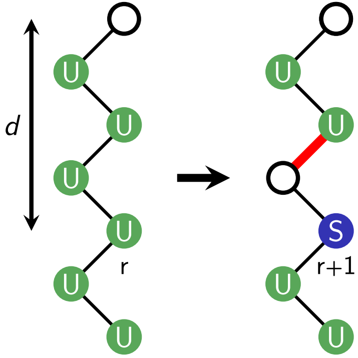

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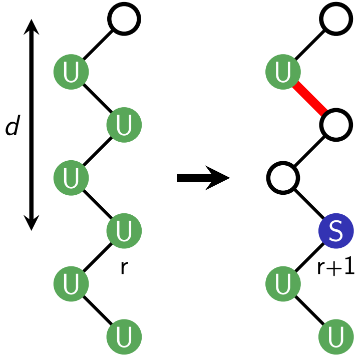

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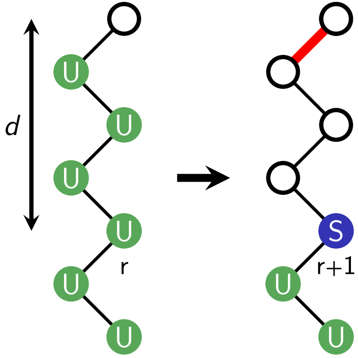

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Top-down insertion algorithm:

1. Look for a **safe** node on your insertion branch.
2. If you succeed quickly, restart from that node. (write cost $= 0$)
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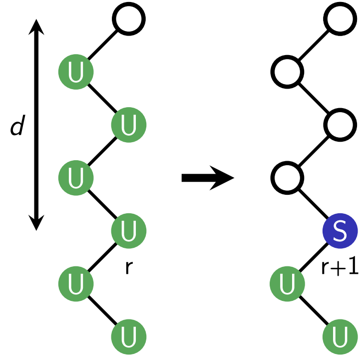
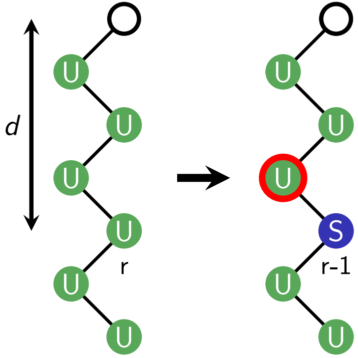


Finding or creating safe nodes (2/2)

Top-down deletion algorithm:

1. Look for a **safe** node on your deletion branch.
2. If you succeed quickly, restart from that node. (write cost $= 0$)
3. If you fail, (write cost $= \mathcal{O}(d+1)$ & $\Delta\text{Pot} \leqslant \mathcal{O}(1) - d$)
   - **demote** an **unsafe** node at depth d & make it **safe**;
   - **propagate** to the top the **four-node** you just created.



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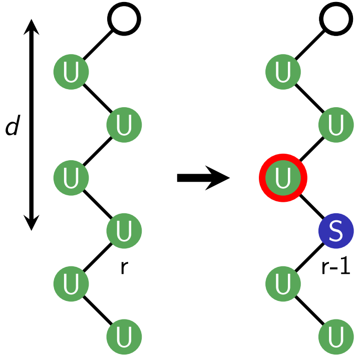
   - **demote** an **unsafe** node at depth d & make it **safe**;
   - **propagate** to the top the **four-node** you just created.

Challenge: Demote deletion-unsafe nodes & make them safe!



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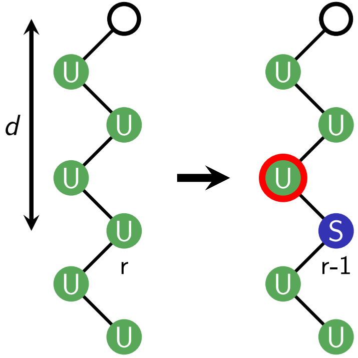

Challenge: Demote deletion-unsafe nodes & make them safe!
Answer: When facing 4 unsafe nodes in a row, you can do it!

Updating AVL trees efficiently top-down

Theorem

Starting from the empty AVL tree, q top-down queries trigger $\mathcal{O}(q)$ write operations.

Proof:

Tree potential decreases with each batch of d transient operations once d is large enough.

With q queries + B batches:

- tree potential increases by $\mathcal{O}(q) - \mathsf{B}$ or less, and
- tree potential remains non-negative, hence
- $\mathsf{B} = \mathcal{O}(q)$.

Contents

Like all over balanced tree structures. . .

AVL trees enjoy:

1. Efficient bottom-up updating algorithms.

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Main take-away:

7. Analyse algorithms and data structures that you love and adapt them!

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MERCI POUR VOTRE ATTENTION !
NE POSEZ PAS DE QUESTIONS TROP DIFFICILES S'IL VOUS PLAÎT !