BIJECTIONS BETWEEN PATTERN-AVOIDING FILLINGS OF YOUNG DIAGRAMS

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Abstract. The pattern-avoiding fillings of Young diagrams we study arose from Postnikov's work on positive Grassman cells. They are called ⅃-diagrams, and are in bijection with decorated permutations. Other closely-related fillings are interpreted as acyclic orientations of some bipartite graphs. The definition of the diagrams is the same but the avoided patterns are different. We give here bijections proving that the number of pattern-avoiding filling of a Young diagram is the same, for these two different sets of patterns. The result was obtained by Postnikov via a recurrence relation. This relation was extended by Spiridonov to obtain more general results about other patterns and other polyominoes than Young diagrams, and we show that our bijections also extend to more general polyominoes.

1. Introduction

In his work on the combinatorics of the totally non-negative part of the Grassmanian and its cell decomposition, Postnikov [4] introduced some diagrams called ⅃-diagrams.

Definition 1. Let λ be a Young diagram, in English notation. A *⅃-diagram* T of shape λ is a filling of every entry of λ with a 0 or a 1, such that for any 0 in T, all entries to its left (in the same row) are 0's, or all entries above it (in the same column) are 0's.

This definition is equivalent to the following pattern-avoidance condition: a diagram filled with 0's and 1's is a ⅃-diagram, if and only if it avoids the patterns $\begin{smallmatrix}1&1\\1&0\end{smallmatrix}$ and $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$. The definition of "pattern-avoidance" is the following:

Definition 2. Let λ be a Young diagram. We call *diagram* of shape λ a filling of entries of λ with 0's and 1's. For every square matrix M of size 2, we say that a diagram D of shape λ *avoids* the pattern M if there is no submatrix of D equal to M.

In [4], Postnikov gives bijections between ⅃-diagrams, and various combinatorial objects: decorated permutations, some matroids called positroids, Grassmann necklaces. He also shows that the number of ⅃-diagram of shape λ is equal to the number of acyclic orientations of some graph G_λ. For a Young diagram $\lambda \subset (n-k)^k$, the graph G_λ is the bipartite graph on the vertices $1 \dots k$ and $1' \dots (n-k)'$, with edges (i,j') corresponding to cells (i,j) of the Young diagram λ.

The proof goes recursively. Let $f_\lambda(j)$ be the number of ⅃-diagrams of shape λ filled with j ones and let $F_\lambda(q)$ be the polynomial $\sum_k f_\lambda(j)q^j$. The polynomial $F_\lambda(q)$ satisfies a very simple recurrence derived by Williams in [8] for a fixed corner and generalized by Postnikov [4]. Indeed let us pick a corner box x of the Young diagram of shape λ. Let $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$ and $\lambda^{(4)}$, be the Young diagrams obtained from λ by removing, respectively, the box x, the row containing x, the column containing x, the column and the row containing x. Then it is easy to see that

$F_\lambda(q) = 1$ if $|\lambda| = 0$ and

$$F_\lambda(q) = qF_{\lambda^{(1)}}(q) + F_{\lambda^{(2)}}(q) + F_{\lambda^{(3)}}(q) - F_{\lambda^{(4)}}(q) \tag{1}$$

otherwise.

Let $\chi_\lambda(t)$ be the chromatic polynomial of the graph G_λ. Postnikov [4] establishes that $\chi_\lambda(t) = 1$ if $|\lambda| = 0$ and

$$\chi_\lambda(t) = \chi_{\lambda^{(1)}}(t) - \chi_{\lambda^{(2)}}(t) - \chi_{\lambda^{(3)}}(t) + t\chi_{\lambda^{(4)}}(t) \tag{2}$$

otherwise (the identity in [4] is slightly different because of our conventions: Postnikov only deletes edges and no vertex of the graphs, whereas here our graphs $G_{\lambda^{(i)}}$ have respectively n, $n-1$, $n-1$ and $n-2$ vertices for $i = 1,2,3,4$). According to [6], the value $(-1)^n\chi_\lambda(-1)$ equals the number ao_λ of acyclic orientations of the graph G_λ. Specializing equation (1) at $q = 1$ and (2) at $t = -1$, one obtains that ao_λ and $F_\lambda(1)$ satisfy the same recurrence and have the same boundary condition, and are therefore equal for any λ.

The acyclic orientations of the graph G_λ are in bijection with some fillings of the diagram of λ with 0's and 1's which we call *X-diagrams*. A diagram is said to be an *X-diagram*, if it avoids the patterns $\substack{10\\01}$ and $\substack{01\\10}$. This bijection is very simple: the filling of a cell (i,j) is 0 (resp. 1) if and only if the orientation of the edge (i,j') is $i \to j'$ (resp. $i \leftarrow j'$). One can check that the pattern-avoidance for the X-diagrams is equivalent to the cycle avoidance for the orientation. Details can be found in [4, 5].

Therefore X-diagrams and ⅃-diagrams are equivalent in the following sense:

Proposition 1. [4, 5] *For every Young diagram λ, the number of X-diagrams of shape λ is equal to the number of ⅃-diagrams of shape λ.*

Postnikov [4] and Burstein [1] noticed that ⅃-diagrams are also equivalent to the diagrams avoiding $\substack{01\\11}$ and $\substack{11\\11}$. Spiridonov [5] then made an extensive study on which pairs of patterns are equivalent, for more general polyominoes than Young diagrams. He proved that many other patterns follow similar recurrence relations as in [4], and he proved the equivalence of X-diagrams and ⅃-diagrams for a whole class of polyominoes, containing for example skew shapes (in French notation).

In this article, we give bijective proofs for the equivalence of the two main families of diagrams:

- the ⅃-diagrams which avoids the patterns $\substack{11\\10}$ and $\substack{01\\10}$,
- and the X-diagrams which avoids the patterns $\substack{10\\01}$ and $\substack{01\\10}$.

This is firstly done in the case of Young diagram, but we extend the bijection to other polyominoes.

Definition 3. A polyomino S is ⅃-*complete* if satisfies the following conditions: for any $i < j$ and $k < \ell$, if $(j,k), (j,\ell), (i,\ell)$ are cells in S then the cell (i,k) is also in S. Otherly said, for any three cells arranged as a ⅃, there is a 2×2-submatrix of S containing these three cells.

See Figure 1 further in the article for an illustration of our conventions. This is equivalent to the recursive condition "2×2-connected bottom-right CR-erasable" from [5]. We give a bijection between X-diagrams and ⅃-diagrams for any ⅃-complete polyomino. There are some parameters preserved by this bijection.

Definition 4. We call *zero-row* (resp. *zero-column*) a row (resp. column) filled with 0's. A row of a diagram is *restricted*, if it contains a 0 having a 1 above it in the same column. Otherwise it is called *unrestricted*.

The first bijection we describe preserves every zero-row and zero-column. With a slightly different construction, we have a bijection preserving every zero-column and unrestricted row.

As the bijection preserves zero-columns, we get a direct corollary linking *permutation tableaux* [1, 2, 7] (which are Ꞁ-diagrams with no zero column) and X-diagrams with no zero-column:

Corollary 1. *There exists a bijection between X-diagrams of shape λ with no zero-column and k unrestricted rows and permutation tableaux of shape λ with k unrestricted rows.*

The *length* of a diagram is its number of columns plus its number of rows. Permutation tableaux of length n with k unrestricted rows (allowing rows of size 0) are known to be enumerated by the Stirling numbers of the first kind [2]. We immediately get that:

Corollary 2. *The number of X-diagrams of length n with no zero column and k unrestricted rows is equal to the number of permutations of $\{1, \dots, n\}$ with k cycles.*

This article is organized as follows. Section 2 contains elementary results that are direct consequences of the previous definitions. In Section 3 we define the main bijection of this article, which answers the original problem given by Postnikov. In Section 4, we extend this bijection to more general polyominoes than Young diagrams, proving bijectively some results of Spiridonov. In Section 5, we discuss various possible generalizations of the present work.

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2. First results on the structure of X-diagrams

We give here several easy but useful lemmas to characterize X-diagrams. Moreover this will give a raw outline of the methods we use throughout this article.

Lemma 1. *For any diagram T, the following conditions are equivalent:*

(1) *T is an X-diagram.*
(2) *For every rectangular submatrix M of T, two rows of M having the same number of 1's are equal.*
(3) *For any rectangular submatrix of T having two rows, the index set of 1's in the first row contains, or is contained in, the index set of 1's in the second row.*
(4) *For every rectangular submatrix M of T, the following condition holds: if an entry of M contains a 0 and belongs to a row with a maximal number of 1's, then this entry belongs to a zero-column.*

Proof. We show that the first condition implies the last one, all other implications are proved with similar arguments. So let M be a rectangular submatrix of an X-diagram (in particular M is also an X-diagram). Let $x, y > 0$ be such that $M(x, y) = 0$ and the xth row of M has a maximal number of 1's. Now, if there is z such that $M(z, y) = 1$, by the pattern-avoidance condition we have $M(x, t) = 1 \Longrightarrow M(z, t) = 1$ for any $t \neq y$. This means that the zth row has strictly more 1's than the xth, which contradicts the definition of x. ☐

From these characterizations, we obtain the following statement.

Corollary 3. *Let T be an X-diagram of rectangular shape.*

- *If T' is obtained from T by permuting rows, T' is also an X-diagram.*
- *If T' is obtained from T by replacing a row with a copy of another row, T' is also an X-diagram.*
- *If T and T' have the same set of distinct rows, T' is also an X-diagram.*

There is also a similar statement with columns instead of rows. Note that an X-diagram of rectangular shape, by permuting its rows and its columns, can be arranged such that the 1's and the 0's are two complementary Young diagrams.

3. The bijection Φ between Ⅎ-diagrams and X-diagrams

This bijection is defined recursively with respect to the number of rows. We first define a bijection ϕ between X-diagrams and *mixed diagrams* (see definition below). Once ϕ is defined there is a short recursive definition of Φ.

Throughout this section we only consider diagrams whose shape is a Young diagram in English notation. We use the convention that the top-left corner is the entry $(1, 1)$, the bottom-left corner is the entry $(k, 1)$, the top-right corner is the entry $(1, \lambda_1)$, and so on. For any diagram T, we denote by $T(i, j) \in \{0, 1\}$ the number in the entry (i, j) of T.

3.1. The bijection ϕ between X-diagrams and mixed diagrams.

Throughout this section, let λ be a Young diagram, in English notation, and k its number of rows. The row lengths are a weakly decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_k > 0$.

Definition 5. A *mixed diagram* of shape λ is a diagram of shape λ with the following properties:

- The $k - 1$ top rows are an X-diagram,
- For any 0 in the kth row, there is no 1 above it in the same column, or there is no 1 to its left in the same row.

The goal of this section is to prove the following:

Proposition 2. *There exists a bijection ϕ between X-diagrams of shape λ and mixed diagrams of shape λ. Moreover for any X-diagram T, T and $\phi(T)$ have the same set of zero-columns and zero-rows.*

The essential tool for this bijection is given by the following definition.

Definition 6. We call *pivot column* of an X-diagram of shape λ, a column among the λ_k first ones (*i.e.* it is a column of maximal size k) such that:

- there is a 1 in bottom position,
- it contains a maximal number of 0's among columns satisfying the previous property,
- it is in leftmost position among columns satisfying the previous two properties.

If such a column exists, it is uniquely defined (by the third property). There is no such column only in the case where the bottom row is a zero-row.

We now describe the bijection ϕ. Let T be an X-diagram of shape λ. In the case where there is no pivot column in T, or equivalently the bottom row is a zero-row, T is also a mixed-diagram and we set $\phi(T) = T$. Otherwise, we define $\phi(T)$ as the result of the following column-by-column transformation of T, where j is the index of its pivot column:

- In a column of index i with $i < j$, we change the bottom entry into a 0.
- A column of index i with $j < i \leq \lambda_k$, which is identical to the pivot column, is changed into a column having a 1 in bottom position and 0's elsewhere.
- In a column of index i with $j < i \leq \lambda_k$, which is not identical to the pivot column and is not a zero-column, we change the bottom entry into a 1.

Note that this transformation only modifies the λ_k leftmost columns of T. For example, an X-diagram T and its image $\phi(T)$ are given by:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&0&0&\mathbf{0}&1&0&0&0&0&0&0&0\\ \hline 1&1&1&\mathbf{1}&1&1&1&0&1&1&1&0\\ \hline 1&1&0&\mathbf{0}&1&0&1&0&0&1&0&0\\ \hline 1&0&0&\mathbf{0}&0&0&0&0&0&0&0\\ \cline{1-11} 1&1&0&\mathbf{1}&1&0&1&0&1\\ \cline{1-9}\end{array}\ , \qquad \phi(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&0&0&\mathbf{0}&1&0&0&0&0&0&0&0\\ \hline 1&1&1&\mathbf{1}&1&1&1&0&0&1&1&0\\ \hline 1&1&0&\mathbf{0}&1&0&1&0&0&1&0&0\\ \hline 1&0&0&\mathbf{0}&0&0&0&0&0&0&0\\ \cline{1-11} 0&0&0&\mathbf{1}&1&1&1&0&1\\ \cline{1-9}\end{array}\ .$$

Here the pivot column is the 4th one, with bold numbers. We can see that the 9th column is identical to the pivot column. So it is replaced with a column having a single 1 in bottom position.

Now that the map is described, we prove that ϕ is indeed a bijection. We begin with two lemmas that will be helpful to define ϕ^{-1}.

Lemma 2. *Let $U = \phi(T)$, and $j > 0$. Then the following conditions are equivalent:*

- *The pivot column of T is the jth column,*
- *We have $U(k, j) = 1$, and $U(k, i) = 0$ for any $i < j$.*

Proof. Each statement is either not satisfied by any j, or satisfied by a unique j. So we just have to prove the direct implication. If the pivot column of T is the jth column, it is not modified so we have $U(k, j) = 1$. And for any $i < j$, to obtain U we put a 0 in the entry (k, i) of T. Thus, if the first condition is true, so is the second, hence the equivalence. □

Lemma 3. *Let j be the index of the pivot column of T. Then for any $i < j$ the following conditions are equivalent:*

- *The $k - 1$ top entries of the ith column contain strictly less 0's than the $k - 1$ top entries of the jth column,*
- *$T(k, i) = 1$.*

Proof. If the $k - 1$ top entries of the ith column contains strictly more 0's than the $k - 1$ top entries of the jth column, we have $T(k, i) = 0$ since otherwise it would contradict the second point in the definition of the pivot column (maximum number of 0's).

If the $k-1$ top entries of the ith column contains exactly as many 0's as the $k-1$ top entries of the jth column, then $T(k, i) = 0$ since otherwise it would contradict the third point in the definition of the pivot column (recall that two columns having the same number number of 1's are identical in an X-diagram).

Eventually the last case to check is when the $k-1$ top entries of the ith column contains strictly less 0's than the $k-1$ top entries of the jth column. Then $T(k,i) = 1$ since otherwise there would be an occurrence of the pattern $\begin{smallmatrix}10\\01\end{smallmatrix}$. □

The same method also gives a proof of:

Lemma 4. *Let j be the index of the pivot column of T. Then for any i such that $j < i \leq \lambda_k$ we have:*

- *If the $k-1$ top entries of the ith column contain strictly more 1's than the $k-1$ top entries of the jth column, then $T(k,i) = 1$.*
- *If the $k-1$ top entries of the ith column contain strictly less 1's than the $k-1$ top entries of the jth column, then $T(k,i) = 0$.*

Remark. In the previous lemma there is nothing about the case of equality. Indeed, it is possible that some columns are identical to the pivot column, but some other columns are identical except that the bottom entry is different. To prove that ϕ is a bijection it is important to note that in any case, two differently-filled columns of T give rise to differently-filled columns in $\phi(T)$. An essential remark is that if there is in T a column with a single 1 in bottom position and 0's elsewhere, then it is necessarly identical to the pivot column, so that there is no confusion possible.

With these lemmas we have all information to define the inverse map of ϕ. Let U be a mixed diagram. If the bottom row is a zero-row, U is also a X-diagram and we have $\phi(U) = U$. Otherwise, we can find an X-diagram T such that $\phi(T) = U$. Indeed, T can be defined from U with the following column-by-column transformation, where j is the index of the leftmost column in U having a 1 in bottom position:

- In a column of index i such that $i < j$, and where the $k-1$ top entries contains strictly less 0's than the jth column, the bottom entry is replaced with a 1.
- A column of index i such that $j < i \leq \lambda_k$, which has a 1 in bottom position and 0's elsewhere, is replaced with a copy of the jth column.
- Eventually, consider a column of index i such that $j < i \leq \lambda_k$, which does not satisfy the previous condition. If the $k-1$ top entries of this column contains at most as many 1's as the jth column, then the last entry is replaced with a 0.

This transformation is the step-by-step inverse process of the transformation defining ϕ. So when composing the two maps, with ϕ in first (resp. in last), we get the identity on X-diagrams (resp. mixed diagrams). So we are able to prove Proposition 2.

Proof. It remains only to check that T and $U = \phi(T)$ have the same zero-columns and zero-rows. The fact about columns is clear since the algorithm is a column-by-column transformation. Now, let $i < k$ be the index of a non-zero row. The ith row of T may be modified by the algorithm only when $T(i,j) = 1$ where j is the index of the pivot column (when transforming a column equal to the pivot column into a column having a single 1). But then we have also $U(i,j) = 1$, so the ith row of U is non-zero. Hence T and U have the same set of zero-rows. □

3.2. **The bijection Φ between X-diagrams and Ⅎ-diagrams.** Now, we describe the main bijection of this article. Given an X-diagram T of shape λ, $\Phi(T)$ is defined by transforming T with the following algorithm:

For i from k to 1, replace the i top rows of T with their image by ϕ.

Equivalently, we obtain $\Phi(T)$ by applying recursively Φ to the $k-1$ first rows of $\phi(T)$.

Lemma 5. *If the ith column of T is a zero-column, then the ith column of $\Phi(T)$ is also a zero-column.*

Proof. It is a direct consequence of the fact that ϕ preserves every zero-column. □

Lemma 6. *The diagram $\Phi(T)$ is a ⅂-diagram.*

Proof. We obtain $\Phi(T)$ by applying Φ to the $k-1$ first rows of $\phi(T)$, so we can prove the result recursively with respect to the number of rows. For $k=1$, every diagram with just one row is an X-diagram and a ⅂-diagram, and Φ is the identity in this case.

For $k>1$ and under the recurrence assumption, a 0 in the $k-1$ top rows of $\Phi(T)$ cannot have a 1 to its left and a 1 above it. Now, suppose there is a 0 in position (k,i), (*i.e.* in the kth row) having a 1 to its left. We have to prove that the ith column of $\Phi(T)$ is a zero-column.

Since the kth row of $\Phi(T)$ is the same as in $\phi(T)$, we have $\phi(T)(k,i)=0$ and this 0 has a 1 to its left. But since $\phi(T)$ is a mixed diagram, it implies that the ith column of $\phi(T)$ is a zero-column. And since Φ preserves every zero-column, it implies that the ith column of $\Phi(T)$ is also a zero-column. This completes the proof. □

Proposition 3. *The map Φ is a bijection between X-diagrams of shape λ and ⅂-diagrams of shape λ. Moreover this bijection preserves the set of zero-rows and the set of zero-columns.*

Proof. At every step, the upper part of T still avoids the patterns $\begin{smallmatrix}10\\01\end{smallmatrix}$ and $\begin{smallmatrix}01\\10\end{smallmatrix}$, so that it makes sense to take its image by ϕ. Now, for every ⅂-diagram U of shape λ, define $\Phi^{-1}(U)$ by transforming U with the following algorithm:

For i from 1 to k, replace the i top rows of U with their image by ϕ^{-1}.

Equivalently, we can replace the $k-1$ top rows of U with their image by Φ^{-1} and then apply ϕ^{-1} to obtain $\Phi^{-1}(U)$. With Lemma 6 and knowing that ϕ is bijective, it is clear that $\Phi^{-1}\circ\Phi$ is the identity on X-diagrams and $\Phi\circ\Phi^{-1}$ is the identity on ⅂-diagrams, so that Φ is bijective and Φ^{-1} as we defined it is indeed its inverse. Moreover, the fact that ϕ and ϕ^{-1} preserve every zero-column, directly implies the same fact for Φ and Φ^{-1}. Similarly, the fact that ϕ and ϕ^{-1} preserves the set of zero-rows implies the same property for Φ and Φ^{-1}. □

3.3. **Examples.** This example contains no zero-column or zero-row, since they would be unchanged at any step of the process. We start from an X-diagram T, and compute step by step its image by Φ. So each step gives an example of a diagram and its image by ϕ. The thick lines indicate where is the upper-left part, which is to be replaced with its image by ϕ. Bold numbers indicate the pivot column. Suppose that we have:

$T=$

1	1	1	1	1	1
0	0	0	1	0	0
1	1	0	1	0	
1	0	0	1		
1	1	1			

.

In the first step, the index of the pivot column is $j = 3$. We have to put 0's in the $j - 1$ leftmost entries of the bottom row. Since there is no column to the right of the pivot column, there is nothing else to do. So the transformation is:

1	1	**1**	1	1	1
0	0	**0**	1	0	0
1	1	**0**	1	0	
1	0	**0**	1		
1	1	**1**			

$\longrightarrow$

1	1	1	1	1	1
0	0	0	1	0	0
1	1	0	1	0	
1	0	0	1		
0	0	1			

In the second step, the index of the pivot column is $j = 1$. So the transformation is:

1	1	1	1	1	1
0	0	0	1	0	0
1	1	0	1	0	
1	0	0	1		
0	0	1			

$\longrightarrow$

1	1	1	1	1	1
0	0	0	1	0	0
1	1	0	1	0	
1	1	1	1		
0	0	1			

In the third step, we have $j = 1$. Here the second column is identical to the pivot column so the transformation is:

1	1	1	1	1	1
0	0	0	1	0	0
1	1	0	1	0	
1	1	1	1		
0	0	1			

$\longrightarrow$

1	0	1	1	1	1
0	0	0	1	0	0
1	1	1	1	1	
1	1	1	1		
0	0	1			

In the fourth step, the index of the pivot column is $j = 4$.

1	0	1	**1**	1	1
0	0	0	**1**	0	0
1	1	1	1	1	
1	1	1	1		
0	0	1			

$\longrightarrow$

1	0	1	1	1	1
0	0	0	1	1	1
1	1	1	1	1	
1	1	1	1		
0	0	1			

So finally, we have:

$\Phi(T) =$

1	0	1	1	1	1
0	0	0	1	1	1
1	1	1	1	1	
1	1	1	1		
0	0	1			

which is easily checked to be a Ⅎ-diagram.

4. Generalization to polyominoes other than Young diagrams

Now, besides Young diagrams in English notation, we consider more general polyominoes. In this section we show that X-diagrams and Ⅎ-diagrams are equivalent for any Ⅎ-*complete* polyomino (see Definition 7).

First we generalize the bijection ϕ for diagrams whose shape is a Young diagram in French notation. Then we derive the generalization of the bijection Φ for any Ⅎ-complete polyomino, and we end this section with some examples.

Definition 7. A polyomino S is Ⅎ-*complete* if satisfies the following conditions: for any $i < j$ and $k < \ell$, if $(j, k), (j, \ell), (i, \ell)$ are cells in S then the cell (i, k) is also in S. In other words, for any three cells arranged as a Ⅎ, there is a 2×2-submatrix of S containing these three cells.

This is equivalent to the recursive condition "2×2-connected bottom-right CR-erasable" from [5]. The first examples of ⅃-complete polyominoes are Young diagrams in English notation, so that this section is indeed a generalization of the previous one. Other examples are skew shapes in the french convention (also known as paralellogram polyominoes), as in the right part of Figure 1.

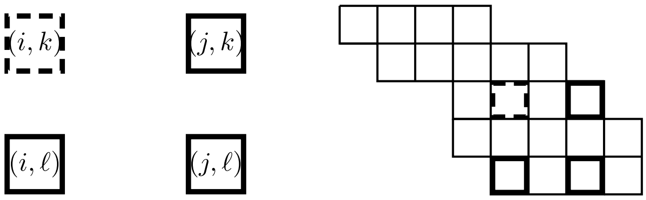


FIGURE 1. ⅃-completeness. If a ⅃-complete polyomino contains the three squares with continuous thick borders, it must contain the square with dashed thick borders too. On the right, we have an example of skew shape (in French convention).

4.1. **The generalization of ϕ.** We now suppose that S is a young diagram, in French notation. Let r be the number of distinct parts of the corresponding partition. Let $c_1, \dots, c_r$ be the top right corners of S, sorted by growing abscissas. And for any top-right corner c_i, let R_i be the rectangular subset of S formed by all cells that are in the bottom-left quarter-plane of origin c_i. See Figure 2 for an example. For any X-diagram T of shape S we denote by $T|_{R_i}$ the subdiagram of T obtained by selecting the cells in R_i.

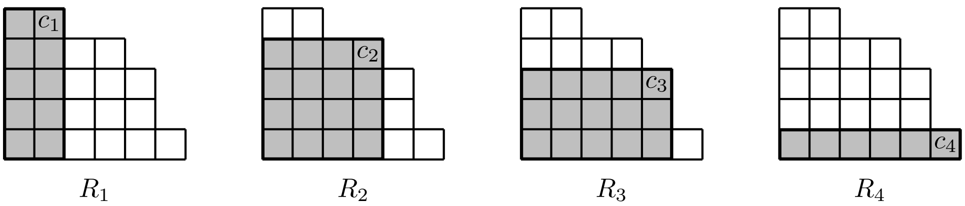


FIGURE 2. Example of rectangles R_i.

Definition 8. We call *pivot column* of a diagram T of shape S a column of index j such that:

- there is an i, such that the jth column of $T|_{R_i}$ is the pivot column of $T|_{R_i}$, and such that j is strictly greater than the column index of c_{i-1} (the last condition is automatically satisfied if $i = 1$).
- it is in righmost position among columns satisfying the first property (in other words, the first property is satisfied with the greatest i possible).

If such a column exists, it is uniquely defined. There is no such column only in the case where the bottom row is a zero-row.

The last affirmation is not as obvious as in the previous section, so we give some precisions. If there is a 1 in the bottom row, we can consider the smallest i such that the bottom row of $T|_{R_i}$ contains a 1, and then the pivot column of $T|_{R_i}$ satisfies the first condition, so there is a pivot column in T. Reciprocally, if there is a pivot column, it contains a 1 in bottom position so the bottom row is not a zero-row. Hence as announced, there is no pivot column in T only in the case where the bottom row is a zero-row.

Now, if we have an X-diagram T the mixed diagram $\phi(T)$ is defined as in the previous case by a column-by-column transformation, but with one difference. Indeed, in the previous case a column identical to the pivot column was transformed into a column having a single 1 in bottom position. Now, the modified columns may have different size, and the condition is the following. If a column to the right of the pivot column is identical to the pivot column except that some top cells are removed, it is transformed into a column having a single 1 in bottom position. This is illustrated in the following example:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 \\ \cline{1-2} 1 & 0 & 0 & \mathbf{0} \\ \cline{1-4} 1 & 1 & 1 & \mathbf{1} & 0 & 1 & 1 \\ \cline{1-7} 1 & 0 & 1 & \mathbf{0} & 0 & 0 & 0 & 0 \\ \cline{1-8} 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & \mathbf{1} & 0 & 1 & 0 & 0 & 1 \\ \hline \end{array}\,, \qquad \phi(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 \\ \cline{1-2} 1 & 0 & 0 & \mathbf{0} \\ \cline{1-4} 1 & 1 & 1 & \mathbf{1} & 0 & 0 & 1 \\ \cline{1-7} 1 & 0 & 1 & \mathbf{0} & 0 & 0 & 0 & 0 \\ \cline{1-8} 1 & 1 & 1 & \mathbf{1} & 1 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}\,.$$

Here, the respective pivot columns of $T|_{R_1}, \dots, T|_{R_5}$ have indices 1, 4, 4, 4, and 1. The indices i satisfying the first condition of Definition 8 are 1 and 2. So the pivot column of R is the one of T_{R_2}, *i.e.* the 4th one. Now, we can see that the 6th and 9th column of T are identical to the pivot column, except that some top cells are removed. So the 6th column of $\phi(T)$, and the 9th column too, have a single 1 in bottom position.

In the previous section, we saw that two differently-filled columns of T give rise to two differently-filled columns in $\phi(T)$, provided they are both strictly to the left (or to the right) of the pivot column. This is also true in the present case. The main point is the following lemma, where we really use Definition 8.

Lemma 7. *Let j be the index of the pivot column of T. Let k and ℓ be such that $j < k < \ell$, the kth column and the ℓth column have the same size, and they are identical except that their bottom entries are different. Then one of the kth and ℓth columns is identical to the pivot column (except that some top cells are removed).*

For example, the 6th and 7th columns in the previous example T satisfy this condition and indeed the 6th column is identical to the pivot column, except that the top cell is removed.

Proof. Note that if the kth and ℓth columns have the same size as the pivot column then it is just a consequence of Lemma 4. Now, we suppose that the kth and ℓth columns are strictly smaller than the pivot column. Let i be the size of the kth and ℓth columns. We distinguish two cases.

- We suppose first that the $i-1$ top cells of the kth column contain strictly more 1's than the corresponding $i-1$ cells of the pivot column. Then, the bottom entry of the kth and ℓth columns should be a 1, otherwise there would be an occurrence of $\substack{01\\10}$. This contradicts the fact that these entries are different.
- We suppose then that the $i-1$ top cells of the kth column contain strictly more 0's than the corresponding $i-1$ cells of the pivot column (say this number of 0's in the pivot column is x). Then, let h be such that the kth and ℓth columns are columns of the rectangle R_h. The pivot column of $T|_{R_h}$ contains at least x 0's. Because of the second point in the Definition 8, it is possible to find in $T|_{R_{h-1}}$ a column having a 1 in bottom position and at least x 0's in the i bottom entries. If this column is also the pivot column of T there is a contradiction, otherwise we can find a column $T|_{R_{h-2}}$ having a 1 in bottom position and at least x 0's in the i bottom entries, so that after some steps we eventually obtain a contradiction.

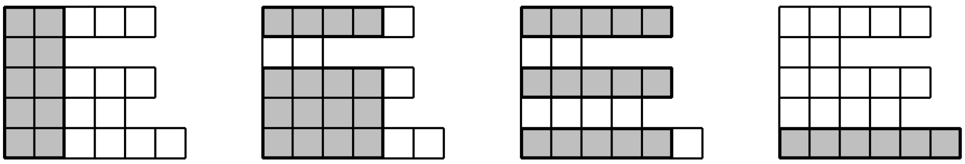
This shows that the $i-1$ top entries of the kth or ℓth column contain as many 0's as the corresponding entries in the pivot column. One of these two columns has a 1 in bottom position, and this one is identical to the i bottommost entries of the pivot column. □

Apart from the situation in the previous lemma, we can see that the bottom entry of any column is determined by the other entries and the fact that this column is to the left or to the right of the pivot column. This is similar to the Lemmas 3 and 4. So we have proved:

Proposition 4. *Proposition 2 remains true when the shape of the diagrams is a Young diagram in French notation.*

The last step of this section is to see that the bijection ϕ for Young diagrams in French notation has an immediate generalization to other polyominoes: they are the ones obtained by permuting the $k-1$ top rows of a Young diagram in French notation with k rows. Let S be such a polyomino.

In this case, we also define rectangular subsets R_i by the following construction. Let us consider the columns of S that contain the righmost cell of some row. We denote by $C_1, \dots, C_r$ these columns, numbered from left to right. Then let R_i be the set of cells that are to the left (in the same row) of some cell of C_i. For an example see Figure 3.



R_1 R_2 R_3 R_4

FIGURE 3. Example of the rectangles R_i. The columns C_i have indices 2,4,5, and 6.

There is also a bijection between X-diagrams and mixed diagrams of shape S, preserving any zero-row or zero-column. It is defined exactly the same way as in the case where S is a Young diagram in French notation.

4.2. **The generalization of Φ.** As in the previous case, Φ is recursively defined with respect to the number of rows of the diagrams. The only thing to check is that the polyominoes we can obtain are precisely the ⅃-complete ones.

Lemma 8. *For any polyomino S the following conditions are equivalent:*

- *Every column of S intersects the bottom row, and S is ⅃-complete.*
- *The polyomino S is obtained by permuting the $k-1$ top rows of a Young diagram in French notation with k rows.*

Proof. An example is given in Figure 3. Suppose that the first condition is true. Then the bottom row is a set of contiguous cells, and it is of maximal size. Since the polyomino is ⅃-complete, we obtain that each other row is also a set of contiguous cells and has the same left extremity as the bottom row. This implies the second condition. The converse is obtained as easily. □

For any ⅃-complete polyomino S, we can consider the subset S' of all cells that are above a cell of the bottom row. Then S' satisfies the condition of the previous Lemma. Then for any X-diagram T of shape S, we define $\phi(T)$ by replacing $T|_{S'}$

with its image by ϕ. And $\Phi(T)$ is defined recursively as in the previous case, by replacing the $k-1$ top rows of $\phi(T)$ with their image by Φ. Although the details may seem messy because of the more general polyominoes, the ideas are really the same as in the previous case.

Proposition 5. *Proposition 3 remains true with diagrams whose shape is a ⅃-complete polyomino.*

An example of a ⅃-complete polyomino S, the subset S' and the rectangles R_i is given in Figure 4.

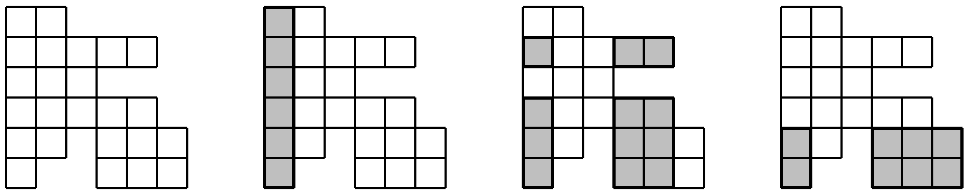


FIGURE 4. Example of a ⅃-complete polyomino S, and rectangles R_i. The subset S' is obtained by removing the second and third column.

5. Overview of related results and conclusion

Many questions may arise about how to generalize this work, as can be seen in [5]. We can consider many kinds of avoided patterns, and each of them raises the question of which classes of polyominoes have interesting properties. In this section, we sketch some consequences of the results of the previous sections and other related results.

First we show that, as announced in the introduction, we can obtain a bijection between X-diagrams and ⅃-diagrams preserving the set of zero-columns and the set of unrestricted rows. This is done by the following modification of the definition of the pivot column:

Definition 9. We call *pivot column* of an X-diagram T a non-zero column among the λ_k first ones (in other words, it is a column of maximal size) such that:

- there is a 0 in bottom position,
- it has a maximal number of 1's among columns satisfying the previous property,
- it is in leftmost position among columns satisfying the two previous properties.

If such a column exists, it is uniquely defined (by the third property). There is no such column only in the case where the bottom row is unrestricted.

With this new definition, we can define new bijections ϕ_2 and Φ_2 the same way we defined ϕ and Φ. Indeed, for any X-diagram, if the bottom row is a zero-row we let $\phi_2(T) = T$, otherwise $\phi_2(T)$ is obtained with the column-by-column transformation described in the three points after Definition 6 (only the definition of the pivot column is changed). Then Φ_2 is defined recursively with respect to the number of rows from Φ_2, as was the case from ϕ and Φ. We obtain the following:

Proposition 6. *For any ⅃-complete polyomino S, the map Φ_2 is a bijection between X-diagrams of shape S and ⅃-diagrams of shape S, and this bijection preserves the zero-columns and the restricted rows.*

Now, we consider a column-convex polyomino S such that the top of each column are at the same height, as in Figure 5 (such objects are sometimes called *stalactite polyominoes*). It is ⅃-complete. If we have an X-diagram of shape S, we can permute the columns of the polyomino and get another X-diagram. It is a consequence of the symmetry in the pattern pair $(\begin{smallmatrix}10\\01\end{smallmatrix}, \begin{smallmatrix}01\\10\end{smallmatrix})$. So with the previous bijection, we obtain that the number of ⅃-diagram of shape S only depends on the column lengths.

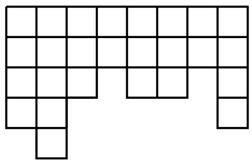


FIGURE 5. Example of a stalactite polyomino.

It is interesting to note that this is very close to some results of [3] (we thank Mireille Bousquet-Mélou for communicating this reference). Indeed, one of the objects studied in this reference are the fillings of stalactite polyominoes by 0's and 1's, avoiding a pattern which is the identity matrix of a given size, and with a maximal number of 1's. Jonsson shows that the number of such fillings only depends on the column lengths of the polyomino, which is a similarity with the case of ⅃-diagrams.

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