Operads of musical phrases and generation

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Outline

1. Operads and constructions

2. The music box operad

3. Bud generating systems

Outline

1. Operads and constructions

Nonsymmetric operads

A **nonsymmetric operad** is a triple $(\mathcal{O}, \circ, \mathbb{1})$ where

- $\mathcal{O}$ is a graded set $\mathcal{O} = \bigsqcup_{n \geqslant 1} \mathcal{O}(n)$;
- $\circ : \mathcal{O}(n) \times (\mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n)) \to \mathcal{O}(m_1 + \cdots + m_n)$ is a map called **full composition**;
- $\mathbb{1}$ is an element of $\mathcal{O}(1)$ called **unit**.

The following relations have to hold:

- for all $x \in \mathcal{O}(n)$, $y_i \in \mathcal{O}(m_i)$, and $z_{i,j} \in \mathcal{O}$, $i \in [n]$, $j \in [m_i]$,

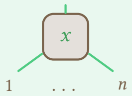
$$(x \circ y_1 \dots y_n) \circ z_{1,1} \dots z_{1,m_1} \dots z_{n,1} \dots z_{n,m_n} = x \circ (y_1 \circ z_{1,1} \dots z_{1,m_1}) \dots (y_n \circ z_{n,1} \dots z_{n,m_n});$$

- for all $x \in \mathcal{O}(n)$,

$$\mathbb{1} \circ x = x = x \circ \mathbb{1}^n.$$

Intuition

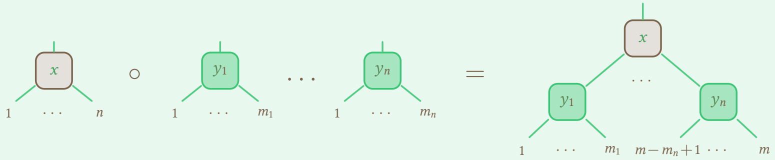
Each element x of $\mathcal{O}(n)$ can be seen as a **planar operator**, that is an entity having n inputs and a single output:



The **arity** $|x|$ of x is its number n of inputs, numbered from 1 to n from left to right.

The full composition $x \circ y_1 \ldots y_n$ consists in **grafting** the output of each y_i onto the i-th input of x.

This produces a new operator



of arity $m := m_1 + \cdots + m_n$.

Partial and homogeneous composition maps

Let $\mathcal{O}$ be an operad.

The **partial composition map** of $\mathcal{O}$ is the map $\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n+m-1)$ defined, for any $x \in \mathcal{O}(n)$, $i \in [n]$, and $y \in \mathcal{O}(m)$, by

$$x \circ_i y := x \circ \mathbb{1}^{i-1}\, y\, \mathbb{1}^{n-i}.$$

Conversely, we recover $\circ$ from the $\circ_i$ by setting, for any $x \in \mathcal{O}(n)$ and $y_1 \dots y_n \in \mathcal{O}^n$,

$$x \circ y_1 \dots y_n := (\dots (x \circ_n y_n) \circ_{n-1} y_{n-1} \dots) \circ_1 y_1.$$

The **homogeneous composition map** of $\mathcal{O}$ is the map $\odot : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(nm)$ defined, for any $x \in \mathcal{O}(n)$ and $y \in \mathcal{O}(m)$, by

$$x \odot y := x \circ y^n.$$

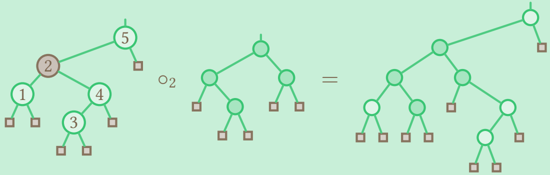
The triple $(\mathcal{O}, \odot, \mathbb{1})$ is a monoid.

Example: the duplicial operad

The **duplicial operad** [**Loday**, 2008] is the operad $(\mathbf{Dup}, \circ_i, \mathbb{1})$ such that

- for any $n \geqslant 1$, $\mathbf{Dup}(n)$ is the set of all binary trees with n internal nodes;
- for any $\mathfrak{t} \in \mathbf{Dup}(n)$, $i \in [n]$, and $\mathfrak{s} \in \mathbf{Dup}$, $\mathfrak{t} \circ_i \mathfrak{s}$ is the binary tree obtained by replacing the i-th internal node u of $\mathfrak{t}$ (for the infix traversal) by $\mathfrak{s}$ and by grafting onto the leftmost (resp. rightmost) leaf of $\mathfrak{s}$ the left (resp. right) child of u;
- $\mathbb{1}$ is the binary tree with exactly one internal node.

– Example –



The construction T

Let $(\mathcal{M}, \star, e)$ be a monoid and let $(\mathbf{T}\mathcal{M}, \circ_i, \mathbb{1})$ be the triple such that

- for any $n \geqslant 1$, $\mathbf{T}\mathcal{M}(n)$ is the set $\mathcal{M}^n$;
- for any $u \in \mathbf{T}\mathcal{M}(n)$, $i \in [n]$, and $v \in \mathbf{T}\mathcal{M}$,

$$u \circ_i v := u(1, i-1) \ \ (u(i) \star v(1)) \dots (u(i) \star v(\ell(v))) \ \ u(i+1, \ell(u));$$

- $\mathbb{1}$ is the element e seen as a word of length 1.

– Examples –

Set $\mathcal{M} := (\{\mathbf{a}, \mathbf{b}\}^*, ., \epsilon)$. In $\mathbf{T}\mathcal{M}$,

$$(\mathbf{aa}, \epsilon, \mathbf{bab}, \epsilon, \mathbf{b}) \in \mathbf{T}\mathcal{M}(5)$$

and

$$(\mathbf{b}, \mathbf{ab}, \epsilon, \mathbf{a}) \circ_2 (\epsilon, \mathbf{a}, \mathbf{aa}) = (\mathbf{b}, \mathbf{ab}.\epsilon, \mathbf{ab}.\mathbf{a}, \mathbf{ab}.\mathbf{aa}, \epsilon, \mathbf{a}) = (\mathbf{b}, \mathbf{ab}, \mathbf{aba}, \mathbf{abaa}, \epsilon, \mathbf{a}).$$

– Theorem [G., 2015] –

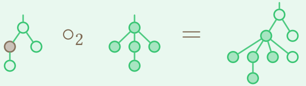
For any monoid $\mathcal{M}$, $\mathbf{T}\mathcal{M}$ is an operad.

Operads from the construction T

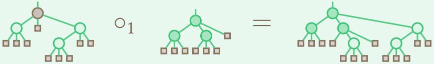
The operads $\mathbf{T}\mathcal{M}$ are large enough to contain a lot of suboperads realizable in combinatorial terms. As main examples:

- For any $m \geqslant 0$, with $\mathcal{M} := (\mathbb{N}, +, 0)$,
  - $\mathbf{PRT}_m$, generated by $\{01, \dots, 0m\}$, on primitive m-Dyck paths;
  - $\mathbf{FCat}_m$, gen. by $\{00, 01, \dots, 0m\}$, on m-trees;
  - $\mathbf{Schr}_m$, gen. by $\{01, \dots, 0m, 00, m0, \dots, 10\}$, on some Schröder trees;
  - $\mathbf{Motz}_m$, gen. by $\{00, 000, 010, \dots, 0m0\}$, on colored Motzkin paths.
- For any $m \geqslant 0$, with $\mathcal{M} := (\mathbb{Z}/(m+1)\mathbb{Z}, +, 0)$,
  - $\mathbf{Comp}_m$, gen. by $\{00, 01, \dots, 0m\}$, on m-words;
  - $\mathbf{DA}_m$, gen. by $\{00, 01, \dots, 0(m-1)\}$, on some directed animals.
- For any $m \geqslant 0$, $\mathcal{M} := (\mathbb{N}, \max, 0)$,
  - $\mathbf{Dias}_m$, gen. by $\{01, \dots, 0m, m0, \dots, 10\}$, is the m-pluriassociative operad [**Loday**, 2001] [**G.**, 2016];
  - $\mathbf{Trias}_m$, gen. by $\{01, \dots, 0m, 00, m0, \dots, 10\}$, is the m-pluritriassociative operad [**Loday, Ronco**, 2004] [**G.**, 2016].

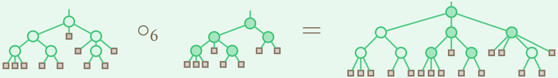
Some partial compositions on combinatorial objects



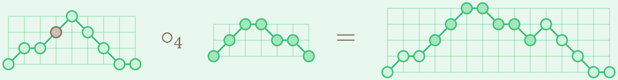
(in $\mathbf{PRT}_1$)



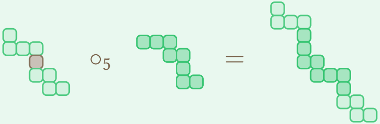
(in $\mathbf{FCat}_2$)



(in $\mathbf{Schr}_1$)

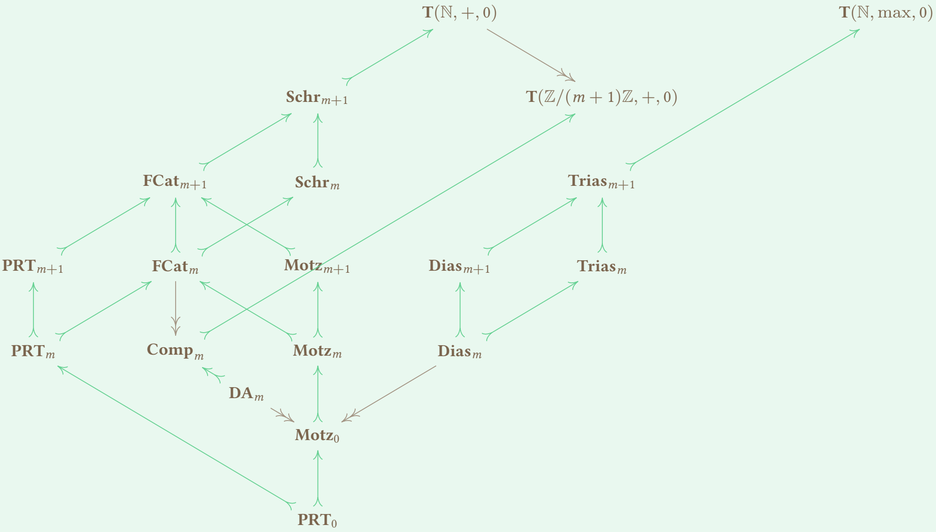


(in $\mathbf{Motz}_1$)



(in $\mathbf{Comp}_1$)

Full diagram



The construction U

Let $(\mathcal{M}, \star, e)$ be a monoid and let $(\mathbf{U}\mathcal{M}, \circ_i, \mathbb{1})$ be the triple such that

- for any $n \geqslant 1$, $\mathbf{U}\mathcal{M}(n)$ is the set $\mathcal{M}^{n+1}$;
- for any $u \in \mathbf{U}\mathcal{M}(n)$, $i \in [n]$, and $v \in \mathbf{U}\mathcal{M}$,

$$u \circ_i v := u(1, i-1) \;\; (u(i) \star v(1)) \;\; v(2, \ell(v)-1) \;\; (v(\ell(v)) \star u(i+1)) \;\; u(i+2, \ell(u));$$

- $\mathbb{1}$ is the element ee.

– Examples –

Set $\mathcal{M} := (\{\mathbf{a}, \mathbf{b}\}^*, ., \epsilon)$. In $\mathbf{U}\mathcal{M}$,

$$(\mathbf{aa}, \epsilon, \mathbf{bab}, \epsilon, \mathbf{b}) \in \mathbf{U}\mathcal{M}(4)$$

and

$$(\mathbf{ba}, \mathbf{aa}, \mathbf{b}, \epsilon, \mathbf{a}) \circ_2 (\mathbf{a}, \mathbf{bb}, \mathbf{b}) = (\mathbf{ba}, \mathbf{aa}.\mathbf{a}, \mathbf{bb}, \mathbf{b}.\mathbf{b}, \epsilon, \mathbf{a}) = (\mathbf{ba}, \mathbf{aaa}, \mathbf{bb}, \mathbf{bb}, \epsilon, \mathbf{a}).$$

– Theorem [G., 2021–] –

For any monoid $\mathcal{M}$, $\mathbf{U}\mathcal{M}$ is an operad.

2. The music box operad

Degree patterns

A **degree** is an integer.

A **degree pattern** is a nonempty word $\mathbf{d}$ of degrees. The **arity** $|\mathbf{d}|$ of $\mathbf{d}$ is its length.

Let $\mathcal{N}$ be the set of all musical notes n_k where n is the pitch class and k is the octave of the note.

A **degree interpretation** is a map $\rho : \mathbb{Z} \to \mathcal{N}$ sending each degree to a note. The ρ-**interpretation** $\rho(\mathbf{d})$ of $\mathbf{d}$ is the sequence $\rho(\mathbf{d}(1)) \dots \rho(\mathbf{d}(|\mathbf{d}|))$ of notes.

– Examples –

Let the degree pattern

$$\mathbf{d} := 10\bar{2}\bar{3}507.$$

A negative value has a bar above its absolute value. The arity of $\mathbf{d}$ is 7.

If ρ sends 0 to 0_4 and the other degrees according with the minor pentatonic scale,

$$\rho(\mathbf{d}) = 3_4\ 0_4\ 7_3\ 5_3\ 0_5\ 0_4\ 5_5.$$

Instead, if ρ sends 0 to 0_4 and the other degrees according with the major natural scale,

$$\rho(\mathbf{d}) = 2_4\ 0_4\ 9_3\ 7_3\ 9_4\ 0_4\ 0_5.$$

Rhythm patterns

A **rhythm pattern** is a nonempty word $\mathbf{r}$ on the alphabet $\{\square, \blacksquare\}$ having at least one occurrence of $\blacksquare$. The **arity** $|\mathbf{r}|$ of $\mathbf{r}$ is its number of occurrences of $\blacksquare$.

The **duration sequence** of a rhythm pattern $\mathbf{r}$ is the unique sequence $\sigma := \sigma(1) \dots \sigma(|\mathbf{r}| + 1)$ of nonnegative integers such that

$$\mathbf{r} = \square^{\sigma(1)} \; \blacksquare \; \square^{\sigma(2)} \dots \; \blacksquare \; \square^{\sigma(|\mathbf{r}|+1)} .$$

A **rhythm interpretation** is a positive integer value δ specifying the duration of the unit of time. The **δ-interpretation** $\delta(\mathbf{r})$ of $\mathbf{r}$ is the sequence $\sigma(1)\delta, (\sigma(2) + 1)\delta, \dots, (\sigma(|\mathbf{r}| + 1) + 1)\delta$. It specifies the duration of the initial rest and the durations of the other beats.

– Examples –

Let the rhythm pattern

$$\mathbf{r} := \square\square\blacksquare\blacksquare\square\blacksquare\square\square\square .$$

The arity of $\mathbf{r}$ is 3.

The duration sequence of $\mathbf{r}$ is 2013 so that $\mathbf{r}$ specifies the rhythm consisting in a rest lasting 2 units of time, a note lasting 1 unit of time, a note lasting 2 units of time, and a note lasting 4 units of time.

Patterns

A **pattern** $\mathbf{p}$ is a pair $(\mathbf{d}, \mathbf{r})$ such that $|\mathbf{d}| = |\mathbf{r}|$.

The **arity** $|\mathbf{p}|$ of $\mathbf{p}$ is the arity of $\mathbf{d}$ (or of $\mathbf{r}$).

– Example –

The pattern $\mathbf{p} := (1\bar{1}2, \blacksquare\blacksquare\square\square\square\blacksquare\square)$ is of arity 3.

Patterns are denoted concisely by replacing each $\blacksquare$ by the corresponding degree. In this way, patterns are words on the alphabet $\{\square\} \cup \mathbb{Z}$.

– Example –

The previous patterns writes as $\mathbf{p} = 1\bar{1}\square\square\square2\square$.

An **interpretation** is a pair (ρ, δ) such that ρ is a degree interpretation and δ is a rhythm interpretation. The **(ρ, δ)-interpretation** of a pattern $\mathbf{p}$ is the sequence of notes with their durations obtained from the ρ-interpretation of $\mathbf{p}$ and the δ-interpretation of $\mathbf{p}$.

By convention, in the following musical scores, each unit of time lasts $\frac{1}{8}$ the duration of a whole note.

– Example –

Let the pattern $\mathbf{p} := 0\square12\bar{1}\square01\bar{2}\square\bar{1}\bar{0}0\square\square\square$. By setting ρ as the degree interpretation specifying the 9_3 harmonic minor scale and δ as the rhythm interpretation specifying 192 as tempo, we obtain the musical phrase

Multi-patterns

A **multi-pattern** is a nonempty sequence $\mathbf{m}$ of patterns such that for all $i, j \in [\ell(\mathbf{m})]$, $|\mathbf{m}(i)| = |\mathbf{m}(j)|$ and $\ell(\mathbf{m}(i)) = \ell(\mathbf{m}(j))$.

The **arity** $|\mathbf{m}|$ of $\mathbf{m}$ is the arity of any $\mathbf{m}(i)$.

The **length** $\ell(\mathbf{m})$ of $\mathbf{m}$ is the length of any $\mathbf{m}(i)$.

The **multiplicity** $\mathfrak{m}(\mathbf{m})$ of $\mathbf{m}$ is $\ell(\mathbf{m})$.

Multi-patterns are denoted as matrices by stacking their patterns.

– Example –

Let the multi-pattern

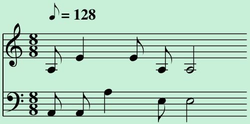
$$\mathbf{m} := \begin{bmatrix} 0 & 4 & \square & 4 & 0 & 0 & \square & \square & \square \\ \bar{7} & \bar{7} & 0 & \square & \bar{3} & \bar{3} & \square & \square & \square \end{bmatrix}.$$

The arity of $\mathbf{m}$ is 5, its length is 9, and its multiplicity is 2.

By interpreting each pattern of a multi-pattern through an interpretation, a multi-pattern specifies a musical phrase consisting in stacked voices.

– Example –

The previous multi-pattern, interpreted through the degree interpretation specifying the 9_3 harmonic minor scale and the rhythm interpretation specifying 128 as tempo gives the musical phrase



Operads of degree patterns

A **degree monoid** is a monoid $(D, \star, e)$ such that $D \subseteq \mathbb{Z}$.

The D-**degree pattern operad** is the operad $\mathbf{DP}^D := \mathbf{T}D$. The elements of $\mathbf{DP}^D$ are degree patterns on D.

– Examples –

By denoting by $\mathbb{Z}$ the monoid $(\mathbb{Z}, +, 0)$, we have in $\mathbf{DP}^{\mathbb{Z}}$,

$$01\mathbf{2}\bar{1} \circ_3 024 = 01\ 246\ \bar{1},$$

$$012\bar{1} \odot 024 = 024\ 135\ 246\ \bar{1}13.$$

By denoting, for any $k \geqslant 1$, by $\mathbb{C}_k$ the cyclic monoid $(\mathbb{Z}/k\mathbb{Z}, +, 0)$, we have in $\mathbf{DP}^{\mathbb{C}_3}$,

$$200\mathbf{1}0 \circ_4 2120 = 200\ 0201\ 0.$$

By denoting, for any subset Z of $\mathbb{Z}$ having a lower bound z, by $\mathbb{M}_Z$ the monoid $(Z, \max, z)$, we have in $\mathbf{DP}^{\mathbb{M}_{[0,2]}}$,

$$200\mathbf{1}0 \circ_4 2120 = 200\ 2121\ 0.$$

Operad of rhythm patterns

Let $\mathbb{N}$ be the monoid $(\mathbb{N}, +, 0)$.

The **rhythm pattern operad** is the operad $\mathbf{RP} := \mathsf{U}\mathbb{N}$. The elements of $\mathbf{RP}$ are duration sequences.

– Example –

In **RP**,

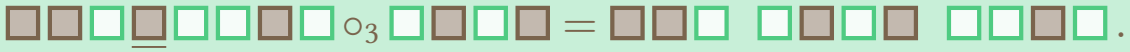
$$001\_21 \circ_3 110 = 00\,212\,1.$$

Since duration sequences and rhythm patterns are in one-to-one correspondence, **RP** can be seen as an operad on rhythm patterns.

On rhythm patterns, the partial composition of **RP** expresses as follows: if $\mathbf{r}$ and $\mathbf{r}'$ are two rhythm patterns, then $\mathbf{r} \circ_i \mathbf{r}'$ is obtained by replacing the i-th beat of $\mathbf{r}$ by $\mathbf{r}'$.

– Example –

The previous composition, seen on rhythm patterns, translates as



Operads of patterns

The **Hadamard product** of two operads $\mathcal{O}$ and $\mathcal{O}'$ is the operad $\mathcal{O} \boxtimes \mathcal{O}'$ such that for any $n \geqslant 1$,

$$(\mathcal{O} \boxtimes \mathcal{O}')(n) := \mathcal{O}(n) \times \mathcal{O}'(n)$$

and for any $(x, x'), (y, y') \in \mathcal{O} \boxtimes \mathcal{O}'$,

$$(x, x') \circ_i (y, y') = (x \circ_i y, x' \circ_i y').$$

Let D be a degree monoid.

The D-**pattern operad** is the operad $\mathbf{P}^D := \mathbf{DP}^D \boxtimes \mathbf{RP}$. The elements of $\mathbf{P}^D$ are pairs $(\mathbf{d}, \mathbf{r})$ such that $|\mathbf{d}| = |\mathbf{r}|$. Therefore, the elements of $\mathbf{P}^D$ are patterns.

– Examples –

In $\mathbf{P}^{\mathbb{Z}}$, we have

$$(\bar{2}31, \square\blacksquare\blacksquare\square\blacksquare) \circ_2 (0\bar{1}, \blacksquare\square\blacksquare) = (\bar{2}\;32\;1, \square\blacksquare\;\;\blacksquare\square\blacksquare\;\;\square\blacksquare)$$

which translates through the concise notation for patterns as

$$\square\,\bar{2}3\,\square\,1 \circ_2 0\,\square\,\bar{1} = \square\,\bar{2}3\,\square\,2\,\square\,1.$$

Operads of multi-patterns

Let D be a degree monoid and $m \geqslant 1$. Let $\mathbf{P}_m^{D\prime}$ be the operad defined as

$$\mathbf{P}_m^{D\prime} := \underbrace{\mathbf{P}^D \boxtimes \cdots \boxtimes \mathbf{P}^D}_{m \text{ terms}}.$$

Let also $\mathbf{P}_m^D$ be the subset of the underlying graded set of $\mathbf{P}_m^{D\prime}$ restrained to the sequences $\mathbf{m}(1) \dots \mathbf{m}(m)$ such that $\ell(\mathbf{m}(1)) = \cdots = \ell(\mathbf{m}(m))$.

– Theorem [G., 2021–] –

For any degree monoid D and any positive integer m, $\mathbf{P}_m^D$ is an operad.

We call $\mathbf{P}_m^D$ the D-**music box operad**.

– Example –

In $\mathbf{P}_2^{\mathbb{Z}}$,

$$\begin{bmatrix} \square & \bar{2} & \bar{1} & \square & 0 \\ 0 & 1 & \square & \square & 1 \end{bmatrix} \circ_2 \begin{bmatrix} 1 & \square \\ \bar{3} & \square \end{bmatrix} = \begin{bmatrix} \square & \bar{2} & 0 & \square & \square & 0 \\ 0 & \bar{2} & \square & \square & \square & 1 \end{bmatrix}.$$

Operations on musical phrases

Every element of $\mathbf{P}_m^D(n)$ can be seen as an operator of arity n acting on multi-patterns.

Here are some examples.

- If $\mathbf{m}$ is an arpeggio shape and $\mathbf{p}$ is a pattern, $\mathbf{p}' \odot \mathbf{m}$ is an arpeggiation of $\mathbf{p}$, where $\mathbf{p}'$ is the multi-pattern obtained by stacking $\mathfrak{m}(\mathbf{m})$ equal voices from $\mathbf{p}$.

– Example –

If $\mathbf{p} := 210\square 1\square$ and $\mathbf{m} := \begin{bmatrix} 0 & \square & \square \\ \square & 4 & \square \\ \square & \square & 2 \end{bmatrix}$, we obtain $\begin{bmatrix} 2 & \square & \square & 1 & \square & \square & 0 & \square & \square & \square & 1 & \square & \square & \square \\ \square & 6 & \square & \square & 5 & \square & \square & 4 & \square & \square & \square & 5 & \square & \square \\ \square & \square & 4 & \square & \square & 3 & \square & \square & 2 & \square & \square & \square & 3 & \square \end{bmatrix}$.

- If $\mathbf{m}$ is a chord shape and $\mathbf{p}$ is a pattern, $\mathbf{p}' \odot \mathbf{m}$ is an harmonization of $\mathbf{p}$, where $\mathbf{p}'$ is the multi-pattern obtained by stacking $\mathfrak{m}(\mathbf{m})$ equal voices from $\mathbf{p}$.

– Example –

If $\mathbf{p} := 210\square 1\square$ and $\mathbf{m} := \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, we obtain $\begin{bmatrix} 2 & 1 & 0 & \square & 1 & \square \\ 6 & 5 & 4 & \square & 5 & \square \end{bmatrix}$.

3. Bud generating systems

Colored operads

Let $\mathfrak{C}$ be a nonempty set, called **set of colors**.

A $\mathfrak{C}$**-colored operad** is a triple $(\mathcal{C}, \circ, \mathbb{1})$ where

- $\mathcal{C}$ is a set
$$\mathcal{C} = \bigsqcup_{\substack{a \in \mathfrak{C} \\ u \in \mathfrak{C}^+}} \mathcal{C}(a, u);$$
- $\circ : \mathcal{C}(a, u) \times \left(\mathcal{C}(u(1), v_1) \times \cdots \times \mathcal{C}\left(u(\ell(u)), v_{\ell(u)}\right)\right) \to \mathcal{C}\left(a, v_1 \dots v_{\ell(u)}\right)$ is a map called **colored full composition**;
- $\mathbb{1} : \mathfrak{C} \to \mathcal{C}(a, a)$ is a map called **colored unit**.

This data satisfies similar relations than the ones of operads.

Intuitively, in a colored operad, each element has an output color and a color for each input. The composition is defined only when colors match.

Bud operads

Let $(\mathcal{O}, \circ, \mathbb{1})$ be an operad and $\mathfrak{C}$ be a set of colors.

Let $(\mathrm{B}_{\mathfrak{C}}\mathcal{O}, \circ', \mathbb{1}')$ be the triple such that

- for any $a \in \mathfrak{C}$ and $u \in \mathfrak{C}^+$,

$$(\mathrm{B}_{\mathfrak{C}}\mathcal{O})(a, u) := \{(a, x, u) : x \in \mathcal{O}(\ell(u))\};$$

- for any $(a, x, u) \in \mathrm{B}_{\mathfrak{C}}\mathcal{O}$ and $(u(i), y_i, v_i) \in \mathrm{B}_{\mathfrak{C}}\mathcal{O}$, $i \in [\ell(u)]$,

$$(a, x, u) \circ' (u(1), y_1, v_1) \dots \left(u(\ell(u)), y_{\ell(u)}, v_{\ell(u)}\right) = \left(a, x \circ y_1 \dots y_{\ell(u)}, v_1 \dots v_{\ell(u)}\right);$$

- for any $c \in \mathfrak{C}$, $\mathbb{1}'(c) := (c, \mathbb{1}, c)$.

– Proposition [G., 2019] –

For any operad $\mathcal{O}$ and any set of colors $\mathfrak{C}$, $\mathrm{B}_{\mathfrak{C}}\mathcal{O}$ is a $\mathfrak{C}$-colored operad.

Bud generating systems

A **bud generating system** is a tuple $\mathcal{B} := (\mathcal{O}, \mathfrak{C}, \mathcal{R}, \mathbf{b})$ where

- $(\mathcal{O}, \circ_i, \mathbb{1})$ is an operad, called **ground operad**;
- $\mathfrak{C}$ is a finite set of colors;
- $\mathcal{R}$ is a finite subset of $\mathrm{B}_{\mathfrak{C}}(\mathcal{O})$, called **set of rules**;
- $\mathbf{b}$ is a color of $\mathfrak{C}$, called **initial color**.

Let $\overset{\circ}{\to}$ be the binary relation on $\mathrm{B}_{\mathfrak{C}}\mathcal{O}$ such that $(a, x, u) \overset{\circ}{\to} (a, y, v)$ if there are rules $r_1, \ldots, r_{|x|} \in \mathcal{R}$ such that

$$(a, y, v) = (a, x, u) \circ r_1 \ldots r_{|x|}.$$

An element x of $\mathcal{O}$ is **fully generated** by $\mathcal{B}$ if there is an element $(\mathbf{b}, x, u)$ such that $(\mathbf{b}, \mathbb{1}, \mathbf{b})$ is in relation with $(\mathbf{b}, x, u)$ w.r.t. the reflexive and transitive closure of $\overset{\circ}{\to}$.

Bud generating systems

– Example –

Let the bud generating system $\mathcal{B} := \left(\mathbf{P}_2^{\mathbb{Z}}, \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}, \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5\}, \mathbf{b}_1\right)$ where

$$\mathbf{c}_1 := \left(\mathbf{b}_1, \begin{bmatrix} 0 & 2 & \square & 1 & \square & 0 & 4 \\ \bar{5} & \square & \square & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b}_3\mathbf{b}_2\mathbf{b}_1\mathbf{b}_1\mathbf{b}_3\right), \quad \mathbf{c}_2 := \left(\mathbf{b}_1, \begin{bmatrix} 1 & \square & 0 \\ 0 & \square & 1 \end{bmatrix}, \mathbf{b}_1\mathbf{b}_1\right),$$

$$\mathbf{c}_3 := \left(\mathbf{b}_2, \begin{bmatrix} \bar{1} \\ \bar{1} \end{bmatrix}, \mathbf{b}_1\right), \quad \mathbf{c}_4 := \left(\mathbf{b}_2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{b}_1\mathbf{b}_1\right), \quad \mathbf{c}_5 := \left(\mathbf{b}_3, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_3\right).$$

Since

$$\left(\mathbf{b}_1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_1\right) \xrightarrow{\circ} \left(\mathbf{b}_1, \begin{bmatrix} 0 & 2 & \square & 1 & \square & 0 & 4 \\ \bar{5} & \square & \square & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b}_3\mathbf{b}_2\mathbf{b}_1\mathbf{b}_1\mathbf{b}_3\right)$$

$$\xrightarrow{\circ} \left(\mathbf{b}_1, \begin{bmatrix} 0 & 1 & \square & 2 & \square & 1 & \square & 0 & 2 & \square & 1 & \square & 0 & 4 & 4 \\ \bar{5} & \square & \square & \bar{1} & 0 & \square & 1 & \bar{5} & \square & \square & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b}_3\mathbf{b}_1\mathbf{b}_1\mathbf{b}_1\mathbf{b}_3\mathbf{b}_2\mathbf{b}_1\mathbf{b}_1\mathbf{b}_3\mathbf{b}_3\right),$$

the multi-pattern

$$\begin{bmatrix} 0 & 1 & \square & 2 & \square & 1 & \square & 0 & 2 & \square & 1 & \square & 0 & 4 & 4 \\ \bar{5} & \square & \square & \bar{1} & 0 & \square & 1 & \bar{5} & \square & \square & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is fully generated by $\mathcal{B}$.

A random generation algorithm

For any color $a \in \mathfrak{C}$, we shall denote by $\mathcal{R}_a$ the set of all rules of $\mathcal{R}$ having a as output color.

If S is a nonempty finite set, $\mathrm{random}(S)$ returns an element of S picked uniformly at random.

Let us consider the following random generation algorithm.

Data: A bud generating system $\mathcal{B} := (\mathcal{O}, \mathfrak{C}, \mathcal{R}, \mathbf{b})$ and an integer $k \in \mathbb{N}$.

Result: A randomly generated element of $\mathcal{O}$.

begin
- $(a, x, u) \leftarrow (\mathbf{b}, \mathbb{1}, \mathbf{b})$
- **for** $j \in [k]$ **do**
  - $R \leftarrow \mathcal{R}_{u(1)} \times \cdots \times \mathcal{R}_{u(\ell(x))}$
  - **if** $R \neq \emptyset$ **then**
    - $(a, x, u) \leftarrow (a, x, u) \circ \mathrm{random}(R)$
- **return** x

A random generation algorithm

– Example –

Let the bud generating system $\mathcal{B} := \left(\mathbf{P}_2^{\mathbb{Z}}, \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}, \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5\}, \mathbf{b}_1\right)$ where

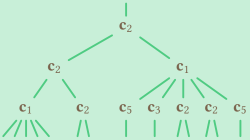
$$\mathbf{c}_1 := \left(\mathbf{b}_1, \begin{bmatrix} 0 & 2 & \square & 1 & \square & 0 & 4 \\ \bar{5} & \square & \square & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b}_3\mathbf{b}_2\mathbf{b}_1\mathbf{b}_1\mathbf{b}_3\right), \quad \mathbf{c}_2 := \left(\mathbf{b}_1, \begin{bmatrix} 1 & \square & 0 \\ 0 & \square & 1 \end{bmatrix}, \mathbf{b}_1\mathbf{b}_1\right),$$

$$\mathbf{c}_3 := \left(\mathbf{b}_2, \begin{bmatrix} \bar{1} \\ \bar{1} \end{bmatrix}, \mathbf{b}_1\right), \quad \mathbf{c}_4 := \left(\mathbf{b}_2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{b}_1\mathbf{b}_1\right), \quad \mathbf{c}_5 := \left(\mathbf{b}_3, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_3\right).$$

The previous random generation algorithm run with this bud generating system and $k := 2$ generates the multi-pattern

$$\begin{bmatrix} 2 & 4 & \square & 3 & \square & 2 & 6 & \square & 2 & \square & 1 & \square & 0 & 1 & \square & 2 & \square & 1 & \square & 1 & \square & 0 & 4 \\ \bar{5} & \square & \square & 0 & 0 & 0 & 0 & \square & 1 & \square & 2 & \square & \bar{4} & \square & \square & 0 & 1 & \square & 2 & 1 & \square & 2 & 1 \end{bmatrix}.$$

This pattern is obtained from the underlying random syntax tree



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The Bud Music Box program

An implementation of these concepts can be found at

https://github.com/SameleGiraudo/Bud-Music-Box

Here is a Bud Music Box program:

```
scale 2 1 4 1 4
root 57
tempo 192
sounds 3 1

multi_pattern mpat_1 0 2 4 * ; -5 * 0 -1
multi_pattern mpat_2 0 * ; * 0
multi_pattern mpat_3 0 ; 0

colorize cpat_1 mpat_1 c1 c3 c1 c3
colorize cpat_2 mpat_1 c1 c3 c2 c3
colorize cpat_3 mpat_2 c2 c1
colorize cpat_4 mpat_3 c3 c3

generate mpat_3 full 8 c1 cpat_1 cpat_2 cpat_3 cpat_4
show
play mpat_3
```

Here is a randomly generated pattern:

```
0 2 4 6 8 10 12 * 14 * 12 * 10 * 8 * * 6 * 4 * ;
-5 * -5 * * -5 * -5 * -5 * -5 * * 0 -1 -1 -1 -1 -1 -1
```

Interpreted in the Hirajoshi scale.

Here is another one:

```
0 2 4 6 8 10 12 * * 10 * * 8 * 6 * * 4 * ;
-5 * * -5 * -5 * * -5 * * -5 * 0 -1 -1 -1 -1 -1
```

Interpreted in the Hirajoshi scale.