THE MATRIX ANSATZ, ORTHOGONAL POLYNOMIALS, AND PERMUTATIONS

SYLVIE CORTEEL, MATTHIEU JOSUAT-VERGÈS AND LAUREN K. WILLIAMS

ABSTRACT. In this paper we outline a *Matrix Ansatz* approach to some problems of combinatorial enumeration. The idea is that many interesting quantities can be expressed in terms of products of matrices, where the matrices obey certain relations. We illustrate this approach with applications to moments of orthogonal polynomials, permutations, signed permutations, and tableaux.

To Dennis Stanton with admiration.

1. INTRODUCTION

The aim of this article is to explain a *Matrix Ansatz* approach to some problems of combinatorial enumeration. The idea is that many interesting enumerative quantities can be expressed in terms of products of matrices satisfying certain relations. In such a situation, this Matrix Ansatz can be useful for a variety of reasons:

- having explicit matrix expressions gives rise to explicit formulas for the quantities of interest;
- finding combinatorial objects which obey the same relations gives rise to a combinatorial formula for the quantities of interest;
- finding two combinatorial solutions to the same set of relations identifies the generating functions for the two sets of combinatorial objects.

This Matrix Ansatz approach is not particularly new and has appeared in various contexts, notably in statistical physics (see [HaNa83], [Kl91], [Fan92], and [DEHP93] for a few examples). We will take this opportunity to illustrate its utility for some problems in combinatorics.

1.1. The Matrix Ansatz and the ASEP.

The inspiration for this article comes from an important paper of Derrida, Evans, Hakim, and Pasquier [DEHP93], in which the authors described a Matrix Ansatz approach to the stationary distribution of the *asymmetric exclusion process* (ASEP), a model from statistical physics. This model can be described as a Markov chain on 2^n states – all words of length n in 0 and 1 – where a 1 in the ith position represents a particle in the ith position of a one-dimensional lattice of n sites. In this Markov chain, a new particle may enter the lattice at the left with probability $\frac{\alpha}{n+1}$, a particle may exit the lattice to the right with probability $\frac{\beta}{n+1}$, and a particle may hop to an empty site to its right or left with probabilities $\frac{1}{n+1}$ and $\frac{q}{n+1}$ respectively. See [DEHP93] for more details. A main result of [DEHP93] was the following.

Theorem 1. [DEHP93] *Suppose that D and E are matrices and $\langle W|$ and $|V\rangle$ are row and column vectors (not necessarily finite-dimensional), respectively, such that:*

$$(1) \qquad DE = qED + D + E, \qquad \alpha\langle W|E = \langle W|, \qquad \beta D|V\rangle = |V\rangle, \qquad \langle W|V\rangle = 1.$$

Then the steady state probability that the ASEP with n sites is in state $(\tau_1, \ldots, \tau_n)$ is equal to

$$\frac{\langle W| \prod_{i=1}^n (\tau_i D + (1 - \tau_i)E)|V\rangle}{Z_n},$$

where $Z_n = \langle W|(D + E)^n|V\rangle$.

Note that one does not lose any information if one forgets the denominator Z_n, and simply keeps track of the *un-normalized steady state probability* $\langle W|\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E)|V\rangle$.

As an example, given $D, E, \langle W|, |V\rangle$ satisfying the conditions of this Matrix Ansatz, the un-normalized steady state probability that the ASEP with 5 sites is in state $(1, 0, 0, 1, 1)$ is $\langle W|DEEDD|V\rangle$.

There are several remarks worth noting about Theorem 1. First, the theorem does not guarantee *a priori* the existence of such matrices and vectors, nor does it give a method for finding them. Second, one may find several very different solutions to the Matrix Ansatz. Third, in this case one does not need to find solutions to (1) in order to compute steady state probabilities: given any word X in D and E, one can keep applying the first relation of (1) to express X as a sum of words of the form $E^k D^\ell$, then use the other relations of (1) to evaluate this sum. (That being said, explicit solutions to (1) can be useful for a variety of reasons, as we will see later.)

As we will explain in this article, there are many other interesting quantities that can be computed via relations similar to those of (1). In a previous article [CoWi07a], the first and third authors explored a combinatorial solution to (1), finding an explicit solution $D, E, \langle W|, |V\rangle$ such that expressions of the form $\langle W|X|V\rangle$ enumerate certain tableaux of a fixed shape corresponding to X. As a consequence, this identified steady state probabilities of the ASEP with generating functions for tableaux. In the remainder of the introduction, we will explain this example in more detail. Subsequent sections will describe variations of the Matrix Ansatz and other combinatorial enumeration problems which fit into this framework.

1.2. **The Matrix Ansatz and permutation tableaux.**

Definition 1. A *permutation tableau* is a Young diagram whose boxes are filled with 0's and 1's such that:
- each column contains at least one 1, and
- there is no box containing a 0 which has both a 1 above it (in the same column) and a 1 to its left (in the same row).

We always take the English convention for Young diagrams. Permutation tableaux are a distinguished subset of Postnikov's Г-diagrams [Pos06], introduced in connection to the totally non-negative part of the Grassmannian. More specifically, if we drop the first condition above, we recover Postnikov's definition of Г-diagram.

Note that we allow our Young diagrams to have rows of length 0, and we define the *length* of a permutation tableau to be the sum of its number of rows and columns. Alternatively, this is the length of the southeast border of its Young diagram. The permutation tableaux of length n are in bijection with permutations on n letters [Pos06, StWi07, Bur07, CoNa09]. See Figure 1 for two examples of permutation tableaux of length 8.
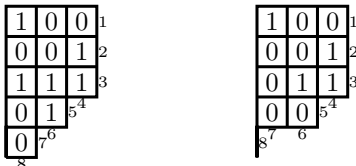

FIGURE 1. Examples of permutation tableaux

Before explaining the connection between the Matrix Ansatz and permutation tableaux given in [CoWi07a], we need to define some statistics. A 0 in a permutation tableau is called *restricted* if it has a 1 above it in the same column. An *unrestricted row* is a row that does not contain any restricted 0. A *topmost* 1 is a 1 which is topmost in its column, and a *superfluous* 1 is a 1 that is not topmost. If T is a permutation tableau, let $wt(T)$ be the number of superfluous 1's, let $f(T)$ be the number of 1's in the first row, and let $u(T)$ be the number of unrestricted rows minus 1.

Label the southeast border of a permutation tableaux with D's and E's going from north-east to south-west such that each south step is labelled by D and each west step is labelled by E. As the first step of the border is always south, we can ignore it, and encode the border of a permutation tableau of length $n+1$ with a word in the alphabet $\{D, E\}$ of length n. The shape of a tableau of length $n+1$ is the corresponding word in $\{D, E\}^n$. Each corner of the tableau corresponds to a pattern DE in the word. For example, the word associated to the diagram at the left of Figure 2 is $DDEDEED$.

If X is a word in D and E, let

$$\mathrm{Tab}(X) = \sum_T q^{wt(T)} \alpha^{-f(T)} \beta^{-u(T)},$$

where the sum is over all permutation tableaux T of shape X.

We first claim that for any words X and Y in D and E,

(2) $$\mathrm{Tab}(XDEY) = q\,\mathrm{Tab}(XEDY) + \mathrm{Tab}(XDY) + \mathrm{Tab}(XEY).$$

To see that this is true, look at the content of a corner box of a tableau of shape $XDEY$. It can contain:

- a superfluous 1, in which case the box can be deleted, leaving a permutation tableau of shape $XEDY$.
- a topmost 1, in which case its column may be deleted, leaving a permutation tableau of shape XDY.
- a 0 (necessarily with only 0's to its left), in which case its entire row may be deleted, leaving a permutation tableau of shape XEY.

See Figure 2. In all cases, the monomial associated to the smaller permutation tableau is easily described in terms of the monomial associated to the original tableau, yielding (2).

$$Tab(XDEY) = q\,\mathrm{Tab}(XEDY) + \mathrm{Tab}(XDY) + \mathrm{Tab}(XEY)$$

FIGURE 2. Decomposition of a permutation tableau

We also claim that for any word X in D and E,

(3) $$\alpha\,\mathrm{Tab}(EX) = \mathrm{Tab}(X) \quad \text{and} \quad \beta\,\mathrm{Tab}(XD) = \mathrm{Tab}(X).$$

These relations are clear upon inspection. Comparing equations (2) and (3) to the equations of the Matrix Ansatz (1), we see that the generating functions $\mathrm{Tab}(X)$ for permutation tableaux of shape X obey the same recursions as do the un-normalized steady state probabilities of the ASEP. Hence for each word X of length n in D and E, $\mathrm{Tab}(X)$ computes the (un-normalized) steady state probability of being in the corresponding state of the ASEP with n sites.

Alternatively, one could concretely define the following matrices and vectors, with rows and columns indexed by the non-negative integers. Let D be the (infinite) upper triangular matrix $(D_{i,j})_{i,j \geq 0}$ such that

$$D_{i,i+1} = \beta^{-1}, \qquad D_{i,j} = 0 \qquad \text{otherwise.}$$

Let E be the (infinite) lower triangular matrix $(E_{ij})_{i,j \geq 0}$ such that for $j \leq i$,

$$E_{ij} = \beta^{i-j}\left(\alpha^{-1}q^j \binom{i}{j} + \sum_{r=0}^{j-1} \binom{i-j+r}{r} q^r\right).$$

Otherwise, $E_{ij} = 0$. Also let $\langle W|$ be the (row) vector $(1, 0, 0, \dots)$ and $|V\rangle$ be the (column) vector $(1, 1, 1, \dots)^T$. Then one can check that these matrices and vectors satisfy (1), and hence that $\langle W|X|V\rangle$ is equal to $\mathrm{Tab}(X)$. More specifically, one can think of D and E as operators acting on the infinite-dimensional vector space indexed by all permutation tableaux (of all shapes and lengths), where D acts by adding a new row of length 0 and E acts by adding a new column at the left of an existing tableau, in all possible ways. See [CoWi07a] for details.

The upshot of both arguments is that the steady state probability that the ASEP is in state τ is proportional to the generating function of permutation tableaux of shape $\prod_{i=1}^{n}(\tau_i D + (1 - \tau_i)E)$. Additionally, the second argument provides an explicit formula for this generating function, as a matrix product.

We remark that there is a more general version of the ASEP in which particles enter and exit at both sides of the lattice, with probabilities α, β, γ, and δ, and there is also a tableaux formula for the stationary

distribution of this more general model [CoWi10]. However, this formula is considerably harder to prove than the special case described above; the methods described here are not sufficient.

1.3. The Matrix Ansatz and Motzkin paths.

Another solution to the Matrix Ansatz of Theorem 1 was given in [BCEPR06, BECE00], and is as follows. Let $\tilde{\alpha} = (1-q)\frac{1}{\alpha} - 1$ and $\tilde{\beta} = (1-q)\frac{1}{\beta} - 1$. Define vectors $\langle W| = (1,0,0,\dots)$ and $|V\rangle = (1,0,0,\dots)^T$, and matrices $D = (D_{i,j})_{i,j \geq 0}$ and $E = (E_{i,j})_{i,j \geq 0}$ where all entries are equal to 0 except

(4) $(1-q)D_{i,i} = 1 - \tilde{\beta}q^i, \qquad (1-q)D_{i,i+1} = 1 - \tilde{\alpha}\tilde{\beta}q^i, \qquad (1-q)E_{i,i} = 1 - \tilde{\alpha}q^i, \qquad (1-q)E_{i+1,i} = 1 - q^{i+1}.$

Recall that a *Motzkin path* is a lattice path in the plane with steps east $(1,0)$, northeast $(1,1)$, and southeast $(1,1)$, which starts at the origin, always stays at or above the x-axis, and ends at the x-axis. One often puts weights on the steps of a Motzkin path and then considers the generating function for all Motzkin paths of a fixed length.

Note that if $\langle W| = (1,0,\dots)$, $|V\rangle = (1,0,\dots)^T$, and D and E are tridiagonal matrices (that is, $D_{i,j} = E_{i,j} = 0$ for $|i-j| > 1$), then for any word X in D and E, the quantity $\langle W|X|V\rangle$ is the generating function for all weighted Motzkin paths of length n and *shape* $X = X_1 \dots X_n$. By *shape* X, we mean that if $X_j = D$ then the jth step of the Motzkin path must be a northeast or east step, and if $X_j = E$, then the jth step of the Motzkin path must be a southeast or east step. Furthermore, if the jth step of the Motzkin path starts at height i, its weight is:

- $D_{i,i+1}$ if the step is northeast and $X_j = D$;
- $E_{i,i+1}$ if the step is northeast and $X_j = E$;
- $D_{i,i}$ if the step is east and $X_j = D$;
- $E_{i,i}$ if the step is east and $X_j = E$;
- $D_{i,i-1}$ if the step is southeast and $X_j = D$;
- $E_{i,i-1}$ if the step is southeast and $X_j = E$.

Moreover the quantity $Z_n = \langle W|(D+E)^n|V\rangle$ is the generating function for all weighted Motzkin paths of length n where the weight of a northeast (resp. east, southeast) step starting at height i is $D_{i,i+1} + E_{i,i+1}$, (resp. $D_{i,i} + E_{i,i}$, $D_{i,i-1} + E_{i,i-1}$).

It follows that the above solution $D, E, \langle W|, |V\rangle$ to the Matrix Ansatz identifies steady state probabilities of the ASEP with generating functions for weighted Motzkin paths of the corresponding shape.

Additionally, these Motzkin paths provide a link to orthogonal polynomials, via the combinatorial theory of orthogonal polynomials provided by [Fla82, Vie88]. More specifically, the moments of the weight function of a family of monic orthogonal polynomials can be identified with the generating functions for weighted Motzkin paths, where the weights on the steps of the Motzkin paths come directly from the coefficients of the three-term recurrence for the orthogonal polynomials. This will be explained more carefully in the following section. This allows us to interpret the partition function $Z_n = \langle W|(D+E)^n|V\rangle$ of the ASEP with n sites as the nth moment of a family of orthogonal polynomials. Indeed, the solution of the Matrix Ansatz given here corresponds to the moments of the Al-Salam-Chihara polynomials. The link between the ASEP and orthogonal polynomials was first made in [Sa99]. Most of the time, when we define orthogonal polynomials, we will refer to the survey paper [KoSw98]. All the original articles are cited in [KoSw98].

This article is organized as follows. In Section 2, we recall some elementary facts concerning moments of orthogonal polynomials, and explain how the Matrix Ansatz sheds new light on some known examples, *e.g.* the connection between rook placements and Hermite polynomials. In Section 3, we apply this method to the enumeration of permutations, signed permutations, and type B permutation tableaux [LaWi08], and we make a link with two different kinds of q-Laguerre polynomials. Finally, in Section 4, we show that the Matrix Ansatz gives rise to a new combinatorial interpretation of a generalization of Genocchi numbers defined by Dumont and Foata [DuFo76].

2. Orthogonal polynomials

In this section we review some notions about orthogonal polynomials, in particular a combinatorial interpretation of their moments. We then revisit some classical results about orthogonal polynomials, illustrating the Matrix Ansatz approach. Note that throughout this section, we take $\langle W| = (1, 0, 0, \dots)$ and $|V\rangle = \langle W|^T$.

Definition 2. We say $\{P_k(x)\}_{k \geq 0}$ is a family of *orthogonal polynomials* if there exists a linear functional $f : K[x] \to K$ such that:

- $\deg(P_k) = k$ for $k \geq 0$,
- $f(P_k P_\ell) = 0$ if $k \neq \ell$,
- $f(P_k^2) \neq 0$ for $k \geq 0$.

The *nth moment* of $\{P_k(x)\}_k$ is defined to be $\mu_n = f(x^n)$.

It might seem that μ_n depends on f and not just on $\{P_k(x)\}_{k \geq 0}$. But since $f(P_n) = 0$ for any $n \geq 1$, we can obtain the moments recursively from the initial moment μ_0. By convention we always take $\mu_0 = 1$.

By Favard's Theorem, monic orthogonal polynomials satisfy a three-term recurrence.

Theorem 2. *Let $\{P_k(x)\}_{k \geq 0}$ be a family of monic orthogonal polynomials. Then there exist coefficients $\{b_k\}_{k \geq 0}$ and $\{\lambda_k\}_{k \geq 1}$ in K such that $P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$.*

By work of [Fla82, Vie88], the nth moment of a family of monic orthogonal polynomials can be identified with the generating function for weighted Motzkin paths as follows.

Theorem 3. [Fla82, Vie88] *Consider a family of monic orthogonal polynomials $\{P_k(x)\}_{k \geq 0}$ which satisfy the three-term recurrence $P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$, for $\{b_k\}_{k \geq 0}$ and $\{\lambda_k\}_{k \geq 1}$ in K. Then the nth moment μ_n of $\{P_k(x)\}_{k \geq 0}$ is equal to $\langle W|M^n|V\rangle$, where $M = (m_{ij})_{i,j \geq 0}$ is the tridiagonal matrix with rows and columns indexed by the non-negative integers, such that $m_{i,i-1} = \lambda_i$, $m_{ii} = b_i$, and $m_{i,i+1} = 1$. Equivalently, μ_n is equal to the generating function for weighted Motzkin paths of length n, where the northeast steps have weight 1, the east steps at height i have weight b_i, and the southeast steps starting at height i have weight λ_i.*

The third author would like to thank Ira Gessel for explaining the following simple proof to her.

Proof. Let us re-write the three-term recurrence as $xP_k(x) = P_{k+1}(x) + b_k P_k(x) + \lambda_k P_{k-1}(x)$. Let $P = (P_0(x), P_1(x), P_2(x), \dots)^T$ be the vector whose ith component is $P_i(x)$. Then the fact that the $P_k(x)$ satisfy the three-term recurrence is equivalent to the statement that $MP = xP$.

It follows that $M^n P = x^n P$. Let f be the linear functional associated to $\{P_k(x)\}$. Applying f to both sides gives

$$(5) \qquad\qquad M^n f(P) = f(x^n P).$$

We now analyze the first entry of the vector on either side of (5). Note that $P_0 = 1$ so $f(x^n P_0) = f(x^n) = \mu_n$. And $f(P_0) = 1$, so the first entry of the right-hand side of (5) is equal to $\langle W|M^n|V\rangle$. It is easy to see that this is equal to the generating function for Motzkin paths of length n, where the northeast steps are weighted 1, the east steps at height i are weighted b_i, and the southeast steps starting at height i are weighted λ_i. This completes the proof. □

The following remarks will be useful in subsequent sections.

Remark 1. Given a tridiagonal matrix M, note that if we define another tridiagonal matrix $M' = (m'_{ij})$ such that $m'_{ii} = m_{ii}$ and $m'_{i,i+1}m'_{i+1,i} = m_{i,i+1}m_{i+1,i}$, then $\langle W|M^n|V\rangle = \langle W|M'^n|V\rangle$.

Remark 2. If $\{P_k(x)\}_{k \geq 0}$ is an orthogonal sequence with moments $\{\mu_n\}_{n \geq 0}$, the shifted sequence $\{P_k(ax + b)\}_{k \geq 0}$ with $a, b \in K$ and $a \neq 0$ is also orthogonal, and its nth moment is $\frac{1}{a^n} \sum_{k=0}^{n} \binom{n}{k} \mu_k (-b)^{n-k}$. In particular, if μ_n has the form $\mu_n = \langle W|M^n|V\rangle$, then the moments of the shifted sequence are $\langle W|M'^n|V\rangle$ where $M' = \frac{1}{a}(M - bI)$.

2.1. Rook placements and q-Hermite polynomials. Let $[n]_q = 1 + q + \ldots + q^{n-1}$. One of the simplest classes of orthogonal polynomials are the "continuous big q-Hermite polynomials", defined by the recurrence

$$(6) \qquad xH_n(x|q) = H_{n+1}(x|q) + aq^n H_n(x|q) + [n]_q H_{n-1}(x|q).$$

When $a = 0$ they specialize to the "continuous q-Hermite polynomials", and in this case a combinatorial interpretation of the moments was given in [ISV87] in terms of perfect matchings and crossings. From Theorem 3, the moments are $\mu_n^h = \langle W|M^n|V\rangle$, where the matrix M has coefficients

$$(7) \qquad m_{i+1,i} = 1, \qquad m_{i,i} = aq^i, \qquad m_{i,i+1} = [i]_q, \qquad m_{i,j} = 0 \quad \text{if } |i-j| > 1.$$

We can write $M = F + U$ where F is strictly upper triangular with 0's on the diagonal, and U is lower triangular. We can check that

$$(8) \qquad FU - qUF = I, \qquad \langle W|U = a\langle W|, \qquad F|V\rangle = 0.$$

So we have a "Matrix Ansatz" for the moments μ_n^h. Indeed, just as permutation tableaux are described by the Matrix Ansatz of (1), we can search for tableaux which are enumerated by products $\langle W|X|V\rangle$, where X is a word in F and U, and F and U satisfy the relations of (8). It's easy to see that $\langle W|X|V\rangle$ enumerates rook placements in Young diagrams of "shape" X, where

- there is exactly one rook per row,
- a counts the columns without rook,
- q counts each cell with no rook to its right in the same row, nor below it in the same column.

And then since the moments μ_n^h of the continuous big q-Hermite polynomials are given by $\langle W|(F+U)^n|V\rangle$, we recover results of [Var05]: μ_n^h enumerates such rook placements in all Young diagrams where the number of rows plus the number of columns is n. For example, the Matrix Ansatz for rook placements plays a crucial role in [Jos08b].

There is a simple bijection between these rook placements and involutions on $[n]$. To obtain the involution from the rook placement, we label the steps in the southeast border of the Young diagram, and draw some arches as follows: for each rook lying in column i and row j, we draw an arch joining i and j. See Figure 3; the rooks are represented by $\circ$. Under this bijection, a counts the fixed points, and q keeps track of a statistic called *total crossing number* defined in [MSS07]. This extends the case $a = 0$ given by Ismail, Stanton, and Viennot [ISV87].

We can also consider the Motzkin paths corresponding to the product $\mu_n^h = \langle W|(F+U)^n|V\rangle$. There is a simple bijection between these paths and involutions. This was given in [Pen95] in the case $a = 0$, and in the present case with a general, Penaud's construction can be adapted so that horizontal steps in the Motzkin paths correspond to fixed points in the involution. See Figure 3 for a rook placement, the corresponding involution, and the corresponding Motzkin path. The weighted Motzkin path is obtained from the involution as follows: the ith step is $\nearrow$ if i is the smallest element of an arch, it is $\rightarrow$ with weight aq^k if i is a fixed point with k arches above it, and it is a step $\searrow$ with weight q^k if i is the largest element of an arch (j,i) and there are k arches (u,v) such that $j < u < i < v$.
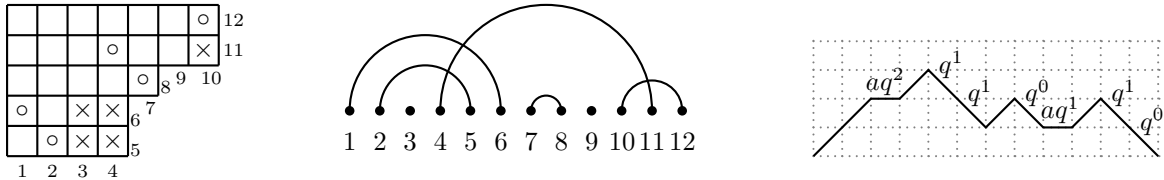

FIGURE 3. A rook placement, the corresponding involution and weighted Motzkin path.

2.2. $0 - 1$ Tableaux and a q-analogue of Charlier polynomials. The Stirling numbers appear as moments of Charlier polynomials. We consider here the q-Charlier polynomials defined by de Médicis, Stanton, and White in [MSW95] as a rescaled version of Al-Salam-Carlitz polynomials, with the recurrence relation

$$(9) \qquad xC_n(x,a;q) = C_{n+1}(x,a;q) + (aq^n + [n]_q)C_n(x,a;q) + a[n]_q q^{n-1}C_{n-1}(x,a;q).$$

From the same reference we know that the nth moment of this sequence is $\sum_{k=1}^{n} a^k S_q(n, k)$ where $S_q(n, k)$ is the Carlitz q-Stirling number, satisfying the relation

$$(10) \qquad S_q(n, k) = S_q(n - 1, k - 1) + [k]_q S_q(n - 1, k).$$

A combinatorial interpretation of $S_q(n, k)$ was given by Leroux in [Ler90]. A $0 - 1$ *tableau* is a filling of a Young diagram with 0's and 1's such that there is exactly one 1 in each column. Then $S_q(n, k)$ is the generating function for $0 - 1$ tableaux with $n - k$ columns and k rows, where q counts the number of 0's above a 1.

Let us explain how we can reproduce Leroux's combinatorial interpretation via the Matrix Ansatz method. From the recurrence relation, the moments of the q-Charlier polynomials are $\mu_n^c = \langle W | M^n | V \rangle$ where the tridiagonal matrix M has coefficients

$$(11) \qquad m_{i+1,i} = [i + 1]_q, \qquad m_{i,i} = aq^i + [i]_q, \qquad m_{i,i+1} = aq^i, \qquad m_{i,j} = 0 \quad \text{if } |i - j| > 1.$$

We can write $M = aX + Y$ where X and Y do not depend on a. These matrices satisfy the relations

$$(12) \qquad XY - qYX = X, \qquad \langle W | Y = 0, \qquad X | V \rangle = | V \rangle.$$

The Matrix Ansatz method shows that with these relations, $\langle W | (aX + Y)^n | V \rangle$ is a generating function of $0 - 1$ tableaux. Indeed there is a recursive decomposition of $0 - 1$ tableaux associated with the relation (12), in the same way that permutation tableaux are associated with the relations (1). So the method explains why these tableaux appear as moments of the q-Charlier polynomials.

2.3. **Rook placements and another q-analogue of Charlier polynomials.** Another q-analogue of Charlier polynomials was defined by Kim, Stanton, and Zeng in [KSZ06] as a rescaled version of Al-Salam-Chihara polynomials. The recurrence is:

$$(13) \qquad x C_n^*(x, a; q) = C_{n+1}^*(x, a; q) + (a + [n]_q) C_n^*(x, a; q) + a [n]_q C_{n-1}^*(x, a; q).$$

The nth moment is $\mu_n^{c^*} = \langle W | M | V \rangle$, where M has coefficients:

$$(14) \qquad m_{i+1,i} = a, \qquad m_{i,i} = a + [i]_q, \qquad m_{i,i+1} = [i + 1]_q, \qquad m_{i,j} = 0 \quad \text{if } |i - j| > 1.$$

Let $F = (f_{i,j})_{i,j \geq 0}$ and $U = (u_{i,j})_{i,j \geq 0}$ be such that $f_{i,i+1} = [i + 1]_q$ and $u_{i+1,i} = 1$, all other coefficients being 0. We can check that $M = (U + I)(F + aI)$ where F and U are the matrices defined above (for the q-Hermite polynomials) satisfying $FU - qUF = I$. From $\mu_n^{c^*} = \langle W | ((U + I)(F + aI))^n | V \rangle$, the Matrix Ansatz method gives a combinatorial interpretation of the moments in terms of rook placements. Indeed, in the expansion of $((U + I)(F + aI))^n$, each term m corresponds to the choice of some columns and rows in the staircase Young diagram, and $\langle W | m | V \rangle$ counts the rook placements with no free row or column of shape m. So we can see that $\mu_n^{c^*}$ counts the rook placements in the staircase Young diagram of length $2n$, possibly with empty rows and columns, and q counts the number of inversions, *i.e.* cells having a rook below and a rook to its left.

A simple bijection links these rook placements with set partitions so that q counts the number of crossings and a counts the number of blocks. This result was first given in [KSZ06]. Motzkin paths in this case are known as *Charlier diagrams* [KaZe06]. See Figure 4 for an example of a rook placement, the corresponding set partition $\pi = (1, 5, 7)(2, 6)(3, 4)$, and the Charlier diagram. The set partition is obtained from the rook placement as follows: we label the inner corner of the Young diagrams with integers, and for any rook at the intersection of a column with label i and a row with label j we draw an arch from i to j. Each block of the set partition is given by a sequence of chained arches. The weighted Motzkin path is obtained from the set partition in the following way. The ith step is $\nearrow$ with weight a if i is the minimum element of a non-singleton block, it is $\rightarrow$ with weight a if i is a singleton block, it is $\rightarrow$ with weight q^k if i is the minimal element of an arch and the maximal element of another arch (j, i), and it is $\searrow$ with weight q^k if i is the maximal element of an arch (j, i) but not the minimal element of another arch. Here k is the number of arches (u, v) such that $j < u < i < v$.
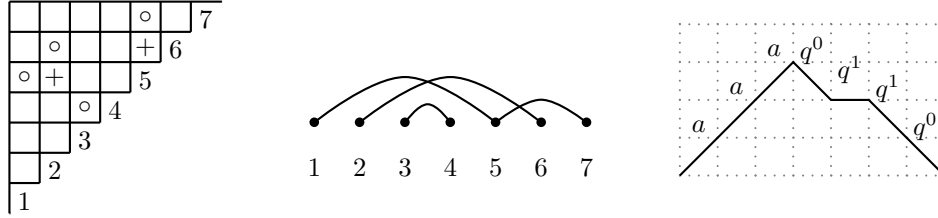
FIGURE 4. A rook placement in a staircase diagram, a set partition, and a Charlier diagram.

3. PERMUTATIONS AND SIGNED PERMUTATIONS

3.1. Permutations and inversions.

De Médicis and Viennot [MeVi94] studied a q-analogue of Laguerre polynomials (with one parameter set to 1), whose nth moment is the classical q-factorial $[n]_q!$. They are a particular case of the little q-Jacobi polynomials. The recurrence relation is:

(15)
$$xL_n(x) = L_{n+1}(x) + q^n([n]_q + [n+1]_q)L_n(x) + q^{2n-1}[n]_q^2 L_{n-1}(x).$$

Recall that an *inversion* of a permutation π is a pair (i,j) with $i < j$ and $\pi(i) > \pi(j)$. We denote the number of inversions of π by $inv(\pi)$. It is well known that

$$F_n(q) := \sum_{\pi \in S_n} q^{inv(\pi)} = [n]_q!.$$

Proposition 1. *Let W and V be vectors, and D and E be matrices, satisfying*

(16)
$$\langle W|V\rangle = 1, \qquad D|V\rangle = E|V\rangle \qquad \langle W|E = \langle W|, \qquad DE = qED + D.$$

Then the generating function $F_n(q)$ is equal to $\langle W|D^n|V\rangle$.

To prove this, we will use certain tableaux contained in shifted Young diagrams.

Definition 3. We equip the set $S := \{(i,j) \in \mathbf{N} \times \mathbf{N} \mid i \le j\}$ with the partial order:

$$(i,j) \ge (i',j') \text{ if and only if } i \ge i' \text{ and } j \ge j'.$$

A finite order ideal of S is called a *shifted Young diagram*.

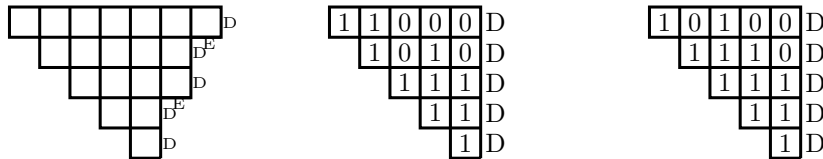See the left of Figure 5 for an example of a shifted Young diagram.

FIGURE 5. A shifted Young diagram and inversion tables

Proof. Consider $0-1$ tableaux contained in shifted Young diagrams such that each cell of the main diagonal contains a 1 and any cell that contains a 0 has only 0's above it. See the diagrams of Figure 5. Then there is a simple recurrence for such tableaux of a given shape, that corresponds to the relations of Proposition 1. Therefore $\langle W|D^n|V\rangle$ enumerates such tableaux of staircase shape $(n, n-1, n-2, \ldots, 1)$. We call such tableaux contained in a staircase shape *inversion tables*.

Label the columns of an inversion table from 1 to n from right to left. Then there is a bijection between inversion tables and permutations, such that the number of superfluous 1's in column i corresponds to the number of inversions of $\pi(i)$ (i.e. the number of $j > i$ such that $\pi(i) > \pi(j)$). Indeed the sequence of the number of superfluous 1's in column $i = 1, \ldots, n$ is the inversion table of the permutation. For example, the tableau in the middle of Figure 5 gives the inversion table $(2, 2, 0, 1, 0)$ and the permutation $(3, 4, 1, 5, 2)$ and the tableau on the right of Figure 5 gives the inversion table $(2, 2, 2, 0, 0)$ and the permutation $(3, 4, 5, 1, 2)$. This proves the proposition.

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Note also that the following vectors and matrices satisfy the Ansatz:

$$E_{i,i} = E_{i,i+1} = [i+1]_q, \qquad E_{i,j} = 0 \text{ otherwise,}$$

$$D_{i,i-1} = q^i[i]_q, \qquad D_{i,i} = q^i([i]_q + [i+1]_q), \qquad D_{i,i+1} = q^i[i+1]_q, \qquad D_{i,j} = 0 \text{ otherwise,}$$

and $\langle W| = (1, 0, 0, \ldots)$ and $|V\rangle = (1, 0, 0, \ldots)^T$. Therefore $\langle W|D^n|V\rangle$ is the generating function of weighted Motzkin paths of length n with weights $\lambda_i = q^{2i-1}[i+1]_q^2$ and $b_i = q^i([i]_q + [i+1]_q)$. This is known to be the generating function of permutations counted by inversions. See for example [Bi93]. □

3.2. **Permutations and crossings.**

Definition 4. [Cor07] A *crossing* in a permutation π is a pair (i, j) such that

- $i < j \leq \pi(i) < \pi(j)$ or
- $i > j > \pi(i) > \pi(j)$.

They appear in the combinatorial interpretation of the moments of Al-Salam-Chihara q-Laguerre polynomials [KSZ08]. These are another q-analogue of Laguerre polynomials whose recurrence relation is:

$$(17) \qquad xL_n^*(x) = L_{n+1}^*(x) + (y[n+1]_q + [n]_q)L_n^*(x) + y[n]_q^2 L_{n-1}^*(x).$$

See [SS96] for some generalizations of these polynomials.

Proposition 2. [CoWi07a] *The generating function of permutations of size n enumerated according to their crossings is*

$$\langle W|(D+E)^n|V\rangle$$

with

$$(18) \qquad \langle W|V\rangle = 1, \qquad \langle W|E = 0, \qquad D|V\rangle = |V\rangle, \qquad DE = qED + D + E.$$

Proof. One can easily check that $\langle W|(D+E)^n|V\rangle$ is the generating function of permutation tableaux of length n where q counts the superfluous 1's, using the same arguments as the ones developed in the introduction. See [CoWi07a], and also [Jos10], for proofs. From [StWi07] we know that permutation tableaux of length n with j superfluous 1's are in bijection with permutations of $[n]$ with j crossings.

One can also prove the result with a solution of the Matrix Ansatz. Indeed the matrices D and E with entries given by

$$D_{i,i} = [i+1]_q, \quad D_{i,i+1} = [i+1]_q, \quad D_{i,j} = 0 \text{ otherwise,}$$

$$E_{i,i} = [i]_q, \qquad E_{i,i-1} = [i]_q, \qquad E_{i,j} = 0 \text{ otherwise,}$$

$$\langle W| = (1, 0, \ldots), \quad |V\rangle = (1, 0, \ldots)^T$$

are a solution. Therefore $\langle W|(D+E)^n|V\rangle$ is the generating function of weighted Motzkin paths of length n with weights $\lambda_i = [i]_q^2$ and $b_i = ([i]_q + [i+1]_q)$. This is known to also be the generating function of permutations counted by crossings [Cor07, SS96]. □

3.3. **Signed permutations.** A signed permutation of $\{1, \ldots, n\}$ is a sequence of integers $(\pi(1), \ldots, \pi(n))$ such that $-n \leq \pi(i) \leq n$ and $\cup_{\ell=1}^n |\pi(\ell)| = \{1, \ldots, n\}$. For example $(2, 3, 7, -5, 6, 1, -4, 8)$ is a signed permutation of $\{1, \ldots, 8\}$. Let B_n be the set of signed permutations of $\{1, \ldots, n\}$.

We extend the definition of crossings of permutations to signed permutations:

Definition 5. A *crossing* of a signed permutation $\pi = (\pi(1), \ldots, \pi(n))$ is a pair (i, j) with $i, j > 0$ such that

- $i < j \leq \pi(i) < \pi(j)$, or
- $-i < j \leq -\pi(i) < \pi(j)$, or
- $i > j > \pi(i) > \pi(j)$.

Remark 3. Note that if for all i, $\pi(i) > 0$, then the crossings of the signed permutation are the same as the crossings in Definition 4 for usual permutations. Moreover if for all i, $\pi(i) < 0$, then the crossings of the signed permutation are the same as the inversions of the permutation $(-\pi(1), \ldots, -\pi(n))$. This notion of crossing has the nice property that the number of signed permutations of $[n]$ with no crossings is $\binom{2n}{n}$, the Catalan number of type B, just as the number of usual permutations of $[n]$ with no crossings is the Catalan number of type A, see [Wil05].

Definition 6. [LaWi08] A *type B permutation tableau* of length n is a filling with 0's and 1's of a shifted Young diagram of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \leq n$, such that:

- Any 0 on the main diagonal has only 0's above it.
- There is no 0 with a 1 above it and a 1 to its left.
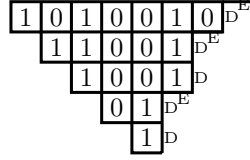- Each row has at least one 1.

FIGURE 6. Permutation tableau of type B

These tableaux are in bijection with signed permutations, and are related to a cell decomposition of the totally non-negative part of a type B Grassmannian [LaWi08].

An example is given in Figure 6. Note that we allow rightmost columns of length 0. We will label the southeast border of a tableau with D's and E's, writing a D (resp. E) whenever we go along a vertical (resp. horizontal) step. This word in D's and E's encodes the *shape* of the tableau.

Let $B_n(q, r)$ be the generating function of permutation tableaux of type B of length n, where q counts the superfluous 1's and r counts the 1's on the main diagonal.

Proposition 3. *The generating function $B_n(q, r)$ is equal to*

$$\langle W | (D + E)^n | V \rangle$$

where

(19) $$\langle W | V \rangle = 1, \qquad \langle W | E = \langle W |, \qquad D | V \rangle = rE | V \rangle, \qquad DE = qED + E + D.$$

Proof. We illustrate a recursion for the generating function for type B permutation tableaux of a fixed shape, which mirrors the relations above. First, the generating function of the tableaux of length 0 is 1 : this corresponds to the fact that $\langle W | V \rangle = 1$. Second, a bottommost row of length one must contain a 1 on the main diagonal. This row can be deleted, which corresponds to $D | V \rangle = rE | V \rangle$. Third, any column of length 0 at the right can be deleted, which corresponds to $\langle W | E = \langle W |$. Finally each corner box (which corresponds to the subword DE in the word encoding the shape of the tableau) contains either:

- a superfluous 1, so this box can be deleted (leaving a tableau whose shape has the DE replaced by ED, and whose weight is equal to the old weight divided by q),
- the unique 1 in its row, and consequently the row and column containing the main diagonal cell to its left can be deleted ($+D$),
- a 0, and consequently its column can be deleted ($+E$).

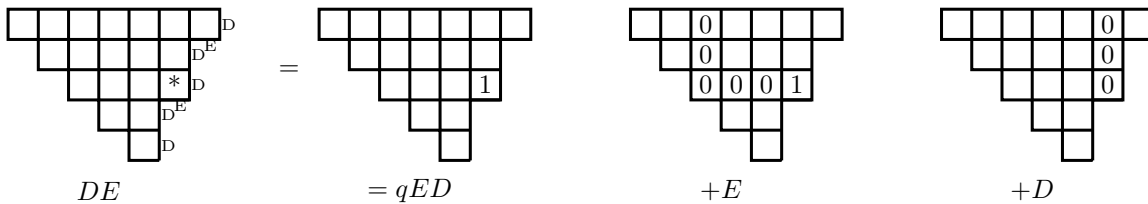See Figure 7 for an example. □

FIGURE 7. Decomposition of permutation tableaux of type B

Theorem 4. *There exists a one-to-one correspondence between signed permutations in B_n with j crossings and ℓ minus signs and permutation tableaux of type B of length n with j superfluous 1's and ℓ 1's on the main diagonal.*

Proof. This result is analogous to a result proved for usual permutations and type A permutation tableaux [Cor07, StWi07], and we can prove it using an argument analogous to that given in [StWi07]. Indeed, we use the bijection from [LaWi08, Section 10], which in turn was an adaptation of the bijection from [StWi07]. We then apply the same arguments that were used in Section 3 of [StWi07] to show how the statistics translate under this bijection.

One can also prove the preceding result with a solution of the Matrix Ansatz. Indeed

$$D_{i,i} = D_{i,i+1} = [i+1]_q, \qquad D_{i,j} = 0 \text{ otherwise};$$

$$E_{i,i-1} = [i]_q(1+rq^i), \qquad E_{i,i} = [i]_q(1+rq^i)+rq^i[i+1]_q, \qquad E_{i,i+1} = rq^i[i+1]_q, \qquad E_{i,j} = 0 \text{ otherwise},$$

$$\bar{W} = (1,0,\ldots), \qquad |V\rangle = (1,0,\ldots)^T,$$

is a solution. Therefore $\langle W|(D+E)^n|V\rangle$ is the generating function of weighted Motzkin paths of length n with weights $\lambda_i = [i+1]_q^2(1+rq^i)(1+rq^{i+1})$ and $b_i = (1+rq^i)([i]_q + [i+1]_q)$. On can adapt the proofs in [Cor07, SS96] to show that this is indeed the generating function of signed permutation in B_n according to crossings [CJKW10]. □

From the weights on the Motzkin paths, we see that the generating function for signed permutations of size n where q counts the number of crossings, is the nth moment of the orthogonal sequence $P_n(x; r \mid q)$ or simply $P_n(x)$ defined by the recurrence:

$$(20) \qquad xP_n(x) = P_{n+1}(x) + \big([n+1]_q + [n]_q\big)(1+rq^n)P_n(x) + (1+rq^n)(1+rq^{n-1})[n]_q^2 P_{n-1}(x),$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$. We have the following two special cases for these orthogonal polynomials (which are the counterpart of combinatorial properties given in Remark 3).

- When $r = 0$, we recover the recurrence relation for the Al-Salam-Chihara q-Laguerre polynomials $L_n^*(x)$,
- When we keep only the terms of maximal degree in r, we recover the recurrence relation for the little q-Jacobi Laguerre polynomials $L_n(x)$.

These $P_n(x)$ can be linked with some classical polynomials called continuous dual q-Hahn polynomials in [KoSw98], denoted by $R_n(x; a, b, c \mid q)$ or simply $R_n(x)$. Their recurrence relation is

$$(21) \qquad 2xR_n(x) = R_{n+1}(x) + \big(a + \tfrac{1}{a} - (A_n + C_n)\big)R_n(x) + A_{n-1}C_n R_{n-1}(x),$$

where

$$(22) \qquad A_n = \tfrac{1}{a}(1-abq^n)(1-acq^n) \qquad \text{and} \qquad C_n = a(1-q^n)(1-bcq^{n-1}).$$

In the case where $a = 1$, $b = -r$ and $c = q$, one has

$$(23) \quad A_n + C_n = -(1-q)\big([n]_q + [n+1]_q\big)(1+rq^n) \quad \text{and} \quad A_{n-1}C_n = (1-q)^2(1+rq^n)(1+rq^{n-1})[n]_q^2.$$

Hence,

$$(24) \qquad P_n(x; r \mid q) = \tfrac{1}{(q-1)^n} R_n\left(x\tfrac{q-1}{2} + 1; 1, -r, q \mid q\right),$$

and we can see $P_n(x)$ as a special case of these continuous dual q-Hahn polynomials.

4. GENOCCHI NUMBERS

Dumont and Foata introduced in [DuFo76] a three-parameter generalization $\{f_n(a,b,c)\}_{n\geq 0}$ of the Genocchi numbers, defined by $f_0(a,b,c) = 1$ and

$$(25) \qquad f_{n+1}(a,b,c) = (a+b)(a+c)f_n(a+1,b,c) - a^2 f_n(a,b,c).$$

When the parameters are set to 1, $f_n(1,1,1)$ is the Genocchi number G_{2n+4} defined by

$$(26) \qquad \sum_{n=1}^{\infty} G_{2n}\frac{x^{2n}}{(2n)!} = x \cdot \tan\left(\frac{x}{2}\right).$$

It is straightforward to see that f_n has non-negative coefficients, but it is more difficult to prove that f_n is symmetric in a, b, and c [DuFo76]. And while several combinatorial interpretations of $f_n(a,b,c)$ have been given [Dum74, DuFo76, Han96, Vie81], none of them readily exhibit the symmetry in a, b, c.

The goal of this section is to give a new combinatorial interpretation of $f_n(a, b, c)$ in terms of *alternative tableaux* [Vie08], and to use this combinatorial interpretation to give a simple proof of the symmetry in a, b and c. This is done by using the link with moments of continuous dual Hahn polynomials, and the Matrix Ansatz for alternative tableaux.

The continuous dual Hahn polynomials $S_n(x)$ are defined by the recurrence :

(27) $$ xS_n(x) = S_{n+1}(x) + (A_n + C_n - a^2)S_n(x) + A_{n-1}C_nS_{n-1}(x), $$

where:

(28) $$ A_n = (n + a + b)(n + a + c) \qquad \text{and} \qquad C_n = n(n + b + c - 1), $$

and $S_{-1}(x) = 0$, $S_0(x) = 1$. (Up to a sign they are the same as the ones defined in [KoSw98].) Zeng [Zen96, Corollaire 2] has given a continued fraction for the generating function $\sum_{n \geq 0} f_n(a, b, c)x^n$. This was conjectured by Dumont and also proved by Randrianarivony, see details in [Zen96]. From this continued fraction and the recurrence (27) we see that $f_n(a, b, c)$ is the nth moment μ_n^s of the orthogonal sequence $\{S_k(x)\}_{k \geq 0}$.

Thus we have $f_n(a, b, c) = \langle W|M^n|V \rangle$ where the matrix M has coefficients:

(29) $$ M_{i,i} = A_i + C_i - a^2, \qquad M_{i+1,i} = A_i, \qquad M_{i-1,i} = C_i. $$

Using the bijection of Corteel and Nadeau [CoNa09], it can be checked that permutation tableaux of staircase shape are linked to *Dumont permutation of the first kind* [Dum74], which were introduced to give a combinatorial interpretation of Genocchi numbers. These permutations are the $\sigma \in \mathfrak{S}_{2n}$ such that $\sigma(i) > \sigma(i+1)$ if and only if $\sigma(i)$ is even for any $1 \leq i \leq 2n$ (with the convention that $\sigma(2n+1) = 2n+1$). Via the bijection, the shape of the tableau is obtained by examining the values of the descent in the permutation so that the result follows easily. Thus, Genocchi numbers count the permutation tableaux of staircase shape. Alternatively, we could use the bijection of Steingrímsson and Williams [StWi07], which links staircase permutation tableaux to *Dumont permutation of the second kind* [Dum74]; indeed, these permutations are the $\sigma \in \mathfrak{S}_{2n}$ such that $\sigma(i) < i$ if and only if i is even, and via the bijection the shape of the tableau is obtained by examining the weak exceedances of the permutations.

So it is natural to compare this matrix M with the product DE of the matrices D and E of the PASEP Matrix Ansatz defined in (4). The limit when $q = 1$ of these matrices are well defined, and a straightforward computation show that $DE + (c-1)(D+E)$ is equal to M, under the condition that $a = \beta^{-1}$ and $b = \alpha^{-1}$. So, knowing that $f_n(a, b, c)$ is symmetric, we have:

$$ f_n(a, b, c) = \langle W|M^n|V \rangle = \langle W|(DE + (c-1)(D+E))^n|V \rangle = \langle W|(ED + cD + cE)^n|V \rangle. $$

The last equality is obtained by using the fact that $DE = ED + D + E$. We can derive a new combinatorial interpretation of $f_n(a, b, c)$ in terms of permutation tableaux.

We will also use *alternative tableaux*, which are slightly different objects. As remarked in [CoNa09], all the entries of a permutation tableau can be recovered if one knows the position of the topmost 1's and the rightmost restricted 0's. Viennot [Vie08, Na09] took this a step further and defined *alternative tableaux*, which are partial fillings of Young diagrams with αs and βs such that any cell to the left of a β (resp. above an α) is empty. There is a direct bijection from permutation tableaux of length $n + 1$ to alternative tableaux of length n (essentially one replaces topmost 1's with αs and rightmost restricted 0's with βs and makes all other boxes empty and deletes the first row). Alternative tableaux are interesting in this context, because they are more symmetric than permutation tableaux, and consequently more adequate to explain the symmetry of the polynomials $f_n(a, b, c)$.

Theorem 5. *The polynomial $(\alpha\beta\gamma)^n f_n(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$ counts the staircase alternative tableaux of length $2n$, where the parameters α, β, γ follow these statistics:*

- *the number of cells containing an α,*
- *the number of cells containing a β,*
- *the number of corners which does not contains an α or a β.*

Equivalently, $(\alpha\beta\gamma)^n f_n(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$ counts the staircase permutation tableaux of length $2n + 2$, where the parameters α, β, γ follow these statistics:

- *the number of 0's in the first row,*
- *the number of restricted rows,*
- *the number of corner which contains a superfluous 1.*

Proof. When $\gamma = 1$, we can use the results from [CoWi07a] which have been recalled in the introduction of this article, and it follows that $\langle W | (DE)^n | V \rangle$ counts the staircase permutation tableaux of length $2n + 2$, where the parameters α^{-1}, β^{-1} counts the number of 1's in the first row and the number of unrestricted row. When we replace DE with $ED + \gamma^{-1}D + \gamma^{-1}E$, in the recurrence relation for permutation tableaux we see that γ^{-1} will count the corners containing a restricted 0 or a topmost 1. So in $(\alpha\beta\gamma)^n f_n(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$, the parameters α, β, and γ counts the complementary statistics which are given in the theorem. $\square$

   In this combinatorial interpretation, the symmetry in α and β is apparent, because we can transpose the tableaux to exchange the two parameters. The symmetry in γ and β is obvious from the recurrence relation (25), so this implies the full symmetry of the three parameters. The symmetry in γ and β can also be proved by an explicit involution [Jos11].
   We can ask if there is a direct bijection between staircase alternative tableaux and other known combinatorial interpretations of $f_n(a, b, c)$. In particular, Viennot [Vie81] gives a combinatorial interpretation which has also an apparent symmetry exchanging two parameters, which can be defined as follows. We consider pairs (f, g) of maps from $[n]$ to $[n]$ such that $f(i) \geq i$ and $g(i) \geq i$ for any i, and such that for any $1 \leq j \leq n$ there is at least an i such that $f(i) = j$ or $g(i) = j$. The three statistics are:

- $u(f, g)$ is the number of i such that $f(i) = i$,
- $v(f, g)$ is the number of i such that $g(i) = i$,
- $w(f, g)$ is the number of i such that $f(i) = n$ plus the number of j such that $g(j) = n$.

The we have $f_n(a, b, c) = \sum_{(f,g)} a^{u(f,g)-1} b^{v(f,g)-1} c^{w(f,g)-2}$. There is an obvious symmetry in a and b but it seems to be different from the symmetry which is apparent on the permutation tableaux.

   If we examine the general case where $q \neq 1$, it appears that $\langle W | (qED + cD + cE)^n | V \rangle$ is not symmetric in α and γ. To obtain a refinement of $f_n(a, b, c)$ or $(\alpha\beta\gamma)^n f_n(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$ it might be interesting to consider the continuous dual q-Hahn polynomials $R_n(x; a, b, c \mid q)$, defined in (21). Let $g_n(a, b, c, q)$ be the nth moment of the polynomials $R_n(\frac{x}{2} - 1; a, b, c \mid q)$. Let

$$\tilde{a} = (1 - q)a - 1, \qquad \tilde{b} = (1 - q)b - 1, \qquad \text{and} \quad \tilde{c} = (1 - q)c - 1.$$

Then we have that $g_n(\tilde{a}, \tilde{b}, \tilde{c}, q)(1 - q)^{-2n}$ is a polynomial, symmetric in a, b and c, which specializes to $f_n(a, b, c)$ when $q = 1$. This just confirms that these q-Hahn polynomials are a q-analog of the Hahn polynomials. Unfortunately the polynomial $g_n(\tilde{a}, \tilde{b}, \tilde{c}, q)(1 - q)^{-2n}$ contains negative terms. However we have the following result when one parameter is set to one, for example $a = 1$.

Theorem 6. *The moment $g_n(-q, \tilde{b}, \tilde{c}, q)(1 - q)^{-2n}$ is the generating function for staircase permutation tableaux of length $2n + 2$, where q counts the superfluous 1's, b counts the number of 1's in the first row except one of them, and c counts the number of unrestricted rows except the first row.*

Proof. This generating function of permutation tableaux is $\langle W | (DE)^n | V \rangle$ where D and E are the solution of the PASEP Matrix Ansatz. By writing the product explicitely we can check that $M = DE$ has coefficients:

$$(1 - q)M_{i,i} = (1 - \tilde{\alpha}q^i)(1 - \tilde{\beta}q^i) + (1 - \tilde{\alpha}\beta q^i)(1 - q^{i+1}),$$

$$(1 - q)M_{i+1,i} = (1 - q^{i+1})(1 - \tilde{\beta}q^{i+1}),$$

$$(1 - q)M_{i,i+1} = (1 - \tilde{\alpha}\tilde{\beta}q^i)(1 - \tilde{\alpha}q^{i+1}).$$

Hence $\langle W | (DE)^n | V \rangle$ is the nth moment of an orthogonal sequence where the three-term relation can be derived from the coefficients of the matrix M, and by comparing it with the coefficients of the recurrence (21) we can derive the result. $\square$

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LIAFA, CNRS and Université Paris-Diderot, Case 7014, 75205 Paris Cedex 13, France

LRI, université Paris-Sud, 91405 Orsay Cedex, France

University of California, Berkeley, CA 94720, USA
E-mail address: corteel@liafa.jussieu.fr, josuat@lri.fr, williams@math.berkeley.edu