

GENERALIZED DYCK TILINGS

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ABSTRACT. Recently, Kenyon and Wilson introduced Dyck tilings, which are certain tilings of the region between two Dyck paths. The enumeration of Dyck tilings is related with hook formulas for forests and the combinatorics of Hermite polynomials. The first goal of this work is to give an alternative point of view on Dyck tilings by making use of the weak order and the Bruhat order on permutations. Then we introduce two natural generalizations: k -Dyck tilings and symmetric Dyck tilings. We are led to consider Stirling permutations, and define an analog of the Bruhat order on them. We show that certain families of k -Dyck tilings are in bijection with intervals in this order. We also enumerate symmetric Dyck tilings.

1. INTRODUCTION

Dyck tilings were recently introduced by Kenyon and Wilson [12] in the study of probabilities of statistical physics model called “double-dimer model”, and independently by Shigechi and Zinn-Justin [16] in the study of Kazhdan-Lusztig polynomials. Dyck tilings also have connection with fully packed loop configurations [7] and representations of the symmetric group [6].

The main purpose of this paper is to give a new point of view on Dyck tilings in terms of the weak order and the Bruhat order on permutations and to consider two natural generalizations of Dyck tilings.

A *Dyck path of length $2n$* is a lattice path consisting of up steps $(0, 1)$ and down steps $(1, 0)$ from the origin $(0, 0)$ to the point (n, n) which never goes strictly below the line $y = x$. We will also consider a Dyck path λ of length $2n$ as the Young diagram whose boundary is determined by λ and the lines $x = 0$ and $y = n$.

Suppose that λ and μ are Dyck paths of length $2n$ with μ weakly above λ . A *Dyck tile* is a ribbon such that the center of the cells form a Dyck path. A *Dyck tiling* of λ/μ is a tiling D of the region between λ and μ with Dyck tiles satisfying the *cover-inclusive property*: if η is a tile of D , then the translation of η by $(1, -1)$ is either completely below λ or contained in another tile of D . See Figure 1 for an example. We denote by $\mathcal{D}(\lambda/\mu)$ the set of Dyck tilings of λ/μ . For $D \in \mathcal{D}(\lambda/\mu)$ we call λ and μ the *lower path* and the *upper path* of D , respectively. Then the set of Dyck tilings with fixed upper path λ is denoted by $\mathcal{D}(\lambda/*)$ and similarly, the set of Dyck tilings with fixed lower path μ is denoted by $\mathcal{D}(*/\mu)$.

For $D \in \mathcal{D}(\lambda/\mu)$ we have two natural statistics: the area $\text{area}(D)$ of the region λ/μ and the number $\text{tiles}(D)$ of tiles of D . We also consider the statistic $\text{art}(D) = (\text{area}(D) + \text{tiles}(D))/2$.

Kenyon and Wilson [12] conjectured the following two formulas:

$$(1) \quad \sum_{D \in \mathcal{D}(\lambda/*)} q^{\text{art}(D)} = \frac{[n]_q!}{\prod_{x \in F} [h_x]_q},$$

$$(2) \quad \sum_{D \in \mathcal{D}(*/\mu)} q^{\text{tiles}(D)} = \prod_{u \in \text{UP}(\mu)} [\text{ht}(u)]_q,$$

where F is the plane forest corresponding to λ and, for a vertex $x \in F$, h_x denotes the hook length of x . The set of up steps of a Dyck path μ is denoted by $\text{UP}(\mu)$ and for $u \in \text{UP}(\mu)$, $\text{ht}(u)$ is the

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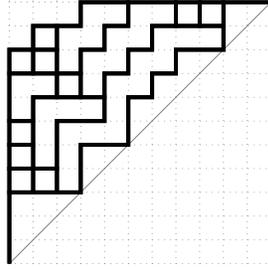


FIGURE 1. An example of Dyck tiling.

number of squares between u and the line $y = x$ plus 1. Here we use the standard notation for q -integers and q -factorials: $[n]_q = 1 + q + q^2 + \dots + q^{n-1}$ and $[n]_q! = [1]_q [2]_q \dots [n]_q$.

Formula (1) was first proved by Kim [13] non-bijectively and then by Kim, Mészáros, Panova, and Wilson [14] bijectively. In [14], they find a bijection between $\mathcal{D}(\lambda/*)$ and increasing labelings of the plane forest corresponding to λ . Kim [13] and Konvalinka independently proved (2) by finding a bijection between $\mathcal{D}(*/\mu)$ and certain labelings of μ called Hermite histories.

Björner and Wachs showed that the right hand side of (1) is the length generating function for permutations in an interval in the weak order, see [3, Theorem 6.8] and [2, Theorem 6.1].

In this paper we first show that, using the results of Björner and Wachs [2, 3], (1) can be interpreted as the length generating function for permutations $\pi \geq_L \sigma$ in the left weak order for a 312-avoiding permutation σ . We also show that (2) is the length generating function for permutations $\pi \geq \sigma$ in the Bruhat order for a 132-avoiding permutation σ . We then consider two natural generalizations of Dyck tilings, namely, k -Dyck tilings and symmetric Dyck tilings.

The first generalization is k -Dyck tiling, where we use k -Dyck paths and k -Dyck tiles with the same cover-inclusive property. We generalize (2) by finding a bijection between k -Dyck tilings and k -Hermite histories. We consider k -Stirling permutations introduced by Gessel and Stanley [9]. We define a k -Bruhat order on k -Stirling permutations and show that k -Dyck tilings with fixed upper path are in bijection with an interval in this order. We also consider a connection with k -regular noncrossing partitions. We generalize (1) to k -Dyck tilings with fixed lower path λ when λ is a zigzag path.

The second generalization is symmetric Dyck tiling, which is invariant under the reflection along a line. We show that symmetric Dyck tilings are in bijection with symmetric matchings and “marked” increasing labelings of symmetric forests.

2. DYCK TILINGS AS INTERVALS OF THE BRUHAT ORDER AND WEAK ORDER

As we have seen in the introduction, the two natural points of view for enumerating Dyck tilings are when we fix the upper path, and when we fix the lower path. We show in this section that both can be interpreted in terms of permutations, using respectively the Bruhat order and the (left) weak order, see [1].

We begin with the case of a fixed upper path, and the Bruhat order.

We denote by \mathfrak{S}_n the set of permutations of $[n] := \{1, 2, \dots, n\}$. An *inversion* of $\pi \in \mathfrak{S}_n$ is a pair (i, j) of integers $1 \leq i < j \leq n$ such that $\pi(i) > \pi(j)$. The number of inversions of π is denoted by $\text{inv}(\pi)$. For a permutation τ , the set of τ -avoiding permutations in \mathfrak{S}_n is denoted by $\mathfrak{S}_n(\tau)$. For example if $\tau = 132$, $\sigma \in \mathfrak{S}_n(132)$ if there is no $i < j < k$ such that $\sigma_i < \sigma_k < \sigma_j$.

We represent a permutation by a diagram with the “matrix convention”, i.e. there is a dot at the intersection of the i th line from the top and the j th column from the left if $\sigma(j) = i$. In these diagrams, we can represent the inversion of a permutation by putting a cross \times in each cell having a dot to its right and a dot below. See the left part of Figure 2. We need a bijection α between 132-avoiding permutations and Dyck paths. It is easy to see that the inversions of a 132-avoiding permutation are top left justified in its diagram. So we can define a path from the bottom left corner to the top right corner by following the boundary of the region filled with \times . This turns



FIGURE 2. The bijection from 132-avoiding permutations to Dyck paths. The crosses represent inversions of the permutation 34215.

out to be a Dyck path and this defines a bijection (this is an easy exercise). The bijection is illustrated in Figure 2.

Definition 2.1. Let μ be a Dyck path.

- A *Hermite history* of shape μ is a labelling of the up steps of μ with integers such that a step starting at height h has a label between 0 and h .
- A *matching* of shape μ is a partition of $[n]$ in 2-element blocks such that $i \in [n]$ is the minimum of a block if and only if the i th step of μ is an up step. A *crossing* of the matching is a pair of blocks $\{i, j\}$ and $\{k, \ell\}$ such that $i < j < k < \ell$.

The following is well known (see for example [13]).

Proposition 2.2. *There is a bijection between Hermite histories of shape μ and matchings of shape μ . It is such that the sum of weights in the Hermite history is the number of crossings in the matching.*

Theorem 2.3. *Let $\sigma \in \mathfrak{S}_n(132)$ and $\mu = \alpha(\sigma)$, then*

$$\sum_{D \in \mathcal{D}(*/\mu)} q^{\text{tiles}(D)} = \sum_{\pi \geq \sigma} q^{\text{inv}(\pi) - \text{inv}(\sigma)},$$

where $\pi \geq \sigma$ is the Bruhat order on \mathfrak{S}_n .

Proof. From [14], we know that Dyck tilings with a fixed upper path μ are in bijection with Hermite histories with shape μ , and the bijection sends the number of tiles to the sum of labels in the Hermite history. Consequently, Dyck tilings with a fixed upper path μ are in bijection with matchings of shape μ , and the bijection sends the number of tiles to the number of crossings in the matching.

To show the proposition, we give a bijection between matchings with the same shape μ , and permutations above σ in the Bruhat order. It is illustrated in Figure 3. The idea is to put dots in the grid as follows: if there is a pair (i, j) in the matching (with $i < j$), the i th step in the Dyck path is vertical and the j th step is horizontal, so row to the left of the i th step intersects the column below the j th step in some cell, and we put a dot in this cell. Then we can read these dots as a permutation (with the matrix convention). In the example in Figure 3 we get 45321. The crossing in the matchings correspond two inversions of the permutations that lay below the Dyck path.

The next step is the following: we can prove the set of permutations where all dots are below the Dyck path μ is precisely the Bruhat interval $\{\pi : \pi \geq \sigma\}$. First, by construction all the dots of σ are below μ . Suppose all the dots of π are below μ and $\pi' \succ \pi$ in the Bruhat order. It means that π' is obtained from π by transforming a pair of dots arranged as $\bullet \bullet$ into a pair of dots arranged as $\bullet \bullet$, and the new dots cannot be above μ . So the interval $\{\pi : \pi \geq \sigma\}$ is included in the set of permutations where all dots are below the Dyck path μ . Reciprocally, let π be a permutation where all dots are below the Dyck path μ . If $\pi \neq \sigma$, consider an inversion of π wich is as low to the right as possible. This inversion is in a pattern $\times \bullet$ and the cross is below μ . By transforming this pattern into $\bullet \bullet$, we obtain π' with $\pi' \prec \pi$ and has still the property that all dots are below μ . By repeating this operation we must arrive at a permutation whose inversions are exactly the cells above μ , i.e. σ . So π is in the interval $\{\pi : \pi \geq \sigma\}$. \square

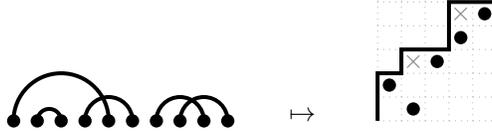
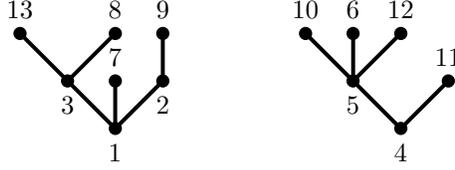


FIGURE 3. The bijection proving Proposition 2.3.

FIGURE 4. The bijection from Dyck paths to plane forests. Here the Dyck path is rotated clockwise by an angle of 45° .FIGURE 5. An increasing labeling L of a forest of size 13. We have $\text{post}(L) = 13, 8, 3, 7, 9, 2, 1, 10, 6, 12, 5, 11, 4$ and $\text{pre}(L) = 1, 3, 13, 8, 7, 2, 9, 4, 5, 10, 6, 12, 11$.

Let us turn to the case of a fixed lower path in Dyck tilings. We can identify a Dyck path λ with a plane forest: it is obtained from λ by “squeezing” the path as shown in Figure 4, a pair of facing up step and down step corresponds to a vertex.

Let F be a plane forest with n vertices. An *increasing labeling* of F is a way of labeling the vertices of F with $1, 2, \dots, n$ so that the label of a vertex is greater than the label of its parent. Let L be an increasing labeling of F . An *inversion* of L is a pair $i < j$ such that j is not a descendant of i and j appears to the left of i . For example, if L is the increasing labeling in Figure 5, then L has many inversions including $(13, 7)$, $(8, 7)$, $(2, 4)$. We denote by $\text{post}(L)$ (respectively, $\text{pre}(L)$) the permutation obtained by reading L from left to right using post-order (respectively, pre-order). See Figure 5. It is easy to see that there is a unique increasing labeling L_0 of F such that $\text{inv}(\text{post}(L_0))$ is minimal. More specifically, L_0 is the increasing labeling of F with $\text{inv}(L_0) = 0$, or equivalently, L_0 is the increasing labeling of F such that $\text{pre}(L_0) = \text{id}_n$, the identity permutation of $[n]$. It is not difficult to see that the permutation $\pi_0 = \text{post}(L_0)$ is 312-avoiding.

Theorem 2.4. *Let λ be a Dyck path with corresponding plane forest F . Let L_0 be the increasing labeling of F such that $\text{pre}(L_0) = \text{id}_n$, and $\pi_0 = \text{post}(L_0)$. Then*

$$\sum_{D \in \mathcal{D}(\lambda/*)} q^{\text{art}(D)} = \sum_{\pi \geq_L \pi_0} q^{\text{inv}(\pi) - \text{inv}(\pi_0)},$$

where \geq_L is the left weak order on \mathfrak{S}_n .

Proof. It is shown in [14] that

$$\sum_{D \in \mathcal{D}(\lambda/*)} q^{\text{art}(D)} = \sum_{L \in \mathcal{L}(F)} q^{\text{inv}(L)},$$

where $\mathcal{L}(F)$ is the set of increasing labelings of F . Thus, it is enough to show that for all $k \geq 0$, there is a bijection between the two sets

$$A_k = \{L \in \mathcal{L}(F) : \text{inv}(L) = k\}, \quad B_k = \{\pi \in \mathfrak{S}_n : \pi \geq_L \pi_0, \text{inv}(\pi) - \text{inv}(\pi_0) = k\}.$$

We will show that the map $L \mapsto \text{post}(L)$ is a bijection from A_k to B_k for all $k \geq 0$ by induction on k . Since $A_0 = \{L_0\}$ and $B_0 = \{\pi_0\}$, it is true when $k = 0$. Suppose that the claimed statement is true for $k \geq 0$. We need to show that the map $L \mapsto \text{post}(L)$ is a bijection from A_{k+1} to B_{k+1} . Let $L \in A_{k+1}$. Since $\text{inv}(L) = k+1 \geq 1$, we can find an integer i such that $(i+1, i)$ is an inversion of L . Since $i+1$ is not a descendant of i in L , the labeling L' obtained from L by exchanging i and

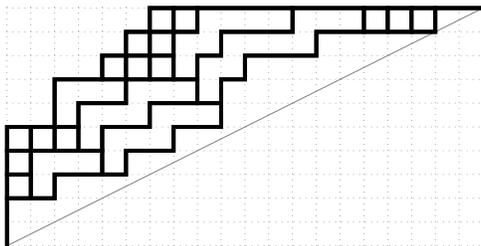


FIGURE 6. An example of 2-Dyck tiling.



FIGURE 7. The up step u on the left has height $\text{ht}(u) = 5$ because there are 4 squares between u and the line $y = x/2$. The diagram on the right is an example of 2-Hermite history.

$i + 1$ is also an increasing labeling of F . Since $\text{inv}(L') = \text{inv}(L) - 1 = k$ we have $L' \in A_k$. By the induction hypothesis, $\pi' = \text{post}(L') \in B_k$. One can easily see that the permutation $\pi = \text{post}(L)$ is obtained from π' by exchanging i and $i + 1$. Since $i + 1$ appears to the left of i in π we have $\text{inv}(\pi) = \text{inv}(\pi') + 1$ and $\pi \in B_{k+1}$. Thus $L \mapsto \text{post}(L)$ is a map from A_{k+1} to B_{k+1} . Similarly, we can show that, for given $\pi \in B_{k+1}$, there is $L \in B_{k+1}$ such that $\text{post}(L) = \pi$. Since L is determined by $\text{post}(L)$ for all $L \in \mathcal{L}(F)$, the map $L \mapsto \text{post}(L)$ is a bijection from A_{k+1} to B_{k+1} . \square

Note that the inversion generating function of increasing labelings of a plane forest is given by a hook length formula [2]. The fact that some intervals for the weak order have a generating function given by a hook length formula follows from [3].

3. k -DYCK TILINGS

For an integer $k \geq 1$, a k -Dyck path is a lattice path consisting of *up steps* $(0, 1)$ and *down steps* $(1, 0)$ from the origin $(0, 0)$ to the point (kn, n) which never goes below the line $y = x/k$. Let $\text{Dyck}^{(k)}(n)$ denote the set of k -Dyck paths from $(0, 0)$ to (kn, n) . It is well known that the cardinal of $\text{Dyck}^{(k)}(n)$ is the Fuss-Catalan number $\frac{1}{kn+1} \binom{(k+1)n}{n}$ (see for example [5]). As in the case of Dyck path, we denote $\text{UP}(\mu)$ the set of up steps of a k -Dyck path μ .

A k -Dyck tile is a ribbon in which the centers of the cells form a k -Dyck path. Let $\lambda, \mu \in \text{Dyck}^{(k)}(n)$ such that μ is weakly above λ . A (*cover-inclusive*) k -Dyck tiling is a tiling D of the region λ/μ between $\lambda, \mu \in \text{Dyck}^{(k)}(n)$ with k -Dyck tiles satisfying the cover-inclusive property: if η is a tile in D , then the translation of η by $(1, -1)$ is either completely below λ or contained in another tile of D . See Figure 6 for an example. We denote by $\mathcal{D}^{(k)}(\lambda/\mu)$ the set of k -Dyck tilings of λ/μ . We also denote by $\mathcal{D}^{(k)}(\lambda/*)$ and $\mathcal{D}^{(k)}(*/\mu)$ the sets of k -Dyck tilings with fixed lower path λ and with fixed upper path μ , respectively.

For $D \in \mathcal{D}^{(k)}(\lambda/\mu)$, there are two natural statistics: the number $\text{tiles}(D)$ of tiles in D and the area $\text{area}(D)$ of the region occupied by D . We also define

$$\text{art}_k(D) = \frac{k \cdot \text{area}(D) + \text{tiles}(D)}{k + 1}.$$

Definition 3.1. For an up step u of a k -Dyck path, we define the *height* $\text{ht}(u)$ of u to be the number of squares between u and the line $y = x/k$ plus 1. A k -Hermite history is a k -Dyck path in which every up step u is labeled with an integer in $\{0, 1, 2, \dots, \text{ht}(u) - 1\}$. See for example Figure 7. Given a k -Dyck path μ , we denote by $\mathcal{H}^{(k)}(\mu)$ the set of k -Hermite histories on μ . The *weight* $\text{wt}(H)$ of a k -Hermite history is the sum of the labels in H .

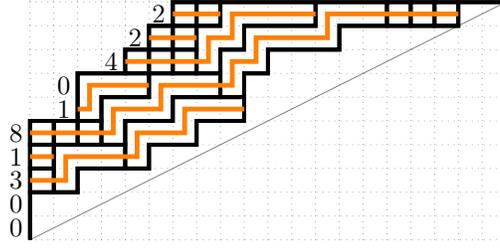


FIGURE 8. An illustration of the bijection between k -Dyck tilings and k -Hermite histories.

The following theorem is a generalization of (2). Our proof generalizes the bijective proof in [13].

Theorem 3.2. *For $\mu \in \text{Dyck}^{(k)}(n)$, we have*

$$(3) \quad \sum_{D \in \mathcal{D}^{(k)}(*/\mu)} q^{\text{tiles}(D)} = \prod_{u \in \text{UP}(\mu)} [\text{ht}(u)]_q.$$

Proof. It suffices to find a bijection $f : \mathcal{D}^{(k)}(*/\mu) \rightarrow \mathcal{H}^{(k)}(\mu)$ such that if $f(D) = H$ then $\text{tiles}(D) = \text{wt}(H)$. We construct such a bijection as follows. Let $D \in \mathcal{D}^{(k)}(*/\mu)$. In order to define the corresponding $f(D) = H \in \mathcal{H}^{(k)}(\mu)$ we only need to define the labels of each up step in μ . For a k -Dyck tile η , the *entry* of η is the left side of the lowest cell in η and the *exit* of η is the right side of the rightmost cell in η . For each up step u in μ , we travel the tiles in D in the following way. If u is the entry of a k -Dyck tile η in D , then enter η at the entry and leave η from the exit. If the exit of η is the entry of another k -Dyck tile of D then travel that tile as well. Continue traveling in this way until we do not reach the entry of any k -Dyck tile in D . Then the label of u is defined to be the number of tiles that we have traveled. See Figure 8 for an example. It is not difficult to see $f(D) \in \mathcal{H}^{(k)}(\mu)$. Clearly, we have $\text{tiles}(D) = \text{wt}(H)$.

It remains to show that f is a bijection. Let $H \in \mathcal{H}^{(k)}(\mu)$. We will find the inverse image $D = f^{-1}(H)$ recursively. Suppose that n is the length of μ and m is the number of cells between μ and the line $y = x/k$. If $n = 0$ or $m = 0$, then both $\mathcal{D}^{(k)}(*/\mu)$ and $\mathcal{H}^{(k)}(\mu)$ have a unique element, and $f^{-1}(H)$ is the unique element in $\mathcal{D}^{(k)}(*/\mu)$ without tiles. Now let $n, m \geq 1$ and suppose that we can find $f^{-1}(H')$ for every $H' \in \mathcal{H}^{(k)}(\mu')$ if the length of μ' is smaller than n or the number of cells between μ' and the line $y = x/k$ is smaller than m . There are two cases.

Case 1: H has an up step with label $\ell \geq 1$ followed by a down step. In this case let H' be the k -Hermite history obtained from H by exchanging the up step and the down step following it and decrease the label ℓ by 1. Then the shape μ' of H' has one less cells between μ' and the line $y = x/k$. By assumption we can find $f^{-1}(\mu')$. Then $f^{-1}(\mu)$ is the k -Dyck tiling obtained from $f^{-1}(\mu')$ by adding the single square $\mu \setminus \mu'$.

Case 2: H has no up step with label $\ell \geq 1$ followed by a down step. Since μ is a k -Dyck path of positive length, we can find an up step u followed by k down steps. Since u is followed by a down step, its label is 0. Let P be the point where u starts. Let μ' be the k -Dyck path obtained from μ by deleting u and the k down steps following u . By assumption, we can find $f^{-1}(\mu')$. Then $f^{-1}(\mu)$ is the k -Dyck tiling obtained from $f^{-1}(\mu')$ by cutting it with the line of slope -1 passing through P and inserting an up step followed by k down steps. For each k -Dyck tile divided by the line, we attach the two divided pieces by connecting the separated points on the border with an up step and k down steps following it. See Figure 9.

This gives the inverse map of f . □

It seems unlikely that there is a hook length formula for $\mathcal{D}^{(k)}(\lambda/*)$ when we fix the lower path λ to be arbitrary. If $n = 6$, $k = 2$, and λ is the following path, then we have $|\mathcal{D}_n^{(k)}(\lambda/*)| = 607$, a

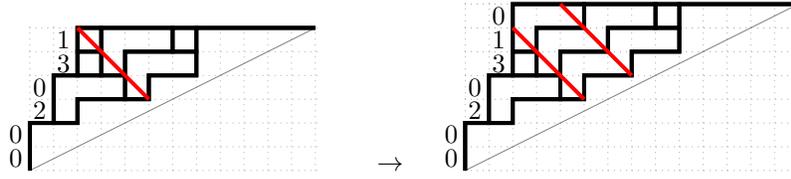
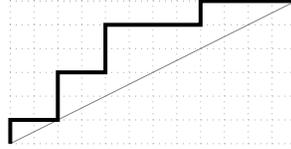
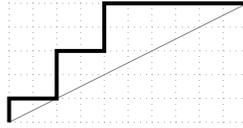


FIGURE 9. An example of cutting a k -Dyck tiling and inserting an up step and k down steps.

prime number.



Also if $|\mathcal{D}_n^{(k)}(\lambda/*)| = 71$ for the following λ with $n = 5, k = 2$.



However, when λ is a zigzag path there is a nice generalization of (1).

Theorem 3.3. *Let λ be the path $u^{n_1}d^{kn_1}u^{n_2}d^{kn_2}\dots u^{n_\ell}d^{kn_\ell}$, where u means an up step and d means a down step. Then we have*

$$\sum_{D \in \mathcal{D}^{(k)}(\lambda/*)} q^{\text{art}_k(D)} = \begin{bmatrix} kn_1 + n_2 \\ n_2 \end{bmatrix}_q \begin{bmatrix} k(n_1 + n_2) + n_3 \\ n_3 \end{bmatrix}_q \dots \begin{bmatrix} k(n_1 + \dots + n_{\ell-1}) + n_\ell \\ n_\ell \end{bmatrix}_q,$$

where $\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]_q!}{[b-a]_q! [b]_q!}$.

Proof. This can be proved using the same idea in the inductive proof in [13]. We will omit the details. \square

Problem 1. Find a bijective proof of Theorem 3.3.

4. k -STIRLING PERMUTATIONS AND THE k -BRUHAT ORDER

In this section we consider k -Stirling permutations which were introduced by Gessel and Stanley [9] for $k = 2$ and studied further for general k by Park [15].

A k -Stirling permutation of size n is a permutation of the multiset $\{1^k, 2^k, \dots, n^k\}$ such that if an integer j appears between two i 's then $i > j$. Let $\mathfrak{S}_n^{(k)}$ denote the set of k -Stirling permutations of size n . We can represent a k -Stirling permutation $\pi = \pi_1\pi_2\dots\pi_{kn}$ as the $n \times kn$ matrix $M = (M_{i,j})$ defined by $M_{i,j} = 1$ if $\pi_j = i$ and $M_{i,j} = 0$ otherwise. Then the entries of M are 0's and 1's such that each column contains exactly one 1, each row contains k 1's, and it does not contain the following submatrix:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For example, see Figure 10.

A k -inversion of $\pi \in \mathfrak{S}_n^{(k)}$ is a pair $(i, j) \in [n] \times [kn]$ such that $\pi_j > i$ and the first i appears after π_j . Equivalently, we will think of a k -inversion as an entry (or a cell) in the matrix of π which has k 1's to the right in the same row and one 1 below in the same column, see Figure 10. We denote the set of k -inversions of π by $\text{INV}_k(\pi)$, and $\text{inv}_k(\pi) = |\text{INV}_k(\pi)|$.

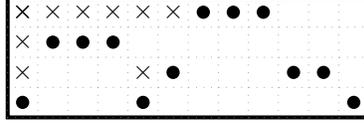


FIGURE 10. The matrix corresponding to $422243111334 \in \mathfrak{S}_4^{(3)}$, where ones are represented as dots. The k -inversions are the cells with crosses.

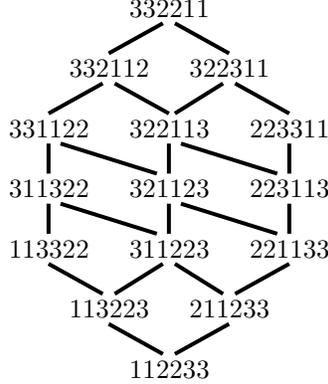


FIGURE 11. The Hasse diagram of $\mathfrak{S}_3^{(2)}$.

Proposition 4.1. *We have*

$$\sum_{\pi \in \mathfrak{S}_n^{(k)}} q^{\text{inv}_k(\pi)} = [k+1]_q [2k+1]_q \cdots [(n-1)k+1]_q.$$

Proof. This is an easy induction. Note that all the 1's in a Stirling permutation form a block of consecutive letters. Starting from $\sigma \in \mathfrak{S}_{n-1}^{(k)}$, we build a Stirling permutation in $\mathfrak{S}_n^{(k)}$ by increasing all letters of σ by 1, and inserting a block 1^k at some position. The positions where we can insert this block give the factor $[(n-1)k+1]_q$. \square

As we can see in the following lemma, a k -Stirling permutation is determined by its k -inversions.

Lemma 4.2. *For $\sigma, \pi \in \mathfrak{S}_n^{(k)}$, if $\text{INV}_k(\sigma) = \text{INV}_k(\pi)$, we have $\sigma = \pi$.*

Proof. Suppose that $\sigma \neq \pi$. Let r be the smallest index satisfying $\sigma_r \neq \pi_r$. We can assume that $\sigma_r < \pi_r$. Let $m = \sigma_r$. Note that $\pi_j = m$ for some $j > r$. Then π does not have the integer m in the first r positions because otherwise $\pi_i = m$ for $i < r$ and we get $\pi_i < \pi_r > \pi_j$ which is forbidden. Then $(m, r) \in \text{INV}_k(\pi)$ but $(m, r) \notin \text{INV}_k(\sigma)$, which is a contradiction. Thus we have $\sigma = \pi$. \square

We are now ready to define the k -Bruhat order on k -Stirling permutations.

Definition 4.3. We define the k -Bruhat order on $\mathfrak{S}_n^{(k)}$ given by the cover relation $\sigma \leq \pi$ if π is obtained from σ by exchanging the two numbers in positions a_1 and a_{k+1} for some integers $a_1 < a_2 < \cdots < a_{k+1}$ satisfying the following conditions:

- (1) $\sigma_{a_1} = \sigma_{a_2} = \cdots = \sigma_{a_k} < \sigma_{a_{k+1}}$, and
- (2) for all $a_k < i < a_{k+1}$ we have either $\sigma_i < \sigma_{a_1}$ or $\sigma_i > \sigma_{a_{k+1}}$.

In fact, if $k \geq 2$, then for all $a_1 < i < a_{k+1}$ with $i \neq a_j$ we always have $\sigma_i < \sigma_{a_1}$, see Lemma 4.4.

Note that the 1-Bruhat order is the usual Bruhat order. Figure 11 illustrates the k -Bruhat order.

Remark 1. The elements of $\mathfrak{S}_n^{(k)}$ where all occurrences of i are consecutive for any i , form a subset which is in natural bijection with \mathfrak{S}_n . In general ($k > 1$ and $n > 2$), the induced order on this subset is strictly contained in the Bruhat order, and strictly contains the left weak order.

Lemma 4.4. *In this lemma we use the notation in Definition 4.3. If $k \geq 2$, then we have $\sigma_i < \sigma_{a_1}$ for all $a_1 < i < a_{k+1}$ with $i \neq a_j$.*

Proof. By the definition of k -Stirling permutation, for all $a_1 < i < a_k$ with $i \neq a_j$ we have $\sigma_i < \sigma_k$, and for all $a_2 < i < a_{k+1}$ with $i \neq a_j$ we have $\pi_i < \pi_k = \sigma_k$. Thus we have $\sigma_i = \pi_i < \sigma_k$ for all $a_1 < i < a_{k+1}$ with $i \neq a_j$. \square

Lemma 4.5. *Let $\sigma \leq \pi$ in $\mathfrak{S}_n^{(k)}$. Then $\text{INV}_k(\pi)$ is obtained from $\text{INV}_k(\sigma)$ by adding one cell and moving some cells (maybe none) to the west or north. Moreover, if a cell is moved to the west (respectively, north), then the new location of the cell is south (respectively, to east) of the newly added cell in the same column (respectively, row).*

Proof. Suppose that π is obtained from σ as described in Definition 4.3. Then $\text{INV}_k(\pi)$ is obtained from $\text{INV}_k(\sigma)$ as follows. We add the inversion (σ_{a_1}, a_1) , and change each k -inversion of the form (r, a_{k+1}) for some $\sigma_{a_{k+1}} < r < \sigma_{a_1}$ to (r, a_1) . Furthermore if $k = 1$, we change each k -inversion of the form $(\sigma_{a_{k+1}}, j)$ for some $j < a_{k+1}$ to (σ_{a_1}, j) . \square

We note that in Lemma 4.5, moving cells to north can happen only when $k = 1$.

By Lemma 4.5 we have $\text{inv}_k(\pi) = \text{inv}_k(\sigma) + 1$ if $\sigma \leq \pi$. This proves:

Proposition 4.6. *Endowed with the k -Bruhat order, $\mathfrak{S}_n^{(k)}$ is a graded poset with rank function inv_k .*

The following generalization of 132-avoiding permutations is straightforward and natural.

Definition 4.7. A k -Stirling permutation $\sigma \in \mathfrak{S}_n^{(k)}$ is 132-avoiding if there is no $1 \leq i < j < k \leq kn$ such that $\sigma_i < \sigma_k < \sigma_j$. Let $\mathfrak{S}_n^{(k)}(132)$ denote the set of 132-avoiding k -Stirling permutations in $\mathfrak{S}_n^{(k)}$.

In a k -Dyck tilings without tiles, the lower path and upper path are the same. So these tilings are trivially in bijection with k -Dyck paths. As for the k -Stirling permutations, we have:

Proposition 4.8. *The inversions of a 132-avoiding k -Stirling permutation σ are arranged as the cells of a top-left justified Ferrers diagram. Define a path $\alpha(\sigma)$ from the bottom left to the top right corner, and following the boundary of this Ferrers diagram with up and right steps. Then α is a bijection between 132-avoiding k -Stirling permutations and k -Dyck paths of length n , in particular both are counted by the Fuss-Catalan numbers $\frac{1}{kn+1} \binom{(k+1)n}{n}$.*

The proof is simple and similar to the case $k = 1$. For example, see Figure 12.

Proposition 4.9. *Suppose that $\sigma \in \mathfrak{S}_n^{(k)}$ is 132-avoiding. Then, for $\pi \in \mathfrak{S}_n^{(k)}$, we have $\sigma \leq \pi$ if and only if $\text{INV}_k(\sigma) \subseteq \text{INV}_k(\pi)$.*

Proof. Since σ is 132-avoiding, $\text{INV}(\sigma)$ is a Ferrers diagram, say λ . By Lemma 4.5 if $\tau \leq \rho$ and $\lambda \subset \text{INV}_k(\tau)$, then we also have $\lambda \subset \text{INV}_k(\rho)$. This implies the ‘‘only if’’ part.

We will prove the ‘‘if’’ part by induction on the number $m = \text{inv}_k(\pi) - \text{inv}_k(\sigma)$. If $m = 0$, it is true. Suppose $m > 0$. Let (i, j) be an element in $\text{INV}_k(\pi) \setminus \text{INV}_k(\sigma)$ with j as large as possible. Then there are integers $a_1 < a_2 < \dots < a_k$ such that $\pi_{a_1} = \pi_{a_2} = \dots = \pi_{a_k} = i$ and $a_1 > j$. Then we have $\pi_t < i$ for all $j < t < a_1$ by the maximality of j . Let π' be obtained from π by exchanging the two integers in positions j and a_k . Then $\pi' \leq \pi$. By Lemma 4.5 $\text{INV}_k(\pi')$ is obtained from $\text{INV}_k(\pi)$ by removing the cell (i, j) and moving some cell located south or east of (i, j) . Thus $\text{INV}_k(\pi')$ still contains $\text{INV}_k(\sigma)$ which is a Ferrers diagram. By induction we have $\sigma \leq \pi'$, which completes the proof. \square

Proposition 4.10. *Let $\sigma \in \mathfrak{S}_n^{(k)}(132)$, and $\mu = \alpha(\sigma)$ the corresponding k -Dyck path. There is a bijection between the interval $\{\pi : \pi \geq \sigma\}$ in $\mathfrak{S}_n^{(k)}$ and k -Hermite histories of shape μ . It is such that $\text{inv}_k(\pi) - \text{inv}_k(\sigma)$ is sent to the sum of weights in the Hermite history.*

FIGURE 12. The bijection α in Proposition 4.8.

Proof. Let $\pi \in \mathfrak{S}_n^{(k)}(132)$ with $\pi \geq \sigma$. Note that by Proposition 4.9, the inversions of σ are inversions of π . Then we define a k -Hermite history as follows. For $i \leq n$, the label of the up step in the i th row from the top is the number of \times 's in the matrix of σ that are in the i th row, and below μ . Since $\text{inv}_k(\sigma)$ is the number of \times 's above the path μ by definition of the bijection in Figure 12, the number $\text{inv}_k(\pi) - \text{inv}_k(\sigma)$ is the sum of weights in the Hermite history. So it remains only to show that these labels define a Hermite history (i.e. they fall in the right range) and that this is a bijection.

Let us start with the first row. If the height of the up step in the first row is h , then there are $h + k$ cells to the right of the up step in the first row. Since there are k consecutive dots in the first row in π , the \times 's are located in the cells after the up step and before the k consecutive dots. Thus the number of \times 's in the first row is among $0, 1, 2, \dots, h$. Now consider the second row. If the height of the up step in the first row is h' , then there are $h' + 2k$ cells to the right of the up step in the second row. By the condition for k -Stirling permutation, if we delete the columns which have a dot in the first row, then the dots in the second row are consecutive, and the \times 's are located in the cells after the up step and before the k consecutive dots. Thus the number of \times 's in the second row is among $0, 1, 2, \dots, h'$. In this manner, we can see that the number of \times 's in each row is at most the height of the up step in the same row. This argument also shows that once the number of \times 's in each row is determined, we can uniquely construct the corresponding permutation π by putting dots starting from the first row. This implies that the map is a desired bijection. \square

We now have a generalization of Proposition 2.3. It is a rewriting of Theorem 3.2 using the bijection from Proposition 4.10.

Theorem 4.11. *If μ is a k -Dyck path corresponding to $\sigma \in \mathfrak{S}_n^{(k)}(132)$, then*

$$\sum_{D \in \mathcal{D}_n^{(k)}(*/\mu)} q^{\text{tiles}(D)} = \sum_{\pi \geq \sigma} q^{\text{inv}_k(\pi) - \text{inv}_k(\sigma)}.$$

5. k -REGULAR NONCROSSING PARTITIONS

In this section, we take another point of view on the k -Stirling permutations studied in the previous section.

Definition 5.1. We denote by $\text{NC}_n^{(k)}$ the set of k -regular noncrossing partitions of size n , i.e. set partitions of $[kn]$ such that each block contains k elements (k -regular), and there are no integers $a < b < c < d$ such that a, c are in one block, and b, d in another block (noncrossing). To each k -regular noncrossing partition π of $[kn]$, we define its *nesting poset* $\text{Nest}(\pi)$ as follows: the elements of the poset are the blocks of π , and $x \leq y$ in the poset when the block x lies between two elements of the block y .

There is a natural way to consider a k -Stirling permutation as a linear extension of the nesting poset of a k -regular noncrossing partition.

Proposition 5.2. *There is a bijection between $\mathfrak{S}_n^{(k)}$ and pairs (π, E) where $\pi \in \text{NC}_n^{(k)}$ and E is a linear extension of the poset $\text{Nest}(\pi)$.*

Proof. Let $\sigma \in \mathfrak{S}_n^{(k)}$. We define π by saying that i and j are in the same block if $\sigma_i = \sigma_j$, and E is defined by saying that the label of a block of π is σ_i where i is any element of the block. We can see that π is noncrossing from the definition of Stirling permutations, and E is a linear extension by

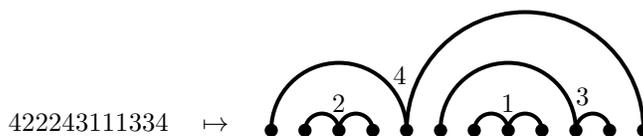


FIGURE 13. The bijection proving Proposition 5.2.

definition of the nesting poset. In the example $\sigma = 422243111334$, we get the noncrossing partition in Figure 13 where the labels define the linear extension of the nesting poset. The inverse bijection is simple to describe: σ_i is equal to j if i is in the block with label j . \square

The poset $\text{Nest}(\pi)$ is always a forest, so it is possible to consider pairs (π, E) as a decreasing forest. Each block of π is a vertex of the forest, and the forest structure is the order $\text{Nest}(\pi)$. Also, the labelling E naturally gives the decreasing labelling of the forest.

Let b be a block of π , the elements of b begin $i_1 < \dots < i_k$. Then the descendants of b in the forest can be distinguished into $k - 1$ categories, depending on the index j such that a descendant of b lies between i_j and i_{j+1} .

So we arrive at the following definition.

Definition 5.3. Let $T_n^{(k-1)}$ denote the set of $(k - 1)$ -ary plane forests on n vertices defined by the following conditions:

- the descendants of a vertex have a structure of a $(k - 1)$ -uple of ordered lists,
- the vertices are labeled with integers from 1 to n and the labels are decreasing from the roots to the leaves.

As a result of the above discussion, we obtain that there is a bijection between $\mathfrak{S}_n^{(k)}$ and $T_n^{(k-1)}$. However we do not insist on this point of view, since the definition of the trees in $T_n^{(k-1)}$ is not particularly natural.

There is a hook length formula for the number of linear extensions of a forest [3]. If $x \in \text{Nest}(\pi)$, let h_x denote the number of elements below x in the nesting poset, then the number of linear extensions of $\pi \in \text{NC}_n^{(k)}$ is

$$\frac{n!}{\prod_{x \in \text{Nest}(\pi)} h_x}.$$

This gives the following formula for the number of k -Stirling permutations.

Proposition 5.4 (Multifactorial hook length formula).

$$1(k + 1)(2k + 1) \dots ((n - 1)k + 1) = \sum_{\pi \in \text{NC}_n^{(k)}} \frac{n!}{\prod_{x \in \text{Nest}(\pi)} h_x}.$$

In the case $k > 2$, it is not clear how to follow the q -statistic, but for $k = 2$ we can use the bijection between noncrossing matchings and plane forests (which is just $\pi \mapsto \text{Nest}(\pi)$), and use the q -hook length formula from [2]. We get a hook length formula for $[1]_q[3]_q \dots [2n - 1]_q$.

Proposition 5.5 (q -double factorial hook length formula). *We have*

$$[1]_q[3]_q \dots [2n - 1]_q = \sum_F [n]_{q^2}! \prod_{v \in F} \frac{q^{h_v - 1}}{[h_v]_{q^2}},$$

where the sum is over all plane forests F with n vertices.

Proof. We know that the left hand side is the inversion generating function of $\mathfrak{S}_n^{(2)}$. It remains to understand what becomes the number of inversion through the bijection which send a 2-Stirling permutation to a noncrossing matching with labeled blocks, or equivalently increasing plane forests.

To this end, we distinguish two kinds of inversions. Let $i < j$ and $\sigma \in \mathfrak{S}_n^{(2)}$. If the four letters i, i, j, j appear in σ in this order, there is no inversion. If they appear in the order j, i, i, j , there is one inversion, and this corresponds to the case where the vertex with label i is below the vertex

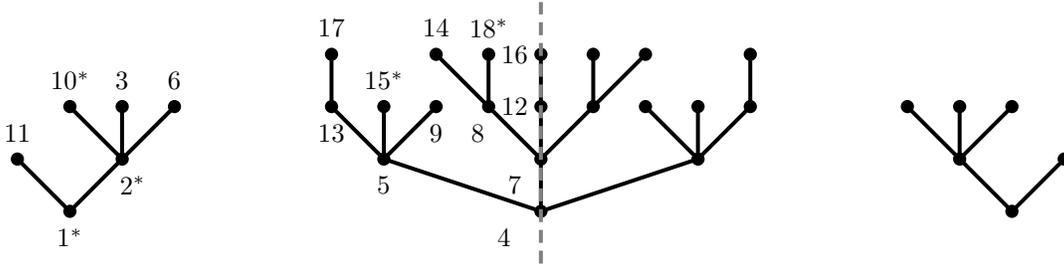


FIGURE 14. A symmetric plane tree of size 18 and a marked increasing labeling of it. The center line is the dashed line.

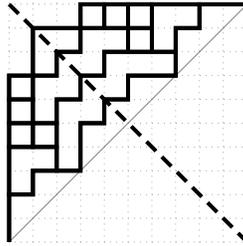


FIGURE 15. An example of symmetric Dyck tiling. It is symmetric with respect to the dashed line.

with label j . It means we have to count one inversion for each pair of comparable vertices, and the number of comparable vertices is clearly $\sum_{x \in \pi} (h_x - 1)$. In particular, it does not depend on the labeling. If they appear in the order j, j, i, i , there are two inversions, and this situation corresponds to the case where the vertices with labels i, j form an inversion in the forest.

So for a particular forest F , we get the term

$$[n]_{q^2}! \prod_{v \in F} \frac{q^{h_v - 1}}{[h_v]_{q^2}}.$$

This completes the proof. \square

6. SYMMETRIC DYCK TILINGS AND MARKED INCREASING FORESTS

A *symmetric plane forest* is a plane forest which is invariant under the reflection about a line, called the *center line*. A *center vertex* is a vertex on the center line. The *left part* of a symmetric plane forest is the subforest consisting of vertices on or to the left of the center line. The *size* of a symmetric plane forest is the number of vertices in the left part of it.

Let F be a symmetric plane forest of size n . A *marked increasing labeling* of F is a labeling of the left part of F with $[n]$ such that the labels are increasing from roots to leaves, each integer appears exactly once, and each non-center vertex may be marked with $*$. See Figure 14 for an example of a marked increasing labeling. We denote the set of marked increasing labelings of F by $\text{INC}^*(F)$.

Let $L \in \text{INC}^*(F)$. We denote by $\text{MARK}(L)$ the set of labels of the marked vertices in L . An *inversion* of L is a pair of vertices (u, v) such that $L(u) > L(v)$, u and v are incomparable, and u is to the left of v . Let $\text{INV}(L)$ denote the set of inversions of L .

A Dyck path λ of length $2n$ is called *symmetric* if it is invariant under the reflection along the line $x + y = n$. For two symmetric Dyck paths λ and μ of length $2n$, a Dyck tiling of λ/μ is called *symmetric* if it is invariant under the reflection along the line $x + y = n$. See Figure 15 for an example of symmetric Dyck tiling. We denote by $\mathcal{D}_{\text{sym}}(\lambda/\mu)$ the set of symmetric Dyck tilings of shape λ/μ .

For a symmetric Dyck tiling D , a *positive tile* is a tile which lies strictly to the left of the center line, and a *zero tile* is a tile which intersects with the center line. We denote by $\text{tiles}_+(D)$ and

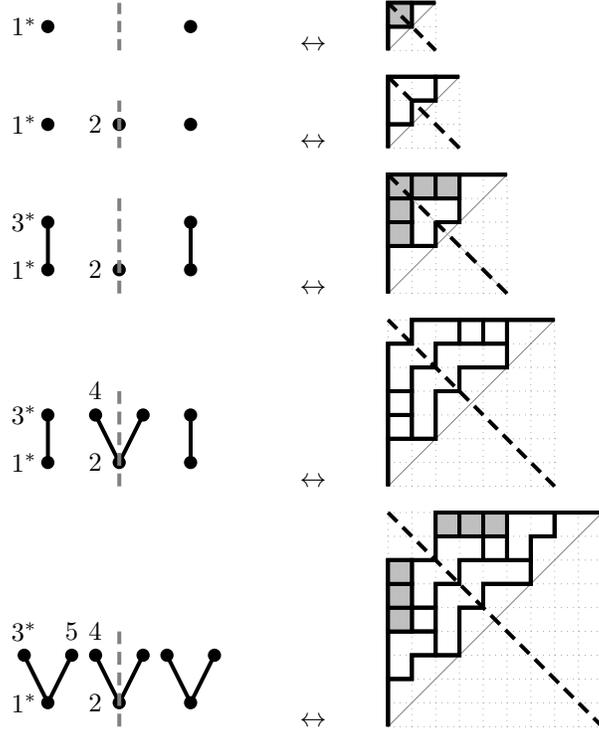


FIGURE 16. An illustration of the bijection between marked increasing labeling of a symmetric forest and symmetric Dyck tilings with fixed lower path. The newly added squares in each step are colored gray.

$\text{tiles}_0(D)$ the number of positive tiles and zero tiles, respectively. Note that the total number of tiles in D is $\text{tiles}(D) = 2 \cdot \text{tiles}_+(D) + \text{tiles}_0(D)$. We also define $\text{area}_+(D)$ and $\text{area}_0(D)$ to be the total area of positive tiles and zero tiles in D , respectively, and

$$\text{art}_+(D) = \frac{\text{area}_+(D) + \text{tiles}_+(D)}{2}, \quad \text{art}_0(D) = \frac{\text{area}_0(D) + \text{tiles}_0(D)}{2}.$$

For example, if D is the symmetric Dyck tiling in Figure 15, we have $\text{tiles}_+(D) = 5$, $\text{tiles}_0(D) = 3$, $\text{area}_+(D) = 7$, $\text{area}_0(D) = 19$, $\text{art}_+(D) = (7 + 5)/2 = 6$, and $\text{art}_0(D) = (19 + 3)/2 = 11$.

Theorem 6.1. *Let F be a symmetric plane forest of size n and λ the corresponding Dyck path. Then there is a bijection $\phi : \text{INC}^*(F) \rightarrow \mathcal{D}_{\text{sym}}(\lambda/*)$ such that if $\phi(L) = D$, then $\text{tiles}_0(D) = |\text{MARK}(L)|$ and*

$$\text{art}_+(D) + \text{art}_0(D) = |\text{INV}(L)| + \sum_{i \in \text{MARK}(L)} (n + 1 - i).$$

Proof. This is a generalization of a bijection in [14] constructed recursively. We “spread” and add “broken strips” to all up steps before the center line and to all down steps after the center line. If a vertex is marked, then we also add a square at the center line. See Figure 16. \square

Corollary 6.2. *Let F be a symmetric plane forest of size n with k center vertices and λ the corresponding Dyck path. Then*

$$|\mathcal{D}_{\text{sym}}(\lambda/*)| = 2^{n-k} \frac{n!}{\prod_{x \in F} h_x}.$$

If $k = 0$, we have

$$\sum_{D \in \mathcal{D}_{\text{sym}}(\lambda/*)} q^{\text{art}_+(D) + \text{art}_0(D)} t^{\text{tiles}_0(D)} = (1 + tq)(1 + tq^2) \cdots (1 + tq^n) \frac{[n]_q!}{\prod_{x \in F} [h_x]_q}.$$

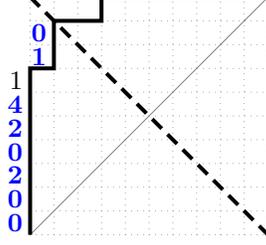


FIGURE 17. An example of symmetric Hermite history. The labels of the unmatched up steps are colored blue. The sequence $(0, 0, 2, 0, 2, 4, 1, 0)$ of the labels of unmatched up steps is an involutive sequence.

7. SYMMETRIC DYCK TILINGS AND SYMMETRIC HERMITE HISTORIES

For a symmetric Dyck path μ of length $2n$, let μ^+ denote the subpath consisting of the first n steps. Note that each up step is matched with a unique down step in a Dyck path. An up step of μ^+ is called *matched* if the corresponding down step in μ lies in μ^+ and *unmatched* otherwise.

A *symmetric Hermite history* is a symmetric Dyck path μ with a labeling of the up steps of μ^+ in such a way that every matched up step of height h has label $i \in \{0, 1, \dots, h-1\}$ and the labels a_1, a_2, \dots, a_k of the unmatched up steps form an involutive sequence. Here, a sequence is called *involutive* if it can be obtained by the following inductive way.

- The empty sequence is defined to be involutive.
- The sequence (0) is the only involutive sequence of length 1.
- For $k \geq 2$, an involutive sequence of length k is either the sequence obtained from an involutive sequence of length $k-1$ by adding 0 at the end or the sequence obtained from an involutive sequence of length $k-2$ by adding an integer r at the end and inserting a 0 before the last r integers, including the newly added integer, for some $1 \leq r \leq k-1$.

From the definition it is clear that the number v_k of involutive sequences of length k satisfies the recurrence $v_k = v_{k-1} + (k-1)v_{k-2}$ with initial conditions $v_0 = v_1 = 1$. Thus v_k is equal to the number of involutions in \mathfrak{S}_n . We denote by $\mathcal{H}_{\text{sym}}(\mu)$ be the set of symmetric Hermite histories on μ . For $H \in \mathcal{H}_{\text{sym}}(\mu)$, let $\|H\|$ be the sum of labels in H and $\text{pos}(H)$ the number of positive labels on unmatched up steps.

Proposition 7.1. *There is a bijection $\psi : \mathcal{H}_{\text{sym}}(\mu) \rightarrow \mathcal{D}_{\text{sym}}(*/\mu)$ such that if $\psi(H) = D$, then $\|H\| = \text{tiles}_+(D) + \text{tiles}_0(D)$ and $\text{pos}(H) = \text{tiles}_0(D)$. Thus,*

$$\sum_{D \in \mathcal{D}_{\text{sym}}(*/\mu)} q^{\text{tiles}_+(D) + \text{tiles}_0(D)} t^{\text{tiles}_0(D)} = \sum_{H \in \mathcal{H}_{\text{sym}}(\mu)} q^{\|H\|} t^{\text{pos}(H)}.$$

Proof. Given a symmetric Dyck tiling D , considering D as a normal Dyck tiling, we can obtain the Hermite history corresponding to D . By taking only the labels of up steps before the center line, we get a symmetric Hermite history. One can check that this gives a desired bijection. \square

Corollary 7.2. *Let μ be a symmetric Dyck path such that μ^+ has k unmatched up steps. Then*

$$\sum_{D \in \mathcal{D}_{\text{sym}}(*/\mu)} q^{\text{tiles}_+(D) + \text{tiles}_0(D)} t^{\text{tiles}_0(D)} = f_k(q, t) \prod_{u \in \text{UP}(\mu^+)} [\text{ht}(u)]_q,$$

where $\text{UP}(\mu^+)$ is the set of matched up steps in μ^+ and $f_k(q, t)$ is defined by $f_0(q, t) = f_1(q, t) = 1$ and $f_k(q, t) = f_{k-1}(q, t) + tq[k-1]_q f_{k-2}(q, t)$ for $k \geq 2$.

A *symmetric matching* is a matching on $[\pm n]$ such that if $\{i, j\}$ is an arc, then $\{-i, -j\}$ is also an arc. We denote by $\mathcal{M}_{\text{sym}}(n)$ the set of symmetric matchings on $[\pm n]$. Note that symmetric matchings are in bijection with fixed-point-free involutions in B_n .

Let $M \in \mathcal{M}_{\text{sym}}(n)$. A *symmetric crossing* of M is a pair of arcs $\{a, b\}$ and $\{c, d\}$ satisfying $a < c < b < d$ and $b, d > 0$. A symmetric crossing $(\{a, b\}, \{c, d\})$ is called *self-symmetric* if

$\{c, d\} = \{-a, -b\}$. We denote by $\text{cr}(M)$ and $\text{sscr}(M)$ the number of symmetric crossings and self-symmetric crossings of M , respectively.

For a symmetric Dyck path μ of length $2n$, let $\mathcal{M}_{\text{sym}}(\mu)$ denote the set of symmetric matchings M on $[\pm n]$ such that the i th smallest vertex of M is a left vertex of an arc if and only if the i th step of μ is an up step.

Proposition 7.3. *We have*

$$\sum_{H \in \mathcal{H}_{\text{sym}}(\mu)} q^{\|H\|} t^{\text{pos}(H)} = \sum_{M \in \mathcal{M}_{\text{sym}}(\mu)} q^{\text{cr}(M)} t^{\text{sscr}(M)}.$$

For $D \in \mathcal{D}_{\text{sym}}(*/\mu)$, let $\text{ht}(D)$ denote the number of unmatched up steps in μ^+ . Using the result in [4] and [10] on a generating function for partial matchings we obtain the following formula.

Proposition 7.4. *We have*

$$\begin{aligned} & \sum_{D \in \mathcal{D}_{\text{sym}}(n)} q^{\text{tiles}_+(D) + \text{tiles}_0(D)} t^{\text{tiles}_0(D)} s^{\text{ht}(D)} \\ &= \sum_{m=0}^n \frac{s^m f_m(q, t)}{(1-q)^{(n-m)/2}} \sum_{k \geq 0} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) (-1)^{(k-m)/2} q^{\binom{k-m}{2} + 1} \left[\frac{k+m}{2} \right]_q, \end{aligned}$$

where $f_m(q, t)$ is defined in Corollary 7.2.

If $t = s = 0$ in the above proposition, then we get the generating function for the usual Dyck tilings according to the number of tiles.

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