Constructing combinatorial operads from monoids

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Contents

Non-symmetric set-operads
  Definitions
  Examples of operads
Operads

A (non-symmetric set-)operad is a triple $(\mathcal{P}, \circ_i, 1)$ where:

- $\mathcal{P}$ is a graded set of the form $\mathcal{P} := \{ n \geq 1 | \mathcal{P}(n) \}$
- $\circ_i$ is a grafting application $\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \rightarrow \mathcal{P}(n + m - 1)$ defined for all $n, m \geq 1$ and $i \in [n]$
- $1$ is an element of $\mathcal{P}(1)$, called unit.

This data has to satisfy some relations.
Operads

A (non-symmetric set-)operad is a triple \((\mathcal{P}, \circ_i, 1)\) where:

- \(\mathcal{P}\) is a graded set of the form

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\mathcal{P} := \bigcup_{n \geq 1} \mathcal{P}(n),
\]

- \(\circ_i\) is a grafting application

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\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n + m - 1),
\]

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This data has to satisfy some relations.
Relations of operads

For all $x \in \mathcal{P}(n)$, $y \in \mathcal{P}(m)$, and $z \in \mathcal{P}(k)$, following relations must be satisfied.

**Associativity relation:**

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z),$$

for all $i \in \mathbb{[}n\mathbb{]}$ and $j \in \mathbb{[}m\mathbb{]}$.

**Commutativity relation:**

$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y,$$

for all $1 \leq i < j \leq n$.

**Unitarity relation:**

$$1 \circ_1 x = x = x \circ_i 1,$$

for all $i \in \mathbb{[}n\mathbb{]}$. 


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Trees and elements of operads

Element of $\mathcal{P}(n) \rightsquigarrow$ operator of arity $n$:

\[ x \] \quad 1 \text{ output}

\[ n \text{ inputs} \]
Trees and elements of operads

Element of $\mathcal{P}(n) \rightsquigarrow$ operator of arity $n$:

\[ \vdots \]

$n$ inputs

1 output

Operator of arity $n \rightsquigarrow$ planar rooted tree with $n$ leaves:

\[ \vdots \]

1

... $n$
Trees and elements of operads

Element of $\mathcal{P}(n)$ $\leadsto$ operator of arity $n$:

Operator of arity $n$ $\leadsto$ planar rooted tree with $n$ leaves:

Grafting application $\leadsto$ grafting of trees:
Trees and elements of operads

Element of $\mathcal{P}(n) \rightsquigarrow$ operator of arity $n$:

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\[(x \circ_i y)\]

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Trees and relations of operads

Associativity relation:

\[
(x \circ_i y) \circ_{i+j-1} z
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Associativity relation:

\[(x \circ_i y) \circ_{i+j-1} z \quad (y \circ_j z)\]

Commutativity relation:

Unitarity relation:
Trees and relations of operads

Associativity relation:

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Typical questions about operads

Let $\mathcal{P}$ be an operad. Usual questions about $\mathcal{P}$ consist in
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1. computing the dimensions of $\mathcal{P}$, that is the sequence

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Typical questions about operads

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1. computing the **dimensions** of $\mathcal{P}$, that is the sequence
   
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1. computing the dimensions of $\mathcal{P}$, that is the sequence

   $\#\mathcal{P}(1), \#\mathcal{P}(2), \#\mathcal{P}(3), \ldots$;

2. finding a set of generators of $\mathcal{P}$;

3. giving a presentation of $\mathcal{P}$ by generators and relations.
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Let \((\text{Assoc}, \circ_i, a_1)\) be the operad defined for all \(n \geq 1\) by

\[
\text{Assoc}(n) := \{a_n\},
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and for all \(n, m \geq 1\) and \(i \in [n]\) by

\[
a_n \circ_i a_m := a_{n+m-1}.
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a_1, \quad a_2, \quad a_3 = a_2 \circ_1 a_2,
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Assoc is generated by \(a_2\):

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a_1, \ a_2, \ a_3 = a_2 \circ_1 a_2, \ a_4 = (a_2 \circ_1 a_2) \circ_1 a_2, \ \ldots
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The generator \(a_2\) is subject to the relation

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a_2 \circ_1 a_2 = a_2 \circ_2 a_2.
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Assoc is generated by \(a_2\):

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The generator \(a_2\) is subject to the relation

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a_2 \circ_1 a_2 = a_2 \circ_2 a_2.
\]

Presentation by generators and relations:

\[
\text{Assoc} = \langle a_2 \mid a_2 \circ_1 a_2 = a_2 \circ_2 a_2 \rangle.
\]
The magmatic operad

Let \((\text{Mag}, \circ_i, \sqcup)\) be the operad defined for all \(n \geq 1\) by

\[
\text{Mag}(n) := \{ T : T \text{ binary tree with } n \text{ leaves} \},
\]

and for all \(n, m \geq 1\) and \(i \in [n]\) by

\[
S \circ_i T := \text{tree obtained by grafting } T \text{ on the } i\text{th leaf of } S.
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Example

\[
\begin{array}{c}
\begin{array}{c}
\circle & \circle \\
\square & \square \\
\end{array} & \begin{array}{c}
\circle & \circle & \circle \\
\square & \square & \square \\
\end{array} \\
1 & 4
\end{array}
\]

Dimensions : 1, 1, 2, 5, 14, 42, ...(Catalan numbers).

\text{Mag} is generated by (proof by induction on the arities).

There is no relation between the generator and itself (\text{Mag} is the free operad on one generator of arity 2).
The magmatic operad

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Example

\[
\begin{array}{ccc}
\text{Example} & \circ_4 & = \\
\begin{array}{c}
\text{Tree 1} \\
\text{Tree 2}
\end{array} & \begin{array}{c}
\text{Tree 3} \\
\text{Tree 4}
\end{array} & \begin{array}{c}
\text{Tree 5} \\
\text{Tree 6}
\end{array}
\end{array}
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\[
\begin{align*}
\circ_4 & \quad \text{Dimensions: } 1, 1, 2, 5, 14, 42, \ldots \text{(Catalan numbers).}
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\[
\begin{array}{ccc}
\circ \circ & \circ \circ & \circ \circ \\
\circ \circ & \circ \circ & \circ \circ \\
\circ \circ & \circ \circ & \circ \circ \\
\end{array}
\quad \circ_4 \quad \begin{array}{ccc}
\circ \circ & \circ \circ & \circ \circ \\
\circ \circ & \circ \circ & \circ \circ \\
\circ \circ & \circ \circ & \circ \circ \\
\end{array}
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\[
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Mag is generated by \(\circ\) (proof by induction on the arities).

There is no relation between the generator \(\circ\) and itself (Mag is the free operad on one generator of arity 2).

Presentation by generators and relations:

\[
\text{Mag} = \langle \circ \mid \rangle.
\]
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The $T$ construction

Let us start with a monoid $(M, \bullet, 1)$. 
The $T$ construction

Let us start with a monoid $(M, \cdot, 1)$.

Let $TM$ be the graded set $TM := \biguplus_{n \geq 1} TM(n)$ where

$TM(n) := \{x_1 \ldots x_n : x_i \in M \text{ for all } i \in [n]\}.$
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Let $M$ and $N$ be two monoids and $\theta : M \rightarrow N$ be a monoid morphism.
The $T$ construction

Let $M$ and $N$ be two monoids and $\theta : M \to N$ be a monoid morphism.

Let $T\theta$ be the application

$$T\theta : TM \to TN,$$

defined for all $x_1 \ldots x_n \in TM(n)$ by

$$T\theta(x_1 \ldots x_n) := \theta(x_1) \ldots \theta(x_n).$$
Some examples of the \( \mathbb{T} \) construction

\[ M := (\mathbb{N}, +) \]. Elements of \( \mathbb{T}M \): words over the alphabet \( \mathbb{N} \).
Some examples of the $T$ construction

$M := (\mathbb{N}, +)$. Elements of $TM$: words over the alphabet $\mathbb{N}$.

Example

$2123 \odot_2 30313$
Some examples of the $T$ construction

$M := (\mathbb{N}, +)$. Elements of $TM$: words over the alphabet $\mathbb{N}$.

Example

$2123 \circ_2 30313 = 24142423$
Some examples of the $T$ construction

$M := (\mathbb{N}, +)$. Elements of $TM$: words over the alphabet $\mathbb{N}$.

<table>
<thead>
<tr>
<th>Example</th>
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<tbody>
<tr>
<td>$2123 \circ_2 30313 = 24142423$</td>
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$N := \{a, b\}^*$. Elements of $TN$: multiwords over the alphabet $\{a, b\}$. 
Some examples of the $T$ construction

$M := (\mathbb{N}, +)$. Elements of $TM$: words over the alphabet $\mathbb{N}$.

Example

$2123 \circ_2 30313 = 24142423$

$N := \{a, b\}^\ast$. Elements of $TN$: multiwords over the alphabet $\{a, b\}$.

Example

\[
\begin{array}{cccccc}
  b & a & a & \varepsilon & b \\
  b & b & \circ_3 & a & \varepsilon & b \\
  a & & & & & b
\end{array}
\]
Some examples of the T construction

\( M := (\mathbb{N}, +) \). Elements of \( TM \): words over the alphabet \( \mathbb{N} \).

Example

\[ 2123 \circ_2 30313 = 24142423 \]

\( N := \{a, b\}^* \). Elements of \( TN \): multiwords over the alphabet \( \{a, b\} \).

Example

\[
\begin{array}{cccc}
  b & a & a & \epsilon & b \\
  b & b & \circ_3 & a & \epsilon & b \\
  a & b & a & a & a & a & \epsilon & b \\
\end{array}
\]

\[ = \]

\[
\begin{array}{cccc}
  b & a & a & a & a & a & \epsilon & b \\
  b & a & b & b & b \\
  a & b \\
\end{array}
\]
Some examples of the $T$ construction

$M := (\mathbb{N}, +)$. Elements of $TM$: words over the alphabet $\mathbb{N}$.

Example

\[
2123 \circ_2 30313 = 24142423
\]

$N := \{a, b\}^\ast$. Elements of $TN$: multiwords over the alphabet $\{a, b\}$.

Example

\[
\begin{align*}
b & \ a & a & \epsilon & b \\
& b & b & \epsilon & a & \epsilon & b \\
& a & b & \circ_3 & b & a & b & b
\end{align*}
\]

\[
= \begin{align*}
& b & a & a & a & a & a & \epsilon & b \\
& b & & & \epsilon & a & \epsilon & b \\
& a & b & \epsilon & b & a & b
\end{align*}
\]

Let $\theta : N \rightarrow M$ be the monoid morphism defined by $\theta(u) := |u|$. 
Some examples of the $T$ construction

$M := (\mathbb{N}, +)$. Elements of $TM$: words over the alphabet $\mathbb{N}$.

Example

$$2123 \circ_2 30313 = 24142423$$

$N := \{a, b\}^\ast$. Elements of $TN$: multiwords over the alphabet $\{a, b\}$.

Example

$$\begin{align*}
\begin{array}{ccccccc}
b & a & a & \epsilon & b \\
 b & b & \circ_3 & \epsilon & a & \epsilon & b \\
 a
\end{array} & = \\
\begin{array}{ccccccc}
b & a & a & a & a & a & \epsilon & b \\
 b & a & b & b
\end{array}
\end{align*}$$

Let $\theta : N \to M$ be the monoid morphism defined by $\theta(u) := |u|$.

Example

$$T\theta \left( \begin{array}{ccccccc}
b & a & a & \epsilon & a & a \\
 b & a \\
 a
\end{array} \right)$$
Some examples of the $T$ construction

$M := (\mathbb{N}, +)$. Elements of $TM$: words over the alphabet $\mathbb{N}$.

**Example**

$$2123 \circ_2 30313 = 24142423$$

$N := \{a, b\}^*$. Elements of $TN$: multiwords over the alphabet $\{a, b\}$.

**Example**

$$\begin{array}{cccc}
  b & a & a & \epsilon & b \\
  b & b & \circ_3 & \epsilon & a & \epsilon & b \\
  b & a & a & a & a & a & \epsilon & b
\end{array}$$

$$\begin{array}{cccc}
  \epsilon & a & \epsilon & b \\
  b & a & b & b
\end{array}$$

Let $\theta : N \rightarrow M$ be the monoid morphism defined by $\theta(u) := |u|$.

**Example**

$$T\theta \begin{pmatrix} b & a & a & \epsilon & a & a \\
  \epsilon & a & \epsilon & b \\
  b & a & b & b \\
  a & a & b
\end{pmatrix} = 131021$$
Properties of the $\mathcal{T}$ construction

**Theorem**

*If $M$ is a monoid, $\mathcal{T}M$ is an operad.*

*If $\theta : M \to N$ is a monoid morphism, $\mathcal{T}\theta$ is an operad morphism.*

*Moreover, $\mathcal{T}$ preserves injections and surjections.*
Properties of the $T$ construction

**Theorem**

*If $M$ is a monoid, $T M$ is an operad.*

*If $\theta : M \to N$ is a monoid morphism, $T \theta$ is an operad morphism.*

*Moreover, $T$ preserves injections and surjections.*

Hence, $T$ is an exact functor from the category of monoids with monoid morphisms to the category of operads with operad morphisms.
Properties of the $T$ construction

The sets $T^M(n)$ are finite iff $M$ is finite. In this case, the dimensions of $T^M$ are

$$m, m^2, m^3, m^4, \ldots$$

where $m := \#M$. 
Properties of the $T$ construction

The sets $T^M(n)$ are finite iff $M$ is finite. In this case, the dimensions of $T^M$ are

$$m, m^2, m^3, m^4, \ldots$$

where $m := \#M$.

$T^M$ is generated by the set

$$G(M) \cup \{1\},$$

where $G(M)$ is a set of generators of $M$ and 1 is its unit.
Properties of the $T$ construction

The sets $TM(n)$ are finite iff $M$ is finite. In this case, the dimensions of $TM$ are

$$m, m^2, m^3, m^4, \ldots$$

where $m := \#M$.

$TM$ is generated by the set

$$G(M) \uplus \{11\},$$

where $G(M)$ is a set of generators of $M$ and 1 is its unit.

Example

$\{1\} \uplus \{00\}$ is a generating set of $T(\mathbb{N}, +)$. For instance,

$$02001 = (((((00 \circ_1 00) \circ_1 00) \circ_1 00) \circ_2 1) \circ_2 1) \circ_5 1.$$
Objectives and goals

Main motivations for introducing the $T$ construction:

1. Give alternative constructions of some well-known operads;
2. Construct new operads.
Objectives and goals

Main motivations for introducing the T construction:

1. Give alternative constructions of some well-known operads;
2. Construct new operads.

The general line is as following:

Choose a monoid

Monoid $M$
Objectives and goals

Main motivations for introducing the $T$ construction:

1. Give alternative constructions of some well-known operads;
2. Construct new operads.

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Objectives and goals

Main motivations for introducing the $T$ construction:

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The general line is as following:

Choose a monoid

Monoid $M$ \quad $T$ \quad Operad $TM$

Choose some elements of $TM$

Set $G$
Objectives and goals

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Objectives and goals
Main motivations for introducing the $T$ construction:
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2. Construct new operads.

The general line is as following:

```
Choose a monoid

Monoid $M$  $\xrightarrow{T}$  Operad $TM$

Choose some elements of $TM$

Set $G$  $\xrightarrow{\text{Operad generated}}$  Operad $\langle G \rangle$
```

We then ask usual questions about operads on $\langle G \rangle$. 
Contents

Applications of the construction
  Survey of the constructed operads
    The operad of Motzkin paths
    The diassociative and triassociative operads
Survey of some obtained operads

Here are the operads obtained using the $T$ construction with some usual monoids:

<table>
<thead>
<tr>
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<th>Operad</th>
<th>Generators</th>
<th>First dimensions</th>
<th>Combinatorial objects</th>
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</thead>
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<tr>
<td>$\mathbb{N}$, $+$</td>
<td>End</td>
<td>—</td>
<td>1, 4, 27, 256, 3125</td>
<td>Endofunctions</td>
</tr>
<tr>
<td>$\mathbb{N}$, $+$</td>
<td>PF</td>
<td>—</td>
<td>1, 3, 16, 125, 1296</td>
<td>Parking functions</td>
</tr>
<tr>
<td>$\mathbb{N}$, $+$</td>
<td>PW</td>
<td>—</td>
<td>1, 3, 13, 75, 541</td>
<td>Packed words</td>
</tr>
<tr>
<td>$(\mathbb{N}, +)$</td>
<td>Per</td>
<td>—</td>
<td>1, 2, 6, 24, 120</td>
<td>Permutations</td>
</tr>
<tr>
<td>$(\mathbb{N}, +)$</td>
<td>PRT</td>
<td>01</td>
<td>1, 1, 2, 5, 14, 42</td>
<td>Planar rooted trees</td>
</tr>
<tr>
<td>$\mathbb{Z}$, $+$</td>
<td>FCat$^k$</td>
<td>00, 01, $\ldots$, 0$k$</td>
<td>Fuß-Catalan num.</td>
<td>Trees of arity $k+1$</td>
</tr>
<tr>
<td>$(\mathbb{Z}, +)$</td>
<td>Schr</td>
<td>00, 01, 10</td>
<td>1, 3, 11, 45, 197</td>
<td>Schröder trees</td>
</tr>
<tr>
<td>$(\mathbb{Z}, +)$</td>
<td>Motz</td>
<td>00, 010</td>
<td>1, 1, 2, 4, 9, 21, 51</td>
<td>Motzkin paths</td>
</tr>
<tr>
<td>$(\mathbb{Z}/2\mathbb{Z}, +)$</td>
<td>Comp</td>
<td>00, 01</td>
<td>1, 2, 4, 8, 16, 32</td>
<td>Int. compo.</td>
</tr>
<tr>
<td>$(\mathbb{Z}/3\mathbb{Z}, +)$</td>
<td>DA</td>
<td>00, 01</td>
<td>1, 2, 5, 13, 35, 96</td>
<td>Directed animals</td>
</tr>
<tr>
<td>$(\mathbb{Z}/3\mathbb{Z}, +)$</td>
<td>SComp</td>
<td>00, 01, 02</td>
<td>1, 3, 27, 81, 243</td>
<td>Segmented int. compo.</td>
</tr>
<tr>
<td>${0, 1}, \times$</td>
<td>Dias</td>
<td>01, 10</td>
<td>1, 2, 3, 4, 5, 6</td>
<td>Words with exactly one 1</td>
</tr>
<tr>
<td>${0, 1}, \times$</td>
<td>Trias</td>
<td>01, 10, 11</td>
<td>1, 3, 7, 15, 31, 63</td>
<td>Words with at least one 1</td>
</tr>
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Experimenting with Sage

Let Motz be the suboperad of $T(\mathbb{N}, +)$ generated by 00 and 010.
Experimenting with Sage

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```
sage: M = AdditiveMonoid()
```
Experimenting with Sage

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```

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sage: G = [Word(M, [0, 0]), Word(M, [0, 1, 0])]
sage: Motz = SubOperad(TM, G)
sage: print [Motz.dimension(n) for n in xrange(1, 10)]
[1, 1, 2, 4, 9, 21, 51, 127, 323]
sage: print Motz.elements(1)
[0]
sage: print Motz.elements(2)
[00]
sage: print Motz.elements(3)
[000, 010]
sage: print Motz.elements(6)
[000000, 000010, 000100, 000110, 001000, 001010, 001100, 001110, 001210, 010000, 010010, 010100, 010110, 011000, 011010, 011100, 011110, 011210, 012100, 012110, 012210]
```
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[000, 010]
```

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[000000, 000010, 000100, 000110, 001000, 001010, 001100, 001110, 001210, 010000, 010010, 010100, 010110, 011000, 011010, 011100, 011110, 011210, 012100, 012110, 012210]
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sage: print [Motz.dimension(n) for n in xrange(1, 10)]
[1, 1, 2, 4, 9, 21, 51, 127, 323]
sage: print Motz.elements(1)
[0]
```
Let Motz be the suboperad of $T(\mathbb{N}, +)$ generated by 00 and 010.

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sage: M = AdditiveMonoid()
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[00]
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Elements of Motz

Proposition

The elements of Motz are exactly the words $x$ on the alphabet $\mathbb{N}$ beginning and ending by 0 and such that, for any factor $ab$ of $x$, $|a - b| \leq 1$. 
Elements of Motz

Proposition

*The elements of Motz are exactly the words* $x$ *on the alphabet* $\mathbb{N}$ *beginning and ending by* $0$ *and such that, for any factor* $ab$ *of* $x$, *$|a - b| \leq 1$.*

Bijection between elements of Motz and Motzkin paths:

$$ab \mapsto \begin{cases} \begin{array}{l} \text{ if } b - a = -1, \\
\text{ if } b - a = 0, \\
\text{ if } b - a = 1. 
\end{array} \end{cases}$$
Elements of Motz

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*The elements of Motz are exactly the words* $x$ *on the alphabet* $\mathbb{N}$ *beginning and ending by* $0$ *and such that, for any factor* $ab$ *of* $x$, $|a - b| \leq 1$.

Bijection between elements of Motz and Motzkin paths:

$$ab \mapsto \begin{cases} 
\square & \text{if } b - a = -1, \\
\bigcirc & \text{if } b - a = 0, \\
\bigcirc & \text{if } b - a = 1.
\end{cases}$$

**Example**

$$001123221010 \mapsto \begin{array}{c}
\text{Motz Path}
\end{array}$$
Elements of Motz

Proposition

The elements of Motz are exactly the words $x$ on the alphabet $\mathbb{N}$ beginning and ending by 0 and such that, for any factor $ab$ of $x$, $|a - b| \leq 1$.

Bijection between elements of Motz and Motzkin paths:

$ab \mapsto \begin{cases} \begin{array}{ll} \text{if } b - a = -1, & \text{square} \\ \text{if } b - a = 0, & \\
\text{if } b - a = 1. & \text{triangle} \end{array} \end{cases}$

Example

$001123221010 \mapsto \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline \hline \end{array}$
Elements of Motz

Proposition

The elements of Motz are exactly the words $x$ on the alphabet $\mathbb{N}$ beginning and ending by 0 and such that, for any factor $ab$ of $x$, $|a - b| \leq 1$.

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$$ab \mapsto \begin{cases} \begin{array}{ll} \begin{array}{c} \begin{array}{c} \text{if } b - a = -1, \end{array} \end{array} \\ \begin{array}{c} \begin{array}{c} \text{if } b - a = 0, \end{array} \end{array} \\ \begin{array}{c} \begin{array}{c} \text{if } b - a = 1. \end{array} \end{array} \end{cases} \end{cases}$$

Example

001123221010 $\mapsto$ \[ \text{Motzkin path} \]
Elements of Motz

Proposition

The elements of Motz are exactly the words \( x \) on the alphabet \( \mathbb{N} \) beginning and ending by 0 and such that, for any factor \( ab \) of \( x \), \(|a - b| \leq 1\).

Bijection between elements of Motz and Motzkin paths:

\[
ab \mapsto \begin{cases} 
& \text{if } b - a = -1, \\
& \text{if } b - a = 0, \\
& \text{if } b - a = 1.
\end{cases}
\]

Example

\[001123221010 \mapsto \]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
& & & & & & & & \\
\hline
& & & & & & & & \\
\hline
& & & & & & & & \\
\hline
\end{array}
\]
Elements of Motz

Proposition

The elements of Motz are exactly the words $x$ on the alphabet $\mathbb{N}$ beginning and ending by 0 and such that, for any factor $ab$ of $x$, $|a - b| \leq 1$.

Bijection between elements of Motz and Motzkin paths:

$ab \mapsto \begin{cases} 
\begin{array}{c} 
\begin{array}{c} 
\end{array} 
\end{array} 
\end{cases}$ if $b - a = -1$, 

$\begin{array}{c} 
\begin{array}{c} 
\end{array} 
\end{array}$ if $b - a = 0$, 

$\begin{array}{c} 
\begin{array}{c} 
\end{array} 
\end{array}$ if $b - a = 1$.

Example

$001123221010 \mapsto$
Elements of Motz

Proposition

The elements of Motz are exactly the words $x$ on the alphabet $\mathbb{N}$ beginning and ending by 0 and such that, for any factor $ab$ of $x$, $|a - b| \leq 1$.

Bijection between elements of Motz and Motzkin paths:

$$ab \mapsto \begin{cases} \square & \text{if } b - a = -1, \\ \circ & \text{if } b - a = 0, \\ \triangle & \text{if } b - a = 1. \end{cases}$$

Example

$$001123221010 \mapsto \begin{array} \text{Motzkin path} \end{array}$$
Elements of Motz

Proposition

The elements of Motz are exactly the words $x$ on the alphabet $\mathbb{N}$ beginning and ending by 0 and such that, for any factor $ab$ of $x$, $|a - b| \leq 1$.

Bijection between elements of Motz and Motzkin paths:

\[
ab \mapsto \begin{cases} 
\bullet & \text{if } b - a = -1, \\
\circ & \text{if } b - a = 0, \\
\triangle & \text{if } b - a = 1.
\end{cases}
\]

Example

001123221010 $\mapsto$

\[\text{Diagram of Motzkin path}\]
Elements of Motz

**Proposition**

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\text{if } b - a = 1.
\end{array}
\end{cases}
\]

**Example**

001123221010 $\mapsto$ 

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\end{array}
\end{cases}
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Example

001123221010 $\mapsto$
Elements of Motz

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The elements of Motz are exactly the words $x$ on the alphabet $\mathbb{N}$ beginning and ending by 0 and such that, for any factor $ab$ of $x$, $|a - b| \leq 1$.

Bijection between elements of Motz and Motzkin paths:

\[
ab \mapsto \begin{cases} 
\quad \text{if } b - a = -1, \\
\quad \text{if } b - a = 0, \\
\quad \text{if } b - a = 1.
\end{cases}
\]

**Example**

\[
001123221010 \mapsto \quad 
\]

$\ldots$
Elements of Motz

**Proposition**

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Bijection between elements of Motz and Motzkin paths:

$$ab \mapsto \begin{cases} 
\text{if } b - a = -1, \\
\text{if } b - a = 0, \\
\text{if } b - a = 1.
\end{cases}$$

**Example**

$$001123221010 \mapsto \text{Motzkin path}$$
Elements of Motz

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The elements of Motz are exactly the words $x$ on the alphabet $\mathbb{N}$ beginning and ending by 0 and such that, for any factor $ab$ of $x$, $|a - b| \leq 1$.

Bijection between elements of Motz and Motzkin paths:

$$ab \mapsto \begin{cases} 
\text{if } b - a = -1, \\
\text{if } b - a = 0, \\
\text{if } b - a = 1.
\end{cases}$$

Example

$$001123221010 \mapsto \begin{array}{c}
\text{Motzkin path}
\end{array}$$
Grafting of Motz

**Proposition**

Let $u$ and $v$ be two Motzkin paths. The grafting $u \circ_i v$ in Motz returns to replace the $i$th point of $u$ by $v$. 

Example:

$01123210 \circ_4 0122110 = 01123443323210 \circ_4 = \frac{25}{33}$
Grafting of Motz

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01123210

01123443323210

$\circ_4$
Grafting of Motz

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Let $u$ and $v$ be two Motzkin paths. The grafting $u \circ_i v$ in Motz returns to replace the $i$th point of $u$ by $v$.

Example

<table>
<thead>
<tr>
<th>01123210</th>
<th>$\circ_4$</th>
<th>0122110</th>
</tr>
</thead>
</table>

![Diagram of Motzkin paths and grafting process]
Proposition

Let $u$ and $v$ be two Motzkin paths. The grafting $u \circ_i v$ in Motz returns to replace the $i$th point of $u$ by $v$.

Example

<table>
<thead>
<tr>
<th>01123210</th>
<th>$\circ_4$</th>
<th>0122110</th>
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</table>

\[0122210 \circ_4 0122110 = 0122210\]

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**Example**

\[
01123210 \circ_4 0122110 = 01123443323210
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![Diagram of grafting example]
Grafting of Motz

**Proposition**

Let $u$ and $v$ be two Motzkin paths. The grafting $u \circ_i v$ in Motz returns to replace the $i$th point of $u$ by $v$.

**Example**

$$01123210 \circ_4 0122110 = 01123443323210$$
Presentation of Motz

Theorem

The operad Motz admits the following presentation

\[ \text{Motz} = \langle \bullet \bullet, \uparrow \uparrow \mid \uparrow \bullet \bullet = \bullet \uparrow \bullet \bullet, \]
\[ \uparrow \bullet \bullet = \bullet \uparrow \bullet \bullet, \]
\[ \uparrow \uparrow \bullet = \uparrow \bullet \bullet \uparrow \bullet \bullet, \]
\[ \uparrow \uparrow \bullet = \uparrow \bullet \bullet \uparrow \bullet \bullet \rangle. \]
Contents

Applications of the construction
Survey of the constructed operads
The operad of Motzkin paths
The diassociative and triassociative operads
The diassociative and triassociative operads

The diassociative operad $\text{Dias}$ [Loday, 2001] is the operad admitting the following presentation:

$$\text{Dias} := \langle \not\!, \top | \not\! \circ_1 \top = \top \circ_2 \not\!, \not\! \circ_1 \top = \not\! \circ_2 \top = \not\! \circ_2 \top, \top \circ_2 \top = \top \circ_1 \top = \top \circ_1 \top \rangle.$$
The diassociative and triassociative operads

The diassociative operad $\text{Dias}$ [Loday, 2001] is the operad admitting the following presentation:

$$\text{Dias} := \langle \vdash, \triangleright \mid \vdash o_1 \triangleright = \vdash o_2 \vdash, \vdash o_1 \vdash = \vdash o_2 \vdash = \vdash o_2 \triangleright, \vdash o_2 \triangleright = \vdash o_1 \triangleright = \vdash o_1 \vdash \rangle.$$ 

The triassociative operad $\text{Trias}$ [Loday, Ronco, 2004] is the operad admitting the following presentation:

$$\text{Trias} := \langle \vdash, \vdash \mid \vdash o_1 \vdash = \vdash o_2 \vdash, \vdash o_1 \vdash = \vdash o_2 \vdash = \vdash o_2 \vdash, \vdash o_2 \vdash = \vdash o_1 \vdash = \vdash o_1 \vdash \rangle.$$
The diassociative and triassociative operads

The diassociative operad Dias [Loday, 2001] is the operad admitting the following presentation:

\[
\text{Dias} := \langle \perp, \top \mid \perp \circ \top = \top \circ \perp, \\
\perp \circ \perp = \perp \circ \perp = \top \circ \top, \\
\top \circ \perp = \top \circ \perp \rangle.
\]

The triassociative operad Trias [Loday, Ronco, 2004] is the operad admitting the following presentation:

\[
\text{Trias} := \langle \perp, \bot, \top \mid \perp \circ \top = \top \circ \perp, \\
\perp \circ \bot = \perp \circ \perp = \top \circ \top, \\
\perp \circ \top = \top \circ \perp, \\
\bot \circ \perp = \bot \circ \bot, \\
\top \circ \top = \bot \circ \bot, \\
\perp \circ \top = \perp \circ \top \rangle.
\]
Let $D$ be the suboperad of $T(\{0, 1\}, \times)$ generated by 01 and 10.
Experimenting with Sage

Let $D$ be the suboperad of $T(\{0, 1\}, \times)$ generated by $01$ and $10$.

```
sage: M = MultiplicativeMonoid()
sage: TM = TConstruction(M)
sage: G = [Word(M, [0, 1]), Word(M, [1, 0])]
sage: D = SubOperad(TM, G)
```

```python
dimension: D.dimension(n) for n in xrange(1, 10)
dimension: [1, 2, 3, 4, 5, 6, 7, 8, 9]
elements: D.elements(5)
elements: [10000, 01000, 00100, 00010, 00001]
```
Let $D$ be the suboperad of $T(\{0, 1\}, \times)$ generated by 01 and 10.

```
sage: M = MultiplicativeMonoid()
sage: TM = TConstruction(M)
sage: G = [Word(M, [0, 1]), Word(M, [1, 0])]
sage: D = SubOperad(TM, G)
sage: print [D.dimension(n) for n in xrange(1, 10)]
[1, 2, 3, 4, 5, 6, 7, 8, 9]
```
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### Proposition

*The elements of $D$ are exactly the words on the alphabet $\{0, 1\}$ which have exactly one occurrence of $1$.***
The operad $D$

**Proposition**

The elements of $D$ are exactly the words on the alphabet $\{0, 1\}$ which have exactly one occurrence of 1.

**Proposition**

The operad $D$ is isomorphic to the operad $\text{Dias}$ through the operad isomorphism $\phi : \text{Dias} \rightarrow D$ defined by

\[
\phi(\vdash) := 10 \quad \text{and} \quad \phi(\dashv) := 01.
\]
The operad $D$

**Proposition**

The elements of $D$ are exactly the words on the alphabet $\{0, 1\}$ which have exactly one occurrence of $1$.

**Proposition**

The operad $D$ is isomorphic to the operad Dias through the operad isomorphism $\phi : \text{Dias} \to D$ defined by

$$\phi(\top) := 10 \quad \text{and} \quad \phi(\bot) := 01.$$ 

Hence, $D$ is a realization of Dias.
Let $T_r$ be the suboperad of $T(\{0, 1\}, \times)$ generated by 01, 10, and 11.
Let $\mathcal{T}r$ be the suboperad of $\mathcal{T}(\{0, 1\}, \times)$ generated by 01, 10, and 11.

```
sage: M = MultiplicativeMonoid()
sage: TM = TConstruction(M)
sage: G = [Word(M, [0, 1]), Word(M, [1, 0]), Word(M, [1, 1])]
sage: Tr = SubOperad(TM, G)
```

```
sage: print [Tr.dimension(n) for n in xrange(1, 10)]
[1, 3, 7, 15, 31, 63, 127, 255, 511]
sage: print Tr.elements(3)
[001, 010, 011, 100, 101, 110, 111]
```
Let $\mathcal{T}$ be the suboperad of $\mathcal{T}(\{0,1\}, \times)$ generated by 01, 10, and 11.

```python
sage: M = MultiplicativeMonoid()
sage: TM = TConstruction(M)
sage: G = [Word(M, [0, 1]), Word(M, [1, 0]), Word(M, [1, 1])]
sage: Tr = SubOperad(TM, G)
sage: print [Tr.dimension(n) for n in xrange(1, 10)]
[1, 3, 7, 15, 31, 63, 127, 255, 511]
```
Let $T_r$ be the suboperad of $T(\{0, 1\}, \times)$ generated by 01, 10, and 11.

```
sage: M = MultiplicativeMonoid()
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```
The operad $\text{Tr}$

**Proposition**

The elements of $\text{Tr}$ are exactly the words on the alphabet $\{0, 1\}$ which have at least one occurrence of 1.
The operad $\mathbf{Tr}$

**Proposition**

*The elements of $\mathbf{Tr}$ are exactly the words on the alphabet $\{0, 1\}$ which have at least one occurrence of $1$.***

**Proposition**

*The operad $\mathbf{Tr}$ is isomorphic to the operad $\mathbf{Trias}$ through the operad isomorphism $\phi : \mathbf{Trias} \to \mathbf{Tr}$ defined by

\[
\phi(\bot) := 11, \quad \phi(\top) := 01, \quad \text{and} \quad \phi(\mid) := 10.
\]
The operad $\text{Tr}$

**Proposition**

The elements of $\text{Tr}$ are exactly the words on the alphabet $\{0, 1\}$ which have at least one occurrence of $1$.

**Proposition**

The operad $\text{Tr}$ is isomorphic to the operad $\text{Trias}$ through the operad isomorphism $\phi : \text{Trias} \rightarrow \text{Tr}$ defined by

\[
\phi(\bot) := 10, \quad \phi(\top) := 01, \quad \text{and} \quad \phi(\bot) := 11.
\]

Hence, $\text{Tr}$ is a realization of $\text{Trias}$.
Survey of some obtained operads

These operads fit into following diagram.

\[ \rightarrow\rightarrow \ (\text{resp. } \rightarrow\rightarrow) \text{ stands for an injective (resp. surjective) operad morphism.} \]

\[
\begin{array}{c}
T(\mathbb{N}, +) \\
\downarrow \\
T(\mathbb{Z}/2\mathbb{Z}, +) \quad \text{End} \\
\downarrow \\
PF \\
\downarrow \\
PW \\
\downarrow \\
\text{Per} \quad \text{Schr} \\
\downarrow \\
\text{FCat}^{(1)} \\
\downarrow \\
\text{FCat}^{(0)} \\
\downarrow \\
\text{Comp} \\
\end{array}
\]

\[
\begin{array}{c}
T(\mathbb{Z}/3\mathbb{Z}, +) \\
\downarrow \\
\text{FCat}^{(3)} \\
\downarrow \\
\text{FCat}^{(2)} \\
\downarrow \\
\text{SComp} \\
\downarrow \\
T(\{0, 1\}, \times) \\
\end{array}
\]

\[
\begin{array}{c}
\text{Trias} \\
\text{Dias} \\
\end{array}
\]