NEW DUMONT PERMUTATIONS

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Abstract. We consider the set of permutations all of whose descents are from an even value to an even value. Proving a conjecture of Kitaev and Remmel, we show that these permutations are enumerated by Genocchi numbers, hence equinumerous to Dumont permutations of the first and second kind, and thus may be called *Dumont permutations of the third kind*. We also define the related *Dumont permutations of the fourth kind*. We prove bijectively that the pattern bistatistic (2-31, 31-2) has the same distribution on Dumont permutations of the first kind as it does on those of the third kind. We then use the properties of that bijection to find certain statistics on Dumont permutations of the first and third kind that generate the Seidel triangle for Genocchi numbers. Finally, we use Laguerre histories to show that the bistatistic (crossings, nestings) has the same distribution on Dumont permutations of the second and fourth kind as does the bistatistic (2-31, 31-2) on Dumont permutations of the first or third kind.

1. Introduction

Dumont [7] showed that the certain sets of permutations are enumerated by Genocchi numbers G_{2n}, which are multiples of Bernoulli numbers B_{2n}, namely $G_{2n} = 2(1-2^{2n})(-1)^n B_{2n}$, so that

$$\sum_{n=1}^{\infty} G_{2n} \frac{x^{2n}}{(2n)!} = x \tan \frac{x}{2}, \qquad \sum_{n=1}^{\infty} (-1)^n G_{2n} \frac{x^{2n}}{(2n)!} = \frac{2x}{e^x+1} - x = -x \tanh \frac{x}{2}.$$

Definition 1.1. A *Dumont permutation of the first kind* (or a *Dumont-1* permutation, for short) is a permutation $\pi \in \mathfrak{S}_{2n}$ in which each even entry begins a descent and each odd entry begins an ascent or ends the string. In other words, for every $i = 1, 2, \dots, 2n$,

$$\pi(i) \text{ is even} \implies i < 2n \text{ and } \pi(i) > \pi(i+1),$$
$$\pi(i) \text{ is odd} \implies i = 2n \text{ or } \pi(i) < \pi(i+1).$$

Definition 1.2. A *Dumont permutation of the second kind* (or *Dumont-2* permutation, for short) is a permutation $\pi \in \mathfrak{S}_{2n}$ in which all entries at even positions are deficiencies and all entries at odd positions are fixed points or excedances. In other words, for every $i = 1, 2, \dots, n$,

$$\pi(2i) < 2i, \qquad \pi(2i-1) \geq 2i-1.$$

Notation 1.3. We denote the set of Dumont permutations of the first (resp. second) kind of length $2n$ by $\mathfrak{D}^1_{2n}$ (resp. $\mathfrak{D}^2_{2n}$). We also let $[m] = \{1, 2, \dots, m\}$ for any positive integer m.

Example 1.4. $\mathfrak{D}^1_2 = \mathfrak{D}^2_2 = \{21\}$, $\mathfrak{D}^1_4 = \{2143, 3421, 4213\}$, $\mathfrak{D}^2_4 = \{2143, 3142, 4132\}$.

Dumont [7] proved that $|\mathfrak{D}^1_{2n}| = |\mathfrak{D}^2_{2n}| = G_{2n+2}$. In this paper, we define two more sets of permutations, $\mathfrak{D}^3_{2n}$ and $\mathfrak{D}^4_{2n}$, whose cardinalities we also prove to be Genocchi numbers. That is, we prove that $|\mathfrak{D}^3_{2n}| = |\mathfrak{D}^4_{2n}| = G_{2n+2}$.

Definition 1.5. Let $\mathfrak{D}^3_{2n}$ be the set of permutations $\pi \in \mathfrak{S}_{2n}$ in which all descents occur only from an even value to an even value; in other words, for every $i = 1, 2, \dots, 2n-1$,

$$\pi(i) > \pi(i+1) \implies \pi(i) \text{ and } \pi(i+1) \text{ are both even.}$$

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Note that this implies that all permutations in $\mathfrak{D}^3_{2n}$ ($n \geq 1$) must start with 1. Also note that the set of permutations in $\mathfrak{D}^3_{2n}$ are almost centrally symmetric: if $\pi \in \mathfrak{D}^3_{2n}$ and $\rho \in \mathfrak{S}_{2n}$ is given by $\rho(1) = 1$, $\rho(i) = 2n + 2 - \pi(2n + 2 - i)$ for $i \geq 2$, then $\rho \in \mathfrak{D}^3_{2n}$ as well. If we form the *diagram* of π by plotting the points $(i, \pi(i))$ in the xy plane, this means that reflecting the diagram through the point $(n+1, n+1)$ (except for the point $(1, 1)$) yields a permutation in $\mathfrak{D}^3_{2n}$. Equivalently, if we strip the initial 1's from both π and ρ, the results are reverse-complements of each other.

Kitaev and Remmel [13, 14] conjectured that the sets of permutations where all descents occur only from an even value to an even value are also enumerated by Genocchi numbers. Our main result is a combinatorial proof of their conjecture.

Theorem 1.6 (Main Theorem)**.** *There is a bijection between sets $\mathfrak{D}^3_{2n}$ and $\mathfrak{D}^1_{2n}$, and hence $|\mathfrak{D}^3_{2n}| = |\mathfrak{D}^1_{2n}|$.*

Definition 1.7. Given the result of our Theorem 1.6, we call permutations in $\mathfrak{D}^3_{2n}$ *Dumont permutations of the third kind* (or *Dumont-3* permutations).

There is a simple natural bijection [7] between Dumont-1 and Dumont-2 permutations. Given a Dumont-1 permutation written in a one-line form, insert parentheses to indicate cycles so that cycles start with each successive left-to-right maximum (an entry $\pi(j)$ for which $\pi(j) > \pi(i)$ whenever $i < j$). This yields a Dumont-2 permutation.

Example 1.8.

$$\begin{aligned}
\mathfrak{D}^1_4 \ni 2143 &\mapsto (21)(43) = 2143 \in \mathfrak{D}^2_4,\\
\mathfrak{D}^1_4 \ni 3421 &\mapsto (3)(421) = 4132 \in \mathfrak{D}^2_4,\\
\mathfrak{D}^1_4 \ni 4213 &\mapsto (4213) \quad = 3142 \in \mathfrak{D}^2_4.
\end{aligned}$$

The same bijection can be applied to Dumont-3 permutations to obtain the set $\mathfrak{D}^4_{2n}$ that we may call *Dumont permutations of the fourth kind*, or *Dumont-4* permutations. One can easily show the following.

Proposition 1.9. *The set $\mathfrak{D}^4_{2n}$ consists exactly of all permutations $\pi \in \mathfrak{S}_{2n}$ all of whose deficiencies occur at even positions and have even values, in other words,*

$$\pi(i) < i \implies i \text{ and } \pi(i) \text{ are both even.}$$

Example 1.10. $\mathfrak{D}^3_2 = \mathfrak{D}^4_2 = \{12\}$, $\mathfrak{D}^3_4 = \{1234, 1342, 1423\}$, $\mathfrak{D}^4_4 = \{1234, 1342, 1432\}$.

$$\begin{aligned}
\mathfrak{D}^3_4 \ni 1234 &\mapsto (1)(2)(3)(4) = 1234 \in \mathfrak{D}^4_4,\\
\mathfrak{D}^3_4 \ni 1342 &\mapsto (1)(3)(42) \quad = 1432 \in \mathfrak{D}^4_4,\\
\mathfrak{D}^3_4 \ni 1423 &\mapsto (1)(423) \quad\;\; = 1342 \in \mathfrak{D}^4_4.
\end{aligned}$$

Note that that all permutations in $\mathfrak{D}^4_{2n}$ ($n \geq 1$) must also start with 1, and that the set of graphs of all $\pi \in \mathfrak{D}^4_{2n}$ with "1" deleted is invariant under reflection across the diagonal $x + y = 2n + 2$. That is, if $\pi \in \mathfrak{D}^4_{2n}$ and $\rho \in \mathfrak{S}_{2n}$ is given by $\rho(1) = 1$, $\rho(i) = 2n + 2 - \pi^{-1}(2n + 2 - i)$ for $i \geq 2$, then $\rho \in \mathfrak{D}^4_{2n}$ as well. Equivalently, if we strip the initial 1's from π and ρ, the results are inverse-reverse-complements of each other.

We will prove bijectively that the pattern bistatistic (2-31, 31-2) has the same distribution on Dumont permutations of the first kind as it does on those of the third kind. See Section 4 for the definition of these statistic, and let us first give some motivation for our interest in the pair of patterns 31-2 and 2-31 and their joint distribution. This pair of patterns was considered by Corteel in [6], who used it to give a combinatorial interpretation of a (p, q)-analog of Eulerian numbers that naturally appears when each integer i is replaced with $[i]_{p,q} = \frac{p^i - q^i}{p - q}$ in a continued fraction representation of generating function of Eulerian numbers. These (p, q)-Eulerian numbers were also studied in [4, 17]. Besides this link with Eulerian numbers, Corteel shows that the two patterns are equidistributed (among all permutations) with a certain pair of statistics called *crossings* and *nestings*. The latter statistics are reminiscent of statistics with the same name on other combinatorial objects such as set partitions and perfect matchings.

Some results are also known about the distribution of same pair of statistics (2-31,31-2) when restricted to certain special subsets of permutations. For example, the case of alternating permutations and Euler numbers was investigated in [18], and it turns out that as in the case of all permutations, the pair of patterns yields a natural (p, q)-analog of Euler numbers such that their generating function have simple continued fraction representation using (p, q)-integers. We will show that the same phenomenon occurs for Dumont permutations, so that these patterns are really natural statistics to consider: the refined Genocchi numbers obtained by counting patterns 31-2 and 2-31 in Dumont permutations yield a nice continued fraction (see Theorem 6.8).

1.1. **Organization of the paper.** We will give several proofs of the Main Theorem. The first proof, in Section 2, is a counting argument that does not include a bijection, but rather partitions the sets $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^3_{2n}$ into subsets that are shown to have the same cardinality.

In Section 3, we explicitly give two bijections from subexcedent functions to permutations of the same length that restrict to bijections from surjective pistols (also enumerated by Genocchi numbers) to $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^3_{2n}$, respectively. Composing these bijections gives an explicit bijection between $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^3_{2n}$.

In Section 4, we give a proof of a refined version of the Main Theorem, namely that the bistatistic $(2\text{-}31, 31\text{-}2)$ of the numbers of occurrences of generalized patterns 2-31 and 31-2 in permutations has the same distribution on $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^3_{2n}$. Then in Section 5 we will use the refined form of our Main Theorem to produce statistics on Dumont permutations that yield the Seidel triangle for Genocchi numbers.

Finally, in Section 6, we will restate the bijection of Section 4 using Laguerre histories, then use the same bijection on Laguerre histories to show that the bistatistic $(\mathrm{cr}, \mathrm{ne})$ of the numbers of crossings and nestings has the same distribution on $\mathfrak{D}^2_{2n}$ and $\mathfrak{D}^4_{2n}$.

2. There are as many Dumont-3 permutations as Dumont-1 permutations

Our initial strategy for proving our main theorem is to define a combinatorial structure called a *signature* for each Dumont-1 permutation, and, in a different way, a signature for each Dumont-3 permutation. We will then count the Dumont-1 permutations and Dumont-3 permutations corresponding to each possible signature. To help with the counting, we define a *signature function* for each signature. We will see that the number of Dumont-1 permutations with a given signature is equal to the product of the values of the associated signature function, and the same is true of Dumont-3 permutations. Since, for each signature, there are as many Dumont-3 permutations with that signature as Dumont-1 permuations with that signature, we conclude that $|\mathfrak{D}^3_{2n}| = |\mathfrak{D}^1_{2n}|$.

2.1. **Signatures and signature functions.** We will first define signatures in the abstract, and then show how to associate signatures with Dumont-1 and Dumont-3 permutations.

Definition 2.1. A *signature* of order k and size $2n$ consists of a set of k even numbers from $[2n]$ called the *peaks* and a set of k odd numbers from $[2n]$ called the *pits*. We also insist that peaks include $2n$ and pits include 1. Given a particular signature, let $\mathrm{pk}(x)$ and $\mathrm{pt}(x)$ be the numbers of peaks and pits, respectively, that are at least x. The associated *signature function* has domain $[2n]$ and is given by

$$f(x) = \mathrm{pk}(x) - \mathrm{pt}(x) + (1 \text{ if } x \text{ is odd}).$$

We are not interested in signatures for which $f(x)$ takes on non-positive values, since they will not correspond to any Dumont-1 or Dumont-3 permutations.

Example 2.2. $(\{6, 8\}, \{1, 3\})$ is a signature of size 8 and order 2. Its signature function is given by

x	1	2	3	4	5	6	7	8
$f(x)$	1	1	2	2	3	2	2	1

The product of these 8 values is 48. Note that $f(1) = 1$ and $f(2n) = 1$ for every signature function f of size $2n$.

2.2. **Signatures of Dumont-1 permutations.** The *peaks* of a Dumont-1 permutation are the even values that do not end descents, and the *pits* are the odd values that do end descents. (In other words: the peaks are the even non-descent-bottoms, and the pits are the odd descent-bottoms. Or, equivalently: the peaks are the local maxima, possibly including the initial entry, and the peaks are the local minima, possibly including the terminal entry.) The sequence from each peak to the next pit is a *descent block*; it can contain only descents, and can contain only even values except for the pit itself. The gaps before, after, and between descent blocks are called *ascent blocks*; they can contain only odd values, and any or all of them may be empty. Note that each descent block includes exactly one peak and one pit. The ascent blocks do not include peaks or pits, and (including the empty ones) there is one more ascent block than there are descent blocks.

Notation 2.3. We denote a descent block starting from a and ending at b by $a \searrow b$.

The *signature* of a Dumont-1 permutation consists of the set of its peaks and the set of its pits, without any indication of their sequence.

Example 2.4. 58364217 is a Dumont-1 permutation. Its peaks are 6 and 8, its pits are 1 and 3, and we may write its signature as $(\{6,8\},\{1,3\})$. The descent blocks are 83 and 6421. The ascent blocks are 5 and 7, plus an empty ascent block between 3 and 6.

Theorem 2.5. *Given a signature of size* $2n$ *and its associated signature function* f*, the number of Dumont-1 permutations with that signature is exactly* $\prod_{x=1}^{2n} f(x)$.

Proof. To construct a Dumont-1 permutation with a given signature, we must first match the pits with the peaks to form descent blocks. Each pit must be matched with a peak larger than itself. We construct the matching beginning with the largest pit, x_1. There are $\mathrm{pk}(x_1)$ peaks to which it might be assigned. That is equal to $f(x_1)$, since there is exactly one pit $\geq x_1$ (x_1 itself) and x_1 is odd.

After picking one of those assignments, let x_2 be the next largest pit. It has $\mathrm{pk}(x_2)$ peaks to which it might be assigned, minus the peak assigned to x_1; the number of choices is therefore $\mathrm{pk}(x_2) - 1$, which is $f(x_2)$.

Continuing to assign the pits in decreasing order, the number of choices we have when dealing with pit x is precisely $\mathrm{pk}(x) - \mathrm{pt}(x) + 1 = f(x)$.

The next step in constructing a Dumont-1 permutation is to put our new descent blocks into some sequence. We start with the block continuing the smallest peak; now call that peak y_1. It can become the first descent block of the permutation, or it can follow any block whose pit is less than y_1 — excluding the pit that has been matched with y_1. The number of choices is therefore

$$1 + (\#\text{pits} < y_1) - 1 = (\#\text{pits} < y_1) = k - \mathrm{pt}(y_1) = \mathrm{pk}(y_1) - \mathrm{pt}(y_1) = f(y_1)$$

since every peak is at least y_1.

Letting y_2 be the next smallest peak, we can place its descent block at the start (if that choice remains) or after any pit that is less than y_2 (excluding the pit matched with y_2 and the pit, if any, followed by y_1's block). The number of choices is

$$1 + (\#\text{pits} < y_2) - 2 = (\#\text{pits} < y_2) - 1 = k - \mathrm{pt}(y_2) - 1 = \mathrm{pk}(y_2) - \mathrm{pt}(y_2) = f(y_2).$$

Continuing to assign the peaks in increasing order, we see that the number of locations available for any peak y is $f(y)$. Making assignments for each descent block determines the sequence of the blocks.

Now we have all of the peaks and pits in order, and we know the bounds of each descent block and each ascent block.

We now assign each even value (other than peaks) to a descent block, and each odd value (other than pits) to an ascent block. These choices can be made independently, and they complete the construction of the permutation, because the order in which values occur within any block is determined by the Dumont definition. The number of descent blocks available to any even value x is $f(x)$, and the number of ascent blocks available to any odd value x is $f(x)$.

We have completed the construction of the permutation. For each $x \in [2n]$, we made a choice from among $f(x)$ alternatives, so the number of Dumont-1 permutations that can be constructed from this signature is exactly the product of the values $f(x)$. □

Example 2.6. There are exactly 48 Dumont-1 permutations in $\mathfrak{D}_8^1$ with signature $(\{6,8\},\{1,3\})$. For example, the permutation 58364217 is constructed as follows:

1. match pit 3 with peak 8 (it could have been matched with 6; $f(3) = 2$)
2. match pit 1 with peak 6 (only choice; $f(1) = 1$)
3. put the descent sequence $6 \searrow 1$ immediately after $8 \searrow 3$ (it could have been put at the start of the permutation; $f(6) = 2$)
4. put the descent sequence $8 \searrow 3$ at the start of the permutation (only choice; $f(8) = 1$)

(5) insert 2 into $6 \searrow 1$ (only choice; $f(2) = 1$)
(6) insert 4 into $6 \searrow 1$ (it could have fit into $8 \searrow 3$; $f(4) = 2$)
(7) insert 5 before $8 \searrow 3$ (it could have gone between $8 \searrow 3$ and $6 \searrow 1$ or after $6 \searrow 1$; $f(5) = 3$)
(8) insert 7 after $6 \searrow 1$ (it could have gone before $8 \searrow 3$; $f(7) = 2$).

2.3. **Signatures of Dumont-3 permutations.** We will define peaks, pits, and signatures differently for Dumont-3 permutations than for Dumont-1 permutations. Some of the definitions may seem peculiar. This is because we are not trying to describe the Dumont-3 permutation itself. Instead, we are trying to identify the ghost of a Dumont-1 permutation that is hidden within it.

The *peaks* of a Dumont-3 permutation are the even values that do not end descents. The *pits* of a Dumont-3 permutation are the even values that do not begin descents. A peak that is also a pit is called a *singleton*.

(In other words: the peaks are the even non-descent-bottoms, and the pits are the even non-descent-tops. The non-singleton peaks are local maxima and the non-singleton pits are local minima, but the singletons need not be maxima or minima.)

The sequence from each peak to the following pit (or to itself if it is a singleton) is called a *descent block*. Note that a singleton is called a descent block even though it contains zero descents. The (possibly empty) sequences before, after, and between the descent blocks are called *ascent blocks*. Descent blocks contain only even values; ascent blocks contain only odd values (if they contain any values at all).

The *signature* of a Dumont-3 permutation consists of

(1) a list of its peaks, and
(2) a list of the values of the form $x - 1$ for each pit x.

The second list contains odd numbers that we call *pit list entries*. (The slippage from x to $x - 1$ is required because we have defined the pits to be even numbers, and the pit list in a signature must consist of odd numbers.)

Example 2.7. 15846237 is a Dumont-3 permutation. Its peaks are 6 and 8. Its pits are 2 and 4, so the pit list entries are 1 and 3, and its signature is $(\{6, 8\}, \{1, 3\})$. The descent blocks are 84 and 62. The ascent blocks are 15 and 37, plus an empty ascent block between 4 and 6.

Theorem 2.8. *Given a signature with size $2n$ and its associated signature function f, the number of Dumont-3 permutations with that signature is exactly $\prod_{x=1}^{2n} f(x)$.*

Proof. We can construct a Dumont-3 permutation as follows.

From the signature, determine the pits. In order from the largest pit to the smallest, assign the pits to peaks to form descent blocks. We only need to make assignments for non-singleton pits since a unique assignment is made by definition in the case of the singletons.

When assigning a pit x, we may choose any peak larger than x, excluding those chosen for pits greater than x. The number of alternatives is therefore

$$\mathrm{pk}(x) - (\mathrm{pt}(x) - 1) = \mathrm{pk}(x) - \mathrm{pt}(x) + 1 = f(x).$$

Note that we are acquiring a factor of $f(x)$ only when x is a non-singleton pit. Singletons do not contribute factors at this stage.

We next assemble the descent blocks in sequence. We proceed in the order from smallest peak to largest. When dealing with a peak x, we can put its descent block at the start of the permutation (if that choice has not been taken) or immediately after any pit that is smaller than x (excluding those already taken by smaller peaks, and the pit matched with x itself). Thus, the number of alternatives is

$$1 + (\#\mathrm{pits} < x) - (\#\mathrm{peaks} < x) - 1 = (\#\mathrm{pits} < x) - (\#\mathrm{peaks} < x) = \mathrm{pk}(x) - \mathrm{pt}(x) = f(x).$$

We have now acquired a factor of $f(x)$ for each peak x, singleton or not.

We now have the full sequence of peaks and pits, including singletons. We can assign the remaining odd numbers and even numbers to ascent blocks and descent blocks exactly as in the case of Dumont-1 permutations, hence the theorem follows. ☐

Example 2.9. There are exactly 48 Dumont-3 permutations in $\mathfrak{D}_8^3$ with signature $(\{6, 8\}, \{1, 3\})$. The permutation 15846237 is constructed as follows:

(1) match pit 4 with peak 8 (it could have been matched with 6; $f(4) = 2$)
(2) match pit 2 with peak 6 (only choice; $f(2) = 1$)
(3) put the descent sequence $6 \searrow 2$ immediately after $8 \searrow 4$ (it could have been put at the start of the permutation; $f(6) = 2$)
(4) put the descent sequence $8 \searrow 4$ at the start of the permutation (only choice; $f(8) = 1$)
(5) insert 1 before $8 \searrow 4$ (only choice; $f(1) = 1$)
(6) insert 3 after $6 \searrow 2$ (it could have fit before $8 \searrow 4$; $f(3) = 2$)
(7) insert 5 before $8 \searrow 4$ (it could have gone between $8 \searrow 4$ and $6 \searrow 2$ or after $6 \searrow 2$; $f(5) = 3$)
(8) insert 7 after $6 \searrow 2$ (it could have gone before $8 \searrow 4$; $f(7) = 2$).

Corollary 2.10. *Given a signature, there are as many Dumont-3 permutations with that signature as Dumont-1 permutations with that signature.*

Corollary 2.11. *For each n, there are as many Dumont-3 permutations of size $2n$ as Dumont-1 permutations of size $2n$. In other words, $|\mathfrak{D}^3_{2n}| = |\mathfrak{D}^1_{2n}|$.*

It would not be hard now to construct a bijection from $\mathfrak{D}^1_{2n}$ to $\mathfrak{D}^3_{2n}$, by making arbitrary matchings within each signature group. Bijections constructed in this way would preserve signatures, but apparently would not have any other interesting properties. Therefore, we will skip this step and construct more natural bijections in the following sections.

3. Dumont-3 permutations and surjective pistols

In this section we give a pair of bijections between subexcedent functions and permutations of size $2n$. One of the bijections maps a certain class of subexcedent functions (the "odd-surjective" subexcedent functions) onto $\mathfrak{D}^1_{2n}$, and the other maps the same class onto $\mathfrak{D}^3_{2n}$. Composing these bijections gives an explicit bijection between $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^3_{2n}$.

Definition 3.1. A *subexcedent function* (or *SE function*) on $[2n]$ is a function $\alpha : [2n] \to [2n]$ such that $\alpha(i) \le i$ for all $i \in [2n]$. Let SE_{2n} be the set of all subexcedent functions on $[n]$.

Definition 3.2. A subexcedent function α is called *odd* if all of its values are odd; that is, if $\alpha(i)$ is odd for each $i \in [2n]$. If, also, α takes on all possible odd values, then α is called *odd-surjective.* Thus, α is odd-surjective if its image is precisely $\{1, 3, 5, \ldots, 2n-1\}$. We denote the set of odd-surjective subexcedent functions on $[2n]$ by OSSE_{2n}.

Odd-surjective SE functions are related to "surjective pistols," which were studied by Dumont and Foata [8] and by Zeng and Zhao [20]. These authors have shown that surjective pistols are also enumerated by the Genocchi numbers.

Definition 3.3. A *surjective pistol* on $[2n]$ is a surjective map $p : [2n] \to 2[n] = \{2, 4, \ldots, 2n\}$ such that $p(i) \ge i$ for each $i \in [2n]$. We denote the set of surjective pistols on $[2n]$ by SP_{2n}.

Odd-surjective SE functions are just reverse-complements of surjective pistols. If p is a surjective pistol on $[2n]$, then the function α on $[2n]$ given by $\alpha(i) = 2n + 1 - p(2n + 1 - i)$ is an odd-surjective subexcedent function. The range of p is $\{2, 4, 6, \ldots, 2n\}$ and the range of α is $\{1, 3, 5, \ldots, 2n - 1\}$.

It is well known that there SE_{2n} is equinumerous to $\mathfrak{S}_{2n}$. The SE functions are used (in many different ways) as recipes for constructing permutations.

Here we give two more bijections:

- $\mathrm{IAX} : \mathrm{SE}_{2n} \to \mathfrak{S}_{2n}$, which maps the odd-surjective SE functions in OSSE_{2n} onto $\mathfrak{D}^1_{2n}$, and
- $\mathrm{IBOP} : \mathrm{SE}_{2n} \to \mathfrak{S}_{2n}$, which maps the odd-surjective SE functions in OSSE_{2n} onto $\mathfrak{D}^3_{2n}$.

The names IAX and IBOP stand for "*insert after, with exchange*" and "*insert before opposite parity,*" respectively.

If $\alpha \in \mathrm{SE}_{2n}$, then construct a permutation $\pi = \mathrm{IAX}(\alpha)$ as follows. For each $i \in [2n]$, in the order from 1 to $2n$:

(1) If $\alpha(i) = i$, insert i at the end.
(2) If $\alpha(i) < i$, then insert i after $\alpha(i)$, *except* if that would put i at the end, insert i at the beginning instead.

(The "exchange" in IAX is that "except" clause, which exchanges what might have been the natural roles of the beginning and end locations.)

If $\alpha \in \mathrm{SE}_{2n}$, then construct a permutation $\pi = \mathrm{IBOP}(\alpha)$ as follows. For each $i \in [2n]$, in the order from 1 to $2n$:

(1) If $\alpha(i)$ is even, then insert the value i before the value $\alpha(i) - 1$. Thus, if $\alpha(i)$ is even, we insert i before an odd value.
(2) If $\alpha(i)$ is odd and the value $\alpha(i)$ happens to precede an even value, then insert the value i between them. Thus, in this case, if $\alpha(i)$ is odd, insert before an even value.
(3) Suppose that $\alpha(i)$ is odd and is not a value that precedes an even value, and in fact, suppose that $\alpha(i)$ is the k-th smallest of such values. Then insert the value i in the k-th leftmost available insertion point, where the available points are
   - before evens that don't follow odds, and
   - at the end.

   Thus, in this case, too, if $\alpha(i)$ is odd, insert before an even or at the end.

(Note that the "opposite parity" in IBOP is the parity opposite to that of $\alpha(i)$, not i. Note also that, while we are interested only in permutations of even length, the definitions of IAX and IBOP would work just as well for permutations of odd length.)

Theorem 3.4.
(1) *IAX and IBOP are indeed bijections from* SE_{2n} *to* $\mathfrak{S}_{2n}$.
(2) *If* $\alpha \in \mathrm{SE}_{2n}$, *then the image of* α *contains exactly all odd numbers less than* $2n$ *if and only if* $\pi = \mathrm{IBOP}(\alpha)$ *is a* $\mathfrak{D}^3$*-permutation. Likewise for* IAX *and* $\mathfrak{D}^1$. *Therefore,* $\mathfrak{D}^3$*-permutations,* $\mathfrak{D}^1$*-permutations, odd-surjective SE functions, and surjective pistols are all equinumerous.*
(3) *The image of* α *contains only odd numbers—that is,* α *is an odd SE function—if and only if* $\pi = IBOP(\alpha)$ *is a potential* $\mathfrak{D}^3$*-permutation, meaning that* π *represents the order of* $1, 2, \ldots, n$ *in some longer* $\mathfrak{D}^3$*-permutation. Likewise for* IAX *and* $\mathfrak{D}^1$.

Example 3.5. IBOP(1133) = 1234, IBOP(1131) = 1423, IBOP(1113) = 1342. Note that 111 does not contain all odd integers in $\{1, 2, 3\}$, so IBOP(111) = 132 is not a $\mathfrak{D}^3$-permutation, but does occur as a subsequence of a larger $\mathfrak{D}^3$-permutation 1342.

Likewise, IAX(1133) = 4213, IAX(1131) = 2143, IAX(1113) = 3421. Note also that IAX(111) = 321 is not a $\mathfrak{D}^1$-permutation, but does occur as a subsequence of a larger $\mathfrak{D}^1$-permutation 3421.

Note that the bijections constructed in this section do not usually preserve signatures, so they are different from the bijections that might have arisen from the methods of the previous section.

4. A bijection between Dumont-1 and Dumont-3 permutations

Here we will consider the distribution of certain pattern statistics on Dumont-1 and Dumont-3 permutations. Recall that a *pattern* is an order-isomorphism type of a subsequence. Thus, for example, an *instance*, or an *occurrence*, of a pattern 231 in a permutation π is a subsequence (a, b, c) of π such that $c < a < b$ and $\pi^{-1}(a) < \pi^{-1}(b) < \pi^{-1}(c)$. A permutation *avoids* a pattern if it contains no instances of it. A *generalized pattern* [1] is a pattern where some consecutive terms of a subsequence must also be consecutive in the whole permutation. For example, an instance of 2-31 in π is an instance of 231 where elements corresponding to "3" and "1" are consecutive in π. We denote the subset of permutations of a set S avoiding a certain pattern τ by $S(\tau)$.

We can now refine our Main Theorem as follows.

Theorem 4.1. *The bistatistic* (2-31, 31-2) *of the numbers of occurrences of patterns* 2-31 *and* 31-2 *has the same distribution on the sets* $\mathfrak{D}^1_{2n}$ *and* $\mathfrak{D}^3_{2n}$ *of Dumont permutations of the first and third kinds of length* $2n$, *for any* $n \geq 0$.

Our strategy is to start with a Dumont-1 permutation and to successively rearrange certain blocks so as to make them "$\mathfrak{D}^3$-legal." Notice that there are only two types of elements that do not occur in both Dumont-1 and Dumont-3 permutations:

- Dumont-1, but not Dumont-3, permutations contain odd descent bottoms (*i.e.* entries that are immediately preceded by a larger value).
- Dumont-3, but not Dumont-1, permutations contain even ascent bottoms (*i.e.* entries that are immediately followed by a larger value) and/or even rightmost entries.

4.1. **The bijection.** Each step ϕ_{2k-1}, $k \in [n]$ of our bijection $\phi^* = \phi_1 \circ \phi_3 \circ \cdots \circ \phi_{2n-1} : \mathfrak{D}^1_{2n} \to \mathfrak{D}^3_{2n}$ will rearrange certain blocks of a given permutation in two cases:

- if $b = 2k - 1$ is an odd descent bottom and $t = 2k$ starts a descent run (a sequence of consecutive descents), *i.e.* t is an ascent top and a descent top.
- if $b = 2k - 1$ is an odd descent bottom and $t = 2k$ is in the middle of a descent run, *i.e.* t is a descent bottom and a descent top.

In addition, we always consider initial entries to be ascent tops and final entries to be ascent bottoms (as if prepending 0 at the beginning and appending ∞ at the end of every permutation).

We define ϕ_b as follows (see Figure 1). Let $\pi \in S_{2n}$ be any permutation. Given a block (substring) S of permutation π define $\ell(S)$ and $r(S)$ to be the leftmost and rightmost elements of S. Also, for a string S and element x, we write $S > x$ (resp. $S < x$) if each element of S is greater (resp. less) than x.

Case 0. If b is not a descent bottom, then $\phi_b(\pi) = \pi$.

Case 1. Suppose b is a descent bottom and $t = b + 1$ starts a descent run, *i.e.* $t = \pi(j) > \pi(j + 1)$ for some $j \in [2n - 1]$, and either $j = 1$ or $\pi(j - 1) < t$. Then either b is to the left of t or t is to the left of b.

Sub-Case 1a. Suppose b is to the left of t. Then π can be represented uniquely as a concatenation of substrings

$$\pi = IT_1bT_2B_1T_3B_2 \dots T_mB_{m-1}tB_mF,$$

where $m \geq 1$, each $T_i > t$, each $B_i < b$, $I = \emptyset$ or $r(I) < b$, $F = \emptyset$ or $\ell(F) > t$, $T_1 \neq \emptyset$, $B_m \neq \emptyset$, and if the sequence $T_2B_1 \dots T_mB_{m-1}$ is nonempty, then all T_i's and B_i's are nonempty. Then we define

$$\phi_b(\pi) = IbT_1B_1T_2B_2T_3 \dots B_{m-1}T_mB_mtF.$$

Note that, defining $B_0 = b$ and $T_{m+1} = t$, we see that ϕ_b exchanges T_{i+1} and B_i for every $i \in [0, m]$.

Sub-Case 1b. Suppose t is to the left of b but not immediately to the left of b. Then π can be represented uniquely as a concatenation of substrings

$$\pi = ItB_1T_1B_2T_2 \dots B_mT_mbF,$$

where $m \geq 1$, each $T_i > t$, each $B_i < b$, $I = \emptyset$ or $r(I) < b$, $F = \emptyset$ or $\ell(F) > t$, and all T_i's and B_i's are nonempty. Then we define

$$\phi_b(\pi) = ItT_1B_1T_2B_2 \dots T_mB_mbF.$$

Note that ϕ_b simply exchanges T_i and B_i for every $i \in [m]$.

Sub-Case 1c. Suppose t is immediately to the left of b. Then

$$\pi = ItbF,$$

where $I = \emptyset$ or $r(I) < b$, and $F = \emptyset$ or $\ell(F) > t$. Then we define

$$\phi_b(\pi) = IbtF.$$

Case 2. Suppose b is a descent bottom and $t = b + 1$ is in the middle of a descent run, *i.e.* $t = \pi(j)$ for some $j \in [2, 2n - 1]$ and $\pi(j - 1) > t > \pi(j + 1)$. Then either b is to the left of t or t is to the left of b.

Sub-Case 2a. Suppose that b is to the left of t. Then π can be represented uniquely as a concatenation of substrings

$$\pi = IbMtBF,$$

where $I \neq \emptyset$, $r(I) > t$, $F = \emptyset$ or $\ell(F) > t$, $M \neq \emptyset$, $B \neq \emptyset$, $\ell(M) > t$, $r(M) > t$, $B < b$. Then we define

$$\phi_b(\pi) = ItMBbF.$$

Sub-Case 2b. Suppose that t is to the left of b. Then π can be represented uniquely as a concatenation of substrings

$$\pi = ITtMbF,$$

where $I = \emptyset$ or $r(I) < b$, $F = \emptyset$ or $\ell(F) > t$, $T \neq \emptyset$, $T > t$, $M = \emptyset$ or $\ell(M) < b$ and $r(M) > t$. Then we define

$$\phi_b(\pi) = IbTMtF.$$

We leave it to the reader to verify the following facts:

- ϕ_b is invertible for any b. We define $\psi_b = \phi_b^{-1}$ and $\psi^* = (\phi^*)^{-1}$.
- ϕ_{b_1} and ϕ_{b_2} commute if $|b_1 - b_2| > 1$. This is because each ϕ_b leaves unchanged the subsequence of elements greater than t and the subsequence of elements less than b.
- after applying $\phi^* = \prod_{i=1}^{n} \phi_{2i-1}$ to a permutation in $\pi \in \mathfrak{D}^1_{2n}$ (where no odd entry is a descent top and only odd entries can be descent bottoms), we obtain a permutation $\phi^*(\pi)$ in which no odd entry is a descent top and, in addition, no odd entry is a descent bottom. Thus, descents in $\phi^*(\pi)$ may occur only from an even value to an even value, so $\phi^*(\pi) \in \mathfrak{D}^3_{2n}$, in other words, ϕ^* is an injection from $\mathfrak{D}^1_{2n}$ to $\mathfrak{D}^3_{2n}$.
- we can similarly prove that ψ^* is an injection from $\mathfrak{D}^3_{2n}$ to $\mathfrak{D}^1_{2n}$, so in fact, both ϕ^* and ψ^* are bijections.

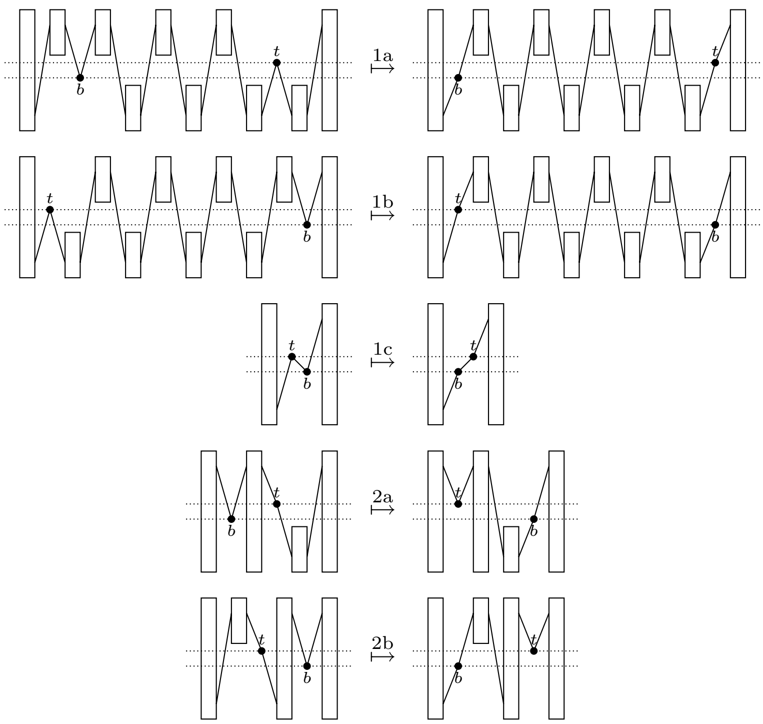

Figure 1. The cases of bijection ϕ_b (from top to bottom: Cases 1a, 1b, 1c, 2a, 2b).

4.2. **The occurrences of 2-31 and 31-2.** As follows from the above observation, the only instances of patterns 2-31 and 31-2 that may occur not in both π and $\phi_b(\pi)$ are those where the "3" is at least t and the "1" is at most b. Thus, we only need to consider subsequences yzx and zxy of π, where $x < y < z$, and z and x are consecutive entries of π such that $z \geq t$ and $x \leq b$. Note that, in Case 0 and Sub-Case 1c, the instances of 2-31 and 31-2 in π and $\phi_b(\pi)$ are exactly the same, so we only need to deal with Sub-Cases 1a, 1b, 2a and 2b.

Even in those remaining cases, if the descent zx is in the substring I or F, then the instances yzx and zxy of patterns 2-31 and 31-2 are the same in π and $\phi_b(\pi)$. Thus, we only need to consider the descents zx not in I or F and such that $z \geq t$ and $x \leq b$.

We will consider Sub-Cases 1a and 2a and leave similar Sub-Cases 1b and 2b to the reader.

Sub-Case 1a. We will use our previous definitions $B_0 = b$ and $T_{m+1} = t$. The only descents of π that are not in $\phi_b(\pi)$ are $r(T_{i+1})\ell(B_i)$ for all $i \in [0, m]$. Likewise, the only descents of $\phi_b(\pi)$ that are not in π are $r(T_i)\ell(B_i)$ for all $i \in [m]$. If yzx (resp. zxy) is an instance of 2-31 (resp. 31-2), $z = r(T_{i+1}) > t$ and $x = \ell(B_i) < b$, then there are two cases.

(1) If $y \geq t$, then the instance $yr(T_{i+1})\ell(B_i)$ (resp. $r(T_{i+1})\ell(B_i)y$) of 2-31 (resp. 31-2) in π corresponds to the instance $yr(T_{i+1})\ell(B_{i+1})$ (resp. $r(T_{i+1})\ell(B_{i+1})y$) of 2-31 (resp. 31-2) in $\phi_b(\pi)$. Note that, in fact, $i < m$, since $r(T_{m+1}) = t \leq y$ cannot be the "3" if y is the "2".

(2) If $y \leq b$, then the instance $yr(T_{i+1})\ell(B_i)$ (resp. $r(T_{i+1})\ell(B_i)y$) of 2-31 (resp. 31-2) in π corresponds to the instance $yr(T_i)\ell(B_i)$ (resp. $r(T_i)\ell(B_i)y$) of 2-31 (resp. 31-2) in $\phi_b(\pi)$. Note that, in fact, $i > 0$, since $\ell(B_1) = b \geq y$ cannot be the "1" if y is the "2".

Sub-Case 2a. The mapping $\pi = IbMtBF \mapsto ItMBbF = \phi_b(\pi)$ can be achieved by exchanging b and t, then exchanging b and B.

Consider the first step, exchanging b and t, $IbMtBF \mapsto ItMbBF$. Since b and t are consecutive integers, we only need to consider the occurrences of 2-31 and 31-2 in π that involve both t and b (in the rest, we simply exchange b for t or t for b). In π, these are $r(I)bt$ (an occurrence of 31-2) and $bt\ell(B)$ (an occurrence of 2-31). Exchanging b and t in π, we lose these subsequences but gain an occurrence $tr(M)b$ of 2-31 in $ItMbBF$. Thus, exchanging t and b preserves the number of occurrences of 2-31 and reduces the number of occurrences of 31-2 by 1.

Now consider the second step, exchanging b and B, $ItMbBF \mapsto ItMBbF$. This means that the consecutive descents $r(M)b$ and $b\ell(B)$ in occurrences of 2-31 and 31-2 in $ItMbBF$ are replaced by a single descent $r(m)\ell(B)$ in the corresponding occurrences of 2-31 and 31-2 in $ItMBbF$. The rest of the occurrences of these patterns remain the same. However, we also gain one extra occurrence $r(M)\ell(B)b$ of 31-2.

Thus, after performing both steps, we see that their composition ϕ_b preserves the number of occurrences of both 2-31 and 31-2.

This ends the proof of Theorem 4.1, a refined version of the Main Theorem.

4.3. **Pattern avoidance on Dumont permutations.** Dumont-1 and Dumont-2 permutations avoiding various small patterns were considered in [2, 3, 15]. We will mention one result in particular:

Theorem 4.2 (Mansour [15])**.** *$|\mathfrak{D}^1_{2n}(\text{2-31})| = |\mathfrak{D}^1_{2n}(\text{31-2})| = C_n$, the n-th Catalan number.*

Theorem 4.1 implies that we have a similar result on $\mathfrak{D}^3$.

Theorem 4.3. *$|\mathfrak{D}^3_{2n}(\text{2-31})| = |\mathfrak{D}^3_{2n}(\text{31-2})| = C_n$.*

Note also that since 2-31 and 31-2 are obviously equidistributed on $\mathfrak{D}^3$ as reverse complements of each other, it implies that 2-31 and 31-2 are also equidistributed on $\mathfrak{D}^1$.

4.4. **Equidistribution of 2-31 and 31-2.** An obvious corollary of the refined Main Theorem is the following.

Corollary 4.4. *The distribution of the number of occurrences of* 2-31 *(resp.* 31-2*) is the same on $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^3_{2n}$.*

In [5], the following fact is noted: every Dumont-3 permutation begins with 1, and if the initial 1 is deleted from every permutation in $\mathfrak{D}^3_{2n}$, the resulting set is centrally symmetric, *i.e.* invariant under the operation of reversal of complement (180° degree rotation). Since 2-31 and 31-2 are 180° rotations of each other, it follows that the distributions of 2-31 and 31-2 are the same on $\mathfrak{D}^3_{2n}$. Thus, Corollary 4.4 implies the following.

Corollary 4.5. *The distributions of the numbers of occurrences of* 2-31 *and* 31-2 *are the same on $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^3_{2n}$.*

Note that this is not an obvious statement on $\mathfrak{D}^1_{2n}$ since the set $\mathfrak{D}^1_{2n}$ is not centrally symmetric.

5. Seidel triangle generation on Dumont-1 and Dumont-3 permutations

Define the following subsets of $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^3_{2n}$ as in [5].

Definition 5.1. Let $\mathcal{H}^3_{2n,2k}$ be the set of permutations $\pi \in \mathfrak{D}^3_{2n}$ such that $\pi(2) = 2k$. Let $\mathcal{G}^3_{2n,2k}$ be the set of permutations $\pi \in \mathfrak{D}^3_{2n}$ whose leftmost even letter is $2k$. Similarly, let $\mathcal{G}^1_{2n,2k}$ be the set of permutations $\pi \in \mathfrak{D}^1_{2n}$ whose leftmost even letter is $2k$, and let $\mathcal{H}^1_{2n,2k}$ be the set of permutations $\pi \in \mathfrak{D}^1_{2n}$ such that $\pi(1) = 2k$ and the descending run starting at $2k$ ends at 1.

Notice that the description of bijections ϕ^* and ψ^* in the previous section implies the following:

Corollary 5.2. *The maps ϕ^* and ψ^* map $\mathcal{G}^1_{2n,2k}$ onto $\mathcal{G}^3_{2n,2k}$ and vice versa. In other words, both maps preserve the value of the leftmost even letter (i.e. the leftmost peak). Furthermore, ϕ^* and ψ^* map $\mathcal{H}^1_{2n,2k}$ onto $\mathcal{H}^3_{2n,2k}$ and vice versa.*

The *Seidel triangle* for Genocchi numbers [5, 9, 10, 19, 20] is a Pascal triangle-type array of integers $(g_{i,j})_{i,j\geq 1}$ that is a refinement of Genocchi numbers. It is defined by the following recursive relation:

$$\begin{cases} g_{2i+1,j} = g_{2i+1,j-1} + g_{2i,j} \text{ for } j = 1, 2, \ldots, i+1, \\ g_{2i,j} = g_{2i,j+1} + g_{2i-1,j} \text{ for } j = i, i-1, \ldots, 1, \end{cases} \tag{5.1}$$

where $g_{1,1} = 1$ and $g_{i,j} = 0$ if $j \leq 0$ or $i \leq 0$ or $i > \lceil j/2 \rceil$. Then

$$g_{2n-1,n-1} = g_{2n-1,n} = g_{2n,n} = G_{2n+2}, \qquad g_{2n-1,1} = g_{2n-2,1} = H_{2n-1},$$

where G_{2n} is the nth Genocchi number, and H_{2n-1} is the nth *median Genocchi number* (or nth *Genocchi number of the second kind*).

The first few values of the Seidel triangle are as follows (numbering rows i from bottom to top and columns j from left to right):

5									155	155
4							17	17	155	310
3					3	3	17	34	138	448
2			1	1	3	6	14	48	104	552
1	1	1	1	2	2	8	8	56	56	608
j/i	1	2	3	4	5	6	7	8	9	10

Note that the odd columns $2n-1$ sum to H_{2n-1}, while even columns $2n$ sum to G_{2n+2}.

It was proven in [5] that

$$|\mathcal{H}^1_{2n,2k}| = g_{2n-1,n-k+1} \quad \text{and} \quad |\mathcal{G}^1_{2n,2k}| = g_{2n,n-k+1}.$$

In particular, $H_{2n-1} = |\mathcal{H}^1_{2n,2n}|$, $H_{2n+1} = |\cup_{k=1}^n \mathcal{H}^1_{2n,2k}| = |\mathcal{H}^1_{2n,\mathrm{even}}|$, $G_{2n} = |\mathcal{G}^1_{2n,2}|$, $G_{2n+2} = |\cup_{k=1}^n \mathcal{G}^1_{2n,2k}| = |\mathcal{G}^1_{2n,\mathrm{even}}| = |\mathfrak{D}^1_{2n}|$.

This, together with Corollary 5.2, implies that

Theorem 5.3. *We have*

$$|\mathcal{H}^3_{2n,2k}| = g_{2n-1,n-k+1} \quad \textit{and} \quad |\mathcal{G}^3_{2n,2k}| = g_{2n,n-k+1}.$$

In particular, $H_{2n-1} = |\mathcal{H}^3_{2n,2n}|$, $H_{2n+1} = |\cup_{k=1}^n \mathcal{H}^3_{2n,2k}| = |\mathcal{H}^3_{2n,\mathrm{even}}|$, $G_{2n} = |\mathcal{G}^3_{2n,2}|$, $G_{2n+2} = |\cup_{k=1}^n \mathcal{G}^3_{2n,2k}| = |\mathcal{G}^3_{2n,\mathrm{even}}| = |\mathfrak{D}^3_{2n}|$.

Then $|\mathcal{H}^1_{2n,2k}| = |\mathcal{H}^3_{2n,2k}|$ and $|\mathcal{G}^1_{2n,2k}| = |\mathcal{G}^3_{2n,2k}|$, so we only need to prove the following theorem.

Theorem 5.4. $|\mathcal{H}^1_{2n,2k}| = g_{2n-1,n-k+1}$ *and* $|\mathcal{G}^1_{2n,2k}| = g_{2n,n-k+1}$. *In particular, $H_{2n-1} = |\mathcal{H}^1_{2n,2n}|$, $H_{2n+1} = |\cup_{k=1}^n \mathcal{H}^1_{2n,2k}| = |\mathcal{H}^1_{2n,\mathrm{even}}|$, $G_{2n} = |\mathcal{G}^1_{2n,2}|$, $G_{2n+2} = |\cup_{k=1}^n \mathcal{G}^1_{2n,2k}| = |\mathcal{G}^1_{2n,\mathrm{even}}| = |\mathfrak{D}^1_{2n}|$.*

Proof. We will prove that that the functions $h(n,k) = |\mathcal{H}^1_{2n,2k}|$ and $g(n,k) = |\mathcal{G}^1_{2n,2k}|$ satisfy the same recurrences as $g_{2n-1,n-k+1}$ and $g_{2n,n-k+1}$, respectively:

$$h(n,k) = h(n,k+1) + \sum_{j=1}^{k-1} h(n-1,j), \tag{5.2}$$

$$g(n,k) = g(n,k-1) + \sum_{j=k-1}^{n-1} g(n-1,j). \tag{5.3}$$

We will now describe two bijections, β_1 and β_2, that prove the recurrences (5.2) and (5.3).
Bijection β_1. Given a permutation that starts with $2k,\dots,2,1$:

(1) exchange $2k$ and $2k+2$ if $2k+2$ is not preceded or followed by $2k+1$,
(2) if $2k+2$ is followed by $2k+1$, remove $2k+2$ and insert it in front of $2k$ (*i.e.* at the beginning),
(3) if $2k+2$ is preceded by $2k+1$, then remove $2k$ (at the beginning), replace the block $2k+1, 2k+2$ with $2k$, and subtract 2 from every letter greater than $2k+2$.

The first two cases yield a permutation of the same size that starts with $2k+2,\dots,2,1$. The last case yields a permutation of size 2 less than the original permutation, that starts with an even letter less than $2k$.
Bijection β_2. Given a permutation with first even letter $2k+2$:

(1) if $2k+2$ is not preceded or followed by $2k+1$ and not followed by $2k$, then exchange $2k$ and $2k+2$,
(2) if $2k+2$ is followed by $2k$ and not preceded by $2k+1$, then remove $2k+2$ and insert it in front of $2k+1$,
(3) if $2k+2$ is followed by $2k+1$, remove $2k+2, 2k+1$ and subtract 2 from every letter greater than $2k+2$.
(4) if $2k+2$ is preceded by $2k+1$ and followed by $2k$, remove $2k+1, 2k+2$ and subtract 2 from every letter greater than $2k+2$.

The first two cases yield a permutation of the same size that with first even letter $2k$. The last two cases yield a permutation of size 2 less than the original permutation, with the first even letter at least $2k$. ☐

We conjecture that Theorem 5.3 can be refined further as follows. Consider $\mathfrak{D}^3$-permutations in the set difference $\mathcal{G}^3_{2n,2k} \setminus \mathcal{H}^3_{2n,2k}$, *i.e.* those whose leftmost even value $2k$ does not occur immediately after 1.

Conjecture 5.5. *Given $m < k \le n$, let $f(n,k,m)$ be the number of permutations in $\mathcal{G}^3_{2n,2k} \setminus \mathcal{H}^3_{2n,2k}$ whose rightmost (alternatively, leftmost after $2k$) even entry less than $2k$ is $2m$. Then $f(n,k,m) = h(n,m) = g_{2n-1,n-m+1}$.*

Note that $f(n,k,m)$ is apparently independent of k. In fact, we conjecture that the overage

$$f(n,k,m) - f(n,k,m+1) = h(n,m) - h(n,m+1) = g_{2n-2,n-m+1} = f(n-1,k-1,m-1)$$

counts permutations where $2k$ is the leftmost even value, $2m$ is the rightmost even value less than $2k$, and $2m-1$ immediately precedes $2m$.

We also conjecture that the even columns of the Seidel triangle can be obtained using a different statistic on $\mathfrak{D}^3$.

Conjecture 5.6. *The number of $\mathfrak{D}^3_{2n}$-permutations where $2k$ immediately follows $2n$ is $g_{2n,k}$. Equivalently, the number of $\mathfrak{D}^3_{2n}$-permutations where $2k$ immediately precedes 2 is $g_{2n,n-k+1}$.*

This last conjecture can be easily restated for $\mathfrak{D}^4$-permutations as follows.

Conjecture 5.7. *The number of $\mathfrak{D}^4_{2n}$-permutations that end on $2k$ is $g_{2n,k}$. Equivalently, the number of $\mathfrak{D}^4_{2n}$-permutations where 2 is in position $2k$ is $g_{2n,n-k+1}$.*

Finally, we make a conjecture on the complementary statistic on $\mathfrak{D}^4$-permutations that generates the odd columns of the Seidel triangle.

Conjecture 5.8. *The number of permutations $\pi \in \mathfrak{D}^4_{2n}$ such that $\pi(2n) = 2$ and $m \in [n-1]$ is the least integer such that $\pi(2m+1) = 2m+2$ is $g_{2n-1,m}$. Also, the number of permutations $\pi \in \mathfrak{D}^4_{2n}$ such that $\pi(2n) = 2$ and there is no $m \in [n-1]$ for which $\pi(2m+1) = 2m+2$ is $g_{2n-1,n-1}$.*

6. Laguerre histories and bijections on Dumont permutations

In this section, we describe how to make use of some weighted Motzkin paths and bijections given by Corteel [6] to give an alternative description of the bijection in Section 4, and obtain further results of equidistribution for some statistics in Dumont permutations. These weighted Motzkin paths were originally studied by Viennot in the general context of combinatorial theory of orthogonal polynomials, and the bijections we use are originally due to Françon and Viennot, Foata and Zeilberger. Corteel showed that several statistics of interest can be followed through these bijections, and we refer to [6] for more precisions.

Definition 6.1. A *Motzkin path* of length n is a path from $(0,0)$ to $(n,0)$ on or above the x-axis with steps $\nearrow = (1,1)$, $\rightarrow = (1,0)$ and $\searrow = (1,-1)$.

Definition 6.2. A *Laguerre history* is a weighted Motzkin path with the following properties:
- the weight of a step $\nearrow$ starting at height h is yp^iq^{h-i} with $0 \le i \le h$,
- the weight of a step $\rightarrow$ starting at height h is either yp^iq^{h-i} with $0 \le i \le h$, or p^iq^{h-1-i} with $0 \le i \le h-1$,
- the weight of a step $\searrow$ starting at height h is p^iq^{h-1-i} with $0 \le i \le h-1$

We call the product of weights of each step *total weight* of the history.

The Françon-Viennot [11] bijection Ψ_{FV} between permutations of S_n and Laguerre histories of size n has the property that for any permutation σ, the total weight of $\Psi_{FV}(\sigma)$ is $y^{\mathrm{asc}(\sigma)}p^{31\text{-}2(\sigma)}q^{2\text{-}31(\sigma)}$, where $\mathrm{asc}(\sigma)$ is the number of ascent (using the convention that n is an ascent of any permutation of $[n]$), and $31\text{-}2(\sigma)$ (respectively, $2\text{-}31(\sigma)$) is the number of occurrences of the pattern 31-2 (respectively, 2-31) in σ. More precisely, the i-th step in a $\Psi_{FV}(\sigma)$ has a weight y if and only if $\sigma^{-1}(n)$ is an ascent of σ, and this property is particularly efficient in characterizing Dumont-1 and Dumont-3 permutations as we will see below.

The Foata-Zeilberger [12] bijection Ψ_{FZ} between permutations of $[n]$ and Laguerre histories of size n has the property that for any permutation of $[n]$, the total weight of $\Psi_{FZ}(\sigma)$ is $y^{\mathrm{wex}(\sigma)}p^{\mathrm{cr}(\sigma)}q^{\mathrm{ne}(\sigma)}$, where we use the following definition from [6].

Definition 6.3. Let σ be a permutation of $[n]$. We call a position i such that $\sigma(i) \ge i$ a *weak excedance* of σ. We call a pair of positions (i,j) such that $i < j \le \sigma(i) < \sigma(j)$ a *crossing* of σ, and a pair of positions (i,j) such that $i < j \le \sigma(j) < \sigma(i)$ a *nesting* of σ. We denote by $\mathrm{wex}(\sigma)$ (respectively, $\mathrm{cr}(\sigma)$ and $\mathrm{ne}(\sigma)$) the number of weak excedances (respectively, crossings and nestings) of σ.

And more precisely, the ith step in $\Psi_{FZ}(\sigma)$ has y as a factor in its weight if and only if i is a weak excedance of σ. From the known properties of the bijections Ψ_{FV} and Ψ_{FZ} we have the following result.

Proposition 6.4. *Let H be a Laguerre history of length $2n$. The following conditions are equivalent:*
- *$H = \Psi_{FV}(\sigma)$ for some permutation $\sigma \in \mathfrak{D}^1_{2n}$,*
- *$H = \Psi_{FZ}(\sigma)$ for some permutation $\sigma \in \mathfrak{D}^2_{2n}$,*
- *every odd step of H has weight yp^iq^j, and every even step of H has weight p^iq^j (in other words, the factor y appears in the weight of every odd step and no even step).*

In a similar manner, these bijections can be used to characterize Dumont-3 and Dumont-4 permutations.

Proposition 6.5. *Let H be a Laguerre history of length $2n$. The following conditions are equivalent:*
- *$H = \Psi_{FV}(\sigma)$ for some permutation $\sigma \in \mathfrak{D}^3_{2n}$,*
- *$H = \Psi_{FZ}(\sigma)$ for some permutation $\sigma \in \mathfrak{D}^4_{2n}$,*
- *every odd step of H is a step $\rightarrow$ with weight yp^iq^j (in other words, every odd step is horizontal and has the factor y in its weight).*

We come now to the main result of this section.

Theorem 6.6. *The following pairs of statistics have the same joint distribution:*
- $(31\text{-}2, 2\text{-}31)$ *on the set $\mathfrak{D}^1_{2n}$,*
- $(\mathrm{cr}, \mathrm{ne})$ *on the set $\mathfrak{D}^2_{2n}$,*
- $(31\text{-}2, 2\text{-}31)$ *on the set $\mathfrak{D}^3_{2n}$,*
- $(\mathrm{cr}, \mathrm{ne})$ *on the set $\mathfrak{D}^4_{2n}$.*

Proof. Proposition 6.4 (respectively, Proposition 6.5) shows the equality between the first and second (respectively, the third and fourth) items in this list. So it remains only to give a bijection between the sets $\Psi_{FV}(\mathfrak{D}^1_{2n})$ and $\Psi_{FV}(\mathfrak{D}^3_{2n})$ so that the total weight in p and q is preserved.

To this end, letting H be in $\Psi_{FV}(\mathfrak{D}^1_{2n})$, *i.e.* the factor y appears in every odd step and no even step, we will construct $\Gamma(H)$ in $\Psi_{FV}(\mathfrak{D}^3_{2n})$ with the same weight in p and q (but possibly with different weight in y). We consider the n pairs of consecutive steps in H such that the first element is an odd step and the second an even step. We obtain $\Gamma(H)$ by acting on successive pairs of consecutive steps of H by the following transformation.

- A pair of consecutive steps $\nearrow\searrow$ with respective weights $yp^{i_1}q^{j_1}$ and $p^{i_2}q^{j_2}$ in H becomes a pair of consecutive steps $\rightarrow\rightarrow$ with weights $yp^{i_1}q^{j_1}$ and $yp^{i_2}q^{j_2}$ in $\Gamma(H)$.
- A pair of consecutive steps $\nearrow\rightarrow$ with weights $yp^{i_1}q^{j_1}$ and $p^{i_2}q^{j_2}$ in H becomes a pair of consecutive steps $\rightarrow\nearrow$ with weights $yp^{i_1}q^{j_1}$ and $yp^{i_2}q^{j_2}$ in $\Gamma(H)$.
- In other cases, the first step of the pair is already a step $\rightarrow$ with weight yp^iq^j, so this pair is not modified in $\Gamma(H)$.

To check that the weights are valid in the obtained path, note that there are two cases where the starting height of a step is changed:

- When $\nearrow\searrow$ becomes $\rightarrow\rightarrow$, but both after and before the transformation the needed criterion is $i_2 + j_2 = h$, where h is the starting height in $\nearrow\searrow$.
- When $\nearrow\rightarrow$ becomes $\rightarrow\nearrow$, but similarly both after and before the transformation the needed criterion is $i_2 + j_2 = h$, where h is the starting height in $\nearrow\rightarrow$.

Once we know that the weights are valid, it is clear that $\Gamma(H)$ is indeed in $\Psi_{FV}(\mathfrak{D}^3_{2n})$, from the very definition of this bijection Γ. The inverse bijection can also be defined by acting on the pairs of consecutive steps. See Figure 2 for an example. $\square$

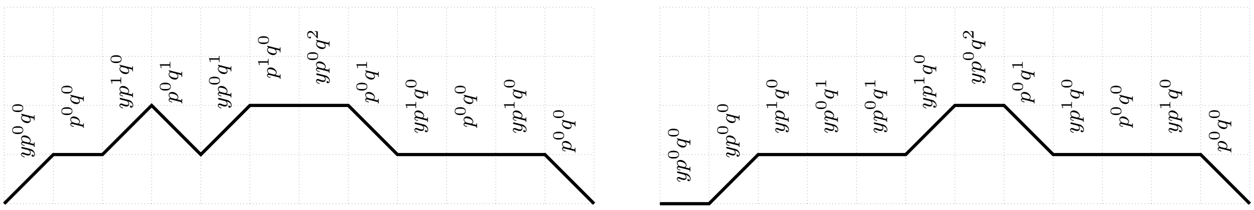

FIGURE 2. Laguerre histories in $\Psi_{FV}(\mathfrak{D}^1_{2n})$ and $\Psi_{FV}(\mathfrak{D}^3_{2n})$.

Remark 6.7. It can be checked that permutations with a given signature, as defined in Section 4, are in bijection with Laguerre histories corresponding to a given Motzkin path. Note that the fact that their number can be expressed as a product is more apparent knowing their interpretation in terms of Laguerre histories.

To end this section, we make a link with continued fractions. In the weighted Motzkin paths considered above, there are some parity conditions on the steps so that it is not immediate to link these paths and continued fractions, but we will give a bijection with weighted Dyck paths such that there is no parity condition, and obtain the theorem below. The continued fraction we obtain is a natural extension of the one given by Viennot [19] for the generating function of G_{2n}.

Theorem 6.8. *Let*

$$G_{2n+2}(p,q) = \sum_{\sigma\in\mathfrak{D}^1_{2n}} p^{\text{2-31}(\sigma)} q^{\text{31-2}(\sigma)}. \tag{6.1}$$

Let $[i]_{p,q} = \frac{p^i - q^i}{p-q}$*, then we have:*

$$\sum_{n=1}^{\infty} G_{2n}(p,q) z^{2n} = \cfrac{z^2}{1 - \cfrac{[1]_{p,q}[1]_{p,q}z^2}{1 - \cfrac{[1]_{p,q}[2]_{p,q}z^2}{1 - \cfrac{[2]_{p,q}[2]_{p,q}z^2}{1 - \cfrac{[2]_{p,q}[3]_{p,q}z^2}{\ddots}}}}}. \tag{6.2}$$

Proof. We can forget the parameter y in a Laguerre history satisfying the criterion in Proposition 6.4 (the Laguerre history can be recovered since the y occurs exactly at odd steps). This shows that $G_{2n}(p,q)$ is the generating function of Motzkin paths such that:

- an odd step starting at height h is $\nearrow$ or $\rightarrow$, and has a weight $[h+1]_{p,q}$,
- an even step starting at height h is $\rightarrow$ or $\searrow$, and has a weight $[h]_{p,q}$.

(To be precise, the weight of one of these Motzkin paths is the generating function of all Laguerre histories of a given "shape".) There is a bijection between these paths, and weighted Dyck paths of same length such that the weight of a step starting at height h is $[i]_{p,q}$ where $i = 1 + \lfloor \frac{h}{2} \rfloor$. Indeed, it suffices to group steps by pairs as in the previous theorem, then each consecutive pair $\nearrow\rightarrow$ (respectively, $\nearrow\searrow$, $\rightarrow\rightarrow$, and $\rightarrow\searrow$) in the Motzkin path, becomes a pair of steps $\nearrow\nearrow$ (respectively, $\nearrow\searrow$, $\searrow\nearrow$, and $\searrow\searrow$) in the Dyck path. See Figure 3 for an example. This transformation of the path is such that the height after $2k$ steps in the second path is twice the height after $2k$ steps in the first path. More precisely:

- a step $\nearrow$ from height h to $h+1$ becomes a step $\nearrow$ from height $2h$ to $2h+1$ (and with weight $[h+1]_{p,q}$),
- an even step $\rightarrow$ at height h becomes a step $\nearrow$ from height $2h-1$ to $2h$ (and with weight $[h]_{p,q}$),
- an odd step $\rightarrow$ at height h becomes a step $\searrow$ from height $2h$ to $2h-1$ (and with weight $[h+1]_{p,q}$),
- a step $\searrow$ from height $h+1$ to h becomes a step $\searrow$ from height $2h$ to $2h-1$ (and with weight $[h+1]_{p,q}$).

From these properties it follows that in the obtained Dyck paths, the weight of each step only depends on its direction and its height. Hence they have a continued fraction as generating function, and it is as given in (6.2). $\square$

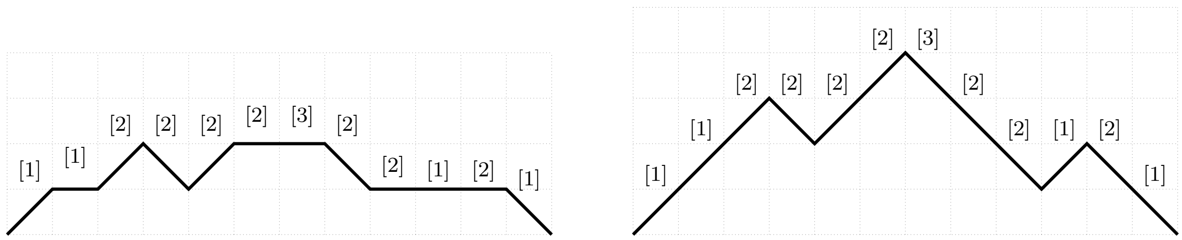

FIGURE 3. A bijection showing the continued fraction for $G_{2n}(p,q)$. We omit indices p,q in the weights $[i]_{p,q}$.

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