

NONCOMMUTATIVE FREE CUMULANTS

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ABSTRACT. The functional equation defining the free cumulants in free probability is lifted successively to the noncommutative Faà di Bruno algebra, and then to the group of a free operad over Schröder trees. This leads to new combinatorial expressions, which remain valid for operator-valued free probability. Specializations of these expressions give back Speicher's formula in terms of noncrossing partitions, and its interpretation in terms of characters due to Ebrahimi-Fard and Patras.

1. INTRODUCTION

1.1. Functional equations and combinatorial Hopf algebras. Recent works on certain functional equations involving reversion of formal power series have revealed that the appropriate setting for their combinatorial understanding involved a series of noncommutative generalizations, ending up as an equation in the group of an operad.

Roughly speaking, this amounts to first interpreting the equation in the Faà di Bruno Hopf algebra, lifting it to its noncommutative version, and then replacing the constant term by a new indeterminate, giving rise to tree expanded series.

This approach to functional inversion has been initiated in [16, 17, 18], and then extended in [13] to deal with the conjugacy equation for formal diffeomorphisms.

Free probability provides other examples of functional equations with a combinatorial solution. The relation between the moments and the free cumulants of a single random variable is just a functional inversion, which can be treated combinatorially by the formalism of [13]. However, the case of several random variables is classically formulated as a triangular system of equations which is solved by Möbius inversion over the lattice of noncrossing partitions [19, 20].

We shall see that this system can be encoded by a single equation in the group of an operad. This version encompasses the case of an operator valued probability. The solution arises as a sum over reduced plane trees which reduces to Speicher's solution in the scalar case. Also, our functional equation gives back that of Ebrahimi-Fard and Patras [7], which interpret the series of the moments as a character of a Hopf algebra, and that of the free cumulants as an infinitesimal character, both related by a dendriform exponential. We have here a similar structure.

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1.2. Classical probability. The free cumulants k_n of a probability measure μ on \mathbb{R} are defined (see *e.g.*, [19]) by means of the generating series of its moments m_n

$$(1) \quad M_\mu(z) := \int_{\mathbb{R}} \frac{\mu(dx)}{z-x} = z^{-1} + \sum_{n \geq 1} m_n z^{-n-1}$$

as the coefficients of its compositional inverse

$$(2) \quad K_\mu(z) := M_\mu(z)^{\langle -1 \rangle} = z^{-1} + \sum_{n \geq 1} k_n z^{n-1}.$$

It is in general instructive to interpret the coefficients of a formal power series as the specializations of the elements of some generating family of the algebra of symmetric functions. In this context, it is the interpretation (in the notation of [12])

$$(3) \quad m_n = \phi(h_n) = h_n(X)$$

which is relevant. Indeed, the process of functional inversion (Lagrange inversion) admits a simple expression within this formalism (see [12], ex. 24 p. 35). If the symmetric functions h_n^* are defined by the equations

$$(4) \quad u = tH(t) \iff t = uH^*(u)$$

where $H(t) := \sum_{n \geq 0} h_n t^n$, $H^*(u) := \sum_{n \geq 0} h_n^* u^n$, then,

$$(5) \quad h_n^*(X) = \frac{1}{n+1} [t^n] E(-t)^{n+1}$$

where $E(t)$ is defined by $E(t)H(t) = 1$. This defines an involution $f \mapsto f^*$ of the ring of symmetric functions. Now, if one sets $m_n = h_n(X)$ as above, then $M_\mu(z) = z^{-1}H(z^{-1}) = u$, so that

$$(6) \quad z = K_\mu(u) = \frac{1}{u} E^*(-u) = u^{-1} + \sum_{n \geq 1} (-1)^n e_n^* u^{n-1},$$

and finally

$$(7) \quad k_n = (-1)^n e_n^*(X).$$

It follows from the explicit formula (see [12] p. 35)

$$(8) \quad -e_n^* = \frac{1}{n-1} \sum_{\lambda \vdash n} \binom{n-1}{l(\lambda)} \binom{l(\lambda)}{m_1, m_2, \dots, m_n} e_\lambda$$

(where $\lambda = 1^{m_1} 2^{m_2} \dots n^{m_n}$) that $-e_n^*$ is Schur positive, and moreover that $-e_n^*$ is the Frobenius characteristic of a permutation representation Π_n , twisted by the sign character. Let

$$(9) \quad (-1)^{(n-1)} k_n = -e_n^* =: \omega(f_n)$$

so that f_n is the character of Π_n . It is proved in [16] that f_n is the characteristic of the action of \mathfrak{S}_n on prime parking functions. These considerations can be extended to noncommutative symmetric functions, the symmetric group being replaced by the 0-Hecke algebra [17].

1.3. Free probability. A *noncommutative probability space* is a pair (A, ϕ) where A is a unital algebra over \mathbb{C} and ϕ a linear form on A such that $\phi(1) = 1$ (see, e.g., [19, 20]).

The *free moments* are the functions m_n defined by

$$(10) \quad m_n[a_1, \dots, a_n] = \phi(a_1 \cdots a_n).$$

The *free cumulants* κ_n are defined by the implicit equations

$$(11) \quad \phi(a_1 \cdots a_n) = \sum_{\pi \in \text{NC}_n} \kappa_\pi[a_1, \dots, a_n]$$

where NC_n is the set of noncrossing partitions of $[n]$ and

$$(12) \quad \kappa_\pi[a_1, \dots, a_n] = \prod_{B \in \pi} \kappa[B] \text{ and for } B = \{b_1 < \dots < b_p\}, \kappa[B] = \kappa[b_1, \dots, b_p].$$

By Möbius inversion over the lattice of noncrossing partitions, this yields

$$(13) \quad \kappa_\pi[a_1 \cdots a_n] = \sum_{\sigma \leq \pi} \mu(\sigma, \pi) \phi_\sigma[a_1, \dots, a_n].$$

where ϕ_π is defined similarly [19].

2. THE FAÀ DI BRUNO HOPF ALGEBRA AND ITS NONCOMMUTATIVE ANALOGUE

2.1. The Faà di Bruno algebra and symmetric functions. The classical algebra of symmetric functions, denoted by Sym or $Sym(X)$, is a free associative and commutative graded algebra with one generator in each degree:

$$(14) \quad Sym = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[e_1, e_2, \dots] = \mathbb{C}[p_1, p_2, \dots]$$

where the h_n are the complete homogeneous symmetric functions, the e_n the elementary symmetric functions, and the p_n the power sums.

Its usual bialgebra structure is defined by the coproduct

$$(15) \quad \Delta_0 h_n = \sum_{i=0}^n h_i \otimes h_{n-i} \quad (h_0 = 1)$$

which allows to interpret it as the algebra of polynomial functions on the multiplicative group

$$(16) \quad G_0 = \{a(z) = \sum_{n \geq 0} a_n z^n \quad (a_0 = 1)\}$$

of formal power series with constant term 1: h_n is the coordinate function

$$(17) \quad h_n : a(z) \longmapsto a_n.$$

Indeed, $h_n(a(z)b(z)) = \Delta_0 h_n(a(z) \otimes b(z))$.

But h_n can also be interpreted as a coordinate on the group

$$(18) \quad G_1 = \{A(z) = \sum_{n \geq 0} a_n z^{n+1} \quad (a_0 = 1)\}$$

of formal diffeomorphisms tangent to identity, under functional composition. Again with $h_n(A(z)) = a_n$ and $h_n(A(z)B(z)) = \Delta_1(A(z) \otimes B(z))$, the coproduct is now

$$(19) \quad \Delta_1 h_n = \sum_{i=0}^n h_i \otimes h_{n-i}((i+1)X) \quad (h_0 = 1)$$

where $h_n(mX)$ is defined as the coefficient of t^n in $(\sum h_k t^k)^m$. The resulting bialgebra is known as the Faà di Bruno algebra [9]. By definition, its antipode sends $H(t)$ to its functional inverse.

2.2. Noncommutative symmetric functions and noncommutative formal diffeomorphisms. These constructions can be repeated word for word with the algebra **Sym** of noncommutative symmetric functions. It is a free associative (and noncommutative) graded algebra with one generator S_n in each degree, which can be interpreted as above if the coefficients a_n belong to a noncommutative algebra. In this case, G_0 is still a group, but G_1 is not, as composition is not anymore associative. However, the coproduct Δ_1

$$(20) \quad \Delta_1 S_n = \sum_{i=0}^n S_i \otimes S_{n-i}((i+1)A) \quad (S_0 = 1)$$

remains coassociative, and **Sym** endowed with this coproduct is a Hopf algebra, known as Noncommutative Formal Diffeomorphisms [1, 17], or as the noncommutative Faà di Bruno algebra [5].

Let $\mathcal{H}_{\text{ncdif}}$ denote this Hopf algebra, and let γ denote its antipode. The image $h = \gamma(\sigma_1)$ of the formal sum of its generators

$$(21) \quad \sigma_1 = \sum_{n \geq 0} S_n$$

is characterized by the functional equation

$$(22) \quad h^{-1} = \sum_{n \geq 0} S_n h^n.$$

The *noncommutative Lagrange series* g is defined by the functional equation

$$(23) \quad g = \sum_{n \geq 0} S_n g^n$$

Recall that for $f \in \mathbf{Sym}$, $f(-A)$ is the image of f by the automorphism $S_n \mapsto (-1)^n \Lambda_n$. It is proved in [17] that $h(A) = g(-A)$, and that

$$(24) \quad g_n = \sum_{\pi \in \text{NDPF}_n} S^{\text{ev}(\pi)}$$

where NDPF_n is the set of nondecreasing parking functions of length n , *i.e.*, nondecreasing words over the positive integers such that $\pi_i \leq i$, and $\text{ev}(\pi) = (|\pi|_i)_{i=1..n}$.

The first terms are

$$\begin{aligned}
(25) \quad g_0 &= 1, & g_1 &= S_1, & g_2 &= S_2 + S^{11}, \\
g_3 &= S_3 + 2S^{21} + S^{12} + S^{111}, \\
g_4 &= S_4 + 3S^{31} + 2S^{22} + S^{13} + 3S^{211} + 2S^{121} + S^{112} + S^{1111}.
\end{aligned}$$

There is a simple bijection between NDPF_n and NC_n , and g_n can as well be written as a sum over noncrossing partitions.

3. NONCOMMUTATIVE FREE CUMULANTS

In the case of a single random variable, the free cumulants κ_n are the images of the noncommutative symmetric functions K_n defined by the functional equation

$$(26) \quad \sigma_1 = \sum_{n \geq 0} K_n \sigma_1^n$$

by the character χ of $\mathcal{H}_{\text{ncdif}}$ such that $\chi(S_n) = m_n = \phi(a^n)$, where a is some element of a noncommutative probability space (A, ϕ) .

This equation is formally similar to (23), so that we can write down immediately an expression of S_n in terms of the basis $K^I := K_{i_1} \cdots K_{i_r}$, by replacing S^I by K^I in the expression of g_n given in (24):

$$(27) \quad S_1 = K_1$$

$$(28) \quad S_2 = K_2 + K^{11}$$

$$(29) \quad S_3 = K_3 + 2K^{21} + K^{12} + K^{111}$$

$$(30) \quad S_4 = K_4 + 3K^{31} + 2K^{22} + 3K^{211} + K^{13} + 2K^{121} + K^{112} + K^{1111}.$$

These expressions are sums over Catalan sets

$$(31) \quad S_n = \sum_{\pi \in \text{NDPF}_n} K^{\text{ev}(\pi)}$$

in the guise of nondecreasing parking functions, instead of noncrossing partitions.

Solving recursively for K_n , we find, in various bases of **Sym**

$$(32) \quad K_1 = S_1 = \Lambda_1$$

$$(33) \quad K_2 = S_2 - S^{11} = -\Lambda_2 = -R_{11}$$

$$(34) \quad K_3 = S_3 - 2S^{21} - S^{12} + 2S^{111} = \Lambda_3 + \Lambda^{21} = R_{12} + 2R_{111}$$

$$(35) \quad K_4 = S_4 - S^{13} - 3(S^{31} - S^{121}) - 2(S^{22} - S^{112}) + 5(S^{211} - S^{1111})$$

$$(36) \quad = -\Lambda_4 - 2\Lambda^{31} - \Lambda^{22} - \Lambda^{211} = -(R_{13} + 2R_{121} + 3R_{112} + 5R_{1111}).$$

On these examples, $(-1)^{n-1}K_n$ appears to be given by the following rule: start from the expression of g_{n-1} on the S^I , add 1 to the first parts, and replace S with Λ . In other words,

$$(37) \quad K_n = -(\Omega g_{n-1})(-A)$$

where Ω is the linear operator defined in [16]

$$(38) \quad \Omega S^{i_1, \dots, i_r} = S^{i_1+1, i_2, \dots, i_r} \quad \text{and} \quad \Omega(1) = S_1.$$

It is proved in [16] that

$$(39) \quad g^{-1} = 1 - \Omega g,$$

and we have indeed

Theorem 3.1.

$$(40) \quad K := 1 + \sum_{n \geq 1} K_n = g^{-1}(-A).$$

Proof – Let γ be the antipode of $\mathcal{H}_{\text{ncdif}}$. It is proved in [16] that $\gamma(\sigma_1) = g(-A)$. Thus, (40) is equivalent to

$$(41) \quad K = \gamma(\lambda_{-1}).$$

We have

$$(42) \quad \begin{aligned} \Delta_1(\lambda_{-1}) &= (\Delta_1 \sigma_1)^{-1} = \left(\sum_{k \geq 0} S_k \otimes \sigma_1^{k+1} \right)^{-1} \\ &= \left(\sum_{k \geq 0} S_k \otimes \sigma_1^k \right)^{-1} (1 \otimes \sigma_1^{-1}) = \sum_{k \geq 0} (-1)^k \Lambda_k \otimes \sigma_1^{k-1} \end{aligned}$$

so that, by definition of an antipode,

$$(43) \quad 1 = \sum_{k \geq 0} (-1)^k \gamma(\Lambda_k) \sigma_1^{k-1}$$

and finally

$$(44) \quad \sigma_1 = \sum_{k \geq 0} (-1)^k \gamma(\Lambda_k) \sigma_1^k$$

which implies the identification

$$(45) \quad K_n = (-1)^n \gamma(\Lambda_n).$$

■

This result implies an expression of $(-1)^{n-1} K_n$ on the ribbon basis: start from the expression of g_{n-1} , *e.g.*, for $n = 4$

$$(46) \quad g_3 = 5R_3 + 3R_{21} + 2R_{12} + R_{111},$$

replace each R_I by $R_{\bar{I}}$ (mirror conjugate composition)

$$(47) \quad g_3 \rightarrow 5R_{111} + 3R_{12} + 2R_{21} + R_3,$$

and insert 1 at the beginning of each composition

$$(48) \quad -K_4 = 5R_{1111} + 3R_{112} + 2R_{121} + R_{13}.$$

On the S -basis, we have for example

$$(49) \quad K_4 = (S^4 - S^{13}) - 3(S^{31} - S^{121}) - 2(S^{22} - S^{112}) + 5(S^{211} - S^{111})$$

which is obtained from (47) or (48) by observing that

$$(50) \quad R_{1I} = S_1 R_I - \Omega R_I.$$

4. FREE CUMULANTS IN THE SCHRÖDER OPERAD

4.1. The Schröder operad and its group. To obtain a combinatorial expression for K_n , one can work in the group of the Schröder operad as in [13, Section 10.2]. This will cover the case of several random variables, hence imply Speicher's formula with noncrossing partitions, as well as the case of an arbitrary operator valued probability ϕ (see, *e.g.*, [3, 20]).

Let PT_n be the set of reduced plane trees, *i.e.*, plane trees for which any internal node has at least two descendants. The *Schröder operad* [13] is the \mathbb{C} -vector space

$$(51) \quad \mathcal{S} = \bigoplus_{n \geq 1} \mathcal{S}_n, \quad \text{where } \mathcal{S}_n = \mathbb{C} \text{PT}_n$$

endowed with the composition operations

$$(52) \quad \mathcal{S}_n \otimes \mathcal{S}_{k_1} \otimes \dots \otimes \mathcal{S}_{k_n} \longrightarrow \mathcal{S}_{k_1 + \dots + k_n} \quad (n \geq 1, k_i \geq 1)$$

which map the tensor product of trees $t_0 \otimes t_1 \otimes \dots \otimes t_n$ to the tree $t_0 \circ (t_1, \dots, t_n)$ obtained by replacing the leaves of t_0 , from left to right, by the trees t_1, \dots, t_n .

The number of leaves of a tree t will be called its degree $d(t)$, and we define the weight $\text{wt}(t)$ of a tree as its degree minus 1.

We can represent trees by noncommutative monomials in indeterminates S_n ($n \geq 0$), by interpreting a node of arity k as a k -ary operator denoted by S_{k-1} , and writing the resulting expression in Polish notation.

For example,

$$(53) \quad \begin{array}{c} \bullet \\ \swarrow \quad \downarrow \quad \searrow \\ \circ \quad \circ \quad \circ \\ \swarrow \quad \downarrow \quad \searrow \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array} = S_2 S_0 S_1 S_3 S_0 S_0 S_0 S_0 S_0 S_1 S_0 S_0 = S^{201300000100}$$

is of degree 8 and weight $7 = 2 + 1 + 3 + 1$. The sequence of exponents I is called a *Schröder pseudocomposition*. The sum of the components of I is therefore the weight of the associated tree. We shall indifferently use the notations S^I or S^t . Let $\hat{\mathcal{S}}$ be the completion of the vector space \mathcal{S} with respect to the weight $\text{wt}(t) = d(t) - 1$. The group of the operad \mathcal{S} is defined as [10, 2]

$$(54) \quad G_{\mathcal{S}} = \left\{ \circ + \sum_{n \geq 2} p_n, \quad p_n \in \mathcal{S}_n \right\} \subset \hat{\mathcal{S}}$$

endowed with the composition product

$$(55) \quad p \circ q = q + \sum_{n \geq 2} p_n \circ \left(\underbrace{q, \dots, q}_n \right) \in G_{\mathcal{S}}$$

for $p = \circ + \sum_{n \geq 2} p_n$ and $q \in G_{\mathcal{S}}$.

Elements of $G_{\mathcal{S}}$ can be described by their coordinates

$$(56) \quad p = \sum_{t \in \text{PT}} p_t t \quad \text{and} \quad q = \sum_{t \in \text{PT}} q_t t \quad (\text{PT} = \cup_{n \geq 1} \text{PT}_n).$$

(with $q_{\circ} = p_{\circ} = 1$) so that the coordinates of $r = p \circ q$ are given by

$$(57) \quad r_t = \sum_{t=t_0 \circ (t_1, \dots, t_n)} p_{t_0} g_{t_1} \cdots g_{t_n}$$

This allows to consider the group $G_{\mathcal{S}}$ as the group of characters of a graded Hopf algebra $H_{\mathcal{S}}$. It is the noncommutative polynomial algebra over reduced plane trees PT (with unit \circ), endowed with the coproduct given by *admissible cuts* (see [4, 13]): an admissible cut of a tree T is a possibly empty subset of internal vertices $c = \{i_1, \dots, i_k\}$ such that along any path from the root to a leaf, there is at most one internal vertex in c . For any such cut, one defines

- $P^c(T) = T_{i_1} \cdots T_{i_k}$ as the product of the subtrees of T having their root in c , ordered as in T from the top and from left to right.
- $R^c(T)$ as the trunk which remains after removing these trees.

The coproduct of $H_{\mathcal{S}}$ is

$$(58) \quad \Delta(T) = \sum_c R^c(T) \otimes P^c(T)$$

so that $H_{\mathcal{S}}$ is a graded Hopf algebra. For instance

$$(59) \quad \Delta \left(\begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} \right) = \circ \otimes \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} \otimes \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} \otimes \circ.$$

The bijection between $G_{\mathcal{S}}$ and the group of characters on $H_{\mathcal{S}}$ is obvious: since $H_{\mathcal{S}}$ is a polynomial algebra, a character χ is entirely determined by its restriction to trees of positive weight, in other words, by its *residue* in the sense of [6] (which can be considered as an infinitesimal character $\text{Res}(\chi)$) and the values of this residue are given by the coordinates in $G_{\mathcal{S}}$.

4.2. Operadic free cumulants. Consider the series of corollas

$$(60) \quad f_c = S_0 + \sum_{n \geq 1} S_n S_0^{n+1}.$$

The inverse of f_c in $G_{\mathcal{S}}$ is, in terms of trees,

$$(61) \quad g_c = \sum_{t \in \text{PT}} (-1)^{i(t)} S^t$$

where $i(t)$ denotes the number of internal notes of t . Indeed, denoting by $\wedge(t_1, \dots, t_n)$ the tree whose subtrees of the root are t_1, \dots, t_n ,

$$(62) \quad g_c = S_0 + \sum_{n \geq 1} \sum_{\substack{t = \wedge(t_1, \dots, t_{n+1}) \\ t_i \in \text{PT}}} (-1)^{i(t)} S^{\wedge(t_1, \dots, t_{n+1})}$$

$$(63) \quad = S_0 + \sum_{n \geq 1} \sum_{\substack{t = \wedge(t_1, \dots, t_{n+1}) \\ t_i \in \text{PT}}} (-1)^{1+i(t_1)+\dots+i(t_{n+1})} S_n S^{t_1} \dots S^{t_{n+1}}$$

$$(64) \quad = S_0 - \sum_{n \geq 1} S_n g_c^{n+1}$$

so that

$$(65) \quad S_0 = g_c + \sum_{n \geq 1} S_n g_c^{n+1} = f_c \circ g_c.$$

Definition 4.1. A Schröder tree is prime if its rightmost subtree is a leaf. We denote by PST_n the set of prime Schröder trees of weight n .

Prime Schröder trees are counted by the large Schröder numbers. Indeed, if $s(x) = 1 + x + 3x^2 + 11x^3 + 45x^4 + \dots$ is the generating series of Schröder trees of weight n , then, that of prime Schröder trees is

$$(66) \quad p(x) = 1 + \frac{xs(x)}{1 - xs(x)} = \frac{4}{3 - x + \sqrt{1 - 6x + x^2}} = 1 + x + 2x^2 + 6x^3 + 22x^4 + \dots$$

The series g_c , introduced in [13, Eq. (158)], projects onto the antipode $g(-A)$ of $\mathcal{H}_{\text{ncdif}}$ by the map $S_0 \mapsto 1$. Imitating the interpretation of the Lagrange series given in [13, Eq. (164)] and exchanging the roles of g_n , K_n and S_n as above, we obtain the following result.

Theorem 4.2. Define η by

$$(67) \quad g_c = \eta \cdot S_0,$$

and κ by

$$(68) \quad \kappa := \eta^{-1} \cdot S_0$$

where the exponent -1 denotes here the multiplicative inverse. Then, the image of κ by the algebra morphism $S_0 \mapsto 1$ is the series K of **Sym**. In terms of trees,

$$(69) \quad \kappa_n = \sum_{t \in \text{PST}_n} (-1)^{i(t)-1} S^t.$$

Proof. For an element f of the Schröder group G_S , write $f = \tilde{f}S_0$, and for $f, g \in G_S$, define

$$(70) \quad f \dashv g = (\tilde{f} \circ g)S_0 = S_0 + ((\tilde{f} - 1) \circ g)S_0$$

This is a partial composition: if

$$(71) \quad g = \circ + \sum_{n \geq 2} g_n \text{ and } f = \circ + \sum_{n \geq 2} f_n,$$

then

$$(72) \quad f \dashv g = \circ + \sum_{n \geq 2} f_n(\underbrace{g, \dots, g}_{n-1}, \circ)$$

From (67) and (4.2), we have

$$(73) \quad \tilde{\kappa}g_c = S_0 \text{ and } (\tilde{\kappa} \circ f_c)S_0 = f_c,$$

so that

$$(74) \quad f_c = \kappa \dashv f_c$$

which implies (69). Indeed, plugging f_c in this expression, we get an alternating sum of trees obtained by grafting corollas to leaves of prime Schröder trees t except to the rightmost one. The sign of the resulting tree t' is $(-1)^{i(t)-1} = (-1)^{i(t')-k-1}$ if k is the number of grafted corollas. Hence, each t' which is not a corolla has coefficient $(1-1)^n$ where n is the number of its internal nodes whose all descendants are leaves.

■

For example

$$(75) \quad \begin{aligned} \kappa &= [1 - (S^{10} + S^{200} - S^{1100} - S^{1010} + S^{3000} - S^{21000} - S^{20100} - S^{20010} - S^{12000} \\ &\quad - S^{10200} + S^{111000} + S^{110100} + S^{101100} + S^{101010} + S^{110010}) + \dots]^{-1} S_0 \\ &= S_0 + S^{100} + S^{2000} - S^{11000} \\ &\quad + S^{30000} - S^{210000} - S^{201000} - S^{120000} + S^{111000} + S^{1101000} + \dots \end{aligned}$$

Our formula is multiplicity-free, and the number of terms is given by the large Schröder numbers. This is also the sum of the absolute values of the Möbius function of the lattice of noncrossing partitions. We shall in the sequel give a combinatorial interpretation of this coincidence, and explain how it allows to recover Speicher's formula.

On the example we shall see that this expression is finer than Speicher's formula as the term $2\phi(a_1)\phi(a_2)\phi(a_3)$ is now separated into two binary trees. Actually, this will allow to take into account the case where ϕ is valued in a noncommutative algebra.

4.3. A dendriform structure arising from the Schröder group. Equation (74) seems to have the same structure as that of [6], which involves dendriform (or shuffle) and codendriform (or unshuffle) algebras.

In terms of the Hopf algebra structure, instead of the convolution of characters, which on trees T (of positive weight) reads as

$$(76) \quad (f * g)(T) = \pi \circ (f \otimes g) \circ \Delta(T) = \sum_{c=\{i_1, \dots, i_k\}} f(R^c(T))g(P^c(T))$$

we get

$$(77) \quad (f \dashv g)(T) = \pi \circ (f \otimes g) \circ \Delta_{\leftarrow}^+(T) = \sum_{c=\{i_1, \dots, i_k\}}^{\leftarrow} f(R^c(T))g(P^c(T))$$

where the sum is restricted to admissible cuts such that the rightmost leaf of T remains in $R^c(T)$ or, equivalently, such that the rightmost subtree in $P^c(T)$ does not contain the rightmost leaf of T .

Extending to Schröder trees the constructions of [6, 7], we can now state:

Theorem 4.3. *For any tree T of positive weight, let*

$$(78) \quad \Delta_{\prec}^+(T) = \sum_{c=\{i_1, \dots, i_k\}}^{\prec} R^c(T) \otimes P^c(T) \text{ and } \Delta_{\succ}^+(T) = \Delta(T) - \Delta_{\prec}^+(T)$$

These maps can be extended to the augmentation ideal H_S^+ of H_S by the rule:

$$(79) \quad \Delta_{\prec}^+(T_1 T_2 \dots T_s) = \Delta_{\prec}^+(T_1) \cdot \Delta(T_2 \dots T_s)$$

$$(80) \quad \Delta_{\succ}^+(T_1 T_2 \dots T_s) = \Delta_{\succ}^+(T_1) \cdot \Delta(T_2 \dots T_s)$$

so that H_S is a codendriform bialgebra.

The last statement means that on H_S^+ , if

$$(81) \quad \Delta(a) = \bar{\Delta}(a) + 1 \otimes a + a \otimes 1$$

$$(82) \quad \Delta_{\prec}^+(a) = \Delta_{\prec}(a) + a \otimes 1$$

$$(83) \quad \Delta_{\succ}^+(a) = \Delta_{\succ}(a) + 1 \otimes a$$

then

$$(84) \quad (\Delta_{\prec} \otimes I) \circ \Delta_{\prec} = (I \otimes \bar{\Delta}) \circ \Delta_{\prec}$$

$$(85) \quad (\Delta_{\succ} \otimes I) \circ \Delta_{\succ} = (I \otimes \Delta_{\prec}) \circ \Delta_{\succ}$$

$$(86) \quad (\bar{\Delta} \otimes I) \circ \Delta_{\succ} = (I \otimes \Delta_{\succ}) \circ \Delta_{\succ}$$

Proof. It is easy to see that, for a forest $F = T_1 \dots T_s$, $\Delta_{\prec}(F)$ is the sum of terms $R^c(F) \otimes P^c(T)$ over the non trivial ($c \neq \emptyset$) admissible cuts of F , such that the rightmost leaf of T_1 is in $R^c(T)$. In the same way, $\Delta_{\succ}(F)$ is the sum of terms $R^c(F) \otimes P^c(T)$ over the non trivial ($P^c(T) \neq 1$) admissible cuts of F such that the rightmost leaf of T_1 is in $P^c(T)$.

The reduced coproduct being coassociative,

$$(87) \quad (\bar{\Delta} \otimes I) \circ \bar{\Delta} = (I \otimes \bar{\Delta}) \circ \bar{\Delta}$$

when applied to a forest $F = T_1 \dots T_s$, this yields a sum of terms $F^{(1)} \otimes F^{(2)} \otimes F^{(3)}$ obtained after two successive nontrivial admissible cuts ($F^{(i)} \in H_S^+$).

As $\bar{\Delta} = \Delta_{\prec} + \Delta_{\succ}$ on H_S^+ ,

$$(88) \quad (\bar{\Delta} \otimes I) \circ \bar{\Delta} = (\Delta_{\prec} \otimes I) \circ \Delta_{\prec} + (\Delta_{\succ} \otimes I) \circ \Delta_{\prec} + (\bar{\Delta} \otimes I) \circ \Delta_{\succ},$$

for $F = T_1 \dots T_s$, the sum of terms $F^{(1)} \otimes F^{(2)} \otimes F^{(3)}$ of $(\bar{\Delta} \otimes I) \circ \bar{\Delta}(F)$ splits into three parts:

- Terms $F^{(1)} \otimes F^{(2)} \otimes F^{(3)}$ contributing to $(\Delta_{\prec} \otimes I) \circ \Delta_{\prec}(F)$, which are those such that the rightmost leaf of T_1 is in $F^{(1)}$.
- Terms $F^{(1)} \otimes F^{(2)} \otimes F^{(3)}$ contributing to $(\Delta_{\succ} \otimes I) \circ \Delta_{\prec}(F)$, which are those such that the rightmost leaf of T_1 is in $F^{(2)}$.

- Terms $F^{(1)} \otimes F^{(2)} \otimes F^{(3)}$ contributing to $(\bar{\Delta} \otimes I) \circ \Delta_{\succ}(F)$, which are those such that the rightmost leaf of T_1 is in $F^{(3)}$.

But this sum is also equal to $(I \otimes \bar{\Delta}) \circ \bar{\Delta}(F)$, and the decomposition

$$(89) \quad (I \otimes \bar{\Delta}) \circ \bar{\Delta} = (I \otimes \bar{\Delta}) \circ \Delta_{\prec} + (I \otimes \Delta_{\prec}) \circ \Delta_{\succ} + (I \otimes \Delta_{\succ}) \circ \Delta_{\succ}$$

must coincide with the previous one. This proves the relations

$$\begin{aligned} (\Delta_{\prec} \otimes I) \circ \Delta_{\prec} &= (I \otimes \bar{\Delta}) \circ \Delta_{\prec} \\ (\Delta_{\succ} \otimes I) \circ \Delta_{\prec} &= (I \otimes \Delta_{\prec}) \circ \Delta_{\succ} \\ (\bar{\Delta} \otimes I) \circ \Delta_{\succ} &= (I \otimes \Delta_{\succ}) \circ \Delta_{\succ} \end{aligned}$$

■

As in [6], the space $\text{Lin}(H_S, k)$, which is a \mathbb{K} -algebra for the convolution product

$$(90) \quad (f * g) = \pi \circ (f \otimes g) \circ \Delta$$

is also a dendriform algebra for left and right half-convolutions

$$(91) \quad (f \prec g) = \pi \circ (f \otimes g) \circ \Delta_{\prec}$$

$$(92) \quad (f \succ g) = \pi \circ (f \otimes g) \circ \Delta_{\succ}.$$

In terms of characters in G_S , the operation \dashv (Equation (77)) coincides on trees with \prec , and the relation between g_c and κ in Theorem 4.2 translates now into the character equation

$$(93) \quad f_c = \epsilon + (\text{Res}(\kappa)) \prec f_c$$

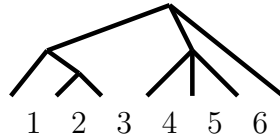
where ϵ is the unit of the group, corresponding to S_0 .

As we shall see, this equation implies [7, Th. 13], and thus also Speicher's formula for the free cumulants. We shall first explain in the forthcoming section how to derive the latter from (69) by a direct combinatorial argument.

5. SPEICHER'S FORMULA

Equation (69) is a formula for the free cumulants in terms of moments, involving prime Schröder trees instead of noncrossing partitions as in Equation (13) (which was originally the definition of free cumulants).

To recover a noncrossing partition from a tree, label the sectors from left to right so as to obtain the identity permutation by flattening the tree:



The blocks of the partition are then formed by the adjacent sectors. Here we obtain $1|2|36|45$.

But one can read more information. If we consider that ϕ is an arbitrary linear map with value in an associative algebra (instead of a linear form), then one reads the expression (compare [20, Ex. 2.1.2])

$$(94) \quad \phi(\phi(a_1\phi(a_2))a_3\phi(a_4a_5)a_6).$$

Denote by $\kappa_n[a_1, \dots, a_n]$ the evaluation of κ_n obtained by this process.

Assuming from now on that ϕ is scalar valued, we can rewrite this evaluation as

$$(95) \quad \kappa_n[a_1, \dots, a_n] = \sum_{t \in \text{PST}_n} (-1)^{i(t)-1} \prod_{v \in \text{int}(T)} \phi \left(\prod_{v \angle i} a_i \right)$$

where in the latter product, $v \angle i$ means that the internal vertex v has a clear view to the i th sector between the i th and $(i+1)$ th leaves. For example, the above tree gives the term $\phi(a_1)\phi(a_2)\phi(a_3a_6)\phi(a_4a_5)$.

Let us show that this expression is equivalent to Speicher's formula, so that $\kappa_n[a_1, \dots, a_n]$ is indeed the value of the free cumulant. For this, we need a natural map from prime Schröder trees to noncrossing partitions. Such a map can be defined in terms for noncrossing arrangements of binary trees:

Definition 5.1. *A noncrossing arrangement of binary trees is a set of binary trees, whose leaves are labeled with integers from 1 to n , in such a way that the canonical drawing of the trees does not create any crossing. Let \mathcal{A}_n denote the set of such objects.*

Proposition 5.2. *There is a bijection between PST_n and \mathcal{A}_n , which can be defined as follows. Let $t \in \text{PST}_n$, then its image is obtained by:*

- removing each middle edge (i.e. an edge which is not leftmost or rightmost among all edges below some internal vertex)
- removing the root and all edges below.

See Figure 1 for an example.



FIGURE 1. The bijection from prime Schröder trees to noncrossing arrangements of binary trees.

Note 5.3. Note that in a prime Schröder tree, we label the sectors, whereas in the arrangement of trees, we label the leaves. This means that through the bijection, we need to shift each label in a sector to the leaf to its left.

To avoid unnecessary notation, we just write $t \mapsto A$ if the bijection defined in the previous proposition sends $t \in \text{PST}_n$ to $A \in \mathcal{A}_n$. Note that there is one less internal

vertex in A than in t . Since the number of internal vertices of a binary tree is its number of leaves minus one,

$$(96) \quad (-1)^{i(t)} = (-1)^{\sum_{a \in A} i(a)} = (-1)^{n-\#A}.$$

Also, $A \in \mathcal{A}_n$ can be sent to a noncrossing partition π as follows, which we also denote $A \mapsto \pi$. It is defined by the condition that two integers are in the same block of π if they are in the same tree in A . For example, the noncrossing arrangement in Figure 1 gives the noncrossing partition $1456|23|78A|9$. The map $A \mapsto \pi$ is not a bijection, but for $\pi \in \text{NC}_n$, we have:

$$(97) \quad \#\{A \in \mathcal{A}_n : A \mapsto \pi\} = \prod_{B \in \pi} C_{\#B-1},$$

where C_n denote the Catalan numbers. Indeed, the construction of A amounts to choosing a binary tree with $\#B$ leaves for each block $B \in \pi$, whence the product of Catalan numbers.

Definition 5.4. *The Kreweras complement of a noncrossing partition $\pi \in \text{NC}_n$ is the noncrossing partition π^c defined by the following process:*

- we draw $2n$ dots representing integers and primed integers $1, 1', 2, 2', \dots, n, n'$ in this order,
- π is drawn as a noncrossing partition on $1, \dots, n$ in the usual way,
- π^c is the coarsest noncrossing partition on $1', \dots, n'$ that can be drawn without crossing π .

Then, π^c is identified with an element of NC_n by removing the prime symbols.

For example, Figure 2 shows that if $\pi = 134|2|57|6|8$, then $\pi^c = 12|3|478|56$. Practically, there is a convenient equivalent definition. If we have a set of noncrossing paths above the horizontal axis linking pairs of integers (as in Figure 2), it defines a noncrossing partition π by taking connected components, and i, j are in a same block of π^c iff there exists a path from i to j that does not cross the other paths.

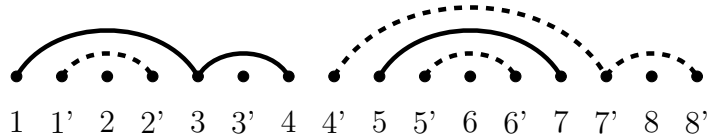


FIGURE 2. The Kreweras complement of a noncrossing partition.

Lemma 5.5. *Suppose that $t \mapsto A \mapsto \pi$ (with the previous notation). Then the noncrossing partition π^c is obtained from t by the following condition: i and j are in the same block of π^c if and only if some internal vertex of t has a clear view to the i th and j th sectors.*

See for example Figure 3 where $\pi = 12|3|46|5$ and $\pi^c = 1|236|45$.

Proof. We place labels $1, 2, \dots, n+1$ at the leaves of t , and labels $1', 2', \dots, n'$ in sectors of t , so that all labels are ordered $1, 1', 2, 2', \dots$ as in the definition of the Kreweras complement. See Figure 3.

Suppose first that $v \in t$ is an internal vertex, and $v \angle i', v \angle j'$. Then it is possible to draw a path from sector i' to sector j' crossing only middle edges below v . This path does not cross the arrangement of binary trees obtained from t by the bijection of Proposition 5.2, since we remove middle edges, and it follows that i and j are in a same block of π^c .

Suppose then that i and j are in a some block of π^c , *i.e.*, there is a path from i' to j' that does not cross the edges in A . So it crosses only middle edges of t . These middle edges are all below some vertex v , because each sector can be connected to a unique internal vertex of A by a path that does not cross edges of A (except the case where these sectors have no internal vertex above, then the path crosses only middles starting from the root of t). ■

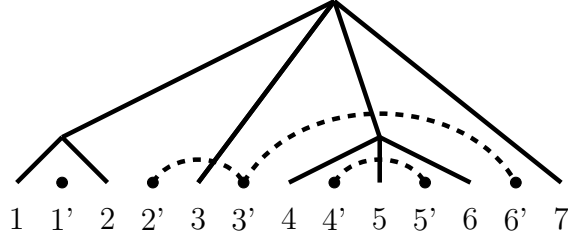


FIGURE 3. The proof of Lemma 5.5.

A consequence of this proposition is that the product $\prod_{v \angle i} a_i$ for an internal vertex v of t , is equal to the product $\prod_{i \in B} a_i$ where B is the appropriate block of π^c . So we can rewrite Equation (95) in terms of arrangements of binary trees, using Equation (96) for the sign:

$$(98) \quad \kappa[a_1, \dots, a_n] = \sum_{A \in \mathcal{A}_n} (-1)^{n-\#A} \phi_{\pi^c}[a_1, \dots, a_n].$$

Note that the sum is over A and that π depends on A by $A \mapsto \pi$. We can rewrite this in terms of noncrossing partitions, using Equation (97) to get the right multiplicities, which yields

$$(99) \quad \kappa[a_1, \dots, a_n] = \sum_{\pi \in \text{NC}_n} (-1)^{n-\#\pi} \left(\prod_{B \in \pi} C_{\#B-1} \right) \phi_{\pi^c}[a_1, \dots, a_n].$$

The product of Catalan numbers in this expression can be identified as a value of the Möbius function of NC_n , as shown by Kreweras [11]:

$$\mu(\hat{0}, \pi) = (-1)^{n-\#\pi} \prod_{B \in \pi} C_{\#B-1}.$$

We also need some further properties of the map $\pi \mapsto \pi^c$. It is an anti-automorphism of the poset NC_n , *i.e.*, an order-reversing bijection. This shows that the interval

edges is attached another corolla on a vertex v_2 with i edges ($i = 4$ and $k = 5$ in the picture).

Denote by $b_{\alpha_1}, \dots, b_{\alpha_{i-1}}, c_{\beta_1}, \dots, c_{\beta_{k-i}}$ the k sectors seen from v . So $b_{\alpha_1}, \dots, b_{\alpha_{i-1}}$ are the sectors seen from v_2 and $c_{\beta_1}, \dots, c_{\beta_{k-i}}$ are those seen from v_1 in u .

In the term indexed by t in the right-hand side of (102), we have a factor

$$(104) \quad \phi(b_{\alpha_1}, \dots, b_{\alpha_{i-1}}, c_{\beta_1}, \dots, c_{\beta_{k-i}}),$$

whereas in the term indexed by u , we have instead the two factors $\phi(b_{\alpha_1}, \dots, b_{\alpha_{i-1}})$ and $\phi(c_{\beta_1}, \dots, c_{\beta_{k-i}})$. By the cluster property, these two terms are equal up to a sign. Since u has one more internal vertex than t , it contributes to the sum in (102) with an opposite sign. So the two terms indexed by t and u cancel each other.

It remains to see how this local move can be used to define the fixed point free involution. Let $t \in \text{PST}_n$, and draw as before a path from the $(j+1)$ th leaf up to the root. Let us follow this path from bottom to top, and stop when finding:

- either middle edge (case 1),
- or a right edge, followed by a left above just above it (case 2).

Note that these two cases correspond to the two sides of Equation (103), so that we can define the involution by performing the local move going from one case to the other.

To see that this map is well-defined, it only remains to see that the two cases are exhaustive. Let $t \in \text{PST}_n$ such that we are not in case 1, *i.e.* the path does not cross any middle edge. Since the tree is prime, the right edge starting from the root arrives at the rightmost leaf (which does not separate two sectors), so the path arrives to the root by the left edge. Also, the path contains at least a right edge (otherwise, it would connect the root to the leftmost edge, which does not separate two sectors). It follows that we can find two edges as in the right part of (103), *i.e.*, we are indeed in case 2. ■

7. THE HOPF ALGEBRA OF DECORATED SCHRÖDER TREES

7.1. Let A be any set (decorations), and $T(A) = \mathbb{K} \otimes (\otimes_{n \geq 1} T_n(A))$ the free associative \mathbb{K} -algebra over A , regarded as the tensor algebra of the linear span of A .

Using the grading of H_S , we can define a decorated version of the algebra H_S

$$(105) \quad H_S(A) = \mathbb{K} \oplus \bigoplus_{n \geq 1} (H_{S,n} \otimes T_n(A)).$$

This space has an obvious algebra structure, and it is also easy to extend the Hopf algebra structure of H_S . Consider a tree $T \in H_{S,n}$ and $w = a_1 \dots a_n$ in $T_n(A)$. Since T has $n+1$ leaves, one can label its sectors from left to right with a_1, \dots, a_n and identify $T \otimes w$ with this decorated tree. For instance

$$(106) \quad \begin{array}{c} \text{Diagram of a tree with 6 leaves labeled } a_1, a_2, a_3, a_4, a_5, a_6 \end{array}$$

In an admissible cut c for such a tree, $P^c(T)$ obviously inherits the letters a_i associated with the subtrees in $P^c(T)$, and $R^c(T)$ keeps the letters which can be viewed from the internal vertices of T that are still in $R^c(T)$. It is clear that $H_S(A)$ is a Hopf algebra, and a straightforward adaptation of the proof of Theorem 4.3 shows that

Theorem 7.1. $H_S(A)$ is a codendriform bialgebra.

The decorated analog of Theorem 4.2 reads on characters

Theorem 7.2. Let ϕ be a linear form on $T(A)$. Extend it to a map $\phi : H_S(A) \rightarrow \mathbb{C}$ sending the decorated corollas to $\phi(w)$ where w is the decorating word and the other trees to 0 (regarded as an infinitesimal character of $\mathcal{H}_S(A)$), and let Φ be its extension to a character of $\mathcal{H}_S(A)$. Then,

$$(107) \quad \Phi = \epsilon + \kappa \prec \Phi$$

where κ is the infinitesimal character on $H_S(A)$ defined by

$$(108) \quad \kappa(T \otimes a_1 \dots a_n) = \begin{cases} (-1)^{i(t)-1} \prod_{v \in \text{int}(T)} \phi \left(\prod_{v \angle i} a_i \right) & \text{if } T \in \text{PST} \\ 0 & \text{otherwise} \end{cases}$$

where $v \angle i$ means that the internal vertex v has a clear view to the i th sector between the i th and $(i+1)$ th leaves.

Proof – As in [6], Φ is necessarily a character. If for a decorated tree $T \otimes a_1 \dots a_n$, T is not in PST, the r.h.s of (107) is necessarily 0. Otherwise, the right-hand side of (107) reads

$$(109) \quad (1-1)^n \prod_{v \in \text{int}(T)} \phi \left(\prod_{v \angle i} a_i \right)$$

where n is the number of its internal nodes whose descendants are leaves. Thus, only the corollas survive, with value $\phi(a_1 \dots a_n)$. ■

When summing over the different prime Schröder trees decorated by the same word, one recovers Speicher's formula

$$(110) \quad \kappa_n[a_1, \dots, a_n] = \sum_{t \in \text{PST}_n} (-1)^{i(t)-1} \prod_{v \in \text{int}(T)} \phi \left(\prod_{v \angle i} a_i \right).$$

Thus, in addition to the combinatorial argument of Section 5, we see that this expression can be recovered algebraically, with the help of morphisms of codendriform bialgebras.

7.2. The formula of Ebrahimi-Fard and Patras. In [6] and [7], free cumulants appear as the solution of a dendriform equation for characters of $T(T_{\geq 1}(A))$. The latter algebra is the free associative algebra over words w in $T_{\geq 1}(A)$, so that its elements can be viewed as linear combinations of segmented words $w_1 | \dots | w_s$. Let us just recall some results of [7, 6], the definition of an codendriform bialgebra having already been recalled in Theorem 4.3.

The algebra $T(T_{\geq 1}(A))$ is endowed with a coproduct defined on words as follows. Consider a word $w = a_1 \dots a_n$ as a ladder tree whose vertices are decorated from the root to the leaf by a_1, \dots, a_n . For any subset $S = \{i_1 < \dots < i_k\}$ of $\{1, \dots, n\}$, denote by $R^S(w) = a_{i_1} \dots a_{i_k} = a_S$ the ladder labeled with the letters a_i corresponding to the subscripts in S .

Once these vertices are removed from the original ladder, there remains some ladders which define a segmented word $P^S(w)$, the tensor (bar) product of these connected components. That is, $P^S(w) = w_S^1 | \dots | w_S^l$ where, in each w_S^j , the subscripts of the letters in each factor are consecutive integers, but the union of the subscripts of two consecutive factors is not an interval.

It is proved in [6] that with the coproduct defined on words by

$$(111) \quad \Delta(w) = \Delta(a_1 \dots a_n) = \sum_{S \subset \{1, \dots, n\}} R^S(w) \otimes P^S(w),$$

$T(T_{\geq 1}(A))$ is a Hopf algebra which is also a codendriform algebra for the splitting of the coproduct

$$(112) \quad \Delta = \Delta_{\succ}^+ + \Delta_{\prec}^+$$

defined on words $w = a_1 \dots a_n$ in $T_{\geq 1}(A)$ by

$$(113) \quad \Delta_{\prec}^+(a_1 \dots a_n) = \sum_{S \subset \{1, \dots, n\}, n \in S} R^S(w) \otimes P^S(w).$$

The relation between free moments and free cumulants is then given by the same equation as in Theorem 7.2, relating a linear map ϕ defined on words of $T_{\geq 1}(A)$ and its extension $\tilde{\Phi}$ as a character of $T(T_{\geq 1}(A))$ to the free cumulants (defining an infinitesimal character $\tilde{\kappa}$ on $T(T_{\geq 1}(A))$) by the same equation as in Theorem 7.2. This also proves Speicher's formula, which can thus be interpreted in terms of morphisms of codendriform bialgebras.

Note that in [6], the codendriform bialgebra structure is defined by splitting the coproduct according to whether 1 is in S instead of n for us. Both structures are obviously isomorphic under reversal of the words $a_1 \dots a_n \in T_{\geq 1}(A) \mapsto a_n \dots a_1$.

7.3. A codendriform Hopf morphism.

Theorem 7.3. *Let ι be the algebra morphism $T(T_{\geq 1}(A)) \rightarrow H_S(A)$ sending a word $w = a_1 \dots a_n$ to the sum of all trees with n sectors decorated from left to right by a_1, \dots, a_n . Then,*

(i) ι is a coalgebra morphism;

(ii) ι is a codendriform morphism:

$$(114) \quad (\iota \otimes \iota) \circ \Delta_{\prec}(w) = \Delta_{\prec} \circ \iota(w).$$

This result implies the formula for free cumulants, as ι send the maps Φ and κ of Theorem 7.2 to the free moments $\tilde{\Phi} = \phi \circ \iota$ and cumulants $\tilde{\kappa} = \kappa \circ \iota$ of [6].

Proof. Consider a word $w = a_1 \dots a_n$, $n \geq 1$. We can assume without loss of generality that the decorations are pairwise distinct. We must first prove that

$$(115) \quad \Delta \circ \iota(w) = (\iota \otimes \iota) \circ \Delta(w).$$

For any subset S of $\{1, \dots, n\}$, there is a unique term

$$(116) \quad R^S(w) \otimes P^S(w) = a_S \otimes w_S^1 \dots |w_S^l$$

corresponding to S in $\Delta(w)$. We shall say that a sequence of trees (T_0, T_1, \dots, T_k) is compatible with S (and write $S \sim (T_0, T_1, \dots, T_k)$) if $k = l$, $\text{wt}(T_0) = |S|$ and for all $1 \leq i \leq l$, $\text{wt}(T_i) = l(w_S^i)$. By definition of ι , the set

$$(117) \quad A(w) = \{(T_0; T_1 \dots T_k; R^S(w); P^S(w)), S \subset \{1, \dots, n\}; S \sim (T_0, T_1, \dots, T_k)\}$$

is such that

$$(118) \quad (\iota \otimes \iota) \circ \Delta(w) = \sum_{(T_0, T_1 \dots T_k; w^0; w^1 \dots |w^k) \in A(w)} (T_0 \otimes w^0) \otimes \prod (T_i \otimes w^i).$$

Let c be an admissible cut of a tree $T \in \text{PT}_{n+1}$. For the decorated tree $T \otimes a_1 \dots a_n$ the decorations of $R^c(T)$ correspond to a unique set $S(c) = \{i_1 < \dots < i_k\}$ of $\{1, \dots, n\}$, so that in the coproduct, we find $R^c(T) \otimes R^{S(c)}(w) = R^c(T) \otimes a_{S(c)}$ on the left-hand side of the term corresponding to this cut. On the right-hand side, if $P^c(T) = T_1 \dots T_k$, the corresponding decorations are the connected components of $P^{S(c)}(w) = w_{S(c)}^1 | \dots | w_{S(c)}^k$: the subscripts of each $w_{S(c)}^i$ form an interval whose length is the weight of T_i . Now, one can observe that, for any tree T , the set

$$(119) \quad A_T(w) = \{(R_c(T); P^c(T); R^{S(c)}(w); P^{S(c)}(w)), c \in \text{cut}(T)\}$$

is a subset of $A(w)$ such that

$$(120) \quad \Delta(T \otimes w) = \sum_{(T_0, T_1 \dots T_k; w^0; w^1 | \dots | w^k) \in A_T(w)} (T_0 \otimes w^0) \otimes \prod (T_i \otimes w^i).$$

There could be an ambiguity in this formula if there were repeated elements in the definition of $A_T(w)$ but this is never the case. For a given tree T , there maybe several cuts that give rise to the same undecorated term but the cuts would be at different positions and the induced decorations would be different.

Similarly, if we consider two different trees $T, T' \in \text{PT}_{n+1}$, then $A_T(w) \cap A_{T'}(w) = \emptyset$. There could be admissible cuts c and c' giving the same undecorated terms in the coproducts of T and T' , but if the associated decorations were the same, the initial ordering of the subscripts in $a_1 \dots a_n$ would impose that $T \otimes w = T' \otimes w$. For instance, if the first interleaves of $R^c(T) = R^{c'}(T')$ have the same decorations, say a_1 and a_5 , then the first factor of $P^c(T) = P^{c'}(T')$ is decorated by $a_2 a_3 a_4$ and the subtree from which one sees $a_1 a_2 a_3 a_4 a_5$ is the same in $T \otimes w$ and $T' \otimes w$. Furthermore, due to the order of the letters in $w = a_1 \dots a_n$, any given element in $\cup_T A_T(w)$ appears in only one of the subsets $A_T(w)$ and corresponds to a unique admissible cut of T .

Thus, the inclusion map from the disjoint union $\cup_T A_T(w)$ to $A(w)$ is injective.

In the same way, one can check that any element of $A(w)$ occurs in a unique $A_T(w)$. As before, one can recover the unique tree and the unique cut such that the letters are in the correct order in $T \otimes w$. This means that the inclusion map is actually a bijection:

$$(121) \quad (\iota \otimes \iota) \circ \Delta(w) = \sum_{(T_0, T_1 \dots T_k; w^0; w^1 | \dots | w^k) \in A(w)} (T_0 \otimes w^0) \otimes \prod (T_i \otimes w^i)$$

$$(122) \quad = \sum_T \sum_{(T_0, T_1 \dots T_k; w^0; w^1 | \dots | w^k) \in A_T(w)} (T_0 \otimes w^0) \otimes \prod (T_i \otimes w^i)$$

$$(123) \quad = \sum_T \Delta(T \otimes w)$$

$$(124) \quad = \Delta \circ \iota(w).$$

The proof for the codendriform structure is similar, noticing that, under restriction to the $A^\prec(w) \subset A(w)$ (resp. $A_T^\prec(w) \subset A_T(w)$) corresponding to the half coproducts, we still have a bijection between $A^\prec(w)$ and the decomposition $\cup_T A_T^\prec(w)$ associated with the equation

$$(\iota \otimes \iota) \circ \Delta_\prec(w) = \Delta_\prec \circ \iota(w).$$

■

8. MISCELLANEOUS REMARKS

We have seen that the large Schröder numbers can be expressed as sums of products of Catalan numbers:

$$(125) \quad \#\text{PST}_n = \sum_{\pi \in \text{NC}_n} \prod_{b \in \pi} C_{\#b-1}.$$

This identity appeared already in the paper by Dykema [3, Corollary 8.4], where it is related to other combinatorial objects called noncrossing linked partitions. Dykema's work being also related to operator-valued free probability theory, it is desirable to look for an explanation of this coincidence. To grasp a better understanding of what is happening, note that in each case, the Schröder numbers appear with a multivariate refinement. Our version is obtained when we remove all signs in the expression of the free cumulants in terms of the moments $(m_i)_{i \geq 1}$ (that we regard as indeterminates). The first values are:

$$(126) \quad m_1,$$

$$(127) \quad m_1^2 + m_2,$$

$$(128) \quad 2m_1^3 + 3m_2m_1 + m_3,$$

$$(129) \quad 5m_1^4 + 6m_2m_1^2 + 2m_2^2 + 8m_3m_1 + m_4.$$

On another hand, the multivariate refinements of Schröder numbers of [3] are obtained by expressing moments in terms of the coefficients of the T-transform, see Proposition

8.1 there. The first values are

$$(130) \quad \alpha_0,$$

$$(131) \quad \alpha_0^2 + \alpha_0\alpha_1,$$

$$(132) \quad \alpha_0^3 + 3\alpha_0^2\alpha_1 + \alpha_0\alpha_1^2 + \alpha_0^2\alpha_2,$$

$$(133) \quad \alpha_0^4 + 6\alpha_0^3\alpha_1 + 6\alpha_0^2\alpha_1^2 + 4\alpha_0^3\alpha_2 + \alpha_0\alpha_1^3 + 3\alpha_0^2\alpha_1\alpha_2 + \alpha_0^3\alpha_3.$$

There does not seem to be any clear link between these families of polynomials.

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