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Operads of packed words, quotients, and monomial bases

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Abstract. The associative operad is a central structure in operad theory, defined on the linear span of permutations. We build two analogs of the associative operad on the linear span of packed words. By seeing a packed word as a surjective map between two finite sets, our first operad is graded by the cardinality of the domain and the second one, by the cardinality of the codomain. In the same way as the associative operad of permutations admits as quotients the duplicial and interstice operads, we derive similar structures for our operads of packed words. Both operadic operations on monomial bases, constructed from partial orders on packed words, are monomial-positive. We propose also an analogue of Dynkin idempotent of Zie algebras in this context of operads of packed words.

Keywords: Operad; Permutation; Packed word; Dynkin idempotent.

Introduction

The associative operad As is an algebraic structure playing a central role in the operad theory. It is defined on the linear span of permutations and its partial composition consists of inserting a permutation into another one, interpreted as permutation matrices. The first reason justifying the importance of As is that it intervenes in a crucial way in the description of the axioms of symmetric operads. A second reason, shown by Aguiar and Livernet [2], relates to its richness from a combinatorial point of view. Indeed, As admits a basis (called *monomial basis*) defined using the left weak order on permutations which has the nice property that the partial composition of two elements of this basis is a sum of an interval of this partial order. Furthermore, this operad enjoys many properties since it admits not only the Lie operad as a suboperad, but also, as quotients, the duplicial operad of binary trees [4] and the interstice operad on two generators of binary words [5, 6].

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These three operads on permutations, binary trees, and binary words form a hierarchy very similar to a well-known hierarchy of combinatorial Hopf algebras involving the same spaces of combinatorial objects, namely, the Malvenuto-Reutenauer Hopf algebra of permutations [17, 7], the Loday-Ronco Hopf algebra of binary trees [15], and the noncommutative symmetric functions Hopf algebra [8]. Interestingly enough, there is a generalization of the Malvenuto-Reutenauer Hopf algebra on the linear span of packed words. This Hopf algebra has been introduced by Hivert [11] when he considered a notion of word quasi-symmetric functions. This construction is natural since while permutations are bijections, packed words are surjections. In this context, the analog of the Loday-Ronco Hopf algebra involves Schröder trees [19] and the analog of the noncommutative symmetric functions Hopf algebra involves ternary words [19]. The starting point of the present work is to explore whether such a natural generalization of As exists and if it leads to a similar hierarchy of operads.

Our main contribution consists of the introduction of two different generalizations of As on the linear span of packed words. By seeing a packed word as its matrix, we obtain a right version $\mathsf{PAs}^{\rightarrow}$ consisting of inserting a packed word matrix at a given position into another one, and a left version $\mathsf{PAs}^{\leftarrow}$ consisting of inserting several copies of a packed word matrix onto given values. These operads also differ in the way the arity of a packed word is defined: in $\mathsf{PAs}^{\rightarrow}$ (resp. $\mathsf{PAs}^{\leftarrow}$), the arity of a packed word is the cardinality of its domain (resp. codomain). As they are not isomorphic as graded spaces, $\mathsf{PAs}^{\rightarrow}$ and $\mathsf{PAs}^{\leftarrow}$ are not isomorphic as operads. Moreover, $\mathsf{PAs}^{\leftarrow}$ is a symmetric operad while $\mathsf{PAs}^{\rightarrow}$ is not. Besides, the operad $\mathsf{PAs}^{\leftarrow}$ is not combinatorial in the sense that it admits infinitely many elements of any given arity $n \geqslant 1$. Nevertheless, despite this fact, this operad is rich from a combinatorial point of view since it admits several quotients using well-known equivalence relations on packed words, like the sylvester [12], hypoplactic [14], or Baxter [9] congruences.

This work is presented as follows. Section 1 contains fundamental notions about the main combinatorial objects and operads appearing in this work. In Section 2, we introduce and construct the operads $\mathsf{PAs}^{\rightarrow}$ and $\mathsf{PAs}^{\leftarrow}$ and present their first properties. Section 3 is devoted to the study of some of the quotients of these two operads of packed words which involve other families of combinatorial objects. After recalling the definition of *left and right weak order for packed words*, we use these orders in Section 4 to construct monomial bases and to study their behavior under our two operads. An analogue identity of the classical Dynkin idempotents for *Zie algebras* [1] in term of monomials is derived.

General notations and conventions. For any integer i, $[i]$ denotes the set $\{1, \ldots, i\}$. For any set A, A^* is the set of words on A. For any $w \in A^*$, $\ell(w)$ is the length of w, and for any $i \in [\ell(w)]$, $w(i)$ is the i-th letter of w. The only word of length 0 is the empty word ϵ. For any $1 \leqslant i \leqslant j \leqslant \ell(w)$, $w(i, j)$ is the word $w(i) \ldots w(j)$. Given two words w and w', the concatenation of w and w' is denoted by ww' or by $w.w'$.

1 Preliminaries

1.1 Packed words

Let $\mathbb{P}$ be the set $\mathbb{N} \setminus \{0\}$. For any $u \in \mathbb{P}^*$ and $\alpha, \beta \in \mathbb{N}$, let $\uparrow_\alpha^\beta (u)$ be the word obtained by incrementing by α the letters of u which are greater than β. For instance, $\uparrow_2^4 (124546) = 124748$. For any $u, v \in \mathbb{P}^*$ and $i \in \mathbb{P}$, let $u \curvearrowleft_i v$ be the word on $\mathbb{P}^*$ obtained by replacing each occurrence of i in u by a copy of v. For instance $2123 \curvearrowleft_2 41 = 411413$ and $21 \curvearrowleft_3 311 = 21$. Let $w \in \mathbb{P}^*$. The *alphabet* of w is the set $\mathrm{Alph}(w)$ formed by the letters appearing in w. An *inversion* of w is a pair (i, j) such that $1 \leqslant i < j \leqslant \ell(w)$ and $w(i) > w(j)$. The *set of inversions* of w is denoted by $\mathrm{Inv}(w)$ and the *number of inversions* of w is $\mathrm{inv}(w) := |\mathrm{Inv}(w)|$. The *standardization map* [13] $\mathrm{st} : \mathbb{P}^n \to \mathfrak{S}_n$ is the function sending w to the unique permutation $\mathrm{st}(w) \in \mathfrak{S}_{\ell(w)}$ such that $\mathrm{Inv}(w) = \mathrm{Inv}(\mathrm{st}(w))$. For instance, if $x < y < z$ are letters of $\mathbb{P}$, then $\mathrm{st}(yyxzzzxzyx) = 451782963 \in \mathfrak{S}_9$.

A *packed word* is a word $w \in \mathbb{P}^*$ satisfying $i - 1 \in \mathrm{Alph}(w)$ whenever $i \in \mathrm{Alph}(w)$ and $i \geqslant 2$. This implies that $\mathrm{Alph}(w) = [k]$ for some $k \geqslant 0$ and that $k \leqslant \ell(w)$. For any $k, n \geqslant 0$, let $\mathfrak{P}_n[k]$ be the set of all packed words of alphabet $[k]$ and length n. When $n = k$, this set coincides with $\mathfrak{S}_n$.

Recall that the symmetric group acts on the vector space of non-commutative polynomials $\mathbb{K}\langle \mathsf{X} \rangle$, for any countable set X, where $\mathbb{K}$ is any field of characteristic zero. When $\mathsf{X} = \mathbb{P}$, this action restricts to packed words: if $w \in \mathfrak{P}_n[k]$ and $\sigma \in \mathfrak{S}_n$, then

$$w \cdot \sigma := w(\sigma(1)) \cdots w(\sigma(n)) \in \mathfrak{P}_n[k]. \tag{1.1}$$

Indeed, if w is a packed word, any permutation of its letters is also a packed word.

For any $w \in \mathfrak{P}_n[k]$, the *composition type* $\chi(w)$ of w is the unique weakly increasing word obtained after rearrangement of w. In particular, $\chi(w) \in \mathfrak{P}_n[k]$. Every weakly increasing packed word $w \in \mathfrak{P}_n[k]$ can be encoded by the integer composition $c_1 c_2 \ldots c_k$ of n such that every i appears c_i times in w. Equivalently, through the classical bijection between subsets of $[n-1]$ and binary strings of length $n - 1$, the composition type $\chi(w)$ of a packed word of length n can be also described as a word of length $n - 1$ on the alphabet $\{1, 2\}$. For instance, $\chi(2311223) = 1122233 \leftrightarrow -+--+- \leftrightarrow 121121$ and $\chi(221) = 122 \leftrightarrow +- \leftrightarrow 21$.

The following lemma, due to Hivert, relates a packed word with its composition type.

Lemma 1.1 (Hivert, [13])**.** *For any $w \in \mathfrak{P}_n[k]$, $w = \chi(w) \cdot st(w)$. Moreover, $st(w)$ is the smallest permutation in $\mathfrak{S}_n$ for the right weak order satisfying the above property.*

This result implies that every packed word w is encoded by its composition type and its standardization: the composition type tells us how many times each letter appears in w, while $\mathrm{st}(w)$ encodes the relative order of the appearance of the letters in w.

1.2 Associative operad and quotients

We follow the usual notations about symmetric unital operads [16] (called simply *operads* here). Here are four important examples of (symmetric or nonsymmetric) operads.

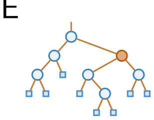
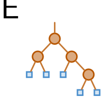
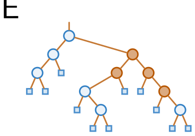
- The *right associative operad* is the symmetric operad $\mathsf{As}^{\rightarrow}$ (also written As) wherein for any $n \geqslant 1$, $\mathsf{As}^{\rightarrow}(n)$ is the linear span of the set $\mathfrak{S}_n$. The set $\{\mathsf{E}_\sigma : \sigma \in \mathfrak{S}_n, n \geqslant 1\}$ is a basis of $\mathsf{As}^{\rightarrow}$. Given $\alpha \in \mathfrak{S}_n$, $\beta \in \mathfrak{S}_m$ and $i \in [n]$, let

$$B_i(\alpha, \beta) := \uparrow_{m-1}^{\alpha(i)} (\alpha(1, i-1)) \,.\, \uparrow_{\alpha(i)-1}^{0} (\beta) \,.\, \uparrow_{m-1}^{\alpha(i)-1} (\alpha(i+1, n)). \tag{1.2}$$

For instance, $B_5(3612457, 231) = 381256479$. The red labels reflect the red permutation 231 inserted into the blue permutation 3612457 onto the letter 4 at the 5-th position. The permutation $B_i(\alpha, \beta)$ is sometimes called the *block permutation* associated to α and β (see [16]). The partial composition of $\mathsf{As}^{\rightarrow}$ satisfies $\mathsf{E}_\sigma \circ_i \mathsf{E}_\nu = \mathsf{E}_{B_i(\sigma,\nu)}$ and the action of the symmetric group $\mathfrak{S}_n$ satisfies $\mathsf{E}_\sigma \cdot \nu = \mathsf{E}_{\sigma \circ \nu}$ where $\circ$ is the composition of permutations.

- The *left associative operad* is the symmetric operad $\mathsf{As}^{\leftarrow}$ defined on the same space as the one of $\mathsf{As}^{\rightarrow}$, seen on the same E-basis. The partial composition of $\mathsf{PAs}^{\leftarrow}$ satisfies $\mathsf{E}_\sigma \circ_i \mathsf{E}_\nu = \mathsf{E}_\pi$ where π is the permutation obtained by replacing in $\uparrow_{m-1}^{i} (\sigma)$ the letter i by $\uparrow_{i-1}^{0} (\nu)$, where m is the greatest letter of ν. The action of the symmetric group $\mathfrak{S}_n$ satisfies $\mathsf{E}_\sigma \cdot \nu = \mathsf{E}_{\nu^{-1} \circ \sigma}$ where $\circ$ is the composition of permutations. The operads $\mathsf{As}^{\rightarrow}$ and $\mathsf{As}^{\leftarrow}$ are isomorphic through the linear map $\phi : \mathsf{As}^{\rightarrow} \to \mathsf{As}^{\leftarrow}$ satisfying $\phi(\mathsf{E}_\sigma) = \mathsf{E}_{\sigma^{-1}}$.

- The *duplicial operad* [4] is the nonsymmetric operad Dup wherein for any $n \geqslant 1$, $\mathsf{Dup}(n)$ is the linear span of the set $\mathfrak{B}_n$ of binary trees with n internal nodes. The set $\{\mathsf{E}_{\mathfrak{t}} : \mathfrak{t} \in \mathfrak{B}_n, n \geqslant 1\}$ is a basis of Dup. The partial composition of Dup satisfies $\mathsf{E}_{\mathfrak{t}} \circ_i \mathsf{E}_{\mathfrak{s}} = \mathsf{E}_{\mathfrak{r}}$ where $\mathfrak{r}$ is the binary tree obtained by replacing the i-th internal node u of $\mathfrak{t}$ for the infix traversal by a copy of $\mathfrak{s}$ and by grafting the left subtree of u on the leftmost leaf of the copy and by grafting the right subtree of u on the rightmost least of the copy. For instance, in Dup, we have

E  $\circ_6$ E  $=$ E  . (1.3)

- For any $s \geqslant 1$, the *s-interstice operad* [5, 6] is the nonsymmetric operad I_s wherein for any $n \geqslant 1$, $\mathsf{I}_s(n)$ is the linear span of the set $[s]^n$. The set $\{\mathsf{E}_u : u \in [s]^n, n \geqslant 1\}$ is a basis of I_s. The partial composition of I_s satisfies $\mathsf{E}_u \circ_i \mathsf{E}_v := \mathsf{E}_{u(1,i-1) \,.\, v \,.\, u(i,\ell(u))}$. For instance, in I_2, we have $\mathsf{E}_{121121} \circ_4 \mathsf{E}_{21} = \mathsf{E}_{121\,21\,121}$.

As shown in [2], Dup and I_2 are nonsymmetric operad quotients of $\mathsf{As}^{\rightarrow}$.

2 Operadic structures on packed words

2.1 Right version

Let $\mathsf{PAs}^{\rightarrow}$ be the graded space such that for any $n \geqslant 1$, $\mathsf{PAs}^{\rightarrow}(n)$ is the linear span of the set $\mathfrak{P}_n$. The set $\{\mathsf{E}_u : u \in \mathfrak{P}_n, n \geqslant 1\}$ is a basis of $\mathsf{PAs}^{\rightarrow}$. Let us endow $\mathsf{PAs}^{\rightarrow}$ with the operations $\circ_i$ defined, for any $u \in \mathfrak{P}_n[r]$, $i \in [n]$, and $v \in \mathfrak{P}_m[s]$, by

$$\mathsf{E}_u \circ_i \mathsf{E}_v := \mathsf{E}_{\uparrow_{s-1}^{u(i)}(u(1,i-1))\,.\,\uparrow_{u(i)-1}^{0}(v)\,.\,\uparrow_{s-1}^{u(i)-1}(u(i+1,n))}. \tag{2.1}$$

For instance, $\mathsf{E}_{2311223} \circ_5 \mathsf{E}_{221} = \mathsf{E}_{241133234}$. Intuitively, the partial composition $\mathsf{E}_u \circ_i \mathsf{E}_v$ of $\mathsf{PAs}^{\rightarrow}$ is similar to the one $\mathsf{As}^{\rightarrow}$ but with the difference that the occurrences of $u(i)$ in u having a position greater than i are incremented by $s-1$ where s is the maximal value of v. In terms of permutation matrices, we have

$$\mathsf{E}_{\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}} \circ_5 \mathsf{E}_{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \mathsf{E}_{\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}. \tag{2.2}$$

Theorem 2.1. *The graded space* $\mathsf{PAs}^{\rightarrow}$*, endowed with the partial composition maps* $\circ_i$*, is a nonsymmetric unital operad.*

We call $\mathsf{PAs}^{\rightarrow}$ the *right associative operad of packed words*. Remark that in [10] and [18], other operads involving the same space of packed words are constructed. The operads $\mathsf{PAs}^{\rightarrow}$ and PW, which is defined in [10], are not isomorphic. Indeed, a simple inspection shows that any minimal generating set of PW contains two elements of arity 3 while any minimal generating set $\mathsf{PAs}^{\rightarrow}$ has no element of this arity.

For any $u \in \mathfrak{P}_k$ and $i \in [k]$, the *record* $\mathrm{rec}_i(u)$ of u at i is $\mathrm{rec}_i(u) := \mathrm{st}(u)(i)$. For example, taking $u = 2311\mathbf{2}23$ and $i = 5$, we have $\mathrm{st}(u) = 3612\mathbf{4}57$, so $\mathrm{rec}_5(u) = \mathrm{st}(u)(5) = 4$. In what follows, we consider that the maps rec and st are extended linearly on the graded space $\mathsf{PAs}^{\rightarrow}$ through its E-basis. In the same way, we extend linearly the action $\cdot$ of (1.1) on $\mathsf{PAs}^{\rightarrow}$ through its E-basis.

As shown by the next result, the operad $\mathsf{PAs}^{\rightarrow}$ relates well with $\mathsf{As}^{\rightarrow}$ and I_2.

Theorem 2.2. *Let* $n \in \mathbb{N}$*,* $u \in \mathfrak{P}_n$*, and* $v \in \mathfrak{P}$*. For any* $i \in [n]$*,*

$$\mathsf{E}_u \circ_i \mathsf{E}_v = \left(\mathsf{E}_{\chi(u)} \circ_{\mathrm{rec}_i(u)} \mathsf{E}_{\chi(v)}\right) \cdot B_i(\mathrm{st}(u), \mathrm{st}(v)), \tag{2.3}$$

where the partial composition $\circ_{\mathrm{rec}_i(u)}$ *is the one of* I_2*. In particular,* $\chi(\mathsf{E}_u \circ_i \mathsf{E}_v) = \mathsf{E}_{\chi(u)} \circ_{\mathrm{rec}_i(u)} \mathsf{E}_{\chi(v)}$ *and* $\mathrm{st}\,(\mathsf{E}_u \circ_i \mathsf{E}_v) = \mathsf{E}_{B_i(\mathrm{st}(u), \mathrm{st}(v))}$*.*

Following with our example let $u = 2311\mathbf{2}23$ and $v = 221$. Notice that $\mathrm{st}(u) = 3612457$, so that $\mathrm{rec}_5(u) = 4$. Also, $\mathrm{st}(v) = 231$. Therefore,

$$\left(\mathsf{E}_{\chi(u)} \circ_{\mathrm{rec}_i(u)} \mathsf{E}_{\chi(v)}\right) \cdot B_i(\mathrm{st}(u), \mathrm{st}(v)) = \mathsf{E}_{112233344} \cdot 381256479 = \mathsf{E}_{241133234}, \tag{2.4}$$

which agrees with our previous example at the beginning of Section 2.1.

2.2 Left version

Let $\mathsf{PAs}^{\leftarrow}$ be the graded space such that for any $n \geqslant 1$, $\mathsf{PAs}^{\leftarrow}(n)$ is the linear span of the set $\mathfrak{P}[n]$. The set $\{\mathsf{E}_u : u \in \mathfrak{P}[n], n \geqslant 1\}$ is a basis of $\mathsf{PAs}^{\leftarrow}$. Let us endow $\mathsf{PAs}^{\leftarrow}$ with the operations $\circ_i$ defined, for any $u \in \mathfrak{P}[n]$, $i \in [n]$, and $v \in \mathfrak{P}[m]$, by

$$\mathsf{E}_u \circ_i \mathsf{E}_v = \mathsf{E}_{\uparrow^i_{m-1}(u) \curvearrowleft_i \uparrow^0_{i-1}(v)}. \tag{2.5}$$

Intuitively, if $\mathsf{E}_w = \mathsf{E}_u \circ_i \mathsf{E}_v$, then w is the word obtained by increasing by $m-1$ the letters of u which are greater than i, and by replacing all its occurrences of i by the word obtained by increasing by $i-1$ all letters of v. For instance, $\mathsf{E}_{231314} \circ_3 \mathsf{E}_{212} = \mathsf{E}_{2\,434\,1\,434\,15}$. In terms of permutation matrices, we have

$$\mathsf{E}_{\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}} \circ_3 \mathsf{E}_{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} = \mathsf{E}_{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}. \tag{2.6}$$

Let us also endow $\mathsf{PAs}^{\leftarrow}$ with the right action $\cdot$ of the symmetric groups such that, for any $u \in \mathfrak{P}[n]$ and $\sigma \in \mathfrak{S}_n$,

$$\mathsf{E}_u \cdot \sigma = \mathsf{E}_{\sigma^{-1}(u(1)) \dots \sigma^{-1}(u(\ell(u)))}. \tag{2.7}$$

For instance, $\mathsf{E}_{1411232} \cdot 3142 = \mathsf{E}_{2322414}$.

Theorem 2.3. *The graded space $\mathsf{PAs}^{\leftarrow}$, endowed with the partial composition maps $\circ_i$ and the action $\cdot$ of the symmetric groups, is a symmetric operad.*

We call $\mathsf{PAs}^{\leftarrow}$ the *left associative operad of packed words*. Observe that this operad is not combinatorial since for any $n \geqslant 1$, there are infinitely many packed words with n as maximal value. However, as we shall see in the next sections, this operad admits interesting quotient operads which are combinatorial.

Given a packed word $w \in \mathfrak{P}_n[k]$, we define the *set-partition type* of w (also called *initial permutation of w* in [3]) as the packed word $\Psi(w) := w \cdot \mathrm{first}_1(w)$, where $\cdot$ is the right action and $\mathrm{first}_1(w)$ is the permutation in $\mathfrak{S}_k$ obtained by keeping from w the first appearance of each letter (see Section 3.2). Our terminology is due to the fact that $\Psi(w)$ can be related to a particular set-partition as follows. A *set-composition* of $[n]$ is a set-partition endowed with a total order on its set of blocks. We denote by $\Sigma[n]$ and $\Pi[n]$ the sets of set-compositions and set-partitions of $[n]$, respectively. There is a map $\mathrm{fgt} : \Sigma[n] \to \Pi[n]$ that forgets the order of the blocks. If $F = (B_1, B_2, \dots, B_k) \in \Sigma[n]$, the map $\Theta : w \mapsto F$ given by the condition $(i \in B_j \leftrightarrow w_i = j)$ defines a bijection $\Theta : \mathfrak{P}_n[k] \to \Sigma_k[n]$. Hence, the set-composition $\Theta(\Psi(w)) = (B_1, B_2, \dots, B_k)$ satisfies $\min(B_1) < \cdots < \min(B_k)$. In this sense, we can identify $\Psi(w)$ with a set-partition.

For instance, if $u = 231314$ and $v = 212$, then $\mathrm{first}_1(u) = 2314$, $\mathrm{first}_1(v) = 21$, so $\Psi(u) = 231314 \cdot 2314 = 123234 \leftrightarrow (1, 24, 35, 6)$ and $\Psi(v) = 212 \cdot 21 = 121 \leftrightarrow (13, 2)$.

The following result is a "leftist-analog" of Hivert's Lemma (1.1).

Lemma 2.4. *For any $w \in \mathfrak{P}_n[k]$, $w = \Psi(w) \cdot \mathrm{first}_1(w)^{-1}$, with respect to the right action of permutations. Moreover, $\mathrm{first}_1(w)$ is the smallest permutation in $\mathfrak{S}_n$ for the left weak order satisfying the above property.*

For any $u \in \mathfrak{P}[n]$ and $i \in [n]$, the *co-record* $\mathrm{crec}_i(u)$ of u at i is $\mathrm{crec}_i(w) := \mathrm{first}_1(u)^{-1}(i)$. For example, taking $u = 231314$ and $i = 3$, we have $\mathrm{first}_1(u) = 2314$, so $\mathrm{crec}_3(u) = 2$. As before, we consider that the maps crec and first_1 are extended linearly on the graded space $\mathsf{PAs}^{\leftarrow}$ through its E-basis. An analogue result to Theorem (2.2) is presented now.

Theorem 2.5. *Let $n \in \mathbb{N}$, $u \in \mathfrak{P}[n]$, and $v \in \mathfrak{P}$. For any $i \in [n]$,*

$$\mathsf{E}_u \circ_i \mathsf{E}_v = \left(\mathsf{E}_{\Psi(u)} \circ_{crec_i(u)} \mathsf{E}_{\Psi(v)}\right) \cdot B_i(\mathrm{first}_1(u)^{-1}, \mathrm{first}_1(v)^{-1}), \tag{2.8}$$

In particular, $\Psi(\mathsf{E}_u \circ_i \mathsf{E}_v) = \mathsf{E}_{\Psi(u)} \circ_{crec_i(u)} \mathsf{E}_{\Psi(v)}$ and $\mathrm{first}_1(\mathsf{E}_u \circ_i \mathsf{E}_v) = \mathsf{E}_{B_i(\mathrm{first}_1(u)^{-1}, \mathrm{first}_1(v))^{-1}}$.

Following with our example let $u = 23\mathbf{1}314$ and $v = 212$. From here,

$$\left(\mathsf{E}_{\Psi(u)} \circ_{\mathrm{crec}_3(u)} \mathsf{E}_{\Psi(v)}\right) \cdot B_3(\mathrm{first}_1(u)^{-1}, \mathrm{first}_1(v)^{-1}) = \mathsf{E}_{1\mathbf{232}423245} \cdot 24315^{-1} = \mathsf{E}_{2434143415}, \tag{2.9}$$

which agrees with our previous example at the beginning of Section 2.2.

3 Operadic quotients

In this section, we regard the operad $\mathsf{PAs}^{\leftarrow}$ as a set-operad through its E-basis. This is possible since the composition of two elements of the E-basis produces a single element of the E-basis. For this reason, we shall write here u instead of E_u for any $u \in \mathfrak{P}$.

3.1 Permutative congruences

Let P be a predicate on $\mathbb{P}^* \times \mathbb{P} \times \mathbb{P} \times \mathbb{P}^*$. From P, we define the binary relation $\leftrightarrow_P$ on $\mathbb{P}^*$ satisfying $uabv \leftrightarrow_P ubav$ for any $u, v \in \mathbb{P}^*$ and $a, b \in \mathbb{P}$ such that $P(u, a, b, v)$ holds. Let also $\equiv_P$ be the reflexive, symmetric, and transitive closure of $\leftrightarrow_P$. The predicate P is

1. *compatible with relabeling* if for any $u, v \in \mathbb{P}^*$ and $a, b \in \mathbb{P}$, $P(u, a, b, v)$ implies that $P(f(u), f(a), f(b), f(v))$ where $f : \mathbb{P} \to \mathbb{P}$ is any strictly monotone map;

2. *compatible with subwords* if for any $u, v \in \mathbb{P}^*$ and $a, b \in \mathbb{P}$, $P(u, a, b, v)$ implies that $P(u', a, b, v')$ for any $u', v' \in \mathbb{P}^*$ such that u is a subword of u' and v is a subword of v'.

Observe in particular that when P is compatible with subwords, $\equiv_P$ is a monoid congruence of the free monoid on $\mathbb{P}$. Here are some examples of predicates compatible with relabeling and subwords, where $u, v \in \mathbb{P}^*$ and $a, b, c, d \in \mathbb{P}$:

- Let P_{F} be the *finest predicate* for which $P_{\mathrm{F}}(u, a, b, v)$ never holds.
- Let P_{C} be the *commutative predicate*, for which $P_{\mathrm{C}}(u, a, b, v)$ always holds.
- Let P_{S} be the *sylvester predicate*, satisfying $P_{\mathrm{S}}(u, a, c, v)$ if there is b in v such that $a \leqslant b < c$. The equivalence relation $\equiv_{P_{\mathrm{S}}}$ is the *sylvester congruence* [12], providing an alternative construction of the Loday-Ronco Hopf algebra.
- Let P_{H} be the *hypoplactic predicate*, satisfying $P_{\mathrm{H}}(u, a, c, v)$ if there is b in u such that $a < b \leqslant c$ or there is b in v such that $a \leqslant b < c$. The equivalence relation $\equiv_{P_{\mathrm{H}}}$ is the *hypoplactic congruence* [14], providing a construction of the Hopf algebra **NCSF** of noncommutative symmetric functions.
- Let P_{B} be the *Baxter predicate*, satisfying $P_{\mathrm{B}}(u, a, c, v)$ if there is b in u and b' in v such that $a \leqslant b' < b \leqslant c$ or $a < b \leqslant b' < c$. The equivalence relation $\equiv_{P_{\mathrm{B}}}$ is the *Baxter congruence* [9], providing a construction of the Baxter Hopf algebra which admits the Loday-Ronco Hopf algebra as quotient.

In contrast, the *plactic predicate* P_{P}, satisfying $P_{\mathrm{P}}(u, a, c, v)$ if v is nonempty and its first letter b is such that $a \leqslant b < c$, or u is nonempty and its last letter b is such that $a < b \leqslant c$, is not compatible with subwords. The equivalence relation $\equiv_{P_{\mathrm{P}}}$ enjoys a lot of properties (see for instance [7] for properties related to the construction of Hopf algebras) but does not play any role in this operadic context.

Theorem 3.1. *If P is a predicate compatible with relabeling and subwords, then $\equiv_P$ is a nonsymmetric operad congruence of* $\mathsf{PAs}^{\leftarrow}$.

When P satisfies the prerequisites of Theorem 3.1, $\equiv_P$ is a *permutative congruence*. Let us denote by $\theta_P : \mathsf{PAs}^{\leftarrow} \to \mathsf{PAs}^{\leftarrow}$ the map sending any $u \in \mathsf{PAs}^{\leftarrow}$ to the minimal word w.r.t. the lexicographic order of the $\equiv_P$-equivalence class of u.

3.2 Quotients forming combinatorial operads

For any $k \geqslant 1$, let $\mathrm{first}_k : \mathsf{PAs}^{\leftarrow} \to \mathsf{PAs}^{\leftarrow}$ be the map sending any $u \in \mathsf{PAs}^{\leftarrow}$ to the packed word obtained by deleting any occurrence of a letter a provided that there are at least k occurrences of a on its left. For instance, $\mathrm{first}_2(421431412) = 4214312$. Let us denote by $\equiv_k$ the equivalence relation on $\mathsf{PAs}^{\leftarrow}$ satisfying, for any $u, v \in \mathsf{PAs}^{\leftarrow}$, $u \equiv_k v$ if $\mathrm{first}_k(u) = \mathrm{first}_k(v)$.

Proposition 3.2. *For any $k \geqslant 1$, the equivalence relation $\equiv_k$ is a nonsymmetric operad congruence of* $\mathsf{PAs}^{\leftarrow}$.

Since for any $k \geqslant 1$ and $n \geqslant 1$, there are finitely packed words having n as maximal value and where each letter appears at most k times, the operad $\mathsf{PAs}^{\leftarrow}/_{\equiv_k}$ is combinatorial. The quotient $\mathsf{PAs}^{\leftarrow}/_{\equiv_1}$ is the left associative operad of permutations.

A predicate P is *compatible with* first_k if for any $u, v \in \mathbb{P}^*$ and $a, b \in \mathbb{P}$, $P(u, a, b, v)$ implies that $P(\mathrm{first}_k(u), a, b, \mathrm{first}_k(v))$. Besides, P is *right-trivial* if for any $u, v \in \mathbb{P}^*$ and $a, b \in \mathbb{P}$, $P(u, a, b, v)$ implies that $P(u, a, b, \epsilon)$.

Theorem 3.3. *Let $k \geqslant 1$. If P is a predicate compatible with subwords and* first_k*, and P is right-trivial, then the maps* first_k *and θ_P commute for any $k \geqslant 1$.*

Let us denote by $\equiv_{P,k}$ the equivalence relation on $\mathsf{PAs}^{\leftarrow}$ satisfying, for any $u, v \in \mathsf{PAs}^{\leftarrow}$, $u \equiv_{P,k} v$ if $\theta_P(\mathrm{first}_k(u)) = \theta_P(\mathrm{first}_k(v))$. By Theorem 3.3, $\equiv_{P,k}$ is a nonsymmetric operad congruence of $\mathsf{PAs}^{\leftarrow}$. By construction, the set $\theta_P(\mathrm{first}_k(\mathsf{PAs}^{\leftarrow}))$ is a set of representatives of the quotient $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P,k}}$. We shall identify $\theta_P(\mathrm{first}_k(\mathsf{PAs}^{\leftarrow}))$ with such set of representatives. Considering this, the partial composition map of $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P,k}}$ satisfies, for any $u, v \in \mathsf{PAs}^{\leftarrow}/_{\equiv_{P,k}}$ and $i \in [|u|]$, $u \circ_i v = \theta_P(\mathrm{first}_k(u \circ_i v))$ where the second occurrence of $\circ_i$ is the partial composition map of $\mathsf{PAs}^{\leftarrow}$.

Let us review some quotients of $\mathsf{PAs}^{\leftarrow}$ obtained via such congruences.

Finest predicate. The finest predicate P_{F} is compatible with first_k for any $k \geqslant 1$ and is right-trivial. In particular, the nonsymmetric operad $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\mathrm{F}},1}}$ is the left associative operad $\mathsf{As}^{\leftarrow}$. For this reason, the operads $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\mathrm{F}},k}}$ are generalizations of the left associative operad. For any $k \geqslant 1$ and any $n \geqslant 1$, $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\mathrm{F}},1}}(n)$ is the set of words such that each letter between 1 and n appears from 1 to k times. The sequences of the dimensions of $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\mathrm{F}},2}}$ begins with 2, 14, 222, 6384, 291720, 19445040, 1781750880 and forms Sequence **A105749** of [20]. The sequences of the dimensions of $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\mathrm{F}},3}}$ begins with 3, 62, 5052, 1087104, 487424520, 393702654960 and forms Sequence **A144422** of [20].

Commutative predicate. The commutative predicate P_{C} is compatible with first_k for any $k \geqslant 1$ and is right-trivial. In particular, the nonsymmetric operad $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\mathrm{C}},1}}$ is the nonsymmetric associative operad. For this reason, the operads $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\mathrm{C}},k}}$ are generalizations of the nonsymmetric associative operad. For any $k \geqslant 1$ and any $n \geqslant 1$, $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\mathrm{C}},k}}(n)$ is the set of weakly increasing words such that each letter between 1 and n appears from 1 to k times. Hence, the dimension of $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\mathrm{C}},k}}(n)$ is k^n.

Sylvester predicate. The sylvester predicate P_{S} is compatible with first_k for any $k \geqslant 1$ but is not right-trivial. Nevertheless, by setting $P_{\overline{\mathrm{S}}}$ as the predicate satisfying $P_{\overline{\mathrm{S}}}(u, a, b, v)$ if and only if $P_{\overline{\mathrm{S}}}(v, a, b, u)$, $P_{\overline{\mathrm{S}}}$ is right-trivial. In particular, the nonsymmetric operad $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\overline{\mathrm{S}}},1}}$ is isomorphic to the duplicial operad Dup. This can be shown by computing the nontrivial relations satisfied by the generators 12 and 21 of $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\overline{\mathrm{S}}},1}}$ and showing that they are analog to the nontrivial relations satisfied by the generators of Dup. For this reason, the operads $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\overline{\mathrm{S}}},k}}$ are generalizations of the duplicial operad. By using some results of [12], for any $k \geqslant 1$ and any $n \geqslant 1$, $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\overline{\mathrm{S}}},k}}(n)$ is the set of words such that each letter between 1 and n appears from 1 to k times and which avoid the patterns 231 and 221. The sequence of the dimensions of $\mathsf{PAs}^{\leftarrow}/_{\equiv_{P_{\overline{\mathfrak{S}}},2}}$ begins with 2, 10, 66, 498, 4066, 34970, 312066, 2862562.

4 Monomial bases and Dynkin idempotents

The weak order on permutations has several analogues for packed words. The *right weak order* for packed words is defined via the following covering relation: for $u, v \in \mathfrak{P}_k$ $u \prec_r v$ if and only if $v = u \cdot \tau$, for some transposition $\tau \in \mathfrak{S}_k$ and $\mathrm{inv}(v) = \mathrm{inv}(u) + 1$. Given $u \in \mathfrak{P}$, the *set of left-inversions* $\mathrm{Inv}_\ell(u)$ of u is the set of pairs (i, j) such that $i < j$ and all appearances of i in u occur after all appearances of j in u. The *left weak order* of packed words is defined via the following covering relation: for $u, v \in \mathfrak{P}[n]$, $u \prec_\ell v$ if and only if $v = \tau \cdot u$, for some transposition $\tau \in \mathfrak{S}_n$ and $|\mathrm{Inv}_\ell(v)| = |\mathrm{Inv}_\ell(u)| + 1$. See [3] for relevant bibliography related to these orders.

Composition of monomial bases. For every packed word u, consider the new elements $\mathsf{F}_u, \mathsf{M}_u$ in $\mathsf{PAs}^{\rightarrow}$ (resp. G_u, N_u in $\mathsf{PAs}^{\leftarrow}$), defined implicitly as:

$$\mathsf{E}_u = \sum_{v \leqslant_r u} \mathsf{F}_v \quad \text{and} \quad \mathsf{F}_u = \sum_{u \leqslant_\ell v} \mathsf{M}_v \quad \left(\text{resp.} \quad \mathsf{H}_u = \sum_{v \leqslant_\ell u} \mathsf{G}_v \quad \text{and} \quad \mathsf{G}_u = \sum_{u \leqslant_r v} \mathsf{N}_v\right). \tag{4.1}$$

For $\mathsf{X} \in \{\mathsf{F}, \mathsf{M}\}$ and $\mathsf{Y} \in \{\mathsf{G}, \mathsf{N}\}$, the sets $\{\mathsf{X}_u : u \in \mathfrak{P}_n, n \geqslant 0\}$ and $\{\mathsf{Y}_u : u \in \mathfrak{P}[n], n \geqslant 0\}$ are linear bases for the Hopf algebras WQSym and WQSym^* of packed words, respectively (see, for example, [3]). The M-basis and N-basis, called *monomial bases*, are of particular interest as they provide a simple description of the primitive spaces of WQSym and WQSym^*. Following the terminology of [1], we call the former primitive space the *space of Zie elements*. In the case of permutations, a *Lie element* is an element of the primitive space of FQSym, the Hopf algebra of permutations [17, 7]. It is straightforward to show that the bases F and G satisfy the same internal operations (2.1) and (2.5) of the E-basis in $\mathsf{PAs}^{\rightarrow}$ and $\mathsf{PAs}^{\leftarrow}$, respectively.

For any $m, n \in \mathbb{N}$ and $i \in [m]$, let the maps $B_i^\ell : \mathfrak{P}[m] \times \mathfrak{P}[n] \to \mathfrak{P}[m+n-1]$ and $B_i^r : \mathfrak{P}_m \times \mathfrak{P}_n \to \mathfrak{P}_{m+n-1}$ defined as the set-theoretical operations coming from the left and right operads (2.5) and (2.1), respectively, defined on the E-basis. Here, the sets $\mathfrak{P}[m] \times \mathfrak{P}[n]$ and $\mathfrak{P}_m \times \mathfrak{P}_n$ are endowed with the partial orders obtained by taking the Cartesian product of the right and left weak order on packed words, respectively.

Theorem 4.1. (a) *The maps B_i^ℓ and B_i^r are order-preserving, with respect to the right and left weak orders, respectively. Moreover, they possess right adjoints.*

(b) *The left (resp. right) operadic composition of the N-basis is N-positive (resp. for the M-basis).*

For instance, $\mathsf{M}_{112} \circ_1 \mathsf{M}_{11} = \mathsf{M}_{1123} + \mathsf{M}_{1132} + \mathsf{M}_{2213} + \mathsf{M}_{3312}$ (which is not an interval for $\leqslant_\ell$), while $\mathsf{N}_{112} \circ_1 \mathsf{N}_{11} = \mathsf{N}_{11112} + \mathsf{N}_{11121}$ (which is an interval for $\leqslant_r$).

Part (b) of the above Theorem is a direct consequence of part (a). In [2], the authors show a stronger result for permutations: the right-operadic composition of the basis M is a sum over an interval for the left-weak order on permutations. This surprising phenomenon is due to the fact that the fiber of the right adjoint for B_i^r in this case is always an interval. For our more general setting to packed words, this is no longer the case for the M-basis. However, computational evidence leads to the following conjecture.

Conjecture 4.2. *The fiber of the right adjoint for the map B_i^ℓ on packed words is always an interval. In particular, the left-operadic composition of the N basis is a sum over an interval of the left weak order for packed words.*

An analogue of Dynkin idempotents to Zie. Given $u, v \in \mathfrak{P}$, define a new operation on $\mathsf{PAs}^{\rightarrow}$ as

$$\{\mathsf{F}_u, \mathsf{F}_v\} := (\mathsf{M}_{12} \circ_2 \mathsf{F}_v) \circ_1 \mathsf{F}_u. \tag{4.2}$$

For instance, $\{\mathsf{F}_{11}, \mathsf{F}_{121}\} = ((\mathsf{F}_{12} - \mathsf{F}_{21}) \circ_2 \mathsf{F}_{121}) \circ_1 \mathsf{F}_{11} = \mathsf{F}_{11232} - \mathsf{F}_{33121}$. Remark that this operation is not the induced Lie bracket of WQSym; while $\{\mathsf{F}_1, \mathsf{F}_1\} = \mathsf{F}_{12} - \mathsf{F}_{21}$, we have $[\mathsf{F}_1, \mathsf{F}_1] = 0$. If $u \in \mathfrak{P}[m]$ and $v \in \mathfrak{P}[n]$, it is straightforward to show that $\{\mathsf{F}_u, \mathsf{F}_v\} = \mathsf{F}_{u/v} - \mathsf{F}_{u\backslash v}$, where $u/v := u \uparrow_m^1 (v)$ and $u\backslash v := \uparrow_n^1 (u)v$.

Now, given $k \geqslant 1$, the *Dynkin idempotent* $\frac{1}{k}\theta_k$ is defined as the unique element in $\mathsf{As}^{\rightarrow}(k)$ for which $\theta_k \cdot a_1a_2a_3\cdots a_k = [\cdots[[a_1, a_2], a_3], \cdots, a_k]$ is the $k-1$ left nested commutator bracket in the vector space $\mathbb{K}\langle\mathbb{N}\rangle$, for any word $a_1a_2a_3\cdots a_k \in \mathbb{K}\langle\mathbb{N}\rangle$. Here, we consider the natural right action of permutations (as in (2.7)). It is shown in [2, Lem. 5.2] that we can express θ_k as $\theta_k = \{\ldots\{\{\mathsf{F}_1, \mathsf{F}_1\}, \mathsf{F}_1\}, \ldots, \mathsf{F}_1\}$ (there are $k-1$ left nested brackets). Given a composition $\mathsf{c} = c_1c_2\cdots c_n \in \mathbb{N}^*$ of sum k, we define the *c-Dynkin's idempotent* $\frac{1}{k}\theta_k^{\mathsf{c}}$, where $\theta_k^{\mathsf{c}} := \{\ldots\{\{\mathsf{F}_{\mathrm{id}_{c_1}}, \mathsf{F}_{\mathrm{id}_{c_2}}\}, \mathsf{F}_{\mathrm{id}_{c_3}}\}, \ldots, \mathsf{F}_{\mathrm{id}_{c_n}}\}$ and id_r denotes the packed word $11\cdots1$ of length r. More generally, let $\mathrm{id}(\mathsf{c}) \in \mathfrak{P}_k[n]$ be the non-decreasing packed word with c_r copies of r, for every $r \in [n]$.

Theorem 4.3. *For any $k \geqslant 1$, θ_k^{c} is a Zie element in* WQSym, *with* M*-expansion given by*

$$\theta_k^{\mathsf{c}} = \sum_{\substack{id(\mathsf{c}) \leqslant_\ell u \\ u = id(c_1)2 \circ_2 u'}} \mathsf{M}_u. \tag{4.3}$$

References

[1] M. Aguiar and S. Mahajan. *Topics in hyperplane arrangements*. Vol. 226. American Mathematical Soc., 2017.

[2] M. Aguiar and M. Livernet. "The associative operad and the weak order on the symmetric groups". *J. Homotopy Relat. Str.* **2**.1 (2007), pp. 57–84.

[3] N. Bergeron, R. S. G. D'León, S. X. Li, C. A. Pang, and Y. Vargas. "Hopf algebras of parking functions and decorated planar trees". *Adv. Appl. Math.* **143** (2023), p. 102436.

[4] C. Brouder and A. Frabetti. "QED Hopf algebras on planar binary trees". *J. Algebra* **no. 1** (2003), pp. 298–322.

[5] F. Chapoton. "Construction de certaines opérades et bigebres associées aux polytopes de Stasheff et hypercubes". *Trans. Am. Math. Soc.* **354**.1 (2002), pp. 63–74.

[6] C. Combe and S. Giraudo. "Cliff operads: a hierarchy of operads on words". *J. Algebr. Comb.* **57** (2022), pp. 239–269.

[7] G. Duchamp, F. Hivert, and J.-Y. Thibon. "Noncommutative Symmetric Functions VI: Free Quasi-Symmetric Functions and Related Algebras". *Int. J. of Algebr. Comput.* **12** (2002), pp. 671–717.

[8] I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, and J.-Y. Thibon. "Noncommutative symmetric functions I". *Adv. Math.* **112** (1995), pp. 218–348.

[9] S. Giraudo. "Algebraic and combinatorial structures on pairs of twin binary trees". *J. Algebra* **360** (2012), pp. 115–157.

[10] S. Giraudo. "Combinatorial operads from monoids". *J. Algebr. Comb.* **41, Issue 2** (2015), pp. 493–538.

[11] F. Hivert. "Combinatoire des fonctions quasi-symétriques". PhD thesis. Université de Marne la Vallée, 1999.

[12] F. Hivert, J.-C. Novelli, and J.-Y. Thibon. "The Algebra of Binary Search Trees". *Theor. Comput. Sc.* **339**.1 (2005), pp. 129–165.

[13] F. Hivert. "An Introduction to Combinatorial Hopf Algebras-—Examples and realizations—". *Physics and theoretical computer science*. IOS Press, 2006, pp. 253–274.

[14] D. Krob and J.-Y. Thibon. "Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at $q = 0$". *Journal Algebr. Comb.* **6** (1997), pp. 339–376.

[15] J.-L. Loday and M. O. Ronco. "Hopf algebra of the planar binary trees". *Adv. Math.* **139**.2 (1998), pp. 293–309.

[16] J.-L. Loday and B. Vallette. *Algebraic operads*. Vol. 346. Springer, 2012.

[17] C. Malvenuto and C. Reutenauer. "Duality between quasi-symmetrical functions and the solomon descent algebra". *J. Algebra* **177**.3 (1995), pp. 967–982.

[18] J. McClure and J. Smith. "Multivariable cochain operations and little n-cubes". *J. Am. Math. Soc.* **16**.3 (2003), pp. 681–704.

[19] J.-C. Novelli and J.-Y. Thibon. "Hopf algebras of m-permutations, $(m+1)$-ary trees, and m-parking functions". *Adv. Appl. Math.* **117** (2020), pp. 102019, 55.

[20] OEIS Foundation Inc. "The On-Line Encyclopedia of Integer Sequences". Published electronically at http://oeis.org. 2023.