

CROSSINGS, MOTZKIN PATHS AND MOMENTS

MATTHIEU JOSUAT-VERGÈS AND MARTIN RUBEY

Dedicated to Jean-Guy Penaud

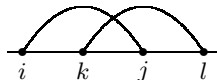
ABSTRACT. Kasraoui, Stanton and Zeng, and Kim, Stanton and Zeng introduced certain q -analogues of Laguerre and Charlier polynomials. The moments of these orthogonal polynomials have combinatorial models in terms of crossings in permutations and set partitions. The aim of this article is to prove simple formulas for the moments of the q -Laguerre and the q -Charlier polynomials, in the style of the Touchard-Riordan formula (which gives the moments of some q -Hermite polynomials, and also the distribution of crossings in matchings).

Our method mainly consist in the enumeration of weighted Motzkin paths, which are naturally associated with the moments. Some steps are bijective, in particular we describe a decomposition of paths which generalises a previous construction of Penaud for the case of the Touchard-Riordan formula. There are also some non-bijective steps using basic hypergeometric series, and continued fractions or, alternatively, functional equations.

1. INTRODUCTION

Our motivation is to derive in a uniform way generating functions for matchings, set partitions and permutations refined by the number of crossings. We achieve this by enumerating certain weighted Motzkin paths, which in turn prompt us to consider these counts as moments of certain families of orthogonal polynomials. In some cases, formulas for these moments are already known. However, the method of proof we present in this algorithm is quite general, and leads to very simple formulas.

Let us first define the notion of crossings in matchings, set partitions and permutations. To do so, it is best to draw the objects we are interested in in a certain standard way. We begin with the set of matchings (or fixed-point free involutions) \mathcal{M}_{2n} of $\{1, \dots, 2n\}$: these are drawn by putting the numbers from 1 to $2n$ in this order on a straight line, and then connecting paired numbers by an arc. Of course, arcs are always drawn in a way such that any two arcs cross at most once, and no more than two arcs intersect at any point, see the first picture in Figure 1 for an example. Then, a *crossing in a matching* is, as one would expect, a pair of matched points $\{i, j\}$ and $\{k, l\}$ with $i < k < j < l$, pictorially:



Indeed, the motivating example for this article is the Touchard-Riordan formula, which gives, for each n , the generating polynomial according to crossings for perfect matchings of the set $\{1, \dots, 2n\}$. Denoting by $\text{cro}(M)$ the number of crossings of the matching M , we have:

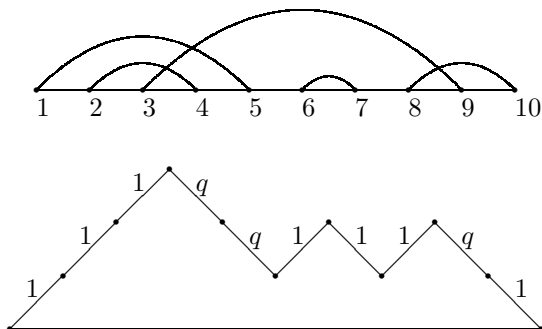


FIGURE 1. A matching of $\{1, \dots, 10\}$ with 3 crossings and the associated “histoire de Hermite”.

Theorem 1.1 (Touchard [25], Riordan [20]).

$$(1) \quad \sum_{M \in \mathcal{M}_{2n}} q^{\text{cro}(M)} = \frac{1}{(1-q)^n} \sum_{k \geq 0} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) q^{\binom{k+1}{2}}.$$

This has been proved in the 1950’s by Touchard, although, curiously, it seems that the formula was not given explicitly. This was later rectified by Riordan.

Quite similar to matchings, a set partition can be depicted by connecting the numbers on the line which are in one block $B = \{b_1 < b_2 < \dots < b_l\}$ by arcs $(b_1, b_2), (b_2, b_3), \dots, (b_{l-1}, b_l)$, see Figure 2 for an example. Again, a *crossing in a set partition* is what one would expect: a pair of arcs $\{i, j\}$ and $\{k, l\}$ with $i < k < j < l$. Denoting the set of set partitions of $\{1, \dots, n\}$ by Π_n , the number of crossings in a set partition π by $\text{cro}(\pi)$ and the number of its blocks by $|\pi|$, we will obtain the following q -analogue of the Stirling numbers of the second kind:

Theorem 1.2.

(2)

$$\sum_{\substack{\pi \in \Pi_n \\ |\pi|=k}} q^{\text{cro}(\pi)} = \frac{1}{(1-q)^{n-k}} \sum_{j=0}^k \sum_{i=j}^{n-k} (-1)^i \left(\binom{n}{k+i} \binom{n}{k-j} - \binom{n}{k+i+1} \binom{n}{k-j-1} \right) \begin{bmatrix} i \\ j \end{bmatrix}_q q^{\binom{j+1}{2}},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^k \frac{[n-k+i]_q}{[i]_q}$ is the q -binomial coefficient, and $[n]_q = 1+q+\dots+q^{n-1}$.

There is an alternative notion of crossings for set partitions, due to Ehrenborg and Readdy [5], coming from juggling patterns. Namely, we introduce an additional infinite arc extending to the right from every maximal element of each block, including singletons, see Figure 3 for an example. Denoting the number of crossings in such a drawing of a set partition π by $\text{cro}^*(\pi)$, we have:

Theorem 1.3 (Gould [9]).

$$(3) \quad \sum_{\substack{\pi \in \Pi_n \\ |\pi|=k}} q^{\text{cro}^*(\pi)} = \frac{1}{(1-q)^{n-k}} \sum_{j=0}^{n-k} (-1)^j \binom{n}{k+j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q.$$

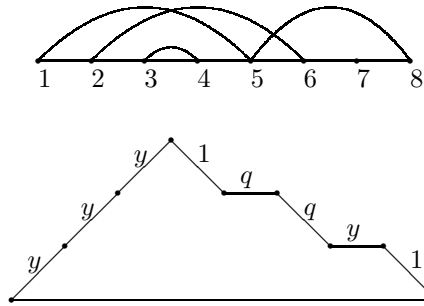


FIGURE 2. A set partition of $\{1, \dots, 8\}$ into 4 blocks with 2 crossings and the associated “histoire de Charlier”.

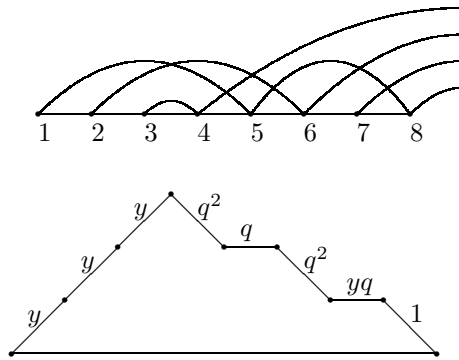
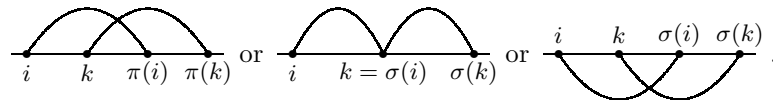


FIGURE 3. A set partition of $\{1, \dots, 8\}$ into 4 blocks with 6 crossings and the associated “histoire de Charlier-*”.

This is not a new result: essentially, this formula was already known to Gould from another definition (the link with crossings is more recent as will appear below).

Finally, to depict a permutation σ , we connect the number i with $\sigma(i)$ with an arc *above* the line, if $i \leq \sigma(i)$, otherwise with an arc *below* the line, as done in Figure 4. The notion of crossing in a permutation was introduced by Corteel [1], and is slightly less straightforward: a pair of numbers (i, k) constitutes a *crossing in a permutation*, if $i < k \leq \sigma(i) < \sigma(k)$ or $\sigma(i) < \sigma(k) < i < k$:



Denoting the set of permutations of $\{1, \dots, n\}$ by \mathfrak{S}_n , and the number of weak exceedances, *i.e.* numbers i with $\sigma(i) \geq i$, of a permutation σ by $\text{wex}(\sigma)$, we have:

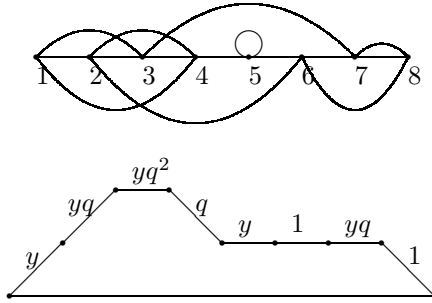


FIGURE 4. A permutation of $\{1, \dots, 8\}$ with 5 weak exceedances and 5 crossings and the associated “histoire de Laguerre”.

Theorem 1.4 (Josuat-Vergès [12], Corteel, Josuat-Vergès, Prellberg, Rubey [2]).

$$\begin{aligned}
 (4) \quad & \sum_{\sigma \in \mathfrak{S}_n} y^{\text{wex}(\sigma)} q^{\text{cro}(\sigma)} \\
 &= \frac{1}{(1-q)^n} \sum_{k=0}^n (-1)^k \left(\sum_{j=0}^{n-k} y^j \left(\binom{n}{j} \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right) \right) \left(\sum_{i=0}^k y^i q^{i(k+1-i)} \right).
 \end{aligned}$$

This theorem recently found a rather different proof by the first author [12]. In the present article we provide an alternative, using a bijective decomposition of weighted Motzkin paths that gives a natural interpretation for the two inner sums.

The rest of this article is organised as follows. In Section 2, we present some background material concerning the combinatorial theory of orthogonal polynomials. In Section 3, we describe the decomposition of weighted Motzkin paths mentioned above, in full generality. Each Motzkin path will be decomposed into a Motzkin prefix and another Motzkin path satisfying certain additional conditions. In Section 4, we enumerate Motzkin prefixes, and in Section 5 the other set of paths appearing in the decomposition are enumerated.

There are three appendices. In the first appendix we give an alternative point of view of the decomposition presented in Section 3, using inverse relations. In the second appendix, we give a bijective proof of the formula for the generating function of the paths appearing in the decomposition in the case of set-partitions, using a sign-reversing involution. It is thus possible to give a fully bijective proof of Theorem 1.2, analogous to Penaud’s proof of the Touchard-Riordan formula. Finally, in the last appendix we sketch a proof showing that one cannot expect closed forms for Motzkin prefixes with weights different from those considered in Section 4.

2. ORTHOGONAL POLYNOMIALS, MOMENTS AND HISTOIRES

Motzkin paths are at the heart of the *combinatorial theory of orthogonal polynomials*, as developed by Flajolet [6] and Viennot [26]. This theory tells us, that the moments of any family of orthogonal polynomials are given by a certain weighted count of Motzkin paths.

More precisely, by Favard's theorem, any monic sequence of orthogonal polynomials $(P_n)_{n \geq 0}$ satisfies a three term recurrence of the form

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + \lambda_n P_{n-1}(x),$$

where b_n and λ_n do not depend on x . Given this recurrence, the n^{th} moment μ_n^P of P can be expressed as the weighted sum of Motzkin paths of length n , that is, paths taking up (\nearrow), down (\searrow) and level (\rightarrow) steps, starting and ending at height 0, and not going below this height, where a horizontal step at height h has weight b_h and a down step starting at height h has weight λ_h .

2.1. Histoires. Three basic examples of families of orthogonal polynomials are given by (rescalings of) the Hermite, Charlier and Laguerre polynomials, where the moments count matchings ($b_n = 0, \lambda_n = n$), set partitions ($b_n = 1 + n, \lambda_n = n$) and permutations ($b_n = 2n + 1, \lambda_n = n^2$) respectively. It turns out that the Hermite, Charlier and Laguerre polynomials indeed have beautiful q -analogues such that the moments count the corresponding objects, and q marks the number of crossings. We want to establish this correspondence via "histoires":

Definition 2.1. Consider a family of orthogonal polynomials with coefficients b_n and λ_n , and fix a_n and c_n such that $\lambda_n = a_{n-1}c_n$ for all n . Suppose that for every fixed n , the coefficients a_n, b_n and c_n are polynomials such that each monomial has coefficient 1 (as will appear shortly, this is general enough in our context).

We then call a weighted Motzkin path *histoire*, when the weight of an up step \nearrow (respectively a level step \rightarrow or a down step \searrow) starting at level h is one of the monomials appearing in a_h (respectively b_h or c_h).

We want to consider four different families of "histoires", corresponding to q -analogues of the Hermite, Charlier and Laguerre polynomials.

Proposition 2.2. *There are weight-preserving bijections between*

- *matchings M with weight $q^{\text{cro}(M)}$, and "histoires de Hermite" defined by $b_n = 0, a_n = 1$ and $c_n = [n]_q$,*
- *set partitions π with weight $y^{|\pi|} q^{\text{cro}(\pi)}$, and "histoires de Charlier" defined by $b_n = y + [n]_q, a_n = y$ and $c_n = [n]_q$,*
- *set partitions π with weight $y^{|\pi|} q^{\text{cro}^*(\pi)}$, and "histoires de Charlier-*" defined by $b_n = yq^n + [n]_q, a_n = yq^n$ and $c_n = [n]_q$, and*
- *permutations σ with weight $y^{\text{wex}(\sigma)} q^{\text{cro}(\sigma)}$, and "histoires de Laguerre" defined by $b_n = yq^n + [n]_q, a_n = yq^n$ and $c_n = [n]_q$.*

These bijections are straightforward modifications of classical bijections used by Viennot [26]. We detail them here for convenience, but also because of their beauty... Examples can be found in Figures 1–4.

Proof. The bijection connecting matchings and "histoires de Hermite", such that crossings are recorded in the exponent of q , goes as follows: we traverse the matching, depicted in the standard way, from left to right, while we build up the Motzkin path step by step, also from left to right. For every arc connecting i and j with $i < j$, we call i an *opener* and j a *closer*. When we have traversed the matching up to and including number ℓ , we call the openers $i \leq \ell$ with corresponding closers $j < \ell$ *active*. Openers are translated into up steps with weight 1. Accordingly, when we encounter a closer ℓ it becomes a down step with weight q^k , where k is

the number of active openers between ℓ and the opener corresponding to ℓ . It is a enjoyable exercise to see that this is indeed a bijection, and that a matching with k crossings corresponds to a Dyck path of weight q^k .

The bijection between set partitions and “histoires de Charlier”, due to Anisse Kasraoui and Jiang Zeng [14], is very similar: in addition to openers and closers, which are the non-maximal and non-minimal elements of the blocks of the set partition, we now also have *singletons*, which are neither openers nor closers. Elements that are openers and closers at the same time are called *transients*. Non-transient openers are translated into up steps with weight y , and singletons are translated into level steps with weight y . Non-transient closers ℓ are translated into down steps both with weight q^k , where k is the number of active openers between ℓ and the opener corresponding to ℓ . Finally, transient closers ℓ become level-steps with weight q^k , with k as before.

To obtain a “histoire de Charlier- $*$ ” of a set partition, using the modified definition of crossings, we only have to multiply the weights of steps corresponding to closers and singletons by q^k , where k is the number of crossings of the infinite arc with other arcs.

It remains to describe the bijection between permutations and “histoires de Laguerre”, due to Dominique Foata and Doron Zeilberger, which is usually done in a different way than in what follows, however. To obtain the Motzkin path itself, we ignore all the arcs below the line and also the loops corresponding to fixed points. What remains can be interpreted as a set partition, and thus determines a Motzkin path. Moreover, the weights of the down steps are computed as in the case of set partitions, except that the weight of each of those steps needs to be multiplied by y . The weights of the level steps that correspond to transients of the set partition are also computed as before, but are then multiplied by yq . Level steps that correspond to fixed points of the permutation get weight y . The weights of the remaining steps are computed by deleting all arcs above the line, and again interpreting what remains as a set partition. However, this set partition has to be traversed from right to left, and weights are accordingly put onto the up steps of the Motzkin path. Later, it will be more convenient to move the factor y that appears in the weight of all the down steps onto the weight of the corresponding up steps, see Figure 4 for an example. \square

2.2. Particular classes of orthogonal polynomials. In this section we relate the families of orthogonal polynomials introduced via their parameters b_n and λ_n in Section 2.1 to classical families. We follow the Askey-Wilson scheme [16] for their definition.

The *continuous q -Hermite polynomials* $H_n = H_n(x|q)$ can be defined [16, Section 3.26] by the recurrence relation

$$2xH_n = H_{n+1} + (1 - q^n)H_{n-1},$$

with $H_0 = 1$.

Theorem 2.3 (Ismail, Stanton and Viennot [11]). *Define rescaled continuous q -Hermite polynomials $\tilde{H}_n = \tilde{H}_n(x|q)$ as*

$$(5) \quad \tilde{H}_n(x|q) = (1 - q)^{-n/2} H_n\left(x \frac{\sqrt{1-q}}{2} | q\right).$$

They satisfy the recurrence relation

$$(6) \quad x\tilde{H}_n = \tilde{H}_{n+1} + [n]_q \tilde{H}_{n-1},$$

and their even moments are given by

$$(7) \quad \mu_{2n}^{\tilde{H}} = \sum_{M \in \mathcal{M}_{2n}} q^{\text{cro}(M)}.$$

The odd moments are all zero.

The *Al-Salam-Chihara* polynomials $Q_n = Q_n(x; a, b|q)$ can be defined [16, Section 3.8] by the recurrence relation

$$2xQ_n = Q_{n+1} + (a+b)q^n Q_n + (1-q^n)(1-abq^{n-1})Q_{n-1},$$

with $Q_0 = 1$. We consider two different specialisations of these polynomials. The first was introduced by Kim, Stanton and Zeng [15], and in their Proposition 5 they also gave a formula for the moments. However, the formula that follows from our Theorem 1.2 appears to be much simpler.

Theorem 2.4 (Kim, Stanton, Zeng [15]). *Define q -Charlier polynomials $\tilde{C}_n = \tilde{C}_n(x; y|q)$ as*

$$(8) \quad \tilde{C}_n(x; y|q) = \left(\frac{y}{1-q}\right)^{n/2} Q_n\left(\sqrt{\frac{1-q}{4y}}\left(x - y - \frac{1}{1-q}\right); \frac{-1}{\sqrt{y(1-q)}}, 0 \mid q\right).$$

They satisfy the recurrence relation

$$(9) \quad x\tilde{C}_n = \tilde{C}_{n+1} + (y + [n]_q)\tilde{C}_n + y[n]_q\tilde{C}_{n-1}$$

and their moments are given by

$$(10) \quad \mu_n^{\tilde{C}} = \sum_{\pi \in \Pi_n} y^{|\pi|} q^{\text{cro}(\pi)}.$$

The other specialisation was introduced by Kasraoui, Stanton and Zeng [13], however, without providing a formula for the moments (these are actually a particular case of octabasic q -Laguerre polynomials from [23]).

Theorem 2.5 (Kasraoui, Stanton, Zeng [13]). *Define q -Laguerre polynomials $\tilde{L}_n = \tilde{L}_n(x; y|q)$ as*

$$(11) \quad \tilde{L}_n(x; y|q) = \left(\frac{\sqrt{y}}{q-1}\right)^n Q_n\left(\frac{(q-1)x+y+1}{2\sqrt{y}}; \frac{1}{\sqrt{y}}, \sqrt{y}q \mid q\right).$$

They satisfy the recurrence relation:

$$x\tilde{L}_n = \tilde{L}_{n+1} + ([n]_q + y[n+1]_q)\tilde{L}_n + y[n]_q^2\tilde{L}_{n-1}.$$

and their moments are given by

$$(12) \quad \mu_n^{\tilde{L}} = \sum_{\sigma \in \mathfrak{S}_n} y^{\text{wex}(\sigma)} q^{\text{cro}(\sigma)}.$$

The *Al-Salam-Carlitz I* polynomials $U_n^{(a)}(x|q)$ can be defined [16, Section 3.24] by the recurrence relation

$$(13) \quad xU_n^{(a)}(x|q) = U_{n+1}^{(a)}(x|q) + (a+1)q^n U_n^{(a)}(x|q) - q^{n-1}a(1-q^n)U_{n-1}^{(a)}(x|q),$$

with $U_0^{(a)}(x|q) = 1$.

Theorem 2.6 (de Médicis, Stanton, White [4]). *Define modified q -Charlier polynomials $\tilde{C}_n^* = \tilde{C}_n^*(x; y|q)$ as:*

$$(14) \quad C_n^*(x; y|q) = y^n U_n\left(\frac{-x}{y(1-q)}\right) \left(\frac{x}{y} - \frac{1}{y(1-q)}|q\right).$$

They satisfy the recurrence relation

$$(15) \quad x\tilde{C}_n^* = \tilde{C}_{n+1}^* + (yq^n + [n]_q)\tilde{C}_n^* + y[n]_q q^{n-1}\tilde{C}_{n-1}^*,$$

and their moments are given by

$$(16) \quad \mu_n^{\tilde{C}^*} = \sum_{\pi \in \Pi_n} y^{|\pi|} q^{\text{cro}^*(\pi)}.$$

The result from [4] was actually stated with another statistic, but both correspond to Carlitz' q -analogue of the Stirling numbers of the second kind $S[n, k]$, which are such that $S[n, k] = S[n-1, k-1] + [k]_q S[n-1, k]$, and

$$\mu_n^{\tilde{C}^*} = \sum_{k=1}^n S[n, k] y^k.$$

3. PENAUD'S DECOMPOSITION

Let us first briefly recall Penaud's strategy to prove the Touchard-Riordan formula for the moments of the rescaled continuous q -Hermite polynomials \tilde{H}_n . As already indicated in the introduction, his starting point was their combinatorial interpretation in terms of weighted Dyck paths, down steps starting at level $h \geq 1$ having weight $[h]_q$, up steps having weight 1.

As the total number of down steps in these paths is n , we may take out a factor $(1-q)^{-n}$, and instead consider paths with down steps having weight $1-q^h$, or, equivalently, consider paths with down steps having weight 1 or $-q^h$.

The next step is to (bijectively) decompose each path into two objects: the first is a left factor of an unweighted Dyck path of length n and final height $n-2k \geq 0$, for some k . The second object, in some sense the remainder, is a weighted Dyck path of length k with the same possibilities for the weights as in the original path, except that peaks (consisting of an up step immediately followed by a down step) of weight 1 are not allowed. This decomposition will be generalised in Lemma 3.2 below.

The left factors are straightforward to count, the result being the ballot numbers $\binom{2n}{n-k} - \binom{2n}{n-k-1}$. For the remainders, Penaud presented a bijective proof that the sum of their weights is given by $(-1)^k q^{\binom{k+1}{2}}$. Summing over all k we obtain the Touchard-Riordan formula (1).

3.1. The general setting.

Definition 3.1. Let $\mathcal{M}_n(a, b, c, d; q)$ be the set of weighted Motzkin paths of length n , such that the weight of

- an up step \nearrow starting at level h is either 1 or $-q^{h+1}$,
- of a level step \rightarrow starting at level h is either d or $(a+b)q^h$,
- a down step \searrow starting at level h is either c or $-abq^{h-1}$.

Furthermore, let $\mathcal{M}_n^*(a, b, c; q) \subset \mathcal{M}_n(a, b, c, d; q)$ be the subset of paths that do not contain any

- level step \rightarrow of weight d ,

- peak $\nearrow\searrow$ such that the up step has weight 1 and the down step has weight c .

Finally, let $\mathcal{P}_{n,k}(c, d)$ be the set of left factors of Motzkin paths of length n and final height k , such that the weight of

- an up step \nearrow is 1,
- a level step \rightarrow is d ,
- a down step \searrow is c .

With these definitions, the decomposition used by Penaud can be generalised in a natural way as follows:

Lemma 3.2. *There is a bijection Δ between $\mathcal{M}_n(a, b, c, d; q)$ and the disjoint union of the sets $\mathcal{P}_{n,k}(c, d) \times \mathcal{M}_k^*(a, b, c; q)$ for $k \in \{0, \dots, n\}$.*

Proof. Let H be a path in $\mathcal{M}_n(a, b, c, d; q)$. Consider the maximal factors f_1, \dots, f_j of H that are Motzkin paths and have up steps of weight 1, level steps of weight d and down steps of weight c . We can thus factorise H as $h_0 f_1 h_1 f_2 \dots f_j h_j$.

Since this factorisation is uniquely determined, we can define $\Delta(H) = (H_1, H_2)$ as follows:

$$H_1 = (\nearrow)^{|h_0|} f_1 (\nearrow)^{|h_1|} f_2 \dots f_j (\nearrow)^{|h_j|} \quad \text{and} \quad H_2 = h_0 \dots h_j.$$

Thus, H_1 is obtained from H by replacing each step in the h_i by an up step \nearrow , and H_2 is obtained from H by deleting the factors f_i . Since the f_i are Motzkin paths, the weight of H is just the product of the weights of H_1 and H_2 . Furthermore, it is clear that H_1 is a path in $\mathcal{P}_{n,k}(c, d)$ with final height $k = |h_0| + |h_1| + \dots + |h_j|$. We observe that the h_i cannot contain a level step \rightarrow of weight d or a peak $\nearrow\searrow$ such that the up step has weight 1 and the down step has weight c , because then the factorisation of H would not have been complete. Thus H_2 is a path in $\mathcal{M}_k^*(a, b, c; q)$.

It remains to verify that Δ is indeed a bijection. To do so, we describe the inverse map: let $(H_1, H_2) \in \mathcal{P}_{n,k}(c, d) \times \mathcal{M}_k^*(a, b, c; q)$ for some $k \in \{0, \dots, n\}$. Thus, there exists a unique factorisation

$$H_1 = (\nearrow)^{u_0} f_1 (\nearrow)^{u_1} f_2 \dots f_j (\nearrow)^{u_j}$$

such that the f_i are Motzkin paths and $k = \sum_{\ell=0}^j u_\ell$. Write H_2 as $h_0 \dots h_j$, where the factor h_ℓ has length u_ℓ . Then $\Delta^{-1}(H_1, H_2) = h_0 f_1 h_1 f_2 \dots f_j h_j$ is the preimage of (H_1, H_2) . \square

3.2. Specialising to matchings, set partitions and permutations. As remarked in the introduction of this section, we begin by multiplying the weighted sum of all Motzkin paths by an appropriate power of $1 - q$. In the case of *matchings* of $\{1, \dots, 2n\}$, we are in fact considering Dyck paths of length $2n$ where a down step starting at height h has weight $[h]_q$. Multiplying the weighted sum with $(1 - q)^n$, or, equivalently, multiplying the weight of each down step by $1 - q$, we thus obtain Dyck paths having down steps starting at height h weighted by $1 - q^h$, which fits well into the model introduced in Definition 3.1: namely, the set $\mathcal{M}_n(a, b, c, d; q)$ with $a = 0$, $b = 0$, $c = 1$ and $d = 0$ consists precisely of these paths – except that they are all reversed.

In the case of *set partitions* of $\{1, \dots, n\}$, multiplying the weighted sum by $(1 - q)^n$ and reversing all paths we see that we need to enumerate the set $\mathcal{M}_n(a, b, c, d; q)$ with $a = 0$, $b = -1$, $c = y(1 - q)$ and $d = 1 + y(1 - q)$. When using the *modified*

definition of crossings in *set partitions*, we obtain surprisingly different parameters, namely $a = -1$, $b = y(1 - q)$, $c = 0$ and $d = 1$. Finally, the case of *permutations* of $\{1, \dots, n\}$ is covered by enumerating the set $\mathcal{M}_n(a, b, 1, d; q)$ with $a = -1$, $b = -yq$, $c = y$ and $d = 1 + y$.

4. COUNTING $\mathcal{P}_{n,k}(c, d)$

In general, formulas for the cardinality of $\mathcal{P}_{n,k}(c, d)$ can be found easily using Lagrange inversion [24]. Consider the generating function $P_k = \sum_n |\mathcal{P}_{n,k}(c, d)| t^n$, we want to determine the coefficient of t^{n+1} in $tP_k = (tP_0)^{k+1}$. Observing the relationship

$$tP_0 = t(1 + d(tP_0) + c(tP_0)^2)$$

we find that $[t^n](tP_0)^k = \frac{k}{n} [z^{n-k}](1 + dz + cz^2)^n$, and thus

$$(17) \quad |\mathcal{P}_{n,k}(c, d)| = \frac{k+1}{n+1} \sum_{l=0}^{n-k} \binom{n+1}{l} \binom{l}{2l-n+k} d^{2l-n+k} c^{n-k-l}.$$

To count matchings, set partitions or permutations according to crossings (modified or not), the only sets of parameters that we need to consider are $(c, d) = (1, 0)$, $(c, d) = (1, 2)$ and $(c, d) = (0, 1)$. Curiously, these are precisely the values for which Equation (17) allows a closed form, *i.e.* can be written as a linear combination of hypergeometric terms. A (sketch of a) justification of this fact is given in Appendix C.

4.1. Matchings. For matchings, we have $(c, d) = (1, 0)$ and we obtain the ballot numbers:

Lemma 4.1. *The cardinality of $\mathcal{P}_{n,n-2k}(1, 0)$, *i.e.* the number of left factors of Dyck paths of length n and final height $n - 2k \geq 0$ is*

$$\binom{n}{k} - \binom{n}{k-1}.$$

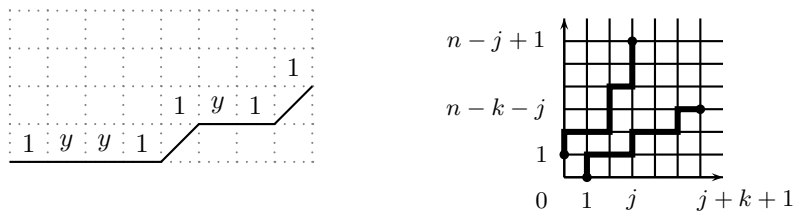
4.2. Set partitions and permutations. For set partitions and permutations, we have $(c, d) = (y, 1 + y)$ and obtain the following:

Lemma 4.2. *The generating function for $\mathcal{P}_{n,k}(y, 1 + y)$ is:*

$$(18) \quad \sum_{j=0}^{n-k} \left(\binom{n}{j} \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right) y^j.$$

Proof. The elements of $\mathcal{P}_{n,k}(y, 1 + y)$ have weight $1 + y$ on each level step. However, it is again more convenient to pretend that there are two different kinds of level steps, with weight 1 and y respectively. Let P be a left factor of a Motzkin path with weight y^j . We then use the following step by step translation to transform it into a pair (C_1, C_2) of non-intersecting paths taking north and east steps, starting at $(0, 1)$ and $(1, 0)$ respectively (see Figure 5 for an example):

i^{th} step of P	i^{th} step of C_1	i^{th} step of C_2
\nearrow	\uparrow	\rightarrow
\rightarrow , weight 1	\uparrow	\uparrow
\rightarrow , weight y	\rightarrow	\rightarrow
\searrow , weight y	\rightarrow	\uparrow

FIGURE 5. A bijection to count $\mathcal{P}_{n,k}(y, 1+y)$.

The condition that a Motzkin path does not go below the x -axis translates into the fact that C_1 and C_2 are non-intersecting. Since P has j steps weighted by y , the path C_1 ends at $(j, n-j+1)$. Since P ends at height k , the number of up steps \nearrow and the number of level steps \rightarrow with weight y add up to $j+k$, so C_2 ends at $(j+k+1, n-j-k)$.

By the Lindström-Gessel-Viennot Lemma [8], these pairs of non-intersecting paths can be counted by a 2×2 -determinant, which gives precisely Formula (18). \square

For $y = 1$ the sum in Equation (18) can be simplified using Vandermonde's identity. Thus, the number of left factors of Motzkin paths of length n and final height k , with weight 2 on every level step is

$$(19) \quad \binom{2n}{n-k} - \binom{2n}{n-k-2}.$$

For $(c, d) = (0, 1)$, that is, $y = 0$, we obtain what we need to count modified crossings in set partitions, namely the binomial coefficient $\binom{n}{k}$.

5. COUNTING $\mathcal{M}_k^*(a, b, c; q)$

In this section we use a continued fraction to find the generating function for the Motzkin paths in $\mathcal{M}_k^*(a, b, c; q)$ (these paths are described in Definition 3.1). It turns out that this continued fraction can be expressed as a basic hypergeometric series, which allows us to compute the coefficients corresponding to paths with given length. Let $K(a, b, c; q)$ be

$$(20) \quad \frac{1}{1 + c - (a + b) - \frac{(c - ab)(1 - q)}{1 + c - (a + b)q - \frac{(c - abq)(1 - q^2)}{1 + c - (a + b)q^2 - \frac{(c - abq^2)(1 - q^3)}{\ddots}}}}.$$

Let us first give a combinatorial interpretation of $K(at, bt, ct^2; q)$ in terms of weighted Motzkin paths. This result is close to those given by Roblet and Viennot [22], who developed a combinatorial theory of T -fractions. These are continued fractions of the form $1/(1 - a_0t - b_0t/(1 - a_1t - b_1t/\dots))$, and they are generating functions of Dyck paths with some weights on the peaks.

Proposition 5.1. *The coefficient of t^k in the expansion of $K(at, bt, ct^2; q)$ is the generating function of $\mathcal{M}_k^*(a, b, c; q)$.*

Proof. The continued fraction $K(at, bt, ct^2; q)$ equals:

$$(21) \quad \frac{1}{1 + ct^2 - (a+b)t - \frac{t^2(c-ab)(1-q)}{1 + ct^2 - (a+b)qt - \frac{t^2(c-abq)(1-q^2)}{1 + ct^2 - (a+b)q^2t - \frac{t^2(c-abq^2)(1-q^3)}{\ddots}}}}.$$

Using the ideas introduced by Flajolet [6, Theorem 1], we thus obtain paths with four types of steps, denoted up \nearrow , down \searrow , level \rightarrow and double-level \longrightarrow , the last type of step simply being twice as long as the usual level step. Moreover, the weight of

- an up step \nearrow starting at height h is either 1 or $-q^{h+1}$,
- a level step \rightarrow starting at height h is $(a+b)q^h$,
- a down step \searrow starting at height h is either c or $-abq^{h-1}$,
- a double-level step \longrightarrow is $-c$.

To prove the statement, it suffices to construct an involution on the paths, such that

- its fixed points are precisely the elements of $\mathcal{M}_k^*(a, b, c; q)$, *i.e.* paths without double-level steps \longrightarrow and without peaks $\nearrow\searrow$ such that the up step has weight 1 and the down step has weight c ,
- the weight of a path that is not fixed under the involution and the weight of its image add to zero.

Such an involution is easy to find: a path that is not in $\mathcal{M}_k^*(a, b, c; q)$, we look for the first occurrence of one of the two forbidden patterns, *i.e.* a double level step \longrightarrow or a peak $\nearrow\searrow$ with steps weighted 1 and c respectively. We then exchange one of the patterns for the other – since the double level step \longrightarrow has weight $-c$, the weights of the two paths add up to zero. \square

As mentioned above, $K(at, bt, ct^2; q)$ can be expressed as a basic hypergeometric series. We use the usual notation for these series, as for example in [7].

Proposition 5.2. *For $A \neq 1$, $B \neq 0$, we have*

$$(22) \quad K(A, B, C; q) = \frac{1}{1-A} \cdot {}_2\phi_1 \left(\begin{matrix} CB^{-1}q, q \\ Aq \end{matrix} \middle| q, B \right).$$

For $A \neq 1$, $B = 0$, we have

$$(23) \quad K(A, 0, C; q) = \frac{1}{1-A} \cdot {}_1\phi_1 \left(\begin{matrix} q \\ Aq \end{matrix} \middle| q, Cq \right).$$

Proof. Consider the following more general continued fraction, containing a new variable z :

$$M(z) = \frac{1}{1 + C - (A+B)z - \frac{(C-ABz)(1-qz)}{1 + C - (A+B)qz - \frac{(C-ABqz)(1-q^2z)}{1 + C - (A+B)q^2z - \frac{(C-ABq^2z)(1-q^3z)}{\ddots}}}}.$$

Following Ismail and Libis [10] (see also Identity 19.2.11a in the Handbook of Continued Fractions for Special Functions [3]), we have:

$$M(z) = \frac{1}{1-z} \cdot {}_2\phi_1 \left(\begin{matrix} A, B \\ Cq \end{matrix} \middle| q, qz \right) \cdot {}_2\phi_1 \left(\begin{matrix} A, B \\ Cq \end{matrix} \middle| q, z \right)^{-1}.$$

To be able to specialise $z = 1$, we can use one of Heine's transformations [7, p.13]. For $B \neq 0$ we obtain

$$\begin{aligned} M(z) &= \frac{1}{1-z} \cdot \frac{(Aqz, B, Cq, z; q)_\infty}{(Az, B, Cq, qz; q)_\infty} \cdot {}_2\phi_1 \left(\begin{matrix} CB^{-1}q, qz \\ Aqz \end{matrix} \middle| q, B \right) \cdot {}_2\phi_1 \left(\begin{matrix} CB^{-1}q, z \\ Az \end{matrix} \middle| q, B \right)^{-1} \\ &= \frac{1}{1-Az} \cdot {}_2\phi_1 \left(\begin{matrix} CB^{-1}q, qz \\ Aqz \end{matrix} \middle| q, B \right) \cdot {}_2\phi_1 \left(\begin{matrix} CB^{-1}q, z \\ Az \end{matrix} \middle| q, B \right)^{-1}. \end{aligned}$$

In case $B = 0$, we have

$$\begin{aligned} M(z) &= \frac{1}{1-z} \cdot \frac{(Aqz, q, Cq, z; q)_\infty}{(Az, q, Cq, qz; q)_\infty} \cdot {}_1\phi_1 \left(\begin{matrix} qz \\ Aqz \end{matrix} \middle| q, Cq \right) \cdot {}_1\phi_1 \left(\begin{matrix} z \\ Az \end{matrix} \middle| q, Cq \right)^{-1} \\ &= \frac{1}{1-az} \cdot {}_1\phi_1 \left(\begin{matrix} qz \\ Aqz \end{matrix} \middle| q, Cq \right) \cdot {}_1\phi_1 \left(\begin{matrix} 1 \\ A \end{matrix} \middle| q, Cq \right)^{-1}. \end{aligned}$$

□

Remark. Although the symmetry in A and B is not apparent in Equation (22), it can be seen using one of Heine's transformations [7, p.13].

In the following, we will always use

$$(24) \quad K(at, bt, ct^2; q) = \frac{1}{1-at} \cdot {}_2\phi_1 \left(\begin{matrix} cb^{-1}qt, q \\ aqt \end{matrix} \middle| q, bt \right).$$

Besides, it is also possible to use a method giving $M(z)$ as a quotient of basic hypergeometric series without knowing *a priori* which identity to use. This method was employed in [2], following Brak and Prellberg [19]. Namely, note that the continued fraction expansion of $M(z)$ is equivalent to the equation:

$$(25) \quad M(z) = \frac{1}{1-c + (a+b)z - (c-abz)(1-qz)M(qz)}.$$

By looking for solutions of the form $M(z) = (1-az)^{-1} \frac{H(qz)}{H(z)}$, we obtain a linear equation in $H(z)$, which gives a recurrence for the coefficients of the Taylor expansion of $H(z)$, which is readily transformed into the explicit form of $H(z)$ as a basic hypergeometric series.

5.1. Matchings. For matchings, we have $(a, b, c) = (0, 0, 1)$ and obtain:

Lemma 5.3.

$$K(0, 0, t^2; q) = \sum_{k=0}^{\infty} (-t^2)^k q^{\binom{k+1}{2}}.$$

Essentially, this was shown by Penaud [17], who enumerated $\mathcal{M}_{2k}^*(0, 0, 1; q)$ by first constructing a bijection with parallelogram polyominoes, passing through several intermediate objects with beautiful names like 'cherry trees'. On the polyominoes he was finally able to construct a weight-preserving, sign-reversing involution,

with the only fixed point having weight $(-1)^k q^{\binom{k+1}{2}}$, corresponding to weighted Dyck paths with a single peak, and all weights maximal.

5.2. Set partitions. For set partitions, we have $(a, b, c) = (0, -1, y(1 - q))$ and can use the following lemma:

Lemma 5.4.

$$(26) \quad K(0, -t, ct^2; q) = \sum_{i=0}^{\infty} \sum_{j=0}^i t^{i+j} c^j (-1)^i q^{\binom{j+1}{2}} \left[\begin{matrix} i \\ j \end{matrix} \right]_q.$$

Proof. Using Equation (24) we find:

$$K(0, -t, ct^2; q) = {}_2\phi_1 \left(\begin{matrix} -cqt, q \\ 0 \end{matrix} \middle| q, -t \right) = \sum_{i=0}^{\infty} (-cqt; q)_i (-t)^i.$$

The proof follows by plugging in the elementary expansion

$$(-cqt; q)_i = \prod_{j=1}^i (1 + q^j ct) = \sum_{j=0}^i q^{\binom{j+1}{2}} \left[\begin{matrix} i \\ j \end{matrix} \right]_q c^j t^j.$$

□

In the appendix, we give a bijective proof of this lemma.

5.3. Set partitions, modified crossings. When using the modified definition of crossings in set partitions, we have $(a, b, c) = (-1, y(1 - q), 0)$ and can use the following lemma:

Lemma 5.5.

$$(27) \quad K(-t, bt, 0; q) = \sum_{i=0}^{\infty} \sum_{j=0}^i t^i b^j (-1)^{i-j} \left[\begin{matrix} i \\ j \end{matrix} \right]_q.$$

Proof. Using Equation (24) we find:

$$K(-t, bt, 0; q) = \frac{1}{1+t} {}_2\phi_1 \left(\begin{matrix} 0, q \\ -qt \end{matrix} \middle| q, bt \right) = \sum_{i=0}^{\infty} \frac{1}{(-t; q)_{i+1}} (bt)^i.$$

The proof follows by plugging in the elementary expansion

$$\frac{1}{(-t; q)_{i+1}} = \prod_{j=1}^i \frac{1}{1 + q^j t} = \sum_{j \geq 0} \left[\begin{matrix} i+j \\ j \end{matrix} \right]_q (-t)^j.$$

□

5.4. Permutations. In the case of permutations, we have $(a, b, c) = (-1, -yq, y)$ and find:

Lemma 5.6.

$$K(-t, -yqt, yt^2; q) = \sum_{k=0}^{\infty} (-t)^k \left(\sum_{i=0}^k y^i q^{i(k+1-i)} \right).$$

Proof. Using Equation (24), we have:

$$\begin{aligned} K(-t, -yqt, yt^2; q) &= \frac{1}{1+t} \cdot {}_2\phi_1 \left(\begin{matrix} -t, q \\ -qt \end{matrix} \middle| q, -yqt \right) = \sum_{i=0}^{\infty} \frac{(-yqt)^i}{1+tq^i} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-yqt)^i (-tq^i)^j = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-t)^{i+j} y^i q^{i(j+1)}. \end{aligned}$$

To finish the proof it only remains to substitute k for $i+j$. \square

6. CONCLUSION

Let us briefly summarize how the four theorems announced in the introduction can be proved using the previous sections. In each case, the enumeration of crossings in combinatorial objects is linked with the enumeration of the weighted Motzkin paths in $\mathcal{M}_n(a, b, c, d; q)$. The bijection Δ shows that the generating function of crossings can be decomposed into the generating functions of the sets $\mathcal{P}_{n,k}(c, d)$ and $\mathcal{M}_n^*(a, b, c; q)$, which in turn have been obtained in the previous two sections. This fulfills our initial objective as stated in the introduction.

APPENDIX A. INVERSE RELATIONS

We would like to mention an interesting non-bijective point of view of the path decomposition given in Section 3, using inverse relations. Given two sequences $\{a_n\}$ and $\{b_n\}$, an inverse relation is an equivalence such as, for example:

$$\forall n \geq 0, \quad a_n = \sum_{k=0}^n \binom{n}{k} b_k \quad \iff \quad \forall n \geq 0, \quad b_n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} a_k.$$

This particular relation is easily proved by checking that the (semi-infinite) lower triangular matrix $((\binom{i}{j})_{i,j \in \mathbb{N}})$ has an inverse, which is $((-1)^{i+j} \binom{i}{j})_{i,j \in \mathbb{N}}$. Other relations of this kind can be found in Chapters 2 and 3 of Riordan's book 'Combinatorial Identities' [21]. To prove the Touchard-Riordan formula (1), let $a_{2n} = (1-q)^n \mu_{2n}^{\tilde{H}}$ and $a_{2n+1} = 0$. We then use the following inverse relation [21, Chapter 2, Equation (12)]:

$$(28) \quad a_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k} - \binom{n}{k-1} \right) b_{n-2k} \quad \iff \quad b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} a_{n-2k}.$$

Again, this can be proved by inverting a lower-triangular matrix. It remains to prove that $b_{2k} = (-1)^k q^{\binom{k+1}{2}}$ and $b_{2k+1} = 0$. To this end, we relate the sequence b_n to Schröder paths:

Definition A.1. A *Schröder path* of length $2n$ is a path in $\mathbb{N} \times \mathbb{N}$ starting at $(0, 0)$, arriving at $(2n, 0)$ with steps $(1, 1)$, $(1, -1)$, or $(2, 0)$.

Lemma A.2. *Suppose that $a_{2n+1} = 0$ and a_{2n} is the generating function of Dyck paths of length $2n$, with weight $1 - q^{h+1}$ on each north-east step starting at height h (this is to say $a_{2n} = (1-q)^n \mu_{2n}^{\tilde{H}}$). Suppose that a_n and b_n are related by (28). Then $b_{2n+1} = 0$, and b_{2n} is the generating function of Schröder paths of length $2n$, with weight -1 on each level step, and $1 - q^{h+1}$ on each north-east step starting at height h .*

Proof. For any even n , consider a Schröder path of length n with k level steps, weighted as described above. This path has $n - k$ non-level steps, and can thus be obtained from a Dyck path of length $n - 2k$ by inserting the level steps. There are $\binom{n-k}{k}$ ways to do so, which implies that the generating function of Schröder paths is indeed equal to $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} a_{n-2k}$, and therefore equal to b_n . \square

With the bijective decomposition of paths in Section 3, we showed that we have to obtain the generating function of the set $\mathcal{M}_k^*(0, 0, 1; q)$ to prove Touchard-Riordan formula. With the inverse relations, we showed that we have to count certain weighted Schröder paths. The fact that both sets have the same generating function follows from the involution given in the proof of Proposition 5.1.

It is also possible to use an inverse relation to obtain the formula in Theorem 1.4 for the q -Laguerre moments. By inverting a lower triangular matrix, one can check that

$$(29) \quad a_n = \sum_{k=0}^n \left(\sum_{j=0}^k y^j \left(\binom{n}{j} \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right) \right) b_k$$

for all n is equivalent to

$$(30) \quad b_n = \sum_{k=0}^n \left(\sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n-j}{n-k-j} \binom{n-k-j}{j} (-y)^j (-1-y)^{n-k-2j} \right) a_k$$

for all n .

Equation (30) can be interpreted as follows: given that a_k counts elements of $\mathcal{M}_n(1, -yq, y, 1 + y; q)$, then b_n count paths of length n , with the same weights as in $\mathcal{M}_n(1, -yq, y, 1 + y; q)$, where we insert some level steps \rightarrow with weight $-1 - y$, and some double level steps \longrightarrow with weight $-y$.

Indeed, suppose that we inserted j double-level steps \longrightarrow , and hence $n - k - 2j$ level steps \rightarrow , starting with a path of length k . This yields the weight $(-y)^j (-1 - y)^{n-k-2j}$ for the inserted steps. The first binomial coefficient, $\binom{n-j}{n-k-j}$, is the number of ways to insert the $n - k - j$ level steps among the $n - j$ steps (the total number of steps being $n - j$ because the length is n , and j steps of double length).

The second binomial coefficient, $\binom{n-k-j}{j}$, is the number of combinations of the j inserted double-level steps and the $n - k - 2j$ inserted level steps. Using the involution given in the proof of Proposition 5.1, we see that b_n counts elements in the set $\mathcal{M}_n^*(1, -yq, y; q)$.

APPENDIX B. A BIJECTIVE PROOF OF LEMMA 5.4

We show in this appendix that Penaud's bijective method of proving Lemma 5.3 can be generalised to prove also Lemma 5.4. Namely, we construct a sign-reversing involution on a set of weighted Motzkin paths, whose fixed points are enumerated by the right-hand side of (26). This involution was essentially given by the first author in [12] in a different context. We take the opportunity to correct some mistakes in this reference.

In the following we fix integers $j, k \geq 0$ and consider the set $\mathcal{C}_{j,k}$ of Motzkin paths of length $k + j$ with $k - j$ level steps \rightarrow , (and hence j up steps \nearrow and j down steps \searrow), satisfying the following conditions:

- the weight of all up steps \nearrow is 1,

- the weight of a level step \rightarrow at height h is $-q^h$,
- the weight of a down step \searrow starting at height h is either 1 or $-q^h$,
- there is no peak $\nearrow\searrow$ such that both the up step and the down step have weight 1.

The generating function of $\mathcal{C}_{j,k}$ is then the coefficient of $a^k t^{k+j}$ in $K(0, -t, at^2; q)$. It thus suffices to prove the following:

Proposition B.1. *There is an involution θ on the set $\mathcal{C}_{j,k}$ such that:*

- *the fixed points are the paths that*
 - *start with j up steps \nearrow ,*
 - *and contain no down steps \searrow of weight 1,*
- *the weight of a path that is not fixed under the involution and the weight of its image add to zero.*

Moreover, the sum of weights of fixed points of the involution is $(-1)^k q^{\binom{j+1}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q$.

Penaud's method consists in introducing several intermediate objects as described in Section 5.1. However, in the case at hand we will not use intermediate objects, but rather construct the involution directly on the paths. What we give is a generalisation of Penaud's construction, which we recover in the case $k = j$.

Proof. Following Penaud [17], we use in this proof a word notation for elements in $\mathcal{C}_{j,k}$. The letters x , z , y , and \bar{y} will respectively denote the steps \nearrow , \rightarrow , \searrow with weight 1, and \searrow with weight $-q^h$. For any word $c \in \mathcal{C}_{j,k}$, we define:

- $u(c)$ as the length of the last sequence of consecutive x ,
- $v(c)$ as the starting height of the last step y , if c contains a y and there is no x after the last y , and j otherwise.

See Figure 6 for an example. The fixed points of θ will be $c \in \mathcal{C}_{j,k}$ such that $u(c) = v(c) = j$, which correspond to the paths described in Proposition B.1.

now, suppose c is such that $u(c) < j$ or $v(c) < j$. We will build θ so that $v(c) \leq u(c)$ if and only if $u(\theta(c)) < v(\theta(c))$. Thus it suffices to define $\theta(c)$ in the case $v(c) \leq u(c)$, and to check that we have indeed $u(\theta(c)) < v(\theta(c))$. So let us suppose $v(c) \leq u(c)$, hence $v(c) < j$.

Since $v(c) < j$, there is at least a letter y in c having no x to its right. Let \tilde{c} be the word obtained from c by replacing the last y with a \bar{y} . There is a unique factorisation

$$\tilde{c} = f_1 x^{u(c)} a y^\ell f_2$$

such that:

- a is either z , or \bar{y} ,
- f_2 begins with z or \bar{y} , contains at least one letter \bar{y} , but contains no x .

Let us explain this factorisation. By definition of $u(c)$, we can write $\tilde{c} = f_1 x^{u(c)} c'$, where c' does not contain any x . In a word $c \in \mathcal{C}_{j,k}$, an x cannot be followed by a y . So we can write $\tilde{c} = f_1 x^{u(c)} a c''$ where a is either z , or \bar{y} . Then, we write $c'' = y^\ell f_2$ with $\ell \geq 0$ maximal, and f_2 satisfy the conditions (f_2 contains indeed a \bar{y} because we transformed a y into a \bar{y}). Uniqueness is immediate.

We set:

$$(31) \quad \theta(c) = f_1 x^{u(c)-v(c)} a y^\ell x^{v(c)} f_2.$$

See Figure 6 for an example with $u(c) = 4$, $v(c) = 2$, $j = 9$ and $k = 12$. We can check that $w(c) = -q^{19} = -w(\theta(c))$, and $u(\theta(c)) = 2$, $v(\theta(c)) = 3$.

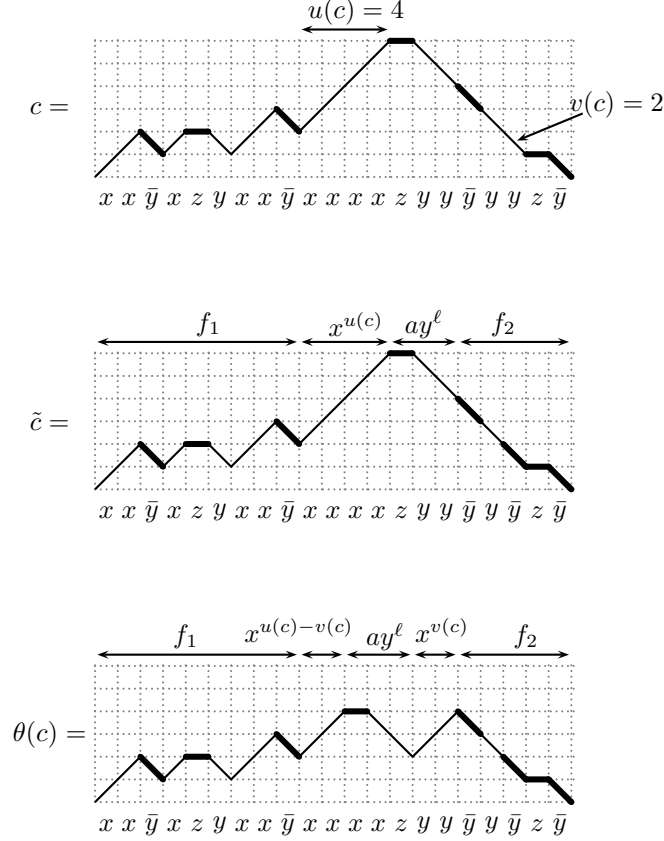


FIGURE 6. An element c in $\mathcal{C}_{j,k}$, and its image $\theta(c)$. Thick steps are the steps \bar{y} and z , *i.e.* steps with weight $-q^h$.

We show the following points:

- The path $\theta(c)$ is a Motzkin path. Indeed, the factor ay^j in c ends at height at least $v(c)$, since the factor f_2 contains a step \bar{y} starting at this height and contains no x . We can thus shift this factor \tilde{c} so that the result is again a Motzkin path.
- The path c and its image $\theta(c)$ have opposite weights. To begin, between c and \tilde{c} , the weight is multiplied by $-q^{v(c)}$, since we have transformed a y into a \bar{y} starting at height $v(c)$. Between \tilde{c} and $\theta(c)$, the height of the factor ay^j has decreased by $v(c)$, so the weight has been divided by $q^{v(c)}$. A factor -1 remains, which proves the claim.
- The path $\theta(c)$ is such that $u(\theta(c)) < v(\theta(c))$. From the definition (31) we see that $u(\theta(c)) = v(c)$. Besides, $v(c) < v(\theta(c))$ since the last step y of c has been transformed into a \bar{y} to obtain \tilde{c} and $\theta(c)$.
- Every path c' with $u(c') < v(c')$ is obtained as a $\theta(c)$ for some other path c with $u(c) \geq v(c)$. Indeed, let \tilde{c}' be the word obtained from c' after

replacing the last \bar{y} at height $u(c')$ with a y . There is a unique factorisation $\tilde{c}' = f_1 a y^j x^{u(c')} f_2$, where a is z or \bar{y} , and f_2 contains no x . Then by construction, $c = f_1 x^{u(c')} a y^j f_2$ has the required properties.

Thus, θ is indeed an involution with the announced fixed points.

It remains only to check that the sum of weights of the fixed points is equal to $(-1)^k q^{\binom{j+1}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q$. A fixed point of θ is specified by the heights h_1, \dots, h_{k-j} of the $k-j$ steps \nearrow . These heights can be any set of values provided that $j \geq h_1 \geq \dots \geq h_{k-j} \geq 0$. The weight of such a fixed point is:

$$(-1)^j q^{\frac{j(j+1)}{2}} \prod_{i=0}^{k-j} (-q^{h_i}) = (-1)^k q^{\frac{j(j+1)}{2}} q^{\sum_{i=0}^{k-j} h_i}.$$

Indeed, the j steps \nearrow have respective weights $-q, -q^2, \dots, -q^j$, which gives a factor $(-1)^j q^{\frac{j(j+1)}{2}}$. Besides, we have:

$$\sum_{j \geq h_1 \geq \dots \geq h_{k-j} \geq 0} q^{\sum h_i} = \begin{bmatrix} k \\ j \end{bmatrix}_q,$$

by elementary property of q -binomial coefficients. This ends the proof. \square

APPENDIX C. CLOSED FORMS FOR $|\mathcal{P}_{n,k}(c, d)|$

In this appendix we give a justification of the fact that there is no (hypergeometric) closed form (in the sense of Petkovšek, Wilf and Zeilberger [18]) for

$$(32) \quad |\mathcal{P}_{n,k}(c, d)| = \frac{k+1}{n+1} \sum_{l=0}^{n-k} \binom{n+1}{l} \binom{l}{2l-n+k} d^{2l-n+k} c^{n-k-l}.$$

except when (c, d) is one of $(1, 0)$, $(0, 1)$ or $(y^2, 2y)$. More precisely, we claim that $|\mathcal{P}_{n,k}(c, d)|$ cannot be written as a linear combination (of a fixed finite number) of hypergeometric terms except in the specified cases. In the following we sketch a straightforward way to check this is using computer algebra.

First we convert the summation into a polynomial recurrence equation. This can be done by using Zeilberger's algorithm (which also proves that the recurrence is correct), for example. Writing $p_n = |\mathcal{P}_{n,k}(c, d)|$ we obtain

$$(33) \quad (4c-d^2)(n+1)(n+2)p_n + d(n+2)(2n+5)p_{n+1} - (n+2-k)(n+4+k)p_{n+2} = 0.$$

Alternatively, one can also find a recurrence for $q_k = |\mathcal{P}_{n,k}(c, d)|$, which is

$$(34) \quad (k+3)(k-n)q_k + d(k+1)(k+3)q_{k+1} + c(k+1)(k+n+4)q_{k+2} = 0.$$

It now remains to show that both equations admit no hypergeometric solutions, except for the values of (c, d) mentioned above. To this end we use Petkovšek's algorithm **hyper**, as described in Chapter 8 of 'A=B' [18]. Unfortunately, this time we cannot use the implementation naively. Namely, *a priori* **hyper** decides only for fixed parameters (c, d) whether a hypergeometric solution exists or not.

However, it is possible to trace the algorithm, and, whenever it has to decide whether a quantity containing c or d is zero or not, do it for the computer. (Of course, it should be possible to actually program this, but that is outside the scope of this article.) We refrain from giving a complete proof, but rather give only a few details to make checking easier.

First of all, let us assume that c , d and $d^2 - 4c$ are all nonzero. Then the degrees of the coefficient polynomials in both recurrence equations are all the same. From now on, the procedure is the same for both recurrence equation, so let us focus on the one for p_n . According to the remark in Example 8.4.2 in [18], we have to consider all monic factors $a(n)$ of the coefficient of p_n , and also the monic factors $b(n)$ of the coefficient of p_{n+2} , such that the degree of $a(n)$ and $b(n)$ coincide. In this case, the characteristic equation, Equation (8.4.5) in [18] one has to solve turns out to be $z^2 - 2dz + d^2 - 4c$. For each of the two solutions in z , one has to check that there is no polynomial solution of the recurrence

$$P_0(n)c_n + zP_1(n)c_{n+1} + z^2P_2(n)c_{n+2} = 0,$$

where the coefficient polynomials $P_0(n)$, $P_1(n)$ and $P_2(n)$ are polynomials derived from the coefficient polynomials of the original recurrence by multiplying with certain shifts of $a(n)$ and $b(n)$.

This can be done with the algorithm **poly**, described in Section 8.3 of [18]. Namely, depending on the degrees of yet another set of polynomials derived from $P_0(n)$, $P_1(n)$ and $P_2(n)$, it computes an upper bound for the degree of a possible polynomial solution. Indeed, the algorithm decides that the degree of such a solution would have to be negative, provided that c , d and $d^2 - 4c$ are nonzero, which is what we assumed.

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