Algebraic structures, series, and enumeration

Samuele Giraudo

LIGM, Université Paris-Est Marne-la-Vallée

Journées nationales 2018 du GDR IM

April 5, 2018

Outline

Combinatorics and enumeration

Formal power series

Operads and composition

Outline

Combinatorics and enumeration

Combinatorial sets

A combinatorial set is a set C endowed with a map

$$|-|: C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : |x| = n\}$ is finite.

For any $x \in C$, we call $|x|$ the size of x.

Combinatorial sets

A combinatorial set is a set C endowed with a map

$$|-|: C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : |x| = n\}$ is finite.

For any $x \in C$, we call $|x|$ the size of x.

Classical questions

1. Enumerate the objects of C of size n.

Combinatorial sets

A combinatorial set is a set C endowed with a map

$$|-|: C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : |x| = n\}$ is finite.

For any $x \in C$, we call $|x|$ the size of x.

Classical questions

1. Enumerate the objects of C of size n.
2. Generate all the objects of C of size n.

Combinatorial sets

A combinatorial set is a set C endowed with a map

$$|-|: C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : |x| = n\}$ is finite.

For any $x \in C$, we call $|x|$ the size of x.

Classical questions

1. Enumerate the objects of C of size n.
2. Generate all the objects of C of size n.
3. Randomly generate an object of C of size n.

Combinatorial sets

A combinatorial set is a set C endowed with a map

$$|-|: C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : |x| = n\}$ is finite.

For any $x \in C$, we call $|x|$ the size of x.

Classical questions

1. Enumerate the objects of C of size n.
2. Generate all the objects of C of size n.
3. Randomly generate an object of C of size n.
4. Establish transformations between C and other combinatorial sets D.

Combinatorial sets

A combinatorial set is a set C endowed with a map

$$|-|: C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : |x| = n\}$ is finite.

For any $x \in C$, we call $|x|$ the size of x.

Classical questions

1. Enumerate the objects of C of size n.
2. Generate all the objects of C of size n.
3. Randomly generate an object of C of size n.
4. Establish transformations between C and other combinatorial sets D.
5. Define statistics on C, that are maps $\mathrm{s} : C \to \mathbb{N}$.

Examples

Words

Let $A := \{\mathsf{a}, \mathsf{b}\}$ be an alphabet and let A^* be the combinatorial set of all words on A where the size of a word is its length.

Examples

Words

Let $A := \{\mathsf{a}, \mathsf{b}\}$ be an alphabet and let A^* be the combinatorial set of all words on A where the size of a word is its length.

Then, $A^*(0) = \{\epsilon\}$, $A^*(1) = \{\mathsf{a}, \mathsf{b}\}$, $A^*(2) = \{\mathsf{aa}, \mathsf{ab}, \mathsf{ba}, \mathsf{bb}\}$.

Examples

Words

Let $A := \{a, b\}$ be an alphabet and let A^* be the combinatorial set of all words on A where the size of a word is its length.

Then, $A^*(0) = \{\epsilon\}$, $A^*(1) = \{a, b\}$, $A^*(2) = \{aa, ab, ba, bb\}$.

Words on A are enumerated by $\#A^*(n) = 2^n$.

Examples

Words

Let $A := \{a, b\}$ be an alphabet and let A^* be the combinatorial set of all words on A where the size of a word is its length.

Then, $A^*(0) = \{\epsilon\}$, $A^*(1) = \{a, b\}$, $A^*(2) = \{aa, ab, ba, bb\}$.

Words on A are enumerated by $\#A^*(n) = 2^n$.

The map s_a sending a word on A to its number of occurrences of a is a statistics on A^*.

Examples

Words

Let $A := \{\mathsf{a}, \mathsf{b}\}$ be an alphabet and let A^* be the combinatorial set of all words on A where the size of a word is its length.

Then, $A^*(0) = \{\epsilon\}$, $A^*(1) = \{\mathsf{a}, \mathsf{b}\}$, $A^*(2) = \{\mathsf{aa}, \mathsf{ab}, \mathsf{ba}, \mathsf{bb}\}$.

Words on A are enumerated by $\#A^*(n) = 2^n$.

The map s_{a} sending a word on A to its number of occurrences of a is a statistics on A^*.

Permutations

Let $\mathfrak{S}$ be the combinatorial set of all permutations where the size of a permutation is its length as a word.

Examples

Words

Let $A := \{\mathsf{a}, \mathsf{b}\}$ be an alphabet and let A^* be the combinatorial set of all words on A where the size of a word is its length.

Then, $A^*(0) = \{\epsilon\}$, $A^*(1) = \{\mathsf{a}, \mathsf{b}\}$, $A^*(2) = \{\mathsf{aa}, \mathsf{ab}, \mathsf{ba}, \mathsf{bb}\}$.

Words on A are enumerated by $\# A^*(n) = 2^n$.

The map s_{a} sending a word on A to its number of occurrences of a is a statistics on A^*.

Permutations

Let $\mathfrak{S}$ be the combinatorial set of all permutations where the size of a permutation is its length as a word.

Then, $\mathfrak{S}(0) = \{\epsilon\}$, $\mathfrak{S}(1) = \{1\}$, $\mathfrak{S}(2) = \{12, 21\}$,
$\mathfrak{S}(3) = \{123, 132, 213, 231, 312, 321\}$.

Examples

Words

Let $A := \{\mathsf{a}, \mathsf{b}\}$ be an alphabet and let A^* be the combinatorial set of all words on A where the size of a word is its length.

Then, $A^*(0) = \{\epsilon\}$, $A^*(1) = \{\mathsf{a}, \mathsf{b}\}$, $A^*(2) = \{\mathsf{aa}, \mathsf{ab}, \mathsf{ba}, \mathsf{bb}\}$.

Words on A are enumerated by $\#A^*(n) = 2^n$.

The map s_{a} sending a word on A to its number of occurrences of a is a statistics on A^*.

Permutations

Let $\mathfrak{S}$ be the combinatorial set of all permutations where the size of a permutation is its length as a word.

Then, $\mathfrak{S}(0) = \{\epsilon\}$, $\mathfrak{S}(1) = \{1\}$, $\mathfrak{S}(2) = \{12, 21\}$,
$\mathfrak{S}(3) = \{123, 132, 213, 231, 312, 321\}$.

Permutations are enumerated by $\#\mathfrak{S}(n) = n!$.

Examples

Words

Let $A := \{\mathsf{a}, \mathsf{b}\}$ be an alphabet and let A^* be the combinatorial set of all words on A where the size of a word is its length.

Then, $A^*(0) = \{\epsilon\}$, $A^*(1) = \{\mathsf{a}, \mathsf{b}\}$, $A^*(2) = \{\mathsf{aa}, \mathsf{ab}, \mathsf{ba}, \mathsf{bb}\}$.

Words on A are enumerated by $\# A^*(n) = 2^n$.

The map s_{a} sending a word on A to its number of occurrences of a is a statistics on A^*.

Permutations

Let $\mathfrak{S}$ be the combinatorial set of all permutations where the size of a permutation is its length as a word.

Then, $\mathfrak{S}(0) = \{\epsilon\}$, $\mathfrak{S}(1) = \{1\}$, $\mathfrak{S}(2) = \{12, 21\}$,
$\mathfrak{S}(3) = \{123, 132, 213, 231, 312, 321\}$.

Permutations are enumerated by $\# \mathfrak{S}(n) = n!$.

A descent of $\sigma \in \mathfrak{S}$ is a position i such that $\sigma(i) > \sigma(i+1)$.

The map s sending a permutation to its number of descents is a statistics on $\mathfrak{S}$. For instance, $\mathrm{s}(35142) = 2$.

Examples

Binary trees

Let $\mathbb{BT}$ be the combinatorial set of all binary trees where the size of a binary tree is its number of internal nodes.

Examples

Binary trees

Let BT be the combinatorial set of all binary trees where the size of a binary tree is its number of internal nodes.

Then, $\mathrm{BT}(0) = \{$  $\}$, $\mathrm{BT}(1) = \{$  $\}$, $\mathrm{BT}(2) = \{$ ,  $\}$.

Examples

Binary trees

Let BT be the combinatorial set of all binary trees where the size of a binary tree is its number of internal nodes.

Then, $\mathrm{BT}(0) = \{$  $\}$, $\mathrm{BT}(1) = \{$  $\}$, $\mathrm{BT}(2) = \{$ ,  $\}$.

Binary trees are enumerated by $\#\mathrm{BT}(n) = \frac{1}{n+1}\binom{2n}{n}$ (Catalan numbers).

Examples

Binary trees

Let BT be the combinatorial set of all binary trees where the size of a binary tree is its number of internal nodes.

Then, $\mathrm{BT}(0) = \{$  $\}$, $\mathrm{BT}(1) = \{$  $\}$, $\mathrm{BT}(2) = \{$ ,  $\}$.

Binary trees are enumerated by $\#\mathrm{BT}(n) = \frac{1}{n+1}\binom{2n}{n}$ (Catalan numbers).

The map s sending a binary tree to its number of leaves oriented to the right is a statistics on BT.

Examples

Binary trees

Let BT be the combinatorial set of all binary trees where the size of a binary tree is its number of internal nodes.

Then, $\mathrm{BT}(0) = \{$  $\}$, $\mathrm{BT}(1) = \{$  $\}$, $\mathrm{BT}(2) = \{$ ,  $\}$.

Binary trees are enumerated by $\#\mathrm{BT}(n) = \frac{1}{n+1}\binom{2n}{n}$ (Catalan numbers).

The map s sending a binary tree to its number of leaves oriented to the right is a statistics on BT.

The map $\mathrm{bst} : \mathfrak{S} \to \mathrm{BT}$ sending a permutation σ to the binary tree obtained by inserting the letters of σ from right to left by following the binary search insertion algorithm is a transformation from $\mathfrak{S}$ to BT.

Examples

Binary trees

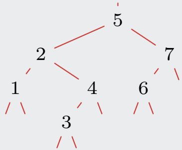
Let BT be the combinatorial set of all binary trees where the size of a binary tree is its number of internal nodes.

Then, $\mathrm{BT}(0) = \{$  $\}$, $\mathrm{BT}(1) = \{$  $\}$, $\mathrm{BT}(2) = \{$ ,  $\}$.

Binary trees are enumerated by $\#\mathrm{BT}(n) = \frac{1}{n+1}\binom{2n}{n}$ (Catalan numbers).

The map s sending a binary tree to its number of leaves oriented to the right is a statistics on BT.

The map $\mathrm{bst} : \mathfrak{S} \to \mathrm{BT}$ sending a permutation σ to the binary tree obtained by inserting the letters of σ from right to left by following the binary search insertion algorithm is a transformation from $\mathfrak{S}$ to BT. For instance,

$6317425 \overset{\mathrm{bst}}{\longmapsto}$ 

Examples

Binary trees

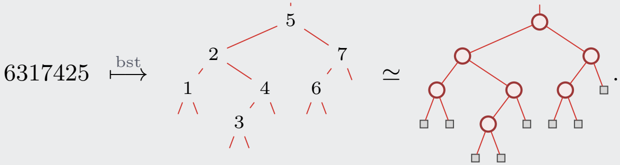
Let BT be the combinatorial set of all binary trees where the size of a binary tree is its number of internal nodes.

Then, $\mathrm{BT}(0) = \{$  $\}$, $\mathrm{BT}(1) = \{$  $\}$, $\mathrm{BT}(2) = \{$  $\}$.

Binary trees are enumerated by $\#\mathrm{BT}(n) = \frac{1}{n+1}\binom{2n}{n}$ (Catalan numbers).

The map s sending a binary tree to its number of leaves oriented to the right is a statistics on BT.

The map $\mathrm{bst} : \mathfrak{S} \to \mathrm{BT}$ sending a permutation σ to the binary tree obtained by inserting the letters of σ from right to left by following the binary search insertion algorithm is a transformation from $\mathfrak{S}$ to BT. For instance,



Generating series

The generating series of a combinatorial set C is

$$\mathcal{G}_C(t) := \sum_{n \in \mathbb{N}} \#C(n) t^n = \sum_{x \in C} t^{|x|}.$$

Generating series are very powerful tools for enumeration. They encode sequences of numbers and support many operations.

Generating series

The generating series of a combinatorial set C is

$$\mathcal{G}_C(t) := \sum_{n\in\mathbb{N}} \#C(n)t^n = \sum_{x\in C} t^{|x|}.$$

Generating series are very powerful tools for enumeration. They encode sequences of numbers and support many operations.

Examples

- $\mathcal{G}_{A^*}(t) = 1 + 2t + 4t^2 + 8t^3 + 16t^4 + 32t^5 + \cdots = \dfrac{1}{1-2t}$

Generating series

The generating series of a combinatorial set C is

$$\mathcal{G}_C(t) := \sum_{n\in\mathbb{N}} \#C(n) t^n = \sum_{x\in C} t^{|x|}.$$

Generating series are very powerful tools for enumeration. They encode sequences of numbers and support many operations.

Examples

- $\mathcal{G}_{A^*}(t) = 1 + 2t + 4t^2 + 8t^3 + 16t^4 + 32t^5 + \cdots = \dfrac{1}{1-2t}$
- $\mathcal{G}_{\mathfrak{S}}(t) = 1 + t + 2t^2 + 6t^3 + 24t^4 + 120t^5 + \cdots$

Generating series

The generating series of a combinatorial set C is

$$\mathcal{G}_C(t) := \sum_{n \in \mathbb{N}} \#C(n) t^n = \sum_{x \in C} t^{|x|}.$$

Generating series are very powerful tools for enumeration. They encode sequences of numbers and support many operations.

Examples

- $\mathcal{G}_{A^*}(t) = 1 + 2t + 4t^2 + 8t^3 + 16t^4 + 32t^5 + \cdots = \dfrac{1}{1-2t}$
- $\mathcal{G}_{\mathfrak{S}}(t) = 1 + t + 2t^2 + 6t^3 + 24t^4 + 120t^5 + \cdots$
- $\mathcal{G}_{\mathrm{BT}}(t) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + \cdots = \dfrac{1 - \sqrt{1-4t}}{2t}$

Formal power series

Generating series forget a lot of information about the underlying combinatorial objects of C.

Formal power series

Generating series forget a lot of information about the underlying combinatorial objects of C.

First idea

Work with formal series of combinatorial objects of C.

Formal power series

Generating series forget a lot of information about the underlying combinatorial objects of C.

First idea

Work with formal series of combinatorial objects of C.

We consider series wherein exponents are combinatorial objects.

Formal power series

Generating series forget a lot of information about the underlying combinatorial objects of C.

First idea

Work with formal series of combinatorial objects of C.

We consider series wherein exponents are combinatorial objects.

Example

We work with the formal power series

instead of the generating series $\mathcal{G}_{\mathrm{BT}}(t)$.

Evaluations of characteristic series

The characteristic series of C is the formal series

$$\mathbf{f}_C := \sum_{x \in C} t^x.$$

Evaluations of characteristic series

The characteristic series of C is the formal series

$$\mathbf{f}_C := \sum_{x \in C} t^x.$$

The evaluation morphism ev is linearly defined by

$$\mathrm{ev}\left(t^x\right) := t^{|x|}.$$

for any $x \in C$.

Evaluations of characteristic series

The characteristic series of C is the formal series

$$\mathbf{f}_C := \sum_{x \in C} t^x.$$

The evaluation morphism ev is linearly defined by

$$\mathrm{ev}\left(t^x\right) := t^{|x|}.$$

for any $x \in C$.

Since

$$\mathrm{ev}(\mathbf{f}_C) = \sum_{x \in C} t^{|x|} = \mathcal{G}_C(t),$$

we recover from the characteristic series of a combinatorial set its generating series.

Evaluations of characteristic series

The characteristic series of C is the formal series

$$\mathbf{f}_C := \sum_{x \in C} t^x.$$

The evaluation morphism ev is linearly defined by

$$\mathrm{ev}\left(t^x\right) := t^{|x|}.$$

for any $x \in C$.

Since

$$\mathrm{ev}(\mathbf{f}_C) = \sum_{x \in C} t^{|x|} = \mathcal{G}_C(t),$$

we recover from the characteristic series of a combinatorial set its generating series.

Example

$$\overset{\mathrm{ev}}{\longmapsto}$$

$$t^0 + t^1 + (t^2 + t^2) + (t^3 + t^3 + t^3 + t^3 + t^3) + \cdots = 1 + t + 2t^2 + 5t^3 + \cdots = \mathcal{G}_{\mathrm{BT}}(t).$$

Operations and generating series

Natural operations on series translate as operations on combinatorial sets.

Let C_1 and C_2 be two combinatorial sets.

Operations and generating series

Natural operations on series translate as operations on combinatorial sets.

Let C_1 and C_2 be two combinatorial sets.

- The addition $\mathcal{G}_{C_1}(t) + \mathcal{G}_{C_2}(t)$ is the generating series of the combinatorial set $C_1 + C_2$ satisfying

$$(C_1 + C_2)(n) := C_1(n) \sqcup C_2(n).$$

Operations and generating series

Natural operations on series translate as operations on combinatorial sets.

Let C_1 and C_2 be two combinatorial sets.

- The addition $\mathcal{G}_{C_1}(t) + \mathcal{G}_{C_2}(t)$ is the generating series of the combinatorial set $C_1 + C_2$ satisfying

$$(C_1 + C_2)(n) := C_1(n) \sqcup C_2(n).$$

- The multiplication $\mathcal{G}_{C_1}(t) \cdot \mathcal{G}_{C_2}(t)$ is the generating series of the combinatorial set $C_1 \times C_2$ satisfying

$$(C_1 \times C_2)(n) := \{(x_1, x_2) : x_1 \in C_1, x_2 \in C_2, |x_1| + |x_2| = n\}.$$

Operations and generating series

Natural operations on series translate as operations on combinatorial sets.

Let C_1 and C_2 be two combinatorial sets.

- The addition $\mathcal{G}_{C_1}(t) + \mathcal{G}_{C_2}(t)$ is the generating series of the combinatorial set $C_1 + C_2$ satisfying

$$(C_1 + C_2)(n) := C_1(n) \sqcup C_2(n).$$

- The multiplication $\mathcal{G}_{C_1}(t) \cdot \mathcal{G}_{C_2}(t)$ is the generating series of the combinatorial set $C_1 \times C_2$ satisfying

$$(C_1 \times C_2)(n) := \{(x_1, x_2) : x_1 \in C_1, x_2 \in C_2, |x_1| + |x_2| = n\}.$$

- The composition $\mathcal{G}_{C_1}(\mathcal{G}_{C_2}(t))$ is the generating series of the combinatorial set $C_1 \circ C_2$ satisfying

$$(C_1 \circ C_2)(n) := \left\{\left(x, y_1, \ldots, y_{|x|}\right) : x \in C_1, y_i \in C_2, |y_1| + \cdots + |y_{|x|}| = n\right\}.$$

Examples

Binary trees

A binary tree can be defined recursively to be either the leaf $\perp$ or an ordered pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ of binary trees.

This leads to the expression $\mathcal{G}_{\mathrm{BT}}(t) = 1 + \mathcal{G}_{\mathrm{BT}}(t)^2$ for the generating series of BT.

Examples

Binary trees

A binary tree can be defined recursively to be either the leaf  or an ordered pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ of binary trees.

This leads to the expression $\mathcal{G}_{\mathrm{BT}}(t) = 1 + \mathcal{G}_{\mathrm{BT}}(t)^2$ for the generating series of BT.

Binary trees of words

A binary tree of words is a binary tree wherein internal nodes are labeled by nonempty words on A. The size of such an object is the sum of the sizes of the words labeling it.

Examples

Binary trees

A binary tree can be defined recursively to be either the leaf $\perp$ or an ordered pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ of binary trees.

This leads to the expression $\mathcal{G}_{\mathrm{BT}}(t) = 1 + \mathcal{G}_{\mathrm{BT}}(t)^2$ for the generating series of BT.

Binary trees of words

A binary tree of words is a binary tree wherein internal nodes are labeled by nonempty words on A. The size of such an object is the sum of the sizes of the words labeling it. These are elements of $\mathrm{BT} \circ A^+$.

Examples

Binary trees

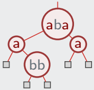
A binary tree can be defined recursively to be either the leaf ⊥ or an ordered pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ of binary trees.

This leads to the expression $\mathcal{G}_{\mathrm{BT}}(t) = 1 + \mathcal{G}_{\mathrm{BT}}(t)^2$ for the generating series of BT.

Binary trees of words

A binary tree of words is a binary tree wherein internal nodes are labeled by nonempty words on A. The size of such an object is the sum of the sizes of the words labeling it. These are elements of $\mathrm{BT} \circ A^+$.

For instance,



is an object of size 7.

Examples

Binary trees

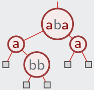
A binary tree can be defined recursively to be either the leaf $\square$ or an ordered pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ of binary trees.

This leads to the expression $\mathcal{G}_{\mathrm{BT}}(t) = 1 + \mathcal{G}_{\mathrm{BT}}(t)^2$ for the generating series of BT.

Binary trees of words

A binary tree of words is a binary tree wherein internal nodes are labeled by nonempty words on A. The size of such an object is the sum of the sizes of the words labeling it. These are elements of $\mathrm{BT} \circ A^+$.

For instance,



is an object of size 7.

This leads to

$$\mathcal{G}_{\mathrm{BT}\circ A^+}(t) = \frac{1 - \sqrt{1 - 4\frac{2t}{1-2t}}}{2\frac{2t}{1-2t}} = 1 + 2t + 12t^2 + 80t^3 + 576t^4 + 4384t^5 + \cdots .$$

Operations and algebraic structures

Second idea

Endow C with operations to form an algebraic structure.

Operations and algebraic structures

Second idea

Endow C with operations to form an algebraic structure.

The study of C as an algebraic structure helps to discover combinatorial properties.

Operations and algebraic structures

Second idea

Endow C with operations to form an algebraic structure.

The study of C as an algebraic structure helps to discover combinatorial properties.

In particular,

1. minimal generating families of C

   $\rightsquigarrow$ highlighting of elementary pieces of assembly;

Operations and algebraic structures

Second idea

Endow C with operations to form an algebraic structure.

The study of C as an algebraic structure helps to discover combinatorial properties.

In particular,

1. minimal generating families of C

   $\rightsquigarrow$ highlighting of elementary pieces of assembly;

2. morphisms involving C

   $\rightsquigarrow$ transformation algorithms and revelation of symmetries;

Operations and algebraic structures

Second idea

Endow C with operations to form an algebraic structure.

The study of C as an algebraic structure helps to discover combinatorial properties.

In particular,

1. minimal generating families of C

   $\rightsquigarrow$ highlighting of elementary pieces of assembly;

2. morphisms involving C

   $\rightsquigarrow$ transformation algorithms and revelation of symmetries;

3. operations on C

   $\rightsquigarrow$ extensions to operations on formal power series.

Outline

Formal power series

Formal power series

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \dots)$ and X be a (combinatorial) set.

Formal power series

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \dots)$ and X be a (combinatorial) set.

A X-series is a map

$$\mathbf{f} : X \to \mathbb{K}.$$

Formal power series

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \dots)$ and X be a (combinatorial) set.

A X-series is a map

$$\mathbf{f} : X \to \mathbb{K}.$$

The coefficient $\mathbf{f}(x)$ of $x \in X$ in $\mathbf{f}$ is denoted by $\langle x, \mathbf{f} \rangle$.

Formal power series

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \dots)$ and X be a (combinatorial) set.

A X-series is a map

$$\mathbf{f} : X \to \mathbb{K}.$$

The coefficient $\mathbf{f}(x)$ of $x \in X$ in $\mathbf{f}$ is denoted by $\langle x, \mathbf{f} \rangle$.

The set of all X-series is $\mathbb{K}\langle\langle X \rangle\rangle$.

Formal power series

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \dots)$ and X be a (combinatorial) set.

A X-series is a map

$$\mathbf{f} : X \to \mathbb{K}.$$

The coefficient $\mathbf{f}(x)$ of $x \in X$ in $\mathbf{f}$ is denoted by $\langle x, \mathbf{f} \rangle$.

The set of all X-series is $\mathbb{K}\langle\langle X \rangle\rangle$.

Endowed with the pointwise addition

$$\langle x, \mathbf{f} + \mathbf{g} \rangle := \langle x, \mathbf{f} \rangle + \langle x, \mathbf{g} \rangle$$

and the pointwise multiplication by a scalar

$$\langle x, \lambda \mathbf{f} \rangle := \lambda \langle x, \mathbf{f} \rangle,$$

the set $\mathbb{K}\langle\langle X \rangle\rangle$ is a vector space.

Formal power series

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \dots)$ and X be a (combinatorial) set.

A X-series is a map

$$\mathbf{f} : X \to \mathbb{K}.$$

The coefficient $\mathbf{f}(x)$ of $x \in X$ in $\mathbf{f}$ is denoted by $\langle x, \mathbf{f} \rangle$.

The set of all X-series is $\mathbb{K}\langle\langle X \rangle\rangle$.

Endowed with the pointwise addition

$$\langle x, \mathbf{f} + \mathbf{g} \rangle := \langle x, \mathbf{f} \rangle + \langle x, \mathbf{g} \rangle$$

and the pointwise multiplication by a scalar

$$\langle x, \lambda \mathbf{f} \rangle := \lambda \langle x, \mathbf{f} \rangle,$$

the set $\mathbb{K}\langle\langle X \rangle\rangle$ is a vector space.

The sum notation of $\mathbf{f}$ is

$$\mathbf{f} = \sum_{x \in X} \langle x, \mathbf{f} \rangle \, x.$$

Formal series and algebraic structures

When X is an algebraic structure, its operations lead to operations on X-series.

Formal series and algebraic structures

When X is an algebraic structure, its operations lead to operations on X-series.

Indeed, if

$$\star : X^k \to X, \qquad k \geqslant 1,$$

is a product of arity k on X, one obtains the product $\bar{\star}$ on $\mathbb{K}\langle\langle X \rangle\rangle$ defined by

$$\langle x, \bar{\star}(\mathbf{f}_1, \dots, \mathbf{f}_k) \rangle := \sum_{\substack{y_1, \dots, y_k \in X \\ x = \star(y_1, \dots, y_k)}} \prod_{1 \leqslant i \leqslant k} \langle y_i, \mathbf{f}_i \rangle .$$

Formal series and algebraic structures

When X is an algebraic structure, its operations lead to operations on X-series.

Indeed, if

$$\star : X^k \to X, \qquad k \geqslant 1,$$

is a product of arity k on X, one obtains the product $\bar{\star}$ on $\mathbb{K}\langle\langle X\rangle\rangle$ defined by

$$\langle x, \bar{\star}(\mathbf{f}_1, \dots, \mathbf{f}_k)\rangle := \sum_{\substack{y_1, \dots, y_k \in X \\ x = \star(y_1, \dots, y_k)}} \prod_{1 \leqslant i \leqslant k} \langle y_i, \mathbf{f}_i\rangle .$$

Example

A binary product $\star : X \times X \to X$ leads to the (possibly partial) product

$$\mathbf{f} \,\bar{\star}\, \mathbf{g} := \sum_{x, y \in X} \langle x, \mathbf{f}\rangle \langle y, \mathbf{g}\rangle \, x \star y$$

on $\mathbb{K}\langle\langle X\rangle\rangle$.

Formal series and algebraic structures

Here are some examples of series on algebraic structures leading to well-known objects.

Structure on X	**Sort of series**
$(\mathbb{N}, +, 0)$	Usual series $\mathbb{K}[[t]]$
Free comm. monoid	Multivariate series $\mathbb{K}[[t_1, t_2, \dots]]$
Free monoid	Noncomm. series $\mathbb{K}\langle\langle t_1, t_2, \dots\rangle\rangle$ [**Eilenberg**, 1974]
Monoid	Series on monoids [**Salomaa, Soittola**, 1978]
Operads	Series on operads [**Chapoton**, 2002, 2008]

Generalizing the product of generating series

Let C be a combinatorial set.

A binary product $\star$ on C is graded if $|x_1 \star x_2| = |x_1| + |x_2|$.

Generalizing the product of generating series

Let C be a combinatorial set.

A binary product $\star$ on C is graded if $|x_1 \star x_2| = |x_1| + |x_2|$.

Proposition

The map ev is an algebra morphism between $(\mathbb{K}\langle\langle C\rangle\rangle, \bar{\star})$ and $(\mathbb{K}\langle\langle t\rangle\rangle, \cdot)$.
Moreover, this morphism is surjective when $C(n) \neq \emptyset$ for all $n \in \mathbb{N}$.

Generalizing the product of generating series

Let C be a combinatorial set.

A binary product $\star$ on C is graded if $|x_1 \star x_2| = |x_1| + |x_2|$.

Proposition

The map ev is an algebra morphism between $(\mathbb{K}\langle\langle C\rangle\rangle, \bar{\star})$ and $(\mathbb{K}\langle\langle t\rangle\rangle, \cdot)$.

Moreover, this morphism is surjective when $C(n) \neq \emptyset$ for all $n \in \mathbb{N}$.

Proof.

$$\begin{aligned}
\mathrm{ev}\left(\mathbf{f}_1 \bar{\star} \mathbf{f}_2\right) &= \sum_{x_1, x_2 \in C} \langle x_1, \mathbf{f}_1\rangle \langle x_2, \mathbf{f}_2\rangle \, \mathrm{ev}(x_1 \star x_2) \\
&= \sum_{x_1, x_2 \in C} \langle x_1, \mathbf{f}_1\rangle \langle x_2, \mathbf{f}_2\rangle \, t^{|x_1|+|x_2|} \\
&= \left(\sum_{x_1 \in C} \langle x_1, \mathbf{f}_1\rangle \, t^{|x_1|}\right) \left(\sum_{x_2 \in C} \langle x_2, \mathbf{f}_2\rangle \, t^{|x_2|}\right) \\
&= \mathrm{ev}(\mathbf{f}_1) \cdot \mathrm{ev}(\mathbf{f}_2)
\end{aligned}$$

□

Generalizing the product of generating series

Let C be a combinatorial set.

A binary product $\star$ on C is graded if $|x_1 \star x_2| = |x_1| + |x_2|$.

Proposition

The map ev is an algebra morphism between $(\mathbb{K}\langle\langle C\rangle\rangle, \bar{\star})$ and $(\mathbb{K}\langle\langle t\rangle\rangle, \cdot)$.

Moreover, this morphism is surjective when $C(n) \neq \emptyset$ for all $n \in \mathbb{N}$.

Proof.

$$\begin{aligned}
\mathrm{ev}(\mathbf{f}_1 \bar{\star} \mathbf{f}_2) &= \sum_{x_1, x_2 \in C} \langle x_1, \mathbf{f}_1\rangle \langle x_2, \mathbf{f}_2\rangle \, \mathrm{ev}(x_1 \star x_2) \\
&= \sum_{x_1, x_2 \in C} \langle x_1, \mathbf{f}_1\rangle \langle x_2, \mathbf{f}_2\rangle \, t^{|x_1|+|x_2|} \\
&= \left(\sum_{x_1 \in C} \langle x_1, \mathbf{f}_1\rangle \, t^{|x_1|}\right) \left(\sum_{x_2 \in C} \langle x_2, \mathbf{f}_2\rangle \, t^{|x_2|}\right) \\
&= \mathrm{ev}(\mathbf{f}_1) \cdot \mathrm{ev}(\mathbf{f}_2)
\end{aligned}$$

□

Therefore, $\bar{\star}$ offers a generalization of the usual product of generating series.

Example: series of paths

A path is an element of $\bigsqcup_{n \geqslant 1} \mathbb{N}^n$.

Example: series of paths

A path is an element of $\bigsqcup_{n \geqslant 1} \mathbb{N}^n$.

Let $\mathcal{P}$ be the (non combinatorial) set of all paths where the size of a path $u_1 \dots u_n$ is $n-1$ (number of steps).

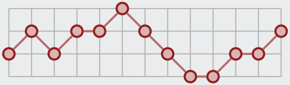
Example: series of paths

A path is an element of $\bigsqcup_{n \geqslant 1} \mathbb{N}^n$.

Let $\mathcal{P}$ be the (non combinatorial) set of all paths where the size of a path $u_1 \dots u_n$ is $n - 1$ (number of steps).

Example

The path 1212232100112 is depicted as



and its size is 12.

Example: series of paths — a monoid of paths

Let $\star$ be the binary product on $\mathcal{P}$ defined by

$$u_1 \dots u_n \star v_1 \dots v_m := \\ \uparrow_{\max(0, v_1 - u_n)} (u_1 \dots u_{n-1}) \max(u_n, v_1) \uparrow_{\max(0, u_n - v_1)} (v_2 \dots v_m)$$

where $\uparrow_i (w)$ is the word obtained by incrementing by i each letter of w.

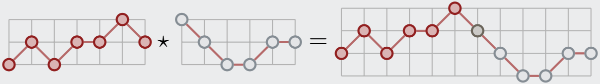
Example: series of paths — a monoid of paths

Let $\star$ be the binary product on $\mathcal{P}$ defined by

$$u_1 \dots u_n \star v_1 \dots v_m := \uparrow_{\max(0, v_1 - u_n)} (u_1 \dots u_{n-1}) \max(u_n, v_1) \uparrow_{\max(0, u_n - v_1)} (v_2 \dots v_m)$$

where $\uparrow_i (w)$ is the word obtained by incrementing by i each letter of w.

Example



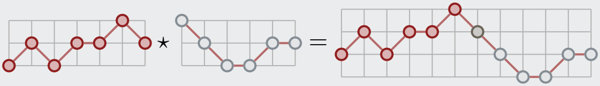
Example: series of paths — a monoid of paths

Let $\star$ be the binary product on $\mathcal{P}$ defined by

$$u_1 \dots u_n \star v_1 \dots v_m :=$$
$$\uparrow_{\max(0, v_1 - u_n)} (u_1 \dots u_{n-1}) \max(u_n, v_1) \uparrow_{\max(0, u_n - v_1)} (v_2 \dots v_m)$$

where $\uparrow_i (w)$ is the word obtained by incrementing by i each letter of w.

Example



This product is associative, admits ● (the path 0) as unit, and is graded.

Hence, $(\mathcal{P}, \star, ●)$ is a graded monoid.

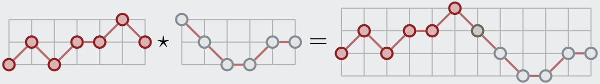
Example: series of paths — a monoid of paths

Let $\star$ be the binary product on $\mathcal{P}$ defined by

$$u_1 \dots u_n \star v_1 \dots v_m :=$$
$$\uparrow_{\max(0, v_1 - u_n)} (u_1 \dots u_{n-1}) \max(u_n, v_1) \uparrow_{\max(0, u_n - v_1)} (v_2 \dots v_m)$$

where $\uparrow_i (w)$ is the word obtained by incrementing by i each letter of w.

Example



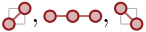
This product is associative, admits ● (the path 0) as unit, and is graded.

Hence, $(\mathcal{P}, \star,$ ●$)$ is a graded monoid.

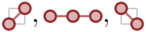
The $\star$-Kleene star of a subset X of $\mathcal{P}$ is

$$X^{\star_*} := \bigsqcup_{\ell \geqslant 0} \underbrace{X \star \dots \star X}_{\times \ell}.$$

Example: series of paths — Schröder paths

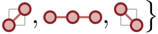
Let Schr be the subset of $\{$  $\}^{\star_*}$ restrained to the paths starting and finishing by 0. We call these Schröder paths.

Example: series of paths — Schröder paths

Let Schr be the subset of $\{$$\}^{\star_*}$ restrained to the paths starting and finishing by 0. We call these Schröder paths.

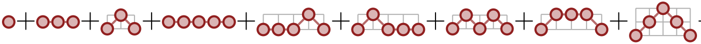
The set Schr is combinatorial and is a submonoid of $\mathcal{P}$

Example: series of paths — Schröder paths

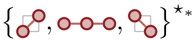
Let Schr be the subset of $\{$$\}^{\star *}$ restrained to the paths starting and finishing by 0. We call these Schröder paths.

The set Schr is combinatorial and is a submonoid of $\mathcal{P}$

The characteristic series of Schr is

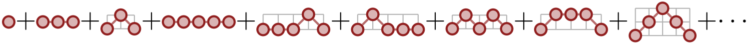
$\mathbf{f}_{\mathsf{Schr}} :=$  $+ \cdots$

Example: series of paths — Schröder paths

Let Schr be the subset of  restrained to the paths starting and finishing by 0. We call these Schröder paths.

The set Schr is combinatorial and is a submonoid of $\mathcal{P}$

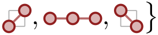
The characteristic series of Schr is

$\mathbf{f}_{\mathsf{Schr}} :=$ 

and, by using the product $\bar{\star}$ on $\mathcal{P}$-series, we obtain the relation

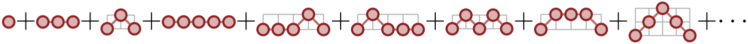
$\mathbf{f}_{\mathsf{Schr}} =$ 

Example: series of paths — Schröder paths

Let Schr be the subset of $\{$$\}^{\star_*}$ restrained to the paths starting and finishing by 0. We call these Schröder paths.

The set Schr is combinatorial and is a submonoid of $\mathcal{P}$

The characteristic series of Schr is

$\mathbf{f}_{\mathsf{Schr}} :=$ 

and, by using the product $\bar{\star}$ on $\mathcal{P}$-series, we obtain the relation

By evaluation, we get the expression

$$\mathcal{G}_{\mathsf{Schr}}(t) = 1 + t^2\mathcal{G}_{\mathsf{Schr}}(t) + t^2\mathcal{G}_{\mathsf{Schr}}(t)^2$$

for the generating series of the Schröder paths.

Example: series of paths — statistics

We can add formal parameters to refine the enumeration by taking into account of statistics. Then,

$\mathbf{g}_{\mathsf{Schr}} = $  $+ \; q_0$  $\bar{\star}\; \mathbf{g}_{\mathsf{Schr}} \; + \; q_1$  $\bar{\star}\; \mathbf{g}_{\mathsf{Schr}} \;\bar{\star}$  $\bar{\star}\; \mathbf{g}_{\mathsf{Schr}}$

is the formal series of Schröder paths where the coefficient of a path u is $q_0^{\alpha_0} q_1^{\alpha_1}$ where α_0 (resp. α_1) is half of the number of  (resp. the number of ) appearing in it.

Example: series of paths — statistics

We can add formal parameters to refine the enumeration by taking into account of statistics. Then,

is the formal series of Schröder paths where the coefficient of a path u is $q_0^{\alpha_0} q_1^{\alpha_1}$ where α_0 (resp. α_1) is half of the number of  (resp. the number of ) appearing in it.

Its evaluation is

$$\mathrm{ev}\left(\mathbf{g}_{\mathsf{Schr}}\right) = 1 + (q_0 + q_1)\, t^2 + \left(q_0^2 + 3q_0q_1 + 2q_1^2\right) t^4 + \left(q_0^3 + 6q_0^2q_1 + 10q_0q_1^2 + 5q_1^3\right) t^6 + \left(q_0^4 + 10q_0^3q_1 + 30q_0^2q_1^2 + 35q_0q_1^3 + 14q_1^4\right) t^8 + \cdots .$$

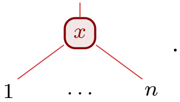
Its specialization $\mathrm{ev}\left(\mathbf{g}_{\mathsf{Schr}}\right)_{|q_1:=1}$ is Triangle **A088617** of OEIS and $\mathrm{ev}\left(\mathbf{g}_{\mathsf{Schr}}\right)_{|q_0:=1}$ is Triangle **A060693** of OEIS.

Outline

Operads and composition

Operators

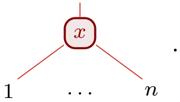
An operator is an entity having $n \geqslant 1$ inputs and a single output:



Its arity is its number n of inputs.

Operators

An operator is an entity having $n \geqslant 1$ inputs and a single output:



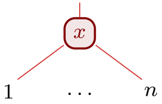
Its arity is its number n of inputs.

Composing two operators x and y consists in

1. selecting an input of x specified by its position i;
2. grafting the output of y onto this input.

Operators

An operator is an entity having $n \geqslant 1$ inputs and a single output:



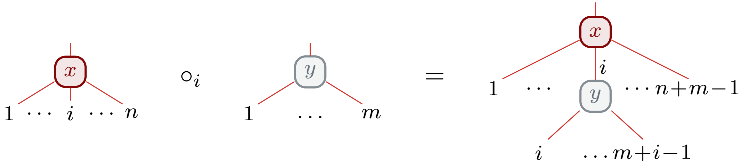
.

Its arity is its number n of inputs.

Composing two operators x and y consists in

1. selecting an input of x specified by its position i;
2. grafting the output of y onto this input.

This produces a new operator $x \circ_i y$ of arity $n + m - 1$:



.

Operads

Operads are algebraic structures formalizing the notion of operators and their composition.

Operads

Operads are algebraic structures formalizing the notion of operators and their composition.

A (nonsymmetric set-theoretic) operad is a triple $(\mathcal{O}, \circ_i, \mathbb{1})$ where

1. $\mathcal{O}$ is a graded set

$$\mathcal{O} := \bigsqcup_{n \geqslant 1} \mathcal{O}(n);$$

Operads

Operads are algebraic structures formalizing the notion of operators and their composition.

A (nonsymmetric set-theoretic) operad is a triple $(\mathcal{O}, \circ_i, \mathbb{1})$ where

1. $\mathcal{O}$ is a graded set

$$\mathcal{O} := \bigsqcup_{n \geqslant 1} \mathcal{O}(n);$$

2. $\circ_i$ is a map, called partial composition map,

$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n + m - 1), \qquad 1 \leqslant i \leqslant n, 1 \leqslant m;$$

Operads

Operads are algebraic structures formalizing the notion of operators and their composition.

A (nonsymmetric set-theoretic) operad is a triple $(\mathcal{O}, \circ_i, \mathbb{1})$ where

1. $\mathcal{O}$ is a graded set
$$\mathcal{O} := \bigsqcup_{n \geqslant 1} \mathcal{O}(n);$$
2. $\circ_i$ is a map, called partial composition map,
$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n+m-1), \qquad 1 \leqslant i \leqslant n, 1 \leqslant m;$$
3. $\mathbb{1}$ is an element of $\mathcal{O}(1)$ called unit.

Operads

Operads are algebraic structures formalizing the notion of operators and their composition.

A (nonsymmetric set-theoretic) operad is a triple $(\mathcal{O}, \circ_i, \mathbb{1})$ where

1. $\mathcal{O}$ is a graded set
$$\mathcal{O} := \bigsqcup_{n \geqslant 1} \mathcal{O}(n);$$
2. $\circ_i$ is a map, called partial composition map,
$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n + m - 1), \qquad 1 \leqslant i \leqslant n, 1 \leqslant m;$$
3. $\mathbb{1}$ is an element of $\mathcal{O}(1)$ called unit.

This data has to satisfy some axioms.

Operad axioms

Associativity:

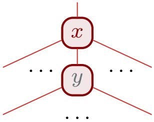
$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$$
$$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$$

Operad axioms

Associativity:

$(x \circ_i y)$

$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$

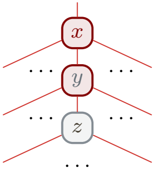


Operad axioms

Associativity:

$(x \circ_i y) \circ_{i+j-1} z$

$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$

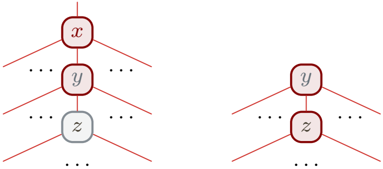


Operad axioms

Associativity:

$(x \circ_i y) \circ_{i+j-1} z \qquad (y \circ_j z)$

$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$

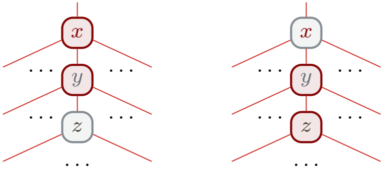


Operad axioms

Associativity:

$(x \circ_i y) \circ_{i+j-1} z \qquad x \circ_i (y \circ_j z)$

$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$

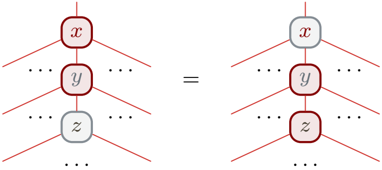


Operad axioms

Associativity:

$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$

$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$

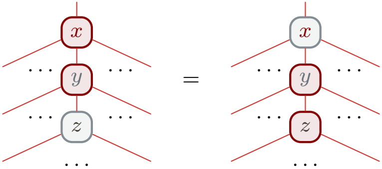


Operad axioms

Associativity:

$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$

$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$



Commutativity:

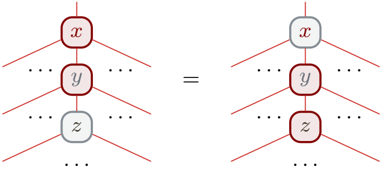
$(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y$

$1 \leqslant i < j \leqslant |x|$

Operad axioms

Associativity:

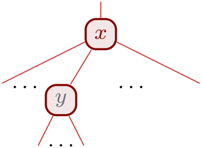
$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$
$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$



Commutativity:

$(x \circ_i y)$

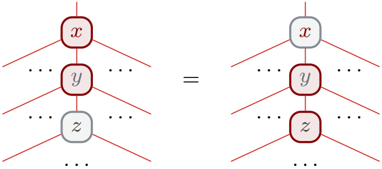
$1 \leqslant i < j \leqslant |x|$



Operad axioms

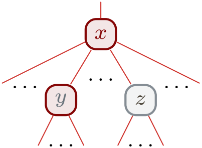
Associativity:

$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$
$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$



Commutativity:

$(x \circ_i y) \circ_{j+|y|-1} z$
$1 \leqslant i < j \leqslant |x|$

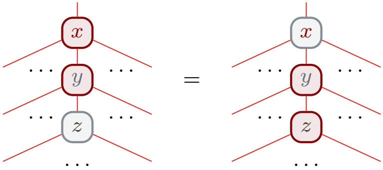


Operad axioms

Associativity:

$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$

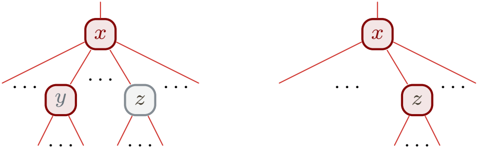
$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$



Commutativity:

$(x \circ_i y) \circ_{j+|y|-1} z \qquad (x \circ_j z)$

$1 \leqslant i < j \leqslant |x|$

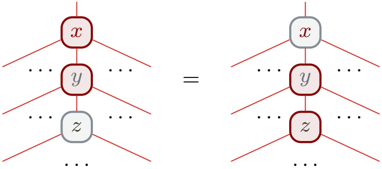


Operad axioms

Associativity:

$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$

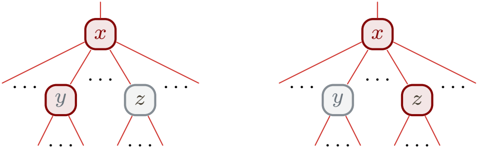
$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$



Commutativity:

$(x \circ_i y) \circ_{j+|y|-1} z \qquad (x \circ_j z) \circ_i y$

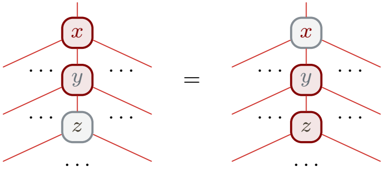
$1 \leqslant i < j \leqslant |x|$



Operad axioms

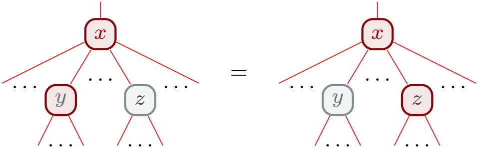
Associativity:

$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$
$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$



Commutativity:

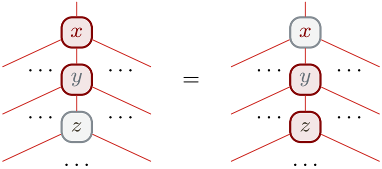
$(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y$
$1 \leqslant i < j \leqslant |x|$



Operad axioms

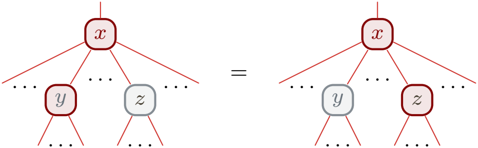
Associativity:

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$$
$$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$$



Commutativity:

$$(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y$$
$$1 \leqslant i < j \leqslant |x|$$



Unitality:

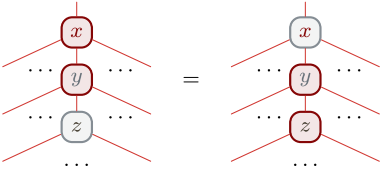
$$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}$$
$$1 \leqslant i \leqslant |x|$$

Operad axioms

Associativity:

$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$

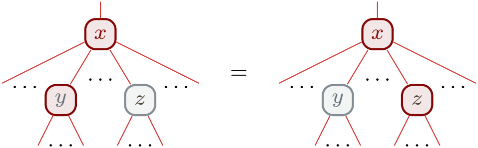
$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$



Commutativity:

$(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y$

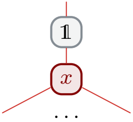
$1 \leqslant i < j \leqslant |x|$



Unitality:

$\mathbb{1} \circ_1 x$

$1 \leqslant i \leqslant |x|$

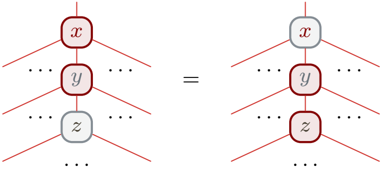


Operad axioms

Associativity:

$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$

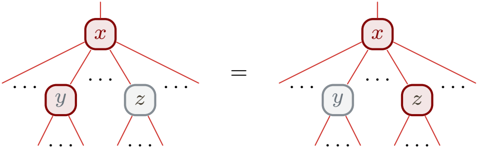
$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$



Commutativity:

$(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y$

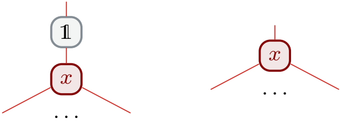
$1 \leqslant i < j \leqslant |x|$



Unitality:

$\mathbb{1} \circ_1 x \quad x$

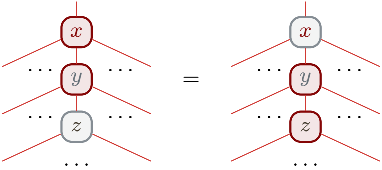
$1 \leqslant i \leqslant |x|$



Operad axioms

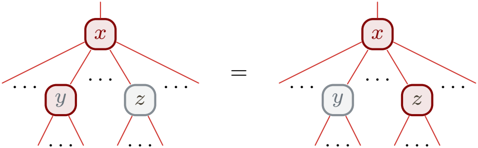
Associativity:

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$$
$$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$$



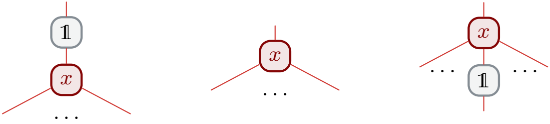
Commutativity:

$$(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y$$
$$1 \leqslant i < j \leqslant |x|$$



Unitality:

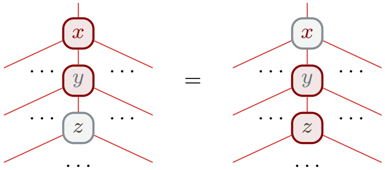
$$\mathbb{1} \circ_1 x \qquad x \qquad x \circ_i \mathbb{1}$$
$$1 \leqslant i \leqslant |x|$$



Operad axioms

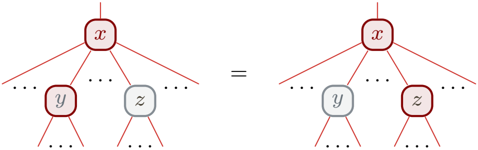
Associativity:

$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$
$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$



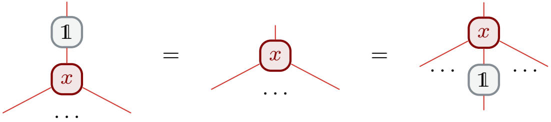
Commutativity:

$(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y$
$1 \leqslant i < j \leqslant |x|$



Unitality:

$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}$
$1 \leqslant i \leqslant |x|$



Complete composition

Let $\mathcal{O}$ be an operad

The complete composition map of $\mathcal{O}$ is the map

$$\circ : \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) \to \mathcal{O}\left(m_1 + \cdots + m_n\right)$$

defined by

$$x \circ [y_1, \ldots, y_n] := (\ldots((x \circ_n y_n) \circ_{n-1} y_{n-1}) \ldots) \circ_1 y_1.$$

Complete composition

Let $\mathcal{O}$ be an operad

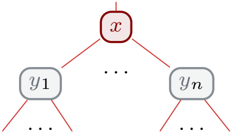
The complete composition map of $\mathcal{O}$ is the map

$$\circ : \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) \to \mathcal{O}\left(m_1 + \cdots + m_n\right)$$

defined by

$$x \circ [y_1, \ldots, y_n] := (\ldots((x \circ_n y_n) \circ_{n-1} y_{n-1}) \ldots) \circ_1 y_1.$$

Then, $x \circ [y_1, \ldots, y_n]$ is the operator

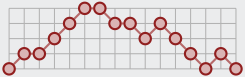


An operad of Motzkin paths

Let Motz be an operad wherein:

- $\mathsf{Motz}(n)$ is the set of all Motzkin paths with n points.

Example



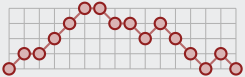
is a Motzkin path of arity 16.

An operad of Motzkin paths

Let Motz be an operad wherein:

- Motz(n) is the set of all Motzkin paths with n points.

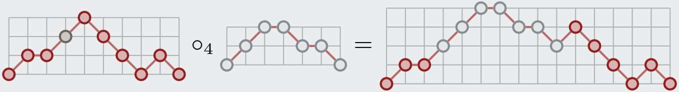
Example



is a Motzkin path of arity 16.

- The partial composition $x \circ_i y$ is computed by replacing the ith point of x by a copy of y.

Example

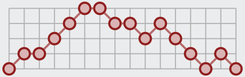


An operad of Motzkin paths

Let Motz be an operad wherein:

- Motz(n) is the set of all Motzkin paths with n points.

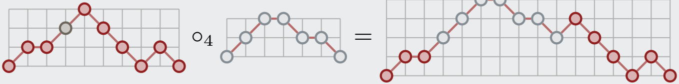
Example



is a Motzkin path of arity 16.

- The partial composition $x \circ_i y$ is computed by replacing the ith point of x by a copy of y.

Example



- The unit is .

A composition product on series on operads

Let $\mathcal{O}$ be an operad.

A composition product on series on operads

Let $\mathcal{O}$ be an operad.

The complete composition map $\circ$ of $\mathcal{O}$ leads to the product

$$\bar{\circ} : \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle \otimes \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle^{\otimes n} \to \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle, \qquad n \geqslant 1,$$

on $\mathcal{O}$-series, satisfying for any $\mathbf{f}, \mathbf{g}_1, \dots, \mathbf{g}_n \in \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle$ and $x \in \mathcal{O}$,

$$\langle x, \mathbf{f} \,\bar{\circ}\, [\mathbf{g}_1, \dots, \mathbf{g}_n] \rangle = \sum_{\substack{y \in \mathcal{O}(n) \\ z_1, \dots, z_n \in \mathcal{O} \\ x = y \circ [z_1, \dots, z_n]}} \langle y, \mathbf{f} \rangle \prod_{1 \leqslant i \leqslant n} \langle z_i, \mathbf{g}_i \rangle .$$

A composition product on series on operads

Let $\mathcal{O}$ be an operad.

The complete composition map $\circ$ of $\mathcal{O}$ leads to the product

$$\bar{\circ} : \mathbb{K}\langle\langle\mathcal{O}\rangle\rangle \otimes \mathbb{K}\langle\langle\mathcal{O}\rangle\rangle^{\otimes n} \to \mathbb{K}\langle\langle\mathcal{O}\rangle\rangle, \qquad n \geqslant 1,$$

on $\mathcal{O}$-series, satisfying for any $\mathbf{f}, \mathbf{g}_1, \dots, \mathbf{g}_n \in \mathbb{K}\langle\langle\mathcal{O}\rangle\rangle$ and $x \in \mathcal{O}$,

$$\langle x, \mathbf{f} \,\bar{\circ}\, [\mathbf{g}_1, \dots, \mathbf{g}_n]\rangle = \sum_{\substack{y \in \mathcal{O}(n) \\ z_1, \dots, z_n \in \mathcal{O} \\ x = y \circ [z_1, \dots, z_n]}} \langle y, \mathbf{f}\rangle \prod_{1 \leqslant i \leqslant n} \langle z_i, \mathbf{g}_i\rangle.$$

The composition product $\odot$ is the product

$$\odot : \mathbb{K}\langle\langle\mathcal{O}\rangle\rangle \times \mathbb{K}\langle\langle\mathcal{O}\rangle\rangle \to \mathbb{K}\langle\langle\mathcal{O}\rangle\rangle$$

defined, for any $\mathbf{f}, \mathbf{g} \in \mathbb{K}\langle\langle\mathcal{O}\rangle\rangle$ and $x \in \mathcal{O}$, by

$$\langle x, \mathbf{f} \odot \mathbf{g}\rangle := \sum_{1 \leqslant n \leqslant |x|} \left\langle x, \mathbf{f} \,\bar{\circ}\, \Big[\underbrace{\mathbf{g}, \dots, \mathbf{g}}_{\times n}\Big]\right\rangle$$

A composition product on series on operads

Let $\mathcal{O}$ be an operad.

The complete composition map $\circ$ of $\mathcal{O}$ leads to the product

$$\bar{\circ} : \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle \otimes \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle^{\otimes n} \to \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle, \qquad n \geqslant 1,$$

on $\mathcal{O}$-series, satisfying for any $\mathbf{f}, \mathbf{g}_1, \dots, \mathbf{g}_n \in \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle$ and $x \in \mathcal{O}$,

$$\langle x, \mathbf{f} \,\bar{\circ}\, [\mathbf{g}_1, \dots, \mathbf{g}_n] \rangle = \sum_{\substack{y \in \mathcal{O}(n) \\ z_1, \dots, z_n \in \mathcal{O} \\ x = y \circ [z_1, \dots, z_n]}} \langle y, \mathbf{f} \rangle \prod_{1 \leqslant i \leqslant n} \langle z_i, \mathbf{g}_i \rangle .$$

The composition product $\odot$ is the product

$$\odot : \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle \times \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle \to \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle$$

defined, for any $\mathbf{f}, \mathbf{g} \in \mathbb{K}\langle\langle \mathcal{O} \rangle\rangle$ and $x \in \mathcal{O}$, by

$$\langle x, \mathbf{f} \odot \mathbf{g} \rangle := \sum_{1 \leqslant n \leqslant |x|} \left\langle x, \mathbf{f} \,\bar{\circ}\, \Big[\underbrace{\mathbf{g}, \dots, \mathbf{g}}_{\times n} \Big] \right\rangle = \sum_{\substack{y \in \mathcal{O} \\ z_1, \dots, z_{|y|} \in \mathcal{O} \\ x = y \circ [z_1, \dots, z_{|y|}]}} \langle y, \mathbf{f} \rangle \prod_{1 \leqslant i \leqslant |y|} \langle z_i, \mathbf{g} \rangle .$$

Generalizing the composition of generating series

Proposition

The map ev is a monoid morphism between $(\mathbb{K}\langle\langle\mathcal{O}\rangle\rangle, \odot, \mathbb{1})$ and $(t\mathbb{K}\langle\langle t\rangle\rangle, \circ, t)$.

Moreover, this morphism is surjective when $\mathcal{O}(n) \neq \emptyset$ for all $n \geqslant 1$.

Generalizing the composition of generating series

Proposition

The map ev is a monoid morphism between $(\mathbb{K}\langle\langle\mathcal{O}\rangle\rangle, \odot, \mathbb{1})$ and $(t\mathbb{K}\langle\langle t\rangle\rangle, \circ, t)$.

Moreover, this morphism is surjective when $\mathcal{O}(n) \neq \emptyset$ for all $n \geqslant 1$.

Proof.

$$\begin{aligned}
\mathrm{ev}(\mathbf{f} \odot \mathbf{g}) &= \sum_{\substack{y \in \mathcal{O} \\ z_1, \dots, z_{|y|} \in \mathcal{O}}} \langle y, \mathbf{f}\rangle \left(\prod_{1 \leqslant i \leqslant |y|} \langle z_i, \mathbf{g}\rangle \right) \mathrm{ev}\left(y \circ \left[z_1, \dots, z_{|y|}\right]\right) \\
&= \sum_{\substack{y \in \mathcal{O} \\ z_1, \dots, z_{|y|} \in \mathcal{O}}} \langle y, \mathbf{f}\rangle \left(\prod_{1 \leqslant i \leqslant |y|} \langle z_i, \mathbf{g}\rangle \right) t^{|z_1| + \dots + |z_{|y|}|} \\
&= \left(\sum_{y \in \mathcal{O}} \langle y, \mathbf{f}\rangle \, t^{|y|} \right) \circ \left(\sum_{z \in \mathcal{O}} \langle z, \mathbf{g}\rangle \, t^{|z|} \right) \\
&= \mathrm{ev}(\mathbf{f}) \circ \mathrm{ev}(\mathbf{g})
\end{aligned}$$

□

Generalizing the composition of generating series

Proposition

The map ev is a monoid morphism between $(\mathbb{K}\langle\langle\mathcal{O}\rangle\rangle, \odot, \mathbb{1})$ and $(t\mathbb{K}\langle\langle t\rangle\rangle, \circ, t)$.

Moreover, this morphism is surjective when $\mathcal{O}(n) \neq \emptyset$ for all $n \geqslant 1$.

Proof.

$$\begin{aligned}
\mathrm{ev}(\mathbf{f} \odot \mathbf{g}) &= \sum_{\substack{y \in \mathcal{O} \\ z_1, \dots, z_{|y|} \in \mathcal{O}}} \langle y, \mathbf{f}\rangle \left(\prod_{1 \leqslant i \leqslant |y|} \langle z_i, \mathbf{g}\rangle \right) \mathrm{ev}\left(y \circ \left[z_1, \dots, z_{|y|}\right]\right) \\
&= \sum_{\substack{y \in \mathcal{O} \\ z_1, \dots, z_{|y|} \in \mathcal{O}}} \langle y, \mathbf{f}\rangle \left(\prod_{1 \leqslant i \leqslant |y|} \langle z_i, \mathbf{g}\rangle \right) t^{|z_1| + \dots + |z_{|y|}|} \\
&= \left(\sum_{y \in \mathcal{O}} \langle y, \mathbf{f}\rangle\, t^{|y|} \right) \circ \left(\sum_{z \in \mathcal{O}} \langle z, \mathbf{g}\rangle\, t^{|z|} \right) \\
&= \mathrm{ev}(\mathbf{f}) \circ \mathrm{ev}(\mathbf{g})
\end{aligned}$$

□

Therefore, $\odot$ offers a generalization of the usual composition of generating series.

Example — Motzkin paths

Proposition [G., 2015]

The set

is the unique minimal generating set of the operad Motz.

Example — Motzkin paths

Proposition [G., 2015]

The set

{, }

is the unique minimal generating set of the operad Motz.

From this (and a reasoning about unambiguous decompositions of Motzkin paths), we obtain that the characteristic series of Motz satisfies

$\mathbf{f}_{\mathsf{Motz}} =$  $+$  $\bar{\circ}$ [, $\mathbf{f}_{\mathsf{Motz}}$] $+$  $\bar{\circ}$ [, $\mathbf{f}_{\mathsf{Motz}}, \mathbf{f}_{\mathsf{Motz}}$] .

Example — Motzkin paths

Proposition [G., 2015]

The set

{ ,  }

is the unique minimal generating set of the operad Motz.

From this (and a reasoning about unambiguous decompositions of Motzkin paths), we obtain that the characteristic series of Motz satisfies

$\mathbf{f}_{\mathsf{Motz}} =$  $+$  $\bar{\circ}$ [, $\mathbf{f}_{\mathsf{Motz}}$] $+$  $\bar{\circ}$ [, $\mathbf{f}_{\mathsf{Motz}}, \mathbf{f}_{\mathsf{Motz}}$] .

Now, the evaluation $\mathcal{G}_{\mathsf{Motz}}(t)$ of $\mathbf{f}_{\mathsf{Motz}}$ satisfies

$$\mathcal{G}_{\mathsf{Motz}}(t) = t + t\mathcal{G}_{\mathsf{Motz}}(t) + t\mathcal{G}_{\mathsf{Motz}}(t)^2$$

and is the generating series of Motzkin paths.

Summary of the main ideas

To understand (enumerate, discover statistics, *etc.*) a combinatorial set C, we

Summary of the main ideas

To understand (enumerate, discover statistics, *etc.*) a combinatorial set C, we

1. work with series on C (objects of $\mathbb{K}\langle\langle C\rangle\rangle$) instead of usual generating series (objects of $\mathbb{K}\langle\langle t\rangle\rangle$);

Summary of the main ideas

To understand (enumerate, discover statistics, *etc.*) a combinatorial set C, we

1. work with series on C (objects of $\mathbb{K}\langle\langle C\rangle\rangle$) instead of usual generating series (objects of $\mathbb{K}\langle\langle t\rangle\rangle$);

2. define operations $\star$ on C;

Summary of the main ideas

To understand (enumerate, discover statistics, *etc.*) a combinatorial set C, we

1. work with series on C (objects of $\mathbb{K}\langle\langle C\rangle\rangle$) instead of usual generating series (objects of $\mathbb{K}\langle\langle t\rangle\rangle$);

2. define operations $\star$ on C;

3. extends these operations to operations $\bar{\star}$ on C-series;

Summary of the main ideas

To understand (enumerate, discover statistics, *etc.*) a combinatorial set C, we

1. work with series on C (objects of $\mathbb{K}\langle\langle C\rangle\rangle$) instead of usual generating series (objects of $\mathbb{K}\langle\langle t\rangle\rangle$);
2. define operations $\star$ on C;
3. extends these operations to operations $\bar{\star}$ on C-series;
4. use these operations to express the characteristic series $\mathbf{f}_C$ of C;

Summary of the main ideas

To understand (enumerate, discover statistics, *etc.*) a combinatorial set C, we

1. work with series on C (objects of $\mathbb{K}\langle\langle C\rangle\rangle$) instead of usual generating series (objects of $\mathbb{K}\langle\langle t\rangle\rangle$);
2. define operations $\star$ on C;
3. extends these operations to operations $\bar{\star}$ on C-series;
4. use these operations to express the characteristic series $\mathbf{f}_C$ of C;
5. evaluate $\mathbf{f}_C$ to obtain a expression for the generating series $\mathcal{G}_C(t)$ of C.

Summary of the main ideas

To understand (enumerate, discover statistics, *etc.*) a combinatorial set C, we

1. work with series on C (objects of $\mathbb{K}\langle\langle C\rangle\rangle$) instead of usual generating series (objects of $\mathbb{K}\langle\langle t\rangle\rangle$);
2. define operations $\star$ on C;
3. extends these operations to operations $\bar{\star}$ on C-series;
4. use these operations to express the characteristic series $\mathbf{f}_C$ of C;
5. evaluate $\mathbf{f}_C$ to obtain a expression for the generating series $\mathcal{G}_C(t)$ of C.

We have in passing defined generalizations of natural operations on generating series:

Operation on C	**Operation on $\mathbb{K}\langle\langle C\rangle\rangle$**
—	Addition $+$
Binary graded product $\star$	Analog $\bar{\star}$ of the multiplication
Operadic product $\circ$	Analogs $\bar{\circ}$ and $\odot$ of the composition