Operads, enumeration, and constructions

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Discrete Structures Days

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Contents

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Combinatorial sets

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The set $\mathfrak{S}$ of the permutations, where the size of a permutation is its length as a word, satisfies

$$\mathfrak{S}(0)=\{\epsilon\},\ \ \mathfrak{S}(1)=\{1\},\ \ \mathfrak{S}(2)=\{12,21\},\ \ \mathfrak{S}(3)=\{123,132,213,231,312,321\},$$

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and its generating series satisfies

$$\mathcal{G}_{\mathfrak{S}}(t) = \sum_{n \in \mathbb{N}} n! t^n = 1 + t + 2t^2 + 6t^3 + 24t^4 + 120t^5 + \cdots .$$

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On $\mathfrak{S}$, let $\mathrm{s}(\sigma)$ be the number of descents of σ (*i.e.*, the number of i such that $\sigma(i) > \sigma(i+1)$). Then,

$$\mathcal{G}_{\mathfrak{S}}^{\mathrm{s}}(t, q) = 1 + t + (1+q)t^2 + (1 + 4q + q^2)t^3 + (1 + 11q + 11q^2 + q^3)t^5 + \cdots .$$

The coefficients of the t^n are the Eulerian polynomials.

Algebraic approach

Aims

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Endow C with the structure of an operad $\mathcal{O}$.

As main benefits,

- the operations of $\mathcal{O}$ lead to operations on series;
- presentations of $\mathcal{O}$ by generators and relations highlight elementary building blocks and branching rules for the objects of C;
- operad morphisms involving $\mathcal{O}$ show connections between the objects of C and other combinatorial sets.

Formal power series

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Endowed with the pointwise addition

$$\langle x, \mathbf{f} + \mathbf{g} \rangle := \langle x, \mathbf{f} \rangle + \langle x, \mathbf{g} \rangle$$

and multiplication by a scalar

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The sum notation of $\mathbf{f}$ is

$$\mathbf{f} = \sum_{x \in X} \langle x, \mathbf{f} \rangle \, x.$$

Formal power series and algebraic structures

When X is endowed with an algebraic structure, its operations transfer on X-series. For instance, if $* : X \times X \to X$ is a binary product on X,

$$\mathbf{f} * \mathbf{g} := \sum_{x,y \in X} \langle x, \mathbf{f} \rangle \langle y, \mathbf{g} \rangle \, x * y.$$

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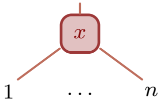
The most common examples are

Structure on X	Series	Reference
$(\mathbb{N}, +, 0)$	Usual series of $\mathbb{K} \langle\langle t \rangle\rangle$	
Free comm. monoid	Multivariate series	
$(A^*, \cdot, \epsilon)$	Noncomm. series	[**Eilenberg**, 1974]
Monoid	Series on monoids	[**Salomaa, Soittola**, 1978]
Operad	Series on operads	[**Chapoton**, 2002, 2008]

Operators

An operator is a device with $n \geqslant 1$ inputs and one output, represented as



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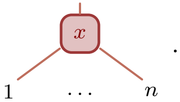

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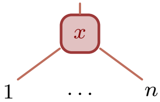

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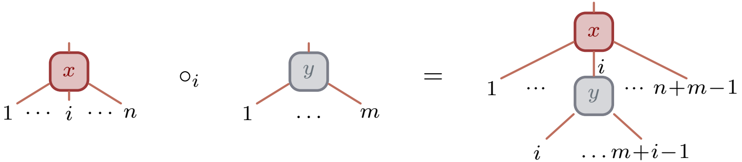
This produces a new operator $x \circ_i y$ of arity $n + m - 1$:



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This data satisfies some axioms.

Axioms of operads

Associativity:

$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$

$x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}$

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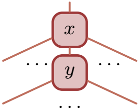

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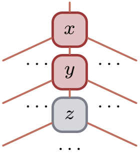

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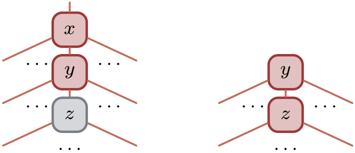

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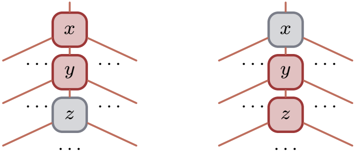

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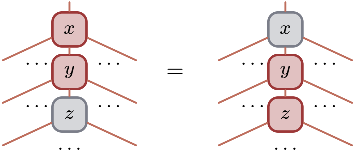

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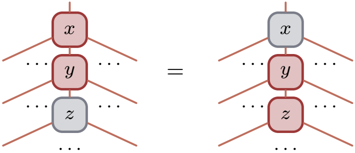

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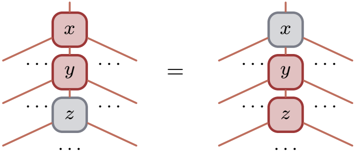

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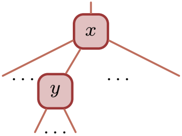

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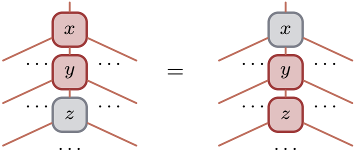

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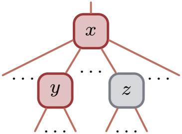

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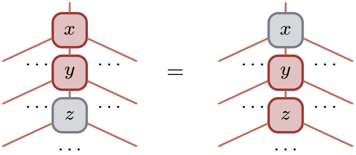

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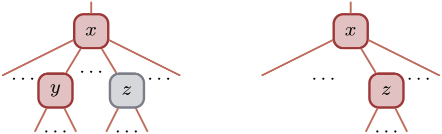

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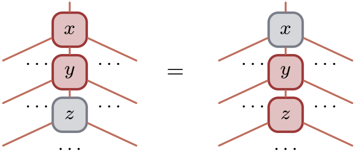

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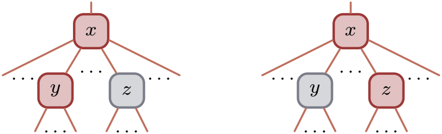

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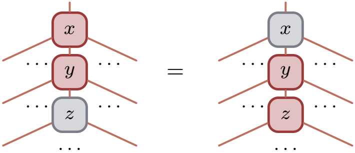

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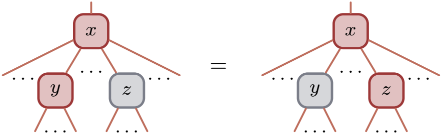

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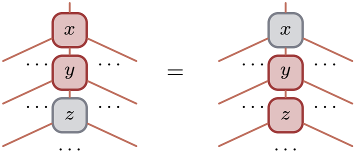

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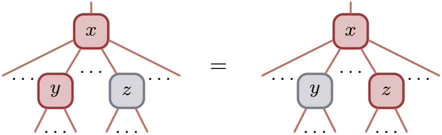

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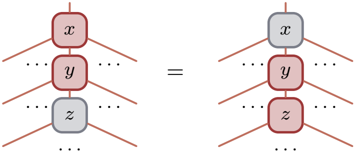

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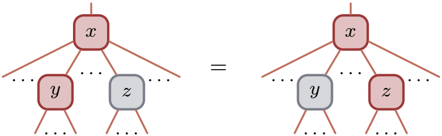

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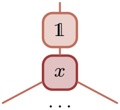

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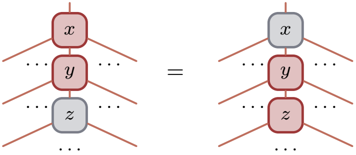
$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$

$x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}$

$i \in [n], j \in [m]$



Commutativity:

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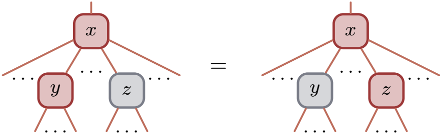

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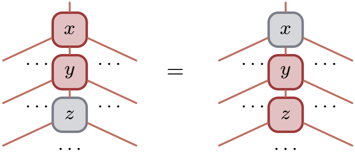

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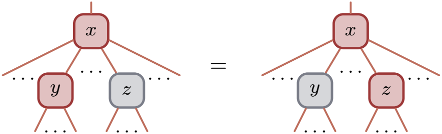

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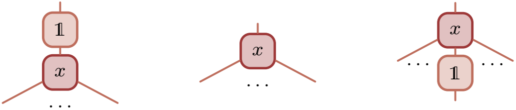

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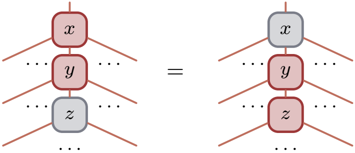

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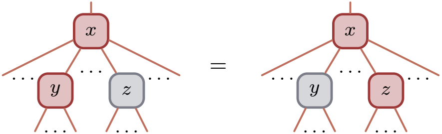

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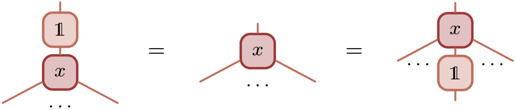

Complete composition

Let $\mathcal{O}$ be an operad.

The complete composition map of $\mathcal{O}$ is the map

$$\circ : \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n)$$

defined by

$$x \circ [y_1, \ldots, y_n] := (\ldots((x \circ_n y_n) \circ_{n-1} y_{n-1}) \ldots) \circ_1 y_1.$$

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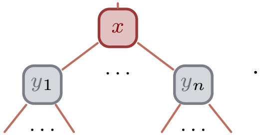
Graphically, $x \circ [y_1, \ldots, y_n]$ reads as



Formal power series on operads

Let $\mathcal{O}$ be an operad.

The characteristic series of $\mathcal{O}$ is the $\mathcal{O}$-series

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Goals

Use the partial and complete compositions of $\mathcal{O}$ to obtain

- an expression for $\mathbf{f}_{\mathcal{O}}$;
- an expression for $\mathcal{G}_{\mathcal{O}}(t)$;
- new statistics on the objects of $\mathcal{O}$.

Free operads

Let $\mathfrak{G} := \sqcup_{n \geqslant 1} \mathfrak{G}(n)$ be a graded set.

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Let $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ with $\mathfrak{G}(2) := \{\mathtt{a}, \mathtt{b}\}$ and $\mathfrak{G}(3) := \{\mathtt{c}\}$.

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- The unit is the tree with exactly one leaf.

Presentations of operads

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- quadratic when $\equiv$ is generated, as an operad congruence, by an equivalence relation involving only syntax trees with exactly 2 internal nodes.

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Let As be the associative operad, defined as the operad having the presentation $(\mathfrak{G}, \equiv)$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\mathtt{a}\}$ and $\equiv$ is the smallest operad congruence satisfying

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A PBW basis of As is the set of the syntax trees on $\mathfrak{G}$ avoiding $\mathtt{a} \circ_1 \mathtt{a}$.

Contents

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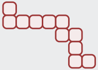
The operad of compositions is the operad Comp such that

- Comp(n) is the set of the ribbon diagrams of compositions of n.

Example



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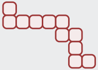

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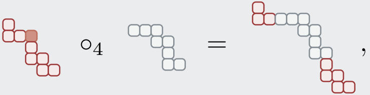

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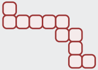 is a ribbon of arity 11.

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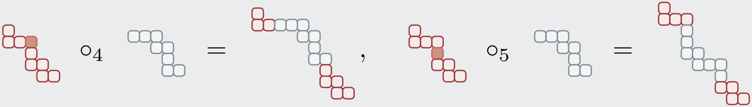

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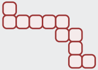

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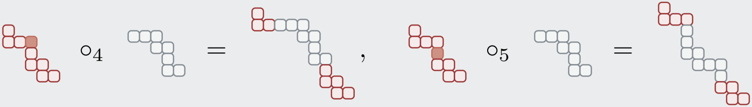
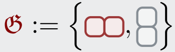


- The unit is .

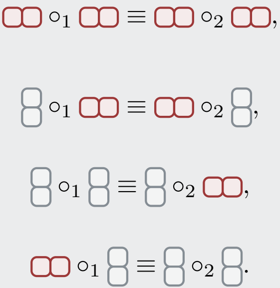
Presentation of Comp

Proposition

The operad Comp admits the presentation $(\mathfrak{G}, \equiv)$ where



and $\equiv$ is the smallest operad congruence satisfying



This presentation is binary and quadratic.

PBW basis of Comp and enumeration

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The set of the syntax trees on $\mathfrak{G}$ avoiding

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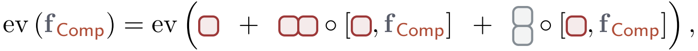

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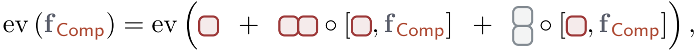

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$$\mathcal{G}_{\mathsf{Comp}}(t) = t + 2t\mathcal{G}_{\mathsf{Comp}}(t) = t + 2t^2 + 4t^3 + 8t^4 + 16t^5 + \cdots .$$

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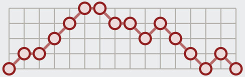
The operad of Motzkin paths is the operad Motz such that

- Motz(n) is the set of the Motzkin paths with $n - 1$ steps.

Example



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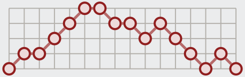

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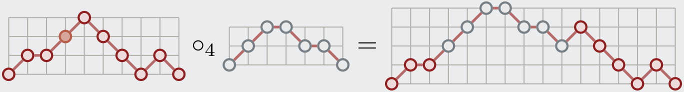

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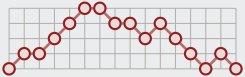 is a Motzkin path of arity 16.

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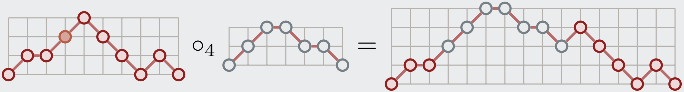

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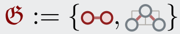

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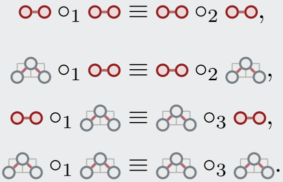

This presentation is not binary but is quadratic.

PBW basis of Motz and enumeration

Proposition

The set of the syntax trees on $\mathfrak{G}$ avoiding

forms a PBW basis of Motz.

PBW basis of Motz and enumeration

Proposition

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This leads to

PBW basis of Motz and enumeration

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The set of the syntax trees on $\mathfrak{G}$ avoiding

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This leads to

Hence,

$$\mathcal{G}_{\mathsf{Motz}}(t) = t + t\mathcal{G}_{\mathsf{Motz}}(t) + t\mathcal{G}_{\mathsf{Motz}}(t)^2.$$

Statistics on Motz

Let us add some parameters in the expression of $\mathbf{f}_{\mathsf{Motz}}$ to obtain

$\mathbf{g}_{\mathsf{Motz}} =$  $+ \quad q_0$  $\circ \,[$  $, \mathbf{g}_{\mathsf{Motz}}] \quad + \quad q_1$  $\circ \,[$  $, \mathbf{g}_{\mathsf{Motz}}, \mathbf{g}_{\mathsf{Motz}}]\,.$

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We obtain

$$\begin{aligned}\mathcal{G}_{\mathsf{Motz}}(t, q_0, q_1) := \mathrm{ev}(\mathbf{g}_{\mathsf{Motz}}) = {} & t + q_0 t^2 + (q_0^2 + q_1)t^3 \\ & + (q_0^3 + 3q_0q_1)t^4 \\ & + (q_0^4 + 6q_0^2q_1 + 2q_1^2)t^5 \\ & + (q_0^5 + 10q_0^3q_1 + 10q_0q_1^2)t^6 \\ & + (q_0^6 + 15q_0^4q_1 + 30q_0^2q_1^2 + 5q_1^3)t^7 \\ & + (q_0^7 + 21q_0^5q_1 + 70q_0^3q_1^2 + 35q_0q_1^3)t^8 \\ & + (q_0^8 + 28q_0^6q_1 + 140q_0^4q_1^2 + 140q_0^2q_1^3 + 14q_1^4)t^9 + \cdots .\end{aligned}$$

Statistics on Motz

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We obtain

$$\begin{aligned}
\mathcal{G}_{\mathsf{Motz}}(t, q_0, q_1) := \mathrm{ev}(\mathbf{g}_{\mathsf{Motz}}) = t + q_0 t^2 &+ (q_0^2 + q_1) t^3 \\
&+ (q_0^3 + 3 q_0 q_1) t^4 \\
&+ (q_0^4 + 6 q_0^2 q_1 + 2 q_1^2) t^5 \\
&+ (q_0^5 + 10 q_0^3 q_1 + 10 q_0 q_1^2) t^6 \\
&+ (q_0^6 + 15 q_0^4 q_1 + 30 q_0^2 q_1^2 + 5 q_1^3) t^7 \\
&+ (q_0^7 + 21 q_0^5 q_1 + 70 q_0^3 q_1^2 + 35 q_0 q_1^3) t^8 \\
&+ (q_0^8 + 28 q_0^6 q_1 + 140 q_0^4 q_1^2 + 140 q_0^2 q_1^3 + 14 q_1^4) t^9 + \cdots .
\end{aligned}$$

This counts Motzkin paths following the number of horizontal steps (parameter q_0) and rising steps (parameter q_1).

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\mathcal{G}_{\mathsf{Motz}}(t, q_0, q_1) := \mathrm{ev}(\mathbf{g}_{\mathsf{Motz}}) = t &+ q_0 t^2 + (q_0^2 + q_1) t^3 \\
&+ (q_0^3 + 3 q_0 q_1) t^4 \\
&+ (q_0^4 + 6 q_0^2 q_1 + 2 q_1^2) t^5 \\
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&+ (q_0^6 + 15 q_0^4 q_1 + 30 q_0^2 q_1^2 + 5 q_1^3) t^7 \\
&+ (q_0^7 + 21 q_0^5 q_1 + 70 q_0^3 q_1^2 + 35 q_0 q_1^3) t^8 \\
&+ (q_0^8 + 28 q_0^6 q_1 + 140 q_0^4 q_1^2 + 140 q_0^2 q_1^3 + 14 q_1^4) t^9 + \cdots .
\end{aligned}
$$

This counts Motzkin paths following the number of horizontal steps (parameter q_0) and rising steps (parameter q_1).

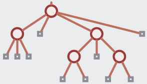
For instance, $\mathcal{G}_{\mathsf{Motz}}(t, 1, q_1)$ is Triangle **A055151**.

Operad of Schröder trees

The operad of Schröder trees is the operad Schr such that

- $\mathsf{Schr}(n)$ is the set of the Schröder trees with $n+1$ leaves.

Example



is a Schröder tree of arity 9.

Operad of Schröder trees

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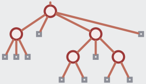
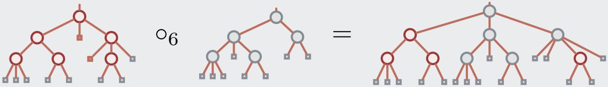


is a Schröder tree of arity 9.

- The partial composition $\mathfrak{t} \circ_i \mathfrak{s}$ is obtained by inserting a copy of $\mathfrak{s}$ between the ith and the $i+1$st leaves of $\mathfrak{t}$.

Example



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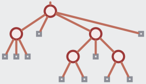 is a Schröder tree of arity 9.

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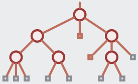
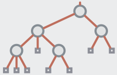
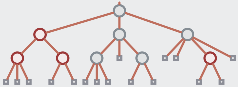
Example

 $\circ_6$  $=$ 

- The unit is .

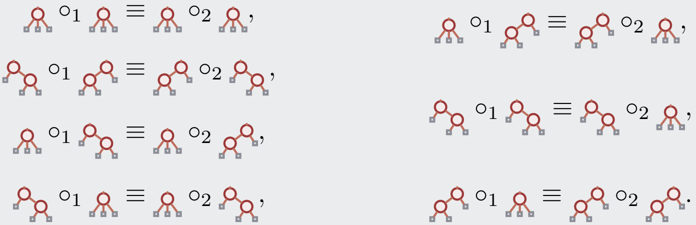
Presentation of Schr

Proposition

The operad Schr admits the presentation $(\mathfrak{G}, \equiv)$ where

$\mathfrak{G} :=$ 

and $\equiv$ is the smallest operad congruence satisfying

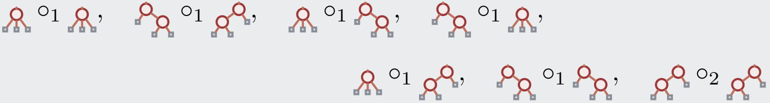


This presentation is binary and quadratic.

PBW basis of Schr and enumeration

Proposition

The set of the syntax trees on $\mathfrak{G}$ avoiding



forms a PBW basis of Schr.

PBW basis of Schr and enumeration

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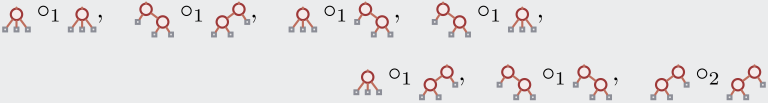

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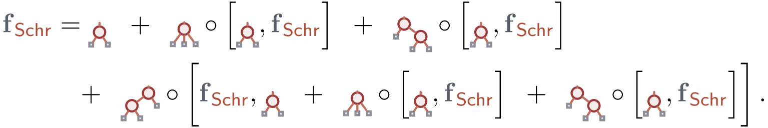
This leads to



PBW basis of Schr and enumeration

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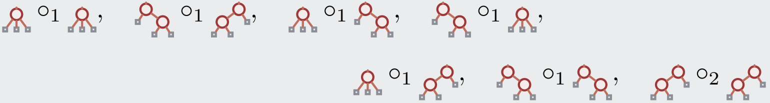

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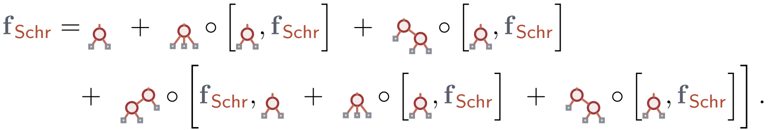
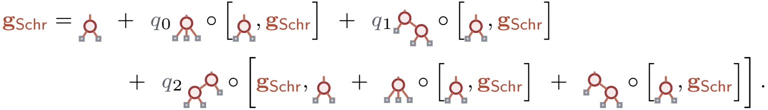


Hence,

$$\mathcal{G}_{\mathsf{Schr}}(t) = t + 3t\mathcal{G}_{\mathsf{Schr}}(t) + 2t\mathcal{G}_{\mathsf{Schr}}(t)^2.$$

Statistics on Schr

Let us add some parameters in the expression of $\mathbf{f}_{\mathsf{Schr}}$ to obtain



Statistics on Schr

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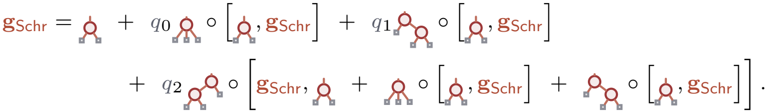

We obtain

$$
\begin{aligned}
\mathcal{G}_{\mathsf{Schr}}(t, q_0, q_1, q_2) := \mathrm{ev}(\mathbf{g}_{\mathsf{Schr}}) = t + (q_0 + q_1 + q_2)t^2 \\
+ (q_0^2 + 2q_0q_1 + q_1^2 + 2q_0q_2 + 2q_1q_2 + q_2^2 + 2q_2)t^3 \\
+ (q_0^3 + 3q_0^2q_1 + 3q_0q_1^2 + q_1^3 + 3q_0^2q_2 + 6q_0q_1q_2 \\
+ 3q_1^2q_2 + 3q_0q_2^2 + 3q_1q_2^2 + q_2^3 + 6q_0q_2 + 6q_1q_2 + 6q_2^2)t^4 + \cdots .
\end{aligned}
$$

Statistics on Schr

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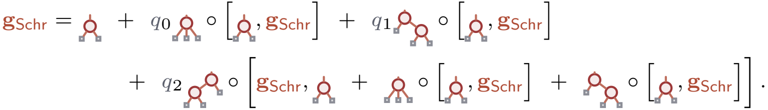


We obtain

$$\begin{aligned}\mathcal{G}_{\mathsf{Schr}}(t, q_0, q_1, q_2) := \mathrm{ev}(\mathbf{g}_{\mathsf{Schr}}) = t &+ (q_0 + q_1 + q_2)t^2 \\ &+ (q_0^2 + 2q_0q_1 + q_1^2 + 2q_0q_2 + 2q_1q_2 + q_2^2 + 2q_2)t^3 \\ &+ (q_0^3 + 3q_0^2q_1 + 3q_0q_1^2 + q_1^3 + 3q_0^2q_2 + 6q_0q_1q_2 \\ &+ 3q_1^2q_2 + 3q_0q_2^2 + 3q_1q_2^2 + q_2^3 + 6q_0q_2 + 6q_1q_2 + 6q_2^2)t^4 + \cdots .\end{aligned}$$

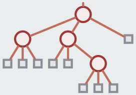
The specialization $\mathcal{G}_{\mathsf{Schr}}(t, 1, 1, q_2)$ is Triangle **A114656** and the specialization $\mathcal{G}_{\mathsf{Schr}}(t, q_0, 1, 1)$ is Triangle **A098473**.

Operad of k-trees

The operad of k-trees is the operad $\mathsf{FCat}^{(k)}$, $k \in \mathbb{N}$, such that

- $\mathsf{FCat}^{(k)}(n)$ is the set of the planar rooted trees with n internal nodes, all having $k+1$ children.

Example



is a 2-tree of arity 4.

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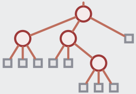
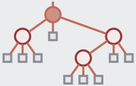


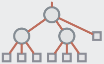
is a 2-tree of arity 4.

- The partial composition $\mathfrak{t} \circ_i \mathfrak{s}$ is obtained by replacing the ith internal node of $\mathfrak{t}$ (with respect to the depth-first traversal) by a copy of $\mathfrak{s}$.

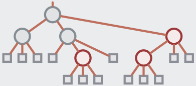
Example



$\circ_1$



$=$



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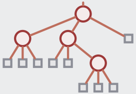
Example



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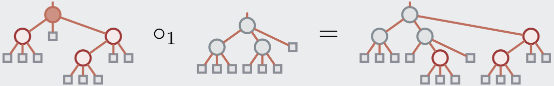
Example

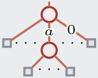


- The unit is  .

Presentation of $\mathsf{FCat}^{(k)}$

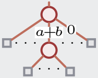
Proposition

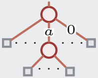
The operad $\mathsf{FCat}^{(k)}$ admits the presentation $(\mathfrak{G}, \equiv)$ where

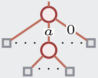
$\mathfrak{G} := \Big\{$ 

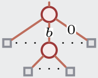
$: 0 \leqslant a \leqslant k+1 \Big\}$

and $\equiv$ is the smallest operad congruence satisfying



$\circ_1$ 

$\equiv$ 

$\circ_2$ 

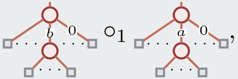
$, \qquad 0 \leqslant a, b, \quad a + b \leqslant k.$

This presentation is binary and quadratic.

PBW basis of FCat$^{(k)}$ and enumeration

Proposition

The set of the syntax trees on $\mathfrak{G}$ avoiding



$0 \leqslant a \leqslant b \leqslant k$

forms a PBW basis of FCat$^{(k)}$.

PBW basis of $\mathsf{FCat}^{(k)}$ and enumeration

Proposition

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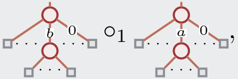


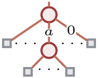
$$0 \leqslant a \leqslant b \leqslant k$$

forms a PBW basis of $\mathsf{FCat}^{(k)}$.

This leads to

$$\mathbf{f}_{\mathsf{FCat}^{(k)}} =$$  $$+ \sum_{0 \leqslant a \leqslant k} \mathbf{f}^{(a)}_{\mathsf{FCat}^{(k)}},$$

where the $\mathbf{f}^{(a)}_{\mathsf{FCat}^{(k)}}$ are the $\mathsf{FCat}^{(k)}$-series satisfying

$$\mathbf{f}^{(a)}_{\mathsf{FCat}^{(k)}} =$$ 

$$\circ \Big[$$  $$+ \sum_{a+1 \leqslant b \leqslant k} \mathbf{f}^{(b)}_{\mathsf{FCat}^{(k)}}, \mathbf{f}_{\mathsf{FCat}^{(k)}} \Big].$$

Enumeration in $\mathsf{FCat}^{(k)}$

By setting

$$\mathcal{G}_{\mathsf{FCat}^{(k)}}(t) := \mathrm{ev}\left(\mathbf{f}_{\mathsf{FCat}^{(k)}}\right)$$

and

$$\mathcal{G}^{(a)}_{\mathsf{FCat}^{(k)}}(t) := \mathrm{ev}\left(\mathbf{f}^{(a)}_{\mathsf{FCat}^{(k)}}\right),$$

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we obtain

$$\mathcal{G}_{\mathsf{FCat}^{(k)}}(t) = t + \sum_{0 \leqslant a \leqslant k} \mathcal{G}^{(a)}_{\mathsf{FCat}^{(k)}}(t)$$

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and

$$\mathcal{G}^{(a)}_{\mathsf{FCat}^{(k)}}(t) = t\mathcal{G}_{\mathsf{FCat}^{(k)}}(t) + \mathcal{G}_{\mathsf{FCat}^{(k)}}(t) \left(\sum_{a+1 \leqslant b \leqslant k} \mathcal{G}^{(b)}_{\mathsf{FCat}^{(k)}}(t) \right).$$

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and

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This leads to

$$\mathcal{G}^{(a)}_{\mathsf{FCat}^{(k)}}(t) = t\mathcal{G}_{\mathsf{FCat}^{(k)}}(t)\left(1 + \mathcal{G}_{\mathsf{FCat}^{(k)}}(t)\right)^{k-a}$$

Enumeration in $\mathsf{FCat}^{(k)}$

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we obtain

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$$\mathcal{G}^{(a)}_{\mathsf{FCat}^{(k)}}(t) = t\mathcal{G}_{\mathsf{FCat}^{(k)}}(t) + \mathcal{G}_{\mathsf{FCat}^{(k)}}(t)\left(\sum_{a+1 \leqslant b \leqslant k} \mathcal{G}^{(b)}_{\mathsf{FCat}^{(k)}}(t)\right).$$

This leads to

$$\mathcal{G}^{(a)}_{\mathsf{FCat}^{(k)}}(t) = t\mathcal{G}_{\mathsf{FCat}^{(k)}}(t)\left(1 + \mathcal{G}_{\mathsf{FCat}^{(k)}}(t)\right)^{k-a}$$

and

$$\mathcal{G}_{\mathsf{FCat}^{(k)}}(t) = t + t\mathcal{G}_{\mathsf{FCat}^{(k)}}(t) \sum_{0 \leqslant a \leqslant k} \left(1 + \mathcal{G}_{\mathsf{FCat}^{(k)}}(t)\right)^{k-a}.$$

Statistics on $\mathsf{FCat}^{(k)}$

Let us add some parameters in the expressions of $\mathbf{f}_{\mathsf{FCat}^{(k)}}$ to obtain

$$\mathbf{g}_{\mathsf{FCat}^{(k)}} =$$

$$+ \sum_{0 \leqslant a \leqslant k} \mathbf{g}^{(a)}_{\mathsf{FCat}^{(k)}},$$

where the $\mathbf{g}^{(a)}_{\mathsf{FCat}^{(k)}}$ are the $\mathsf{FCat}^{(k)}$-series satisfying

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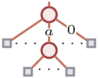

$$\circ \Bigg[$$

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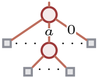
$\mathbf{g}^{(a)}_{\mathsf{FCat}^{(k)}} = q_a$ 

$\circ \left[\right.$  $\left. + \sum_{a+1 \leqslant b \leqslant k} \mathbf{g}^{(b)}_{\mathsf{FCat}^{(k)}}, \mathbf{g}_{\mathsf{FCat}^{(k)}} \right].$

We obtain, when $k := 1$,

$$\begin{aligned}
\mathcal{G}(t, q_0, q_1) := \mathrm{ev}\left(\mathbf{g}_{\mathsf{FCat}^{(1)}}\right) = t &+ (q_0 + q_1)t^2 + (q_0^2 + 3q_0q_1 + q_1^2)t^3 \\
&+ (q_0^3 + 6q_0^2q_1 + 6q_0q_1^2 + q_1^3)t^4 \\
&+ (q_0^4 + 10q_0^3q_1 + 20q_0^2q_1^2 + 10q_0q_1^3 + q_1^4)t^5 \\
&+ (q_0^5 + 15q_0^4q_1 + 50q_0^3q_1^2 + 50q_0^2q_1^3 + 15q_0q_1^4 + q_1^5)t^6 \\
&+ (q_0^6 + 21q_0^5q_1 + 105q_0^4q_1^2 + 175q_0^3q_1^3 + 105q_0^2q_1^4 + 21q_0q_1^5 + q_1^6)t^7 + \cdots .
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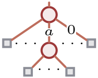
$$\mathbf{g}^{(a)}_{\mathsf{FCat}^{(k)}} = q_a$$ 

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This is Triangle **A001263**, known as the triangle of Narayana numbers.

General result

Theorem

Let $\mathcal{O}$ be an operad admitting a presentation $(\mathfrak{G}, \equiv)$ and $\mathcal{B}$ be a PBW basis of $\mathcal{O}$ for $(\mathfrak{G}, \equiv)$. Let the $\mathcal{O}$-series $\mathbf{f}$ satisfying

$$\mathbf{f} = \mathbb{1} + \sum_{x \in \mathfrak{G}} \mathbf{f}_x,$$

where for any syntax tree $\mathfrak{t}$ on $\mathfrak{G}$ of arity n, $\mathbf{f}_{\mathfrak{t}}$ satisfies

$$\mathbf{f}_{\mathfrak{t}} = \mathfrak{t} \circ \left[\mathbf{f} - \sum_{\mathfrak{t} \circ_1 \mathfrak{s} \notin \mathcal{B}} \mathbf{f}_{\mathfrak{s}}, \; \ldots, \; \mathbf{f} - \sum_{\mathfrak{t} \circ_n \mathfrak{s} \notin \mathcal{B}} \mathbf{f}_{\mathfrak{s}} \right].$$

Then, $\mathbf{f}$ is the characteristic series of $\mathcal{O}$.

Contents

From monoids to operads

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- For any $u \in \mathsf{T}\mathcal{M}(n)$ and $v \in \mathsf{T}\mathcal{M}(m)$,

$$u \circ_i v := u_1 \dots u_{i-1}(u_i * v_1) \dots (u_i * v_m) u_{i+1} \dots u_n.$$

Example

In $\mathsf{T}(\mathbb{N}, +, 0)$,

$$2100213 \circ_5 3001 = 2100522313.$$

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In $\mathsf{T}(\mathbb{N}, +, 0)$,

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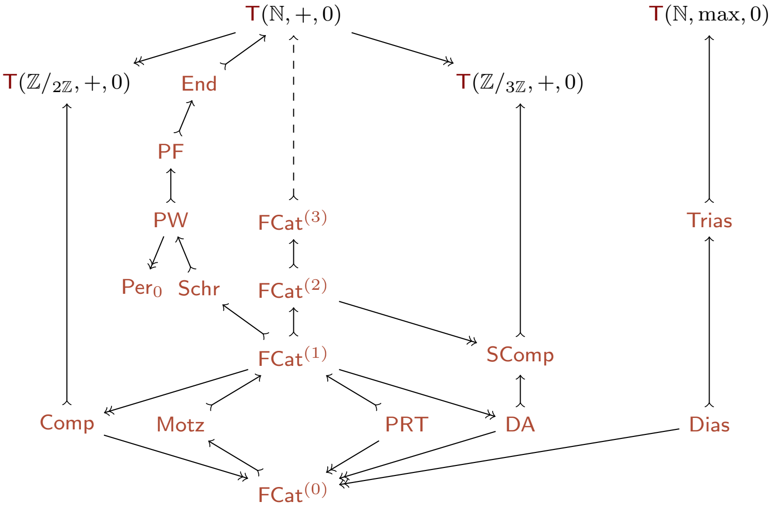
Theorem

For any monoid $\mathcal{M}$, $\mathsf{T}\mathcal{M}$ is an operad.

From monoids to operads — main examples

Monoid	Operad	Generators	First dimensions	Combinatorial objects
$(\mathbb{N}, +, 0)$	End	–	$1, 4, 27, 256, 3125$	Endofunctions
	PF	–	$1, 3, 16, 125, 1296$	Parking functions
	PW	–	$1, 3, 13, 75, 541$	Packed words
	Per_0	–	$1, 2, 6, 24, 120$	Permutations
	PRT	01	$1, 1, 2, 5, 14, 42$	Planar rooted trees
	$\mathsf{FCat}^{(k)}$	$00, 01, \dots, 0k$	Fuß-Catalan numbers	k-trees
	Schr	$00, 01, 10$	$1, 3, 11, 45, 197$	Schröder trees
	Motz	$00, 010$	$1, 1, 2, 4, 9, 21, 51$	Motzkin words
$(\mathbb{Z}/_{2\mathbb{Z}}, +, 0)$	Comp	$00, 01$	$1, 2, 4, 8, 16, 32$	Compositions
$(\mathbb{Z}/_{3\mathbb{Z}}, +, 0)$	DA	$00, 01$	$1, 2, 5, 13, 35, 96$	Directed animals
	SComp	$00, 01, 02$	$1, 3, 27, 81, 243$	Seg. compositions
$(\mathbb{N}, \max, 0)$	Dias	$01, 10$	$1, 2, 3, 4, 5$	Bin. words with exact. one 0
	Trias	$00, 01, 10$	$1, 3, 7, 15, 31$	Bin. words with at least one 0

From monoids to operads — diagram



From posets to operads

Let $(\mathcal{P}, \preccurlyeq)$ be a finite poset. We denote by $\uparrow$ the binary (partial operation) $\min$ on $\mathcal{P}$ with respect to the order relation $\preccurlyeq$.

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and $\equiv$ is the smallest operad congruence satisfying

$$\star_a \circ_1 \star_b \equiv \star_{a \uparrow b} \circ_1 \star_{a \uparrow b} \equiv \star_{a \uparrow b} \circ_2 \star_{a \uparrow b} \equiv \star_a \circ_2 \star_b, \qquad a \preccurlyeq b \text{ or } b \preccurlyeq a.$$

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Example

Let the poset $\mathcal{P} :=$ 

.

Then, $\mathsf{P}\mathcal{P}$ is generated by $\{\star_1, \star_2, \star_3, \star_4\}$ and these generators are subjected to the relations

$$\star_1 \circ_1 \star_1 \equiv \star_1 \circ_1 \star_3 \equiv \star_3 \circ_1 \star_1 \equiv \star_3 \circ_2 \star_1 \equiv \star_1 \circ_2 \star_3 \equiv \star_1 \circ_2 \star_1,$$

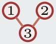
$$\star_2 \circ_1 \star_2 \equiv \star_2 \circ_1 \star_3 \equiv \star_2 \circ_1 \star_4 \equiv \star_3 \circ_1 \star_2 \equiv \star_4 \circ_1 \star_2$$
$$\equiv \star_4 \circ_2 \star_2 \equiv \star_3 \circ_2 \star_2 \equiv \star_2 \circ_2 \star_4 \equiv \star_2 \circ_2 \star_3 \equiv \star_2 \circ_2 \star_2,$$

$$\star_3 \circ_1 \star_3 \equiv \star_3 \circ_2 \star_3, \qquad \star_4 \circ_1 \star_4 \equiv \star_4 \circ_2 \star_4.$$

From posets to operads — PBW basis

Theorem

When $\mathcal{P}$ is a finite poset avoiding the pattern



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the operad $\mathsf{P}\mathcal{P}$ is a Koszul operad and the set of all syntax trees on $\mathfrak{G}$ avoiding

$$\star_a \circ_1 \star_b, \qquad a \preccurlyeq b \text{ or } b \preccurlyeq a$$

and

$$\star_a \circ_2 \star_b, \qquad a \neq b \text{ and } (a \preccurlyeq b \text{ or } b \preccurlyeq a).$$

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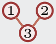

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When the premises of this theorem hold, $\mathsf{P}\mathcal{P}$ can be realized in terms of an operad of Schröder trees labeled on $\mathcal{P}$.

From posets to operads — example

More precisely, the elements of $\mathsf{P}\mathcal{P}(n)$ are Schröder trees with n leaves such that the internal nodes are labeled on $\mathcal{P}$ and the label of a node and any of its child are incomparable in $\mathcal{P}$.

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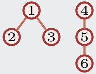
$\mathcal{P} :=$ 

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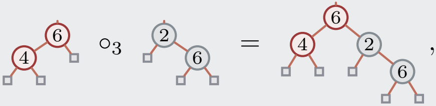
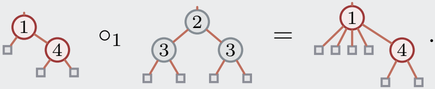
Then, in $\mathsf{P}\mathcal{P}$,





From unitary magmas to operads — cliques

Let $(\mathcal{M}, *, \mathbb{1}_{\mathcal{M}})$ be a unitary magma.

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An $\mathcal{M}$-clique $\mathfrak{p}$ is a clique on $[n+1]$ where each edge (x, y) is labeled by an element $\mathfrak{p}(x, y)$ of $\mathcal{M}$. The arity $|\mathfrak{p}|$ of $\mathfrak{p}$ is n.

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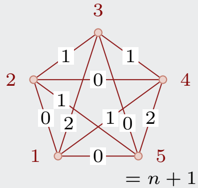
Example

Let $\mathcal{M}$ be the unitary magma $(\mathbb{Z}/_{3\mathbb{Z}}, +, 0)$. Here is an $\mathcal{M}$-clique:



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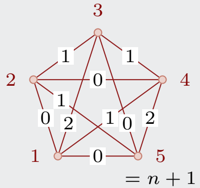

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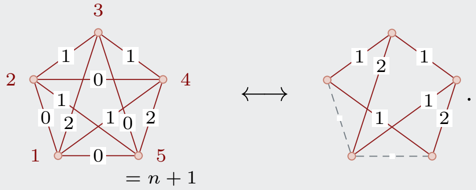

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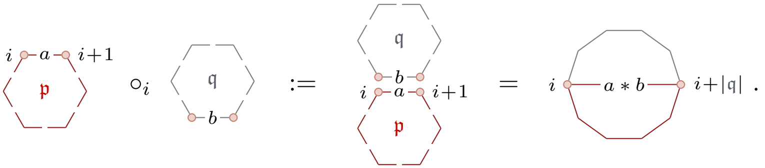

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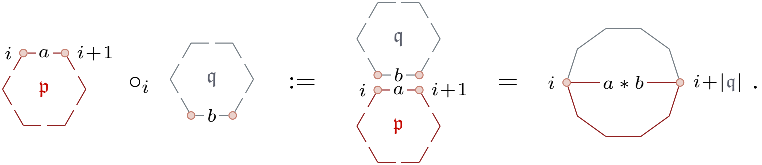

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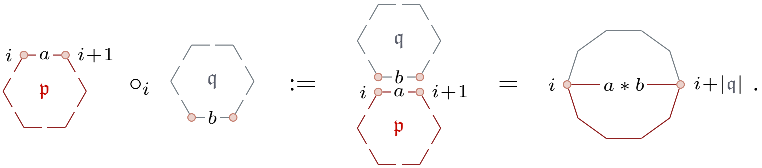

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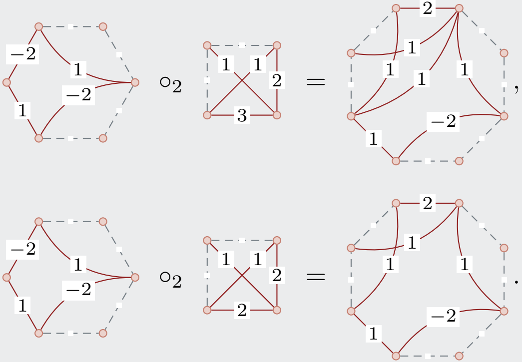
Theorem

For any unitary magma $\mathcal{M}$, $\mathsf{C}\mathcal{M}$ is an operad.

From unitary magmas to operads — example

Exemple

In $\mathsf{C}(\mathbb{Z}, +, 0)$,



From unitary magmas to operads — main examples

The construction C leads to alternative definitions of

- the operad NCT of noncrossing trees [**Chapoton**, 2007];
- the operad FF_4 of some formal fractions [**Chapoton, Hivert, Novelli**, 2016];
- the operad BNC of bicolored noncrossing configurations [**Chapoton, Giraudo**, 2014];
- the operad MT of multi-tildes [**Luque, Mignot, Nicart**, 2013];
- the operad DMT of double multi-tildes [**Giraudo, Luque, Mignot, Nicart**, 2016];
- the dipterous operad [**Loday, Ronco**, 2003];
- the gravity operad [**Getzler**, 1994].