Nonsymmetric operads in combinatorics

Samuele Giraudo

Université Paris-Est, LIGM (UMR 8049), CNRS, ENPC, ESIEE Paris, UPEM, F-77454, Marne-la-Vallée, France

Email address: samuele.giraudo@u-pem.fr

2010 *Mathematics Subject Classification.* 05-00, 05C05, 05E15, 16T05, 18D50.

Key words and phrases. Combinatorics; Algebraic combinatorics; Computer science; Tree; Formal power series; Rewrite system; Operad; Bialgebra; Hopf bialgebra; Pre-Lie algebra; Dendriform algebra.

Abstract. Operads are algebraic devices offering a formalization of the concept of operations with several inputs and one output. Such operations can be naturally composed to form bigger and more complex ones. Coming historically from algebraic topology, operads intervene now as important objects in computer science and in combinatorics. The theory of operads, together with the algebraic setting and the tools accompanying it, promises advances in these two areas. On the one hand, operads provide a useful abstraction of formal expressions, and also, provide connections with the theory of rewrite systems. On the other hand, a lot of operads involving combinatorial objects highlight some of their properties and allow to discover new ones.

This book presents the theory of nonsymmetric operads under a combinatorial point of view. It portrays the main elements of this theory and the links it maintains with several areas of computer science and combinatorics. A lot of examples of operads appearing in combinatorics are studied and some constructions relating operads with known algebraic structures are presented. The modern treatment of operads consisting in considering the space of formal power series associated with an operad is developed. Enrichments of nonsymmetric operads as colored, cyclic, and symmetric operads are reviewed.

This text is addressed to any computer scientist or combinatorist who looks a complete and a modern description of the theory of nonsymmetric operads. Evenly, this book is intended to an audience of algebraists who are looking for an original point of view fitting in the context of combinatorics.

March 5, 2019

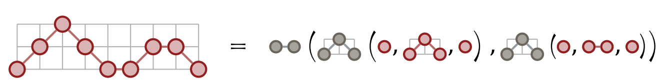
Contents

Introduction

Combinatorics, algebra, and programming

Let us start this text by considering two fields of mathematics: combinatorics and algebra, and a field of computer science: programming. These topics have far more in common than it appears at first glance and we shall explain why.

Combinatorics. One of the most common activities in combinatorics consists in studying families of combinatorial objects such as permutations, words, trees, or graphs. One of the most common activities about such families is to provide formulas to enumerate them according to a decent size notion. A large number of tools exists for this purpose such as manipulation of generating series, use of bijections, and use of decompositions rules to break objects into elementary pieces. This last tool is crucial and relies on the fact that if one knows a way to *compose* objects to build a bigger one, then without much effort one knows a way to *decompose* them. Let us, for instance, consider the celebrated set of Motzkin paths. Recall that a Motzkin path is a walk in the quarter plane connecting the origin to a point on the x-axis by means of steps , , and . Let us denote by (u_1, u_2) the binary composition rule consisting in substituting u_1 (resp. u_2) with the first (resp. second) point of , and by (u_1, u_2, u_3) the ternary composition rule consisting in substituting u_1 (resp. u_2, u_3) with the first (resp. second, third) point of . By using these two operations, one can, for instance, provide the decomposition



(0.0.1)

of the Motzkin path appearing as left member. An easy result says that a nonempty Motzkin path decomposes in a unique way either as a step followed by a Motzkin path, or as a step followed by a Motzkin path, a step , and another Motzkin path. This leads to the expression

(0.0.2)

describing the set $\mathfrak{M}$ of all Motzkin paths. From (0.0.2), one harvests combinatorial data as a formula to count Motzkin paths of a given size (with respect to an adequate notion of size), bijections between Motzkin paths and other families of combinatorial objects admitting similar decompositions (like Motzkin trees or Motzkin configurations of chords in polygons), or even the definition of generalizations of these objects (by adding colors on the steps, or by adding other possible sorts of steps).

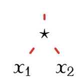
Algebra. On another side, in algebra, the study of algebraic structures consisting in a set of operations and the relations they satisfy is very habitual. To specify such an algebraic structure, one provides the nontrivial relations satisfied by its operations. For instance, any binary operation $\star$ satisfying for any inputs x_1, x_2, and x_3 the relation

$$(x_1 \star x_2) \star x_3 - x_1 \star (x_2 \star x_3) = (x_1 \star x_3) \star x_2 - x_1 \star (x_3 \star x_2) \tag{0.0.3}$$

specifies the class of the so-called pre-Lie algebras. Classical examples include among others monoids, groups, lattices, associative algebras, commutative algebras, and dendriform algebras. An obvious but important fact is that expressions and relations can be *composed*: the variables in a relation refer to any term of the algebraic structure. For instance, in (0.0.3), all the occurrences of x_2 can be replaced by a term $\mathbf{x_4} \star \mathbf{x_5}$, leading to the relation

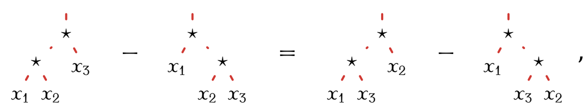
$$(x_1 \star (\mathbf{x_4} \star \mathbf{x_5})) \star x_3 - x_1 \star ((\mathbf{x_4} \star \mathbf{x_5}) \star x_3) = (x_1 \star x_3) \star (\mathbf{x_4} \star \mathbf{x_5}) - x_1 \star (x_3 \star (\mathbf{x_4} \star \mathbf{x_5})) \tag{0.0.4}$$

that still holds. Furthermore, by seeing the binary operation $\star$ as a device



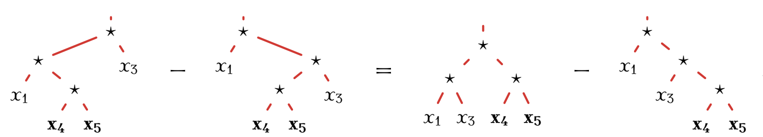
(0.0.5)

having two inputs x_1 and x_2 and one output $x_1 \star x_2$, Relation (0.0.3) reformulates as



(0.0.6)

and Relation (0.0.4) as



(0.0.7)

Therefore, one observes that algebraic expressions translates as syntax trees, and that composition and substitution of expressions translates as grafting of trees.

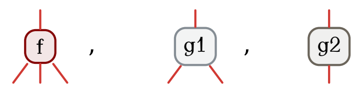
Programming. In programming, and more precisely in the context of functional programming, one encounters expressions of the form

$$\mathsf{f}(\mathsf{g1}(\mathsf{v1}, \mathsf{v2}), \mathsf{g2}(\mathsf{v3}), \mathsf{v4}) \tag{0.0.8}$$

where f, g1, and g2 are function names, and v1, v2, v3, and v4 are value names. This function call can be considered as a program, and the returned value is the value computed by the execution of the program. When our programming language satisfies referential transparency (that is, any expression can be substituted by its value without changing the overall computation), the call (0.0.8) is equivalent to both the calls

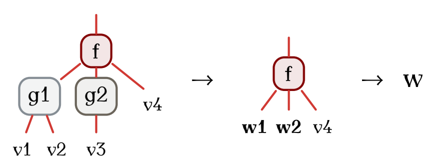
$$\mathsf{f}(\mathbf{w1}, \mathsf{g2}(\mathsf{v3}), \mathsf{v4}) \qquad \text{and} \qquad \mathsf{f}(\mathsf{g1}(\mathsf{v1}, \mathsf{v2}), \mathbf{w2}, \mathsf{v4}) \tag{0.0.9}$$

where $\mathbf{w1}$ is the value of $\mathsf{g1}(\mathsf{v1}, \mathsf{v2})$ and $\mathbf{w2}$ is the value of $\mathsf{g2}(\mathsf{v3})$. As a consequence, this leads to the fact that the order of evaluation of the sub-expressions $\mathsf{g1}(\mathsf{v1}, \mathsf{v2})$ and $\mathsf{g2}(\mathsf{v3})$ does not influence the computation of (0.0.8). Furthermore, by seeing the functions f1, g1, and g2 as black boxes



(0.0.10)

where inputs are depicted below and outputs above the boxes, the program (0.0.8) is made of *compositions* of such black boxes where inputs are completed with values (that are arguments of the function calls). Under this formalism, this program and its evaluation are depicted as



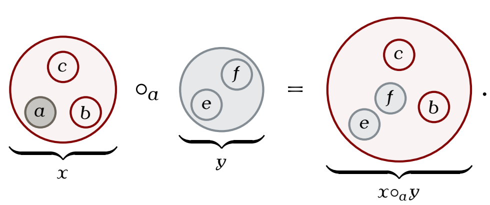
(0.0.11)

where w is the value computed by the program. In this context, the evaluation of a program can be carried out by using rewrite rules on such composition diagrams.

Nonsymmetric operads as a meeting point

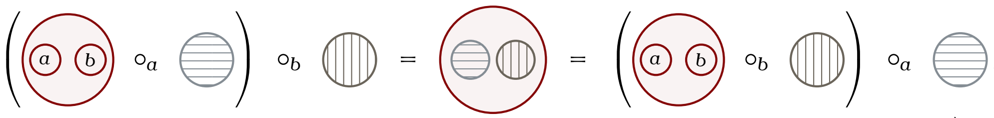
The three previous examples highlight the importance of the notion of composition in combinatorics, algebra, and programming. This is precisely the common point between these three fields we want to emphasize. Let us develop this concept.

Coherent compositions. The compositions of combinatorial objects, of syntax trees of algebraic expressions, and of function calls have to be coherent. Indeed, to lead to interesting and substantial consequences, they have to mimic the usual functional composition. Composing two objects x and y consists in choosing a *substitution sector* a of x and in replacing a it by a copy of y. This composition is denoted by $x \circ_a y$ and is schematically represented, in the following arbitrary example, as



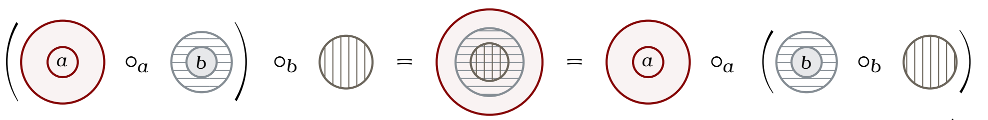
(0.0.12)

To be coherent, this composition has to satisfy the two relations



(0.0.13)

and



(0.0.14)

First relation can be thought as a horizontal compatibility, and the second as a vertical one.

To come back on the previous examples:

(1) In a Motzkin path $\mathfrak{u}$, the substitution sectors are its points, specified by their positions i. The composition $\mathfrak{u} \circ_i \mathfrak{v}$ of two Motzkin paths $\mathfrak{u}$ and $\mathfrak{v}$ consists in replacing the ith point of $\mathfrak{u}$ by a copy of $\mathfrak{v}$.

(2) In a syntax tree $\mathfrak{t}$ of an algebraic expression, the substitution sectors are its leaves, specified by their labels x_i. The composition $\mathfrak{t} \circ_{x_i} \mathfrak{s}$ of two syntax trees $\mathfrak{t}$ and $\mathfrak{s}$ consists in grafting the root of $\mathfrak{s}$ onto the leaf x_i of $\mathfrak{t}$.

(3) In a function f of a functional programming language, the substitution sectors are its parameters x_i provided that f admits the prototype $\mathsf{f}(\mathsf{x}_1, \dots, \mathsf{x}_{i-1}, \mathsf{x}_i, \mathsf{x}_{i+1}, \dots, \mathsf{x}_n)$. The composition $\mathsf{f} \circ_{\mathsf{x}_i} \mathsf{g}$ where f is the function just considered and g is a function admitting the prototype $\mathsf{g}(\mathsf{y}_1, \dots, \mathsf{y}_m)$ is the function with prototype $\mathsf{f}(\mathsf{x}_1, \dots, \mathsf{x}_{i-1}, \mathsf{y}_1, \dots, \mathsf{y}_m, \mathsf{x}_{i+1}, \dots, \mathsf{x}_n)$ defined as the function calling f wherein its ith argument substituting x_i is set to the value returned by g.

It is easy to see that Relations (0.0.13) and (0.0.14) are satisfied in these cases.

Nonsymmetric operads. Nonsymmetric operads are precisely algebraic structures furnishing an abstraction of this concept of generalized compositions. Intuitively, a nonsymmetric operad $\mathcal{O}$ is a set (or a vector space) equipped with a map $|-|$ associating a positive integer with each of its elements, and with composition maps $\circ_i$. Each element x of $\mathcal{O}$ is seen as an operator of arity $|x|$ and the composition maps of $\mathcal{O}$ satisfy the coherence relations stated above. Nonsymmetric operads mimic and generalize in this way the usual functional composition for any families of objects. There exist, for instance, nonsymmetric operads on words, permutations, binary trees, Schröder trees, configurations of chords, and paths.

There are a lot of reasons motivating the study of nonsymmetric operads. Here follow the main ones:

(A) Endowing a set of combinatorial objects with the structure of a nonsymmetric operad provides an algebraic framework for studying it. This framework can potentially stress some of the properties of the combinatorial objects, as for instance, enumerative results and the discovery of hidden symmetries.

(B) Continuing this last point, with any nonsymmetric operad $\mathcal{O}$ defined in the category of sets, it is associated a space of formal power series $\mathbb{K}\langle\langle\mathcal{O}\rangle\rangle$ on the elements of $\mathcal{O}$. This notion of series on nonsymmetric operads generalizes the usual one. Moreover, the extension of the composition maps of $\mathcal{O}$ on $\mathbb{K}\langle\langle\mathcal{O}\rangle\rangle$ leads to generalizations of the multiplication and the composition products of series. All this provides alternative ways to obtain expressions for the generating series of families of combinatorial objects.

(C) Nonsymmetric operads admit close connections with the combinatorics of planar rooted trees. This is due to the fact that free nonsymmetric operads can be constructed on sets of such trees. Moreover, given a nonsymmetric operad $\mathcal{O}$, a classical question consists in exhibiting a presentation by generators and relations of $\mathcal{O}$. Since any nonsymmetric operad can be described as a quotient of a free operad, the computation of a presentation is based upon manipulation of trees. In this context, tools coming from rewrite systems and some of their properties like termination and confluence intervene.

(D) There are a lot of generalizations of nonsymmetric operads, increasing their fields of applications. For instance, in a symmetric operad $\mathcal{O}$, all symmetric groups $\mathfrak{S}(n)$, $n \in \mathbb{N}$, act on the sets of elements of arity n of $\mathcal{O}$ by permuting the inputs of their elements. Besides, in a colored operad $\mathcal{O}$, the inputs and outputs have a color and the composition of two elements is defined only if the colors of the involved input and output match. In fact, our previous example about functional programming and composition of functions lies in this context of colors operads when the language is typed: the colors play the role of data types.

(E) Given a nonsymmetric operad $\mathcal{O}$, there is a notion of algebras over $\mathcal{O}$. More precisely, an algebra over $\mathcal{O}$ is a vector space $\mathcal{A}$ wherein elements of $\mathcal{O}$ behave as operations on $\mathcal{A}$ by respecting the arities and the composition maps of the nonsymmetric operad. For instance, any algebra over the associative operad is an associative algebra, and any algebra over the pre-Lie operad is a pre-Lie algebra.

(F) Related to the previous point, all the algebras over $\mathcal{O}$ form a category of algebras. In this way, by studying $\mathcal{O}$, one obtains general results on all the algebras over $\mathcal{O}$. For instance, it is possible to show that the sum of the two usual generators of the dendriform operad behaves as an associative operation. This implies that the sum of the two operations of any dendriform algebra is associative. Besides, operads furnish here a framework for operation calculus.

(G) Again related to the two previous points, nonsymmetric operads lead to the discovery of link between different categories of algebras. Indeed, if $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ is a morphism between two nonsymmetric operads $\mathcal{O}_1$ and $\mathcal{O}_2$, one can construct from ϕ a functor from the category of algebras over $\mathcal{O}_2$ to the category of algebras over $\mathcal{O}_1$. For instance, there is a morphism from the two-associative operad to the duplicial operad leading to a functor from the category of duplicial algebras (algebras endowed with two binary associative operations satisfying one extra relation) to the category of two-associative algebras (algebras endowed with two binary associative operations).

(H) Given a nonsymmetric operad $\mathcal{O}$ satisfying some precise properties, it is possible to compute a presentation by generators and relations of its so-called Koszul dual $\mathcal{O}^!$. This duality is an extension of the Koszul duality for associative algebras and establishes connections between nonsymmetric operads at first sight very different. For instance, the dendriform operad and the diassociative operad are Koszul dual one of the other. Moreover, linked to this notion of Koszul duality, there is a notion of Koszulity for non-symmetric operads. This property, defined originally algebraically, admits equivalent reformulations in terms of properties of rewrite systems on trees associated with the presentation of the nonsymmetric operads. Moreover, given a Koszul nonsymmetric operad $\mathcal{O}$ admitting an Hilbert series, the alternating version of the Hilbert series of $\mathcal{O}^!$ and the one of $\mathcal{O}$ are inverse one of the other for series composition. This property admits for instance applications for enumerative prospects.

Construction of the book

We now give some practical information about this text.

Point of view. All the algebraic structures considered here are linear spans $\mathbb{K}\langle C \rangle$ of some sets of combinatorial objects C. For this reason, these spaces admit always an explicit basis C. Moreover, to handle families of combinatorial objects, we introduce the notion of collections. A collection is a set of combinatorial objects presented as a disjoint union of subsets of objects satisfying a same property. For instance, a graded collection is a set of combinatorial objects defined as a disjoint union of sets of objects of a same size. We shall consider different sorts of collections as colored, multigraded, or symmetric collections, suited to the study of some particular algebraic structures.

Main purposes and audience. The aim of this text is to give a presentation of the theory of nonsymmetric operads under the context of algebraic combinatorics. We orient our exposition of properties of operads toward enumerative and combinatorial directions. This book is also intended to be an introduction to algebraic combinatorics: the first chapters deal with combinatorics and combinatorics of trees, while next one deals with general properties of algebraic structures on combinatorial families. For instance, Hopf bialgebras, forming a vast and rich topic in algebraic combinatorics, are studied here.

We decide to not overload the text with bibliographic and historical references. For this reason, each chapter ends with a section containing bibliographic material. Besides, to present as much results as possible about combinatorics, algebraic combinatorics, and operads, we decide to not mention their proofs. For a vast majority of them, they can be treated as not so hard exercises.

This book is addressed to any computer scientist or combinatorist who is aiming to establish a first contact with the theory of operads. Evenly, this book is intended to an audience of algebraists who are looking for an original point of view fitting in the context of combinatorics.

Structure. The book contains five chapters. Each chapter depends on the previous ones. They are organized as follows.

(1) Chapter 1 presents general notions about combinatorics and collections. It also provides definitions about collections endowed with a poset structure. These structures intervene in changes of bases of algebraic structures. It presents finally collections endowed with rewrite rules. These collections intervene as tools to establish presentations of nonsymmetric operads.

(2) Chapter 2 is devoted to general treelike structures. It presents syntax trees, that are sorts of trees appearing in the study of nonsymmetric operads. Rewrite systems on syntax trees are considered and tools to prove their termination or confluence are provided.

(3) Chapter 3 concerns algebraic structures defined on the linear span of collections. These vector spaces are called polynomial spaces. It presents also the notion of biproducts on polynomial spaces and of types of bialgebras. Classical types of bialgebras appearing in combinatorics are given: associative, dendriform, and pre-Lie algebras, and Hopf bialgebras.

(4) Chapter 4 presents nonsymmetric operads and related notions. It exposes the notions of algebras over operads, free operads, presentations by generators and relations, Koszul duality, and Koszulity. Several examples of operads appearing in algebraic combinatorics are reviewed.

(5) Chapter 5 contains generalizations, applications, and apertures of the theory of nonsymmetric operads. It reviews three topics in this vein. First, formal power series on nonsymmetric operads are considered and applications to enumeration are provided. Next, enrichments of nonsymmetric operads are discussed: colored operads, cyclic operads, and symmetric operads. Finally, it provides an overview of product categories, a generalization of operads wherein elements can have several outputs.

CHAPTER 1

Enriched collections

This preliminary chapter contains general notions about combinatorics used in the rest of the book. We introduce the notion of collections of combinatorial objects and then the notions of posets and rewrite systems, which are seen as collections endowed with some extra structure.

1. Collections

A collection is a set of combinatorial objects partitioned into subsets of objects sharing a same property. Operations over collections are then introduced and some classical examples of collections are provided.

1.1. Structured collections. There are many kinds of collections such as graded, multigraded, colored, cyclic, and symmetric. Definitions about them are provided here.

1.1.1. *Elementary definitions.* Let I be a nonempty set called *index set*. An *I-collection* is a set C expressible as a disjoint union

$$C = \bigsqcup_{i \in I} C(i) \tag{1.1.1}$$

where all $C(i)$, $i \in I$, are (possibly infinite) sets. All the elements of C (resp. $C(i)$ for an $i \in I$) are called *objects* (resp. *i-objects*) of C. If x is an i-object of C, we say that the *index* $\operatorname{ind}(x)$ of x is i. When for all $i \in I$, all $C(i)$ are finite sets, C is *combinatorial*. Besides, C is *finite* if C is finite as a set. The *empty I-collection* is the set $\emptyset$. When I is a singleton, C is *simple*. Any set can thus be seen as a simple collection and conversely.

A *relation* on C is a binary relation $\mathscr{R}$ on C such that for any objects x and y of C satisfying $x \mathrel{\mathscr{R}} y$, $\operatorname{ind}(x) = \operatorname{ind}(y)$. Let C_1 and C_2 be two I-collections. A map $\phi : C_1 \to C_2$ is an *I-collection morphism* if, for all $x \in C_1$, $\operatorname{ind}(x) = \operatorname{ind}(\phi(x))$. We express by $C_1 \simeq C_2$ the fact that there exists an isomorphism between C_1 and C_2. Besides, if for all $i \in I$, $C_1(i) \subseteq C_2(i)$, C_1 is a *subcollection* of C_2. For any $i \in I$, we can regard each $C(i)$ as a subcollection of C consisting only in all its i-objects. Moreover, for any subset J of I, we denote by $C(J)$ the subcollection of C consisting only in all its j-objects for all $j \in J$.

Let us now consider particular I-collections for precise sets I. Table 1.1 contains an overview of the properties that such collections can satisfy.

<table>
<tr><td colspan="3">Collections
Combinatorial   Finite   Simple   With products</td></tr>
<tr><td colspan="2">k-graded</td><td rowspan="3">Colored
Monochrome   k-colored</td></tr>
<tr><td colspan="2">1-graded
Connected   Augmented   Monatomic</td></tr>
<tr><td>Cyclic</td><td>Symmetric</td></tr>
</table>

TABLE 1.1. The most common sorts of I-collections (in bold) and the properties (in italic) they can satisfy. The inclusions relations between these sorts of collections read from bottom to top. For instance, cyclic collections are particular 1-graded collections which are themselves particular k-graded collections which are themselves particular collections.

1.1.2. *Graded collections.* An $\mathbb{N}$-collection is called a *graded collection*. If C is a graded collection, for any object x of C, the *size* $|x|$ of x is the integer $\operatorname{ind}(x)$. The map $|-| : C \to \mathbb{N}$ is the *size function* of C.

We say that C is *connected* if $C(0)$ is a singleton, and that C is *augmented* if $C(0) = \emptyset$. Moreover, C is *monatomic* if it is augmented and $C(1)$ is a singleton. We denote by $\{\epsilon\}$ the graded collection such that ϵ is an object satisfying $|\epsilon| = 0$. This collection is called the *unit collection*. Observe that $\{\epsilon\}$ is connected, and that C is connected if and only if there is a unique collection morphism from $\{\epsilon\}$ to C. We denote by $\{\bullet\}$ the collection such that $\bullet$ is an *atom*, that is an object satisfying $|\bullet| = 1$. This collection is called the *neutral collection*. Observe that $\{\bullet\}$ is monatomic, and that C is monatomic if and only if C is augmented and there is a unique collection morphism from $\{\bullet\}$ to C. When C is a combinatorial graded collection, the *generating series* of C is the series

$$\mathbb{G}_C(t) := \sum_{n \in \mathbb{N}} \#C(n) t^n, \tag{1.1.2}$$

where t is a formal parameter and $\#S$ denotes the cardinality of any finite set S. This formal power series encodes the *integer sequence* of C, that is the sequence $(\#C(n))_{n \in \mathbb{N}}$. Observe that if C_1 and C_2 are two combinatorial graded collections, $C_1 \simeq C_2$ holds if and only if $\mathbb{G}_{C_1}(t) = \mathbb{G}_{C_2}(t)$.

1.1.3. *Multigraded collections and statistics.* A *k-graded collection* (also called *multigraded collection*) is an $\mathbb{N}^k$-collection for an integer $k \geqslant 1$. To not overload the notation, we denote by $C(n_1, \dots, n_k)$ the subset $C((n_1, \dots, n_k))$ of any k-graded collection C. Recall that a *statistics* on an I-collection C is a map $\mathrm{s} : C \to \mathbb{N}$, associating a nonnegative integer value with any object of C. Multigraded collections are useful to work with objects endowed with many statistics. Indeed, if x is an $(n_1, \dots, n_k)$-object of a k-graded collection C, one sets $\mathrm{s}_j(x) := n_j$ for each $1 \leqslant j \leqslant k$. This defines in this way k statistics $\mathrm{s}_j : C \to \mathbb{N}$, $1 \leqslant j \leqslant k$.

1.1.4. *Colored collections.* Let $\mathfrak{C}$ be a finite set, called *set of colors*. A *$\mathfrak{C}$-colored collection* C is an I-collection such that

$$I := \left\{(a, u) : a \in \mathfrak{C} \text{ and } u \in \mathfrak{C}^{\ell} \text{ for an } \ell \in \mathbb{N}\right\}. \tag{1.1.3}$$

In other terms, any object x of C has an index $(a, u) \in \mathfrak{C} \times \mathfrak{C}^{\ell}$, $\ell \in \mathbb{N}$, called *$\mathfrak{C}$-colored index*. Moreover, the *output color* of x is $\mathrm{out}(x) := a$, and the *word of input colors* of x is $\mathrm{in}(x) := u$. The *jth input color* of x is the jth letter of $\mathrm{in}(x)$, denoted by $\mathrm{in}_j(x)$. To not overload the notation, we denote by $C(a, u)$ the subset $C((a, u))$ of C. We say that C is *monochrome* if $\mathfrak{C}$ is a singleton. For any nonnegative integer k, a *k-colored collection* is a $\mathfrak{C}$-colored collection where $\mathfrak{C}$ is the set of integers $\{1, \dots, k\}$.

1.1.5. *Cyclic collections.* Let C be a graded collection endowed for all $n \in \mathbb{N}$ with maps

$$\circlearrowleft_n : C(n) \to C(n) \tag{1.1.4}$$

such that each $n+1$st functional power $\circlearrowleft_n^{n+1}$ is the identity map on $C(n)$. Then, one says that C is a *cyclic collection* and that the $\circlearrowleft_n$, $n \in \mathbb{N}$, are the *cycle maps* of C. Observe that by setting for any $n \in \mathbb{N}$, $\bullet : \mathbb{Z}/_{(n+1)\mathbb{Z}} \times C(n) \to C(n)$ as the map defined for any $k \in \mathbb{Z}/_{(n+1)\mathbb{Z}}$ and $x \in C(n)$ by $k \bullet x := \circlearrowleft_n^k (x)$, $\bullet$ is a left group action of the cyclic group of order $n+1$ on $C(n)$. The reason why we demand that each $\circlearrowleft_n$ is of order $n+1$ (and not of order n) will appear in the context of cyclic operads.

1.1.6. *Symmetric collections.* Let us first denote by $\mathfrak{S}$ the graded collection of all the bijections on the set $\{1, \dots, n\}$, $n \in \mathbb{N}$, such that the size of a bijection is the cardinality of its domain. Let C be a graded collection endowed, for all $n \in \mathbb{N}$ and $\sigma \in \mathfrak{S}(n)$, with maps

$$\circledcirc_\sigma : C(n) \to C(n) \tag{1.1.5}$$

such that $\circledcirc_{\mathrm{Id}_n}$ is the identity map on $C(n)$, where Id_n denotes the identity map of $\mathfrak{S}(n)$, and $\circledcirc_{\sigma_1} \circ \circledcirc_{\sigma_2} = \circledcirc_{\sigma_2 \circ \sigma_1}$ for any bijections σ_1 and σ_2 of $\mathfrak{S}(n)$. Then, one says that C is a *symmetric collection* and that the $\circledcirc_\sigma$, $\sigma \in \mathfrak{S}(n)$, are the *symmetric maps* of C. Observe that by setting for any $n \in \mathbb{N}$, $\bullet : \mathfrak{S}(n) \times C(n) \to C(n)$ as the map defined for any $\sigma \in \mathfrak{S}(n)$ and $x \in C(n)$ by $\sigma \bullet x := \circledcirc_\sigma(x)$, $\bullet$ is a left group action of the symmetric group of order n on $C(n)$.

1.1.7. *Collections with products.* Let C be an I-collection. A *product* on C is a map

$$\star : C(J_1) \times \dots \times C(J_p) \to C \tag{1.1.6}$$

where $p \in \mathbb{N}$ and J_1, ..., J_p are nonempty subsets of I. The *arity* of $\star$ is p and the *index domain* of $\star$ is the set $J_1 \times \dots \times J_p$. A sequence $(x_1, \dots, x_p)$ of objects of C is a *valid input* for $\star$ whenever $\star(x_1, \dots, x_p)$ is defined, that is, $(\mathrm{ind}(x_1), \dots, \mathrm{ind}(x_p))$ belongs to the index domain of $\star$. When C is endowed with a set of such products, we say that C

is an I-collection *with products*. Such a product $\star$ can be seen as an operation taking p elements of C as input and outputting one element of C. Let us now review some properties a product $\star$ of the form (1.1.6) can satisfy.

First, when the index domain of $\star$ is I^p, $\star$ is *complete*. When $J_1 = \cdots = J_p = \{i\}$ for a certain index i of I, and, for any valid input $(x_1, \dots, x_p)$ for $\star$, $\operatorname{ind}(\star(x_1, \dots, x_p)) = i$, we say that $\star$ is *internal*. Besides, when there is a map $\omega : J_1 \times \cdots \times J_p \to I$ satisfying, for any valid input $(x_1, \dots, x_p)$ for $\star$,

$$\star(x_1, \dots, x_p) \in C(\omega(\operatorname{ind}(x_1), \dots, \operatorname{ind}(x_p))), \tag{1.1.7}$$

we say that $\star$ is *ω-concentrated* (or simply *concentrated* when it not useful to specify ω). In intuitive terms, this means that the index of the result of a product depends only of the indexes of its operands. Finally, in the particular case where C is a graded collection, $\star$ is *graded* if $\star$ is ω-concentrated for the map $\omega : \mathbb{N}^p \to \mathbb{N}$ defined by $\omega(n_1, \dots, n_p) := n_1 + \cdots + n_p$. As a side remark, the cycle maps (resp. symmetric maps) of a cyclic (resp. symmetric) collection C are unary internal products on C.

1.2. Operations over collections. We list here the most important operations that take as input I-collections and output new ones. Some of these are defined only on graded collections. Table 1.2 shows an overview of some properties of these operations. Since combinatorial graded collections have generating series, we provide expressions for the generating series of the collection produced by the exposed operations.

1.2.1. *Sum operation.* Let C_1 and C_2 be two I-collections. The *sum* of C_1 and C_2 is the I-collection $C_1 + C_2$ such that, for all $i \in I$,

$$(C_1 + C_2)(i) := C_1(i) \sqcup C_2(i). \tag{1.2.1}$$

In other words, each object of index i of $C_1 + C_2$ is either an object of index i of C_1 or an object of index i of C_2. Since the sum operation (1.2.1) is defined through a disjoint union, when the sets $C_1(i)$ and $C_2(i)$ are not disjoint, there are in $(C_1 + C_2)(i)$ two copies of each element belonging to the intersection $C_1(i) \cap C_2(i)$, one coming from $C_1(i)$, the other from $C_2(i)$. Moreover, observe that the sum operation admits the empty I-collection $\emptyset$ as unit and that it is associative and commutative. The iterated version of the operation $+$ shall be denoted by $\bigsqcup$ in the sequel.

When C_1 and C_2 are combinatorial, $C_1 + C_2$ is combinatorial. Moreover, when C_1 and C_2 are combinatorial and graded, the generating series of $C_1 + C_2$ satisfies

$$\mathbb{G}_{C_1+C_2}(t) = \mathbb{G}_{C_1}(t) + \mathbb{G}_{C_2}(t). \tag{1.2.2}$$

Name	Arity	Inputs	Output
Sum	2	I-coll. C_1 and C_2	I-coll. $C_1 + C_2$
Casting	1	I-coll C	J-coll. $\mathbf{Cast}^\omega(C)$
Cartesian product	$p \in \mathbb{N}$	I_k-coll. C_k, $1 \leqslant k \leqslant p$	$I_1 \times \cdots \times I_p$-coll. $[C_1, \dots, C_p]_\times$
Hadamard product	$p \in \mathbb{N}$	I-coll. C_1, …, C_p	I-coll. $[C_1, \dots, C_p]_\boxtimes$
List collection	1	I-coll. C	$\mathcal{T}(I)$-coll. $\mathbf{List}(C)$
Multiset collection	1	I-coll. C	$\mathcal{M}(I)$-coll. $\mathbf{MSet}(C)$
Set collection	1	I-coll. C	$\mathcal{S}(I)$-coll. $\mathbf{Set}(C)$
ℓ-suspension	1	graded coll. C	graded coll. $\mathbf{Sus}_\ell(C)$
Augmentation	1	graded coll. C	graded coll. $\mathbf{Aug}(C)$
Composition	2	graded coll. C_1 and C_2	graded coll. $C_1 \odot C_2$
$\mathfrak{C}$-coloration	1	graded coll. C	$\mathfrak{C}$-colored coll. $\mathbf{Col}_\mathfrak{C}(C)$
Cycle	1	graded coll. C	Cyclic coll. $\mathbf{Cyc}(C)$
Regularization	1	graded coll. C	symmetric coll. $\mathbf{Reg}(C)$

TABLE 1.2. Main properties of some operations over collections. Here, I, J, and $I_1, \dots, I_p$, $p \in \mathbb{N}$, are index sets, $\mathfrak{C}$ is a set of colors, $\mathcal{T}(I)$ is the set of all the finite tuples of elements of I, $\mathcal{M}(I)$ is the set of all the finite multisets of elements of I, $\mathcal{S}(I)$ is the set of all the finite subsets of I, and ℓ is an integer.

1.2.2. *Casting operation.* Let C be an I-collection, J be an index set, and $\omega : I \to J$ be a map. The *ω-casting* of C is the J-collection $\mathbf{Cast}^\omega(C)$ defined for any $j \in J$ by

$$(\mathbf{Cast}^\omega(C))(j) := \bigsqcup_{\substack{i \in I \\ \omega(i) = j}} C(i). \tag{1.2.3}$$

In other words, each object of index $j \in J$ of $\mathbf{Cast}^\omega(C)$ comes from an object of index $i \in I$ of C such that $\omega(i) = j$. Observe also that the right member of (1.2.3) is equal to $C\left(\omega^{-1}(j)\right)$.

When the codomain of ω is $\mathbb{N}$, $\mathbf{Cast}^\omega(C)$ is a graded collection called *ω-graduation* of C. Let us detail two particular cases of ω-graduations. When C is a k-graded collection, for any $1 \leqslant p \leqslant k$ we call *p-graduation* of C the π_p-graduation of C for the map $\pi_p : \mathbb{N}^k \to \mathbb{N}$ defined by $\pi_p((n_1, \dots, n_k)) := n_p$. Besides, when C is a $\mathfrak{C}$-colored collection where $\mathfrak{C}$ is a set of colors, we call *graduation* of C the ω-graduation of C for the map ω sending any $\mathfrak{C}$-colored index (a, u) to the length of the tuple u.

Observe that when C is combinatorial and each fiber $\omega^{-1}(j)$ is finite for any $j \in J$, $\mathbf{Cast}^{\omega}(C)$ is combinatorial.

1.2.3. *Cartesian product operation.* Let $p \in \mathbb{N}$, I_1, ..., I_p be index sets, and C_1 be an I_1-collection, ..., C_p be an I_p-collection. The *Cartesian product* of C_1, ..., C_p is the $I_1 \times \cdots \times I_p$-collection $\left[C_1, \dots, C_p\right]_\times$ such that, for all $(i_1, \dots, i_p) \in I_1 \times \cdots \times I_p$,

$$\left[C_1, \dots, C_p\right]_\times \left((i_1, \dots, i_p)\right) := C_1\left(i_1\right) \times \cdots \times C_p\left(i_p\right). \tag{1.2.4}$$

In other words, each object of index $(i_1, \dots, i_p)$ of $\left[C_1, \dots, C_p\right]_\times$ is a tuple $(x_1, \dots, x_p)$ such that for any $1 \leqslant k \leqslant p$, each x_k is an object of index i_k of C_k. As a special but important case, the Cartesian product $[]_\times$ of zero I-collections is the I-collection containing the empty tuple (that is, the unique tuple of length 0). The index of this object is the empty tuple on I (that is, the unique element of I^0). To not overload the notation, we denote by $\left[C_1, \dots, C_p\right]_\times (i_1, \dots, i_p)$ the set $\left[C_1, \dots, C_p\right]_\times \left((i_1, \dots, i_p)\right)$ for any index $(i_1, \dots, i_p)$ of $I_1 \times \cdots \times I_p$.

Observe that when all the C_1, ..., C_p are combinatorial, $\left[C_1, \dots, C_p\right]_\times$ is combinatorial. Besides, when all the C_1, ..., C_p are graded, $\left[C_1, \dots, C_p\right]_\times$ is a p-graded collection.

When J is an index set and $\omega : I_1 \times \cdots \times I_p \to J$ is a map, the *ω-Cartesian product operation* of C_1, ..., C_p is the J-collection

$$\left[C_1, \dots, C_p\right]_\times^{\omega} := \mathbf{Cast}^{\omega}\left(\left[C_1, \dots, C_p\right]_\times\right). \tag{1.2.5}$$

By definition of the casting and the Cartesian product operations, one has, for any $j \in J$,

$$\left[C_1, \dots, C_p\right]_\times^{\omega}(j) = \left\{(x_1, \dots, x_p) \in C_1 \times \cdots \times C_p : \omega\left(\operatorname{ind}(x_1), \dots, \operatorname{ind}(x_p)\right) = j\right\}. \tag{1.2.6}$$

In other words, each object of index $j \in J$ of $\left[C_1, \dots, C_p\right]_\times^{\omega}$ is a tuple $(x_1, \dots, x_p)$ such that for any $1 \leqslant k \leqslant p$, each x_k is an object of C_k, and the image by ω of the tuple formed by the indexes of x_1, ..., x_p is j.

Observe that when all the C_1, ..., C_p are combinatorial and each fiber $\omega^{-1}(j)$ is finite for any $j \in J$, $\left[C_1, \dots, C_p\right]_\times^{\omega}$ is combinatorial. When all the index sets I_1, ..., I_p, and J are equal to $\mathbb{N}$, we denote by $+ : \mathbb{N}^p \to \mathbb{N}$ the map defined by $+\left((n_1, \dots, n_p)\right) := n_1 + \cdots + n_p$. When all the C_1, ..., C_p are combinatorial and graded, $\left[C_1, \dots, C_p\right]_\times^{+}$ is combinatorial and its generating series satisfies

$$\mathbb{G}_{\left[C_1, \dots, C_p\right]_\times^{+}}(t) = \prod_{1 \leqslant k \leqslant p} \mathbb{G}_{C_k}(t). \tag{1.2.7}$$

1.2.4. *Hadamard product operation.* Let $p \in \mathbb{N}$ and C_1, ..., C_p be I-collections. The *Hadamard product* of C_1, ..., C_p is the I-collection $\left[C_1, \dots, C_p\right]_\boxtimes$ such that, for all $i \in I$,

$$\left[C_1, \dots, C_p\right]_\boxtimes(i) := \left[C_1, \dots, C_p\right]_\times \left(\underbrace{i, \dots, i}_{p \text{ terms}}\right). \tag{1.2.8}$$

In other words, each object of index i of $\left[C_1, \dots, C_p\right]_{\boxtimes}$ is a tuple $(x_1, \dots, x_p)$ such that for all $1 \leqslant k \leqslant p$, all the x_k are objects of index i of C_k.

Observe that when all the C_1, …, C_p are combinatorial, $\left[C_1, \dots, C_p\right]_{\boxtimes}$ is combinatorial. When all the C_1, …, C_p are combinatorial and graded, $\left[C_1, \dots, C_p\right]_{\boxtimes}$ is combinatorial and its generating series satisfies

$$\mathbb{G}_{\left[C_1,\dots,C_p\right]_{\boxtimes}}(t) = \sum_{n \in \mathbb{N}} \left(\prod_{1 \leqslant k \leqslant p} \#C_k(n) \right) t^n. \tag{1.2.9}$$

1.2.5. *List collection operation.* Let C be an I-collection. Let us denote by $\mathcal{T}(I)$ the index set of all the finite tuples $(i_1, \dots, i_p)$, $p \in \mathbb{N}$, of elements of I. The *list collection* of C is the $\mathcal{T}(I)$-collection $\mathbf{List}(C)$ such that

$$\mathbf{List}(C) := \bigsqcup_{p \in \mathbb{N}} \left[\underbrace{C, \dots, C}_{p \text{ terms}} \right]_{\times} \tag{1.2.10}$$

In other words, each object of $\mathbf{List}(C)$ is a tuple $(x_1, \dots, x_p)$ of objects of C and its index is the element $(\mathrm{ind}(x_1), \dots, \mathrm{ind}(x_p))$ of $\mathcal{T}(I)$. Moreover, for any subset S of $\mathbb{N}$, let $\mathbf{List}_S(C)$ be the subcollection of $\mathbf{List}(C)$ restrained on tuples that have a length in S.

Observe that when C is combinatorial, $\mathbf{List}(C)$ is combinatorial.

When J is an index set and $\omega : \mathcal{T}(I) \to J$ is a map, the *ω-list collection* of C is the J-collection

$$\mathbf{List}^{\omega}(C) := \mathbf{Cast}^{\omega}(\mathbf{List}(C)). \tag{1.2.11}$$

By definition of the casting and the list collection operations, one has, for any $j \in J$,

$$(\mathbf{List}^{\omega}(C))(j) = \left\{ (x_1, \dots, x_p) \in C^p : p \in \mathbb{N} \text{ and } \omega\left((\mathrm{ind}(x_1), \dots, \mathrm{ind}(x_p))\right) = j \right\}. \tag{1.2.12}$$

In other words, each object of index $j \in J$ of $\mathbf{List}^{\omega}(C)$ is a tuple $(x_1, \dots, x_p)$, $p \in \mathbb{N}$, such that for all $1 \leqslant k \leqslant p$, the x_k are objects of C, and the image by ω of the tuple formed by the indexes of x_1, …, x_p is j. Moreover, for any subset S of $\mathbb{N}$, let $\mathbf{List}^{\omega}_S(C)$ be the subcollection of $\mathbf{List}^{\omega}(C)$ restrained on tuples that have a length in S.

Observe that when C is combinatorial and each fiber $\omega^{-1}(j)$ is finite for any $j \in J$, $\mathbf{List}^{\omega}(C)$ is combinatorial. When I and J are equal to $\mathbb{N}$, we denote by $+ : \mathcal{T}(\mathbb{N}) \to \mathbb{N}$ the map defined by $+\left((n_1, \dots, n_p)\right) := n_1 + \dots + n_p$. When C is combinatorial and graded, $\mathbf{List}^{+}(C)$ is combinatorial if and only if C is augmented. In this case, its generating series satisfies

$$\mathbb{G}_{\mathbf{List}^{+}(C)}(t) = \frac{1}{1 - \mathbb{G}_C(t)}. \tag{1.2.13}$$

1.2.6. *Multiset collection operation.* Let C be an I-collection. Let us denote by $\mathscr{M}(I)$ the index set formed by all finite multisets $\lbag i_1, \dots, i_p \rbag$, $p \in \mathbb{N}$, of elements of I. The *multiset collection* of C is the $\mathscr{M}(I)$-collection $\mathbf{MSet}(C)$ such that, for all $\lbag i_1, \dots, i_p \rbag \in \mathscr{M}(I)$, $p \in \mathbb{N}$,

$$(\mathbf{MSet}(C)) \left(\lbag i_1, \dots, i_p \rbag\right) := \left(\bigcup_{\sigma \in \mathfrak{S}(p)} (\mathbf{List}(C)) \left(i_{\sigma(1)}, \dots, i_{\sigma(p)}\right) \right) /_{\equiv}, \tag{1.2.14}$$

where $\mathfrak{S}$ is the graded collection of bijections defined in Section 1.1.6 and $\equiv$ is the equivalence relation on $\mathbf{List}(C)$ satisfying, for any $(x_1, \dots, x_p) \in \mathbf{List}(C)$, $p \in \mathbb{N}$, and any bijection $\sigma \in \mathfrak{S}(p)$,

$$\left(x_1, \dots, x_p\right) \equiv \left(x_{\sigma(1)}, \dots, x_{\sigma(p)}\right). \tag{1.2.15}$$

In other words, each object of $\mathbf{MSet}(C)$ is an $\equiv$-equivalence class of tuples of $\mathbf{List}(C)$ and such an $\equiv$-equivalence class $\left[\left(x_1, \dots, x_p\right)\right]_{\equiv}$ can be represented by the multiset $\lbag x_1, \dots, x_p \rbag$. Therefore, the objects of $\mathbf{MSet}(C)$ can be regarded as multisets of objects of C.

When J is an index set and $\omega : \mathscr{M}(I) \to J$ is a map, the *ω-multiset collection* of C is the J-collection

$$\mathbf{MSet}^{\omega}(C) := \mathbf{Cast}^{\omega}(\mathbf{MSet}(C)). \tag{1.2.16}$$

By definition of the casting and the multiset collection operations, one has, for any $j \in J$,

$$(\mathbf{MSet}^{\omega}(C))(j) = \left\{ \lbag x_1, \dots, x_p \rbag : p \in \mathbb{N} \text{ and } \omega\left(\lbag \operatorname{ind}(x_1), \dots, \operatorname{ind}\left(x_p\right) \rbag\right) = j \right\}. \tag{1.2.17}$$

In other words, each object of index $j \in J$ of $\mathbf{MSet}^{\omega}(C)$ is a finite multiset $\lbag x_1, \dots, x_p \rbag$, $p \in \mathbb{N}$, such that for all $1 \leqslant k \leqslant p$, the x_k are objects of C, and the image by ω of the multiset formed by the indexes of x_1, ..., x_p is j.

Observe that when C is combinatorial and each fiber $\omega^{-1}(j)$ is finite for any $j \in J$, $\mathbf{MSet}^{\omega}(C)$ is combinatorial. When I and J are equal to $\mathbb{N}$, we denote by $+ : \mathscr{M}(\mathbb{N}) \to \mathbb{N}$ the map defined by $+\left(\lbag n_1, \dots, n_p \rbag\right) := n_1 + \dots + n_p$. When C is combinatorial and graded, $\mathbf{MSet}^{+}(C)$ is combinatorial if and only if C is augmented. In this case, its generating series satisfies

$$\mathbb{G}_{\mathbf{MSet}^{+}(C)}(t) = \prod_{n \in \mathbb{N} \setminus \{0\}} \left(\frac{1}{1 - t^n} \right)^{\#C(n)}. \tag{1.2.18}$$

1.2.7. *Set collection operation.* Let C be an I-collection. Let us denote by $\mathcal{S}(I)$ the index set formed by all finite subsets of elements of I. The *set collection* of C is the $\mathcal{S}(I)$-collection $\mathbf{Set}(C)$ defined as the subcollection of $\mathbf{MSet}(C)$ containing only the multisets having all their elements with multiplicity 1. In this way, the objects of $\mathbf{Set}(C)$ can be represented as finite sets of objects of C.

When J is an index set and $\omega : \mathcal{S}(I) \to J$ is a map, the *ω-set collection* of C is the J-collection

$$\mathbf{Set}^{\omega}(C) := \mathbf{Cast}^{\omega}(\mathbf{Set}(C)). \tag{1.2.19}$$

By definition of the casting and the set collection operations, one has, for any $j \in J$,

$$\mathbf{Set}^{\omega}(C)(j) = \left\{ \{x_1, \dots, x_p\} \subseteq C : p \in \mathbb{N} \text{ and } \omega\left(\{\mathrm{ind}(x_1), \dots, \mathrm{ind}(x_p)\}\right) = j \right\}. \tag{1.2.20}$$

In other words, each object of index $j \in J$ of $\mathbf{Set}^{\omega}(C)$ is a finite set $\{x_1, \dots, x_p\}$, $p \in \mathbb{N}$, such that for all $1 \leqslant k \leqslant p$, the x_k are objects of C, and the image by ω of the set formed by the indexes of x_1, ..., x_p is j.

Observe that when C is combinatorial and each fiber $\omega^{-1}(j)$ is finite for any $j \in J$, $\mathbf{Set}^{\omega}(C)$ is combinatorial. When I and J are equal to $\mathbb{N}$, we denote by $+ : \mathcal{S}(\mathbb{N}) \to \mathbb{N}$ the map defined by $+\left(\{x_1, \dots, x_p\}\right) := x_1 + \dots + x_p$. When C is combinatorial and graded, $\mathbf{Set}^{\omega}(C)$ is combinatorial (without requiring any additional condition contrariwise to the similar cases for the list and multiset collection operations) and its generating series satisfies

$$\mathbb{G}_{\mathbf{Set}^{+}(C)}(t) = \prod_{n \in \mathbb{N} \setminus \{0\}} (1 + t^n)^{\#C(n)}. \tag{1.2.21}$$

1.2.8. *Suspension and augmentation operations.* Let C be a graded collection. For any $\ell \in \mathbb{Z}$, the *ℓ-suspension* of C is the graded collection $\mathbf{Sus}_{\ell}(C)$ such that, for all $n \in \mathbb{N}$,

$$(\mathbf{Sus}_{\ell}(C))(n) := \begin{cases} C(n-\ell) & \text{if } n - \ell \in \mathbb{N}, \\ \emptyset & \text{otherwise.} \end{cases} \tag{1.2.22}$$

Observe that $\mathbf{Sus}_1(\mathbf{Sus}_{-1}(C))$ is the subcollection $C \setminus C(0)$ of C, that is the augmented collection having the objects of C without its objects of size 0. We call this collection the *augmentation* of C and we denote it by $\mathbf{Aug}(C)$.

When C is combinatorial, $\mathbf{Sus}_{\ell}(C)$ and $\mathbf{Aug}(C)$ are combinatorial and their generating series satisfy, respectively,

$$\mathbb{G}_{\mathbf{Sus}_{\ell}(C)}(t) = t^{\ell}\left(\mathbb{G}_C(t) - \mathbb{G}_{C([0,-\ell-1])}(t)\right) \tag{1.2.23}$$

where $[0, -\ell - 1]$ is the set of the integers n satisfying $0 \leqslant n \leqslant -\ell - 1$, and

$$\mathbb{G}_{\mathbf{Aug}(C)}(t) = \mathbb{G}_C(t) - \#C(0). \tag{1.2.24}$$

1.2.9. *Composition operation.* Let C_1 and C_2 be two graded collections. The *composition* of C_1 and C_2 is the graded collection $C_1 \odot C_2$ such that, for all $n \in \mathbb{N}$,

$$(C_1 \odot C_2)(n) := \bigsqcup_{k \in \mathbb{N}} \left[C_1(k), \left(\mathbf{List}^{+}_{\{k\}}(C_2)\right)(n) \right]^{\omega}_{\times} \tag{1.2.25}$$

where $\omega : \mathbb{N}^2 \to \mathbb{N}$ is the map defined by $\omega(n_1, n_2) := n_2$. In other words, each object of size n of $C_1 \odot C_2$ is an ordered pair $(x, (y_1, \dots, y_k))$, $k \in \mathbb{N}$, where x is an object of C_1 of size k, and $(y_1, \dots, y_k)$ is a tuple of objects of C_2 such that the sum of the sizes of the y_j, $1 \leqslant j \leqslant k$ is n. Observe that, if C_3 is a graded collection,

$$C_1 \odot \{\bullet\} \simeq C_1 \simeq \{\bullet\} \odot C_1, \tag{1.2.26a}$$

$$(C_1 \odot C_2) \odot C_3 \simeq C_1 \odot (C_2 \odot C_3). \tag{1.2.26b}$$

When C_1 and C_2 are combinatorial and graded, $C_1 \odot C_2$ is combinatorial if and only if C_2 is augmented. In this case, its generating series satisfies

$$\mathbb{G}_{C_1 \odot C_2}(t) = \mathbb{G}_{C_1}\left(\mathbb{G}_{C_2}(t)\right). \tag{1.2.27}$$

1.2.10. *Coloration operation.* Let C be a graded collection and $\mathfrak{C}$ be a set of colors. The *$\mathfrak{C}$-coloration* of C is the $\mathfrak{C}$-colored collection $\mathbf{Col}_{\mathfrak{C}}(C)$ defined, for all $\mathfrak{C}$-colored indexes $(a, u) \in \mathfrak{C} \times \mathfrak{C}^\ell$, $\ell \in \mathbb{N}$, by

$$\left(\mathbf{Col}_{\mathfrak{C}}(C)\right)(a, u) := \{(a, x, u) : x \in C(\ell)\}. \tag{1.2.28}$$

In other words, each object of $\mathbf{Col}_{\mathfrak{C}}(C)$ is built from an object x of C by equipping it freely with an output color from $\mathfrak{C}$ and a word of input colors from $\mathfrak{C}$ having the size of x as length.

When C is combinatorial, the graduation $\mathbf{Cast}^\omega\left(\mathbf{Col}_{\mathfrak{C}}(C)\right)$ of $\mathbf{Col}_{\mathfrak{C}}(C)$ is combinatorial if and only if the set of colors $\mathfrak{C}$ is finite. In this case, by setting $m := \#\mathfrak{C}$, its generating series satisfies

$$\mathbb{G}_{\mathbf{Cast}^\omega(\mathbf{Col}_{\mathfrak{C}}(C))}(t) = \sum_{n \in \mathbb{N}} \#C(n) m^{n+1}\, t^n. \tag{1.2.29}$$

1.2.11. *Cycle operation.* Let C be a graded collection. The *cycle collection* of C is the graded collection $\mathbf{Cyc}(C)$ defined, for all $n \in \mathbb{N}$, by

$$\left(\mathbf{Cyc}(C)\right)(n) := \{(x, k) : x \in C(n) \text{ and } 0 \leqslant k \leqslant n\}. \tag{1.2.30}$$

In other words, each object of $\mathbf{Cyc}(C)$ is built from an object x of C by equipping it freely with a nonnegative integer nongreater than its size.

Let us observe that by defining, for any $n \in \mathbb{N}$, the map $\circlearrowright_n : (\mathbf{Cyc}(C))(n) \to (\mathbf{Cyc}(C))(n)$ by $\circlearrowright_n ((x, k)) := (x, k + 1 \pmod{n+1})$ for any $(x, k) \in (\mathbf{Cyc}(C))(n)$, each $\circlearrowright_n$ is a cycle map of $\mathbf{Cyc}(C)$. Therefore $\mathbf{Cyc}(C)$ is cyclic.

When C is combinatorial, $\mathbf{Cyc}(C)$ is combinatorial and its generating series satisfies

$$\mathbb{G}_{\mathbf{Cyc}(C)}(t) = \sum_{n \in \mathbb{N}} \#C(n)(n+1)\, t^n. \tag{1.2.31}$$

1.2.12. *Regularization operation.* Let C be a graded collection. The *regularization* of C is the graded collection $\mathbf{Reg}(C)$ defined by

$$\mathbf{Reg}(C) := [C, \mathfrak{S}]_{\boxtimes} \tag{1.2.32}$$

where $\mathfrak{S}$ is the graded collection of bijections defined in Section 1.1.6. In other words, each object of $\mathbf{Reg}(C)$ is built from an object x of C by equipping it freely with a bijection of $\mathfrak{S}(n)$ where n is the size of x.

Let us observe that by defining, for any $\sigma \in \mathfrak{S}(n)$, $n \in \mathbb{N}$, the map $\circledcirc_\sigma : (\mathbf{Reg}(C))(n) \to (\mathbf{Reg}(C))(n)$ by $\circledcirc_\sigma((x, \nu)) := \left(x, \sigma^{-1} \circ \nu\right)$ for any $(x, \nu) \in (\mathbf{Reg}(C))(n)$, each $\circledcirc_\sigma$ is a symmetric map of $\mathbf{Reg}(C)$. Therefore, $\mathbf{Reg}(C)$ is symmetric.

When C is combinatorial, $\mathbf{Reg}(C)$ is combinatorial and its generating series satisfies

$$\mathbb{G}_{\mathbf{Reg}(C)}(t) = \sum_{n \in \mathbb{N}} \#C(n) n! \, t^n. \tag{1.2.33}$$

1.3. Examples. We define, in some cases by using the operations of Section 1.2, some usual graded combinatorial collections. At the same time, we set here our main notations and definitions about their objects.

1.3.1. *Natural numbers.* We can regard the set $\mathbb{N}$ as the graded collection satisfying $\mathbb{N}(n) := \{n\}$ for all $n \in \mathbb{N}$. Hence, $\mathbf{List}^+(\{\bullet\}) \simeq \mathbb{N}$. Moreover, for any $\ell \in \mathbb{N}$, let $\mathbb{N}_{\geqslant \ell}$ be the graded collection defined by

$$\mathbb{N}_{\geqslant \ell} := \mathbf{Sus}_\ell \left(\mathbf{Sus}_{-\ell}(\mathbb{N})\right). \tag{1.3.1}$$

By definition of the suspension operation over graded collections, $\mathbb{N}_{\geqslant \ell}$ is the set of all integers greater than or equal to ℓ. Observe that $\mathbb{N}_{\geqslant 1} = \mathbf{Aug}(\mathbb{N})$. The generating series of $\mathbb{N}_{\geqslant \ell}$ satisfies

$$\mathbb{G}_{\mathbb{N}_{\geqslant \ell}}(t) = \frac{t^\ell}{1 - t} = \sum_{n \in \mathbb{N}_{\geqslant \ell}} t^n. \tag{1.3.2}$$

Observe also that the list collection operation over graded collections can be expressed as a composition involving $\mathbb{N}$ since

$$\mathbf{List}^+(C) \simeq \mathbb{N} \odot C \tag{1.3.3}$$

for any graded collection C. We shall consider in the sequel, for any $x, z \in \mathbb{N}$, the subcollections $[x, z] := \{y \in \mathbb{N} : x \leqslant y \leqslant z\}$, and $[x] := [1, x]$ of $\mathbb{N}$. These examples of graded collections are among the simplest nontrivial ones.

1.3.2. *Words.* Let A be an *alphabet*, that is a set whose elements are called *letters*. One can see A as a graded collection wherein all letters are atoms. In this case, we denote by A^* the graded collection $\mathbf{List}^+(A)$. By definition, the objects of A^* are finite sequences of elements of A. We call *words* on A these objects. When A is finite, A^* is combinatorial and it follows from (1.2.13) that the generating series of A^* satisfies

$$\mathbb{G}_{A^*}(t) = \frac{1}{1 - mt} = \sum_{n \in \mathbb{N}} m^n \, t^n \tag{1.3.4}$$

where $m := \#A$. If $u := (a_1, \dots, a_n)$ is a word on A, it follows from the definition of A^* that the size $|u|$ of u is n. The unique word on A of size 0 is denoted by ϵ and is called *empty word*.

Let $u := (a_1, \dots, a_n)$ be a word on A. The *ith letter* of u is a_i and is denoted by $u(i)$. For any letter $\mathrm{b} \in A$, the *number of occurrences* $|u|_{\mathrm{b}}$ of b in u is the cardinality of the set $\{i \in [|u|] : u(i) = \mathrm{b}\}$. When A is endowed with a total order $\preccurlyeq$ and u is nonempty, $\max_{\preccurlyeq}(u)$ is the greatest letter appearing in u with respect to $\preccurlyeq$. Moreover, an *inversion* of u is a pair (i, j) such that $i < j$, $u(i) \neq u(j)$, and $u(j) \preccurlyeq u(i)$. Given two words u and v on A, the *concatenation* of u and v is the word $u \cdot v$ containing from left to right the letters of u and then the ones of v. The concatenation $\cdot$ is a graded complete product on A^*. If u can be expressed as $u = u_1 \cdot u_2$ where $u_1, u_2 \in A^*$, we say that u_1 is a *prefix* of u and this property is denoted by $u \preccurlyeq_{\mathrm{p}} v$. For any subset $S := \{s_1 \leqslant \dots \leqslant s_k\}$ of $[|u|]$, $u_{|S}$ is the word $u(s_1) \dots u(s_k)$. Moreover, when v is a word such that there exists $S \subseteq [|u|]$ satisfying $v = u_{|S}$, v is a *subword* of u.

A *language* on A is subcollection of A^*. A language $\mathcal{L}$ on A is *prefix* if for all $u \in \mathcal{L}$ and $v \in A^*$, $v \preccurlyeq_{\mathrm{p}} u$ implies $v \in \mathcal{L}$. We denote by A^+ the language $\mathbf{Aug}(A^*)$ containing all nonempty words on A, and, for any $n \in \mathbb{N}$, by A^n the language $A^*(n)$.

1.3.3. *Integer compositions.* By regarding the set $\mathbb{N}$ as a graded collection as explained in Section 1.3.1, let $\mathfrak{Com}$ be the combinatorial graded collection $\mathbf{List}^+(\mathbb{N}_{\geqslant 1})$. It follows from (1.2.13) and (1.3.2) that the generating series of $\mathfrak{Com}$ is

$$\mathbb{G}_{\mathfrak{Com}}(t) = \frac{1 - t}{1 - 2t} = 1 + \sum_{n \in \mathbb{N}_{\geqslant 1}} 2^{n-1} \, t^n. \tag{1.3.5}$$

Hence, the integer sequence of $\mathfrak{Com}$ begins by

$$1, 1, 2, 4, 8, 16, 32, 64, 128 \tag{1.3.6}$$

and is Sequence **A011782** of [**Slo**]. By definition, the objects of $\mathfrak{Com}$ are finite sequences of positive numbers. We call *integer compositions* (or, for short, *compositions*) these objects. If $\boldsymbol{\lambda} := (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k)$ is a composition, it follows from the definition of $\mathfrak{Com}$ that the size $|\boldsymbol{\lambda}|$ of $\boldsymbol{\lambda}$ is $\boldsymbol{\lambda}_1 + \dots + \boldsymbol{\lambda}_k$. The *length* $\ell(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$ is k, and for any $i \in [\ell(\boldsymbol{\lambda})]$, the *$i$th part* of $\boldsymbol{\lambda}$ is $\boldsymbol{\lambda}_i$. The unique composition of size 0 is denoted by ϵ and is called *empty composition* (even if ϵ is already used to express the empty word, this overloading of notation is not a problem in practice).

The *descents set* of $\boldsymbol{\lambda}$ is the set

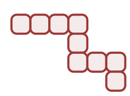
$$\mathrm{Des}(\boldsymbol{\lambda}) := \{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, \dots, \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 + \cdots + \boldsymbol{\lambda}_{k-1}\}. \tag{1.3.7}$$

For instance, $\mathrm{Des}(4131) = \{4, 5, 8\}$. Moreover, for any word u defined on an alphabet A equipped with a total order $\preccurlyeq$, the *composition* $\mathrm{cmp}(u)$ of u is the composition of size $|u|$ defined by

$$\mathrm{cmp}(u) := (|u_1|, \dots, |u_k|), \tag{1.3.8}$$

where $u = u_1 \cdot \ldots \cdot u_k$ is the factorization of u in longest nondecreasing factors (with respect to the order $\preccurlyeq$). For instance, if $u := a_2a_2a_3a_1a_3a_2a_1a_2$ is a word on the alphabet $A := \{a_1, a_2, a_3\}$ ordered by $a_1 \preccurlyeq a_2 \preccurlyeq a_3$, $\mathrm{cmp}(u) = 3212$. When $\#A \geqslant 2$, this map cmp is a surjective collection morphism from A^* to $\mathfrak{Com}$.

Integer compositions are drawn as *ribbon diagrams* in the following way. For each part $\boldsymbol{\lambda}_i$ of $\boldsymbol{\lambda}$, we draw a horizontal line of $\boldsymbol{\lambda}_i$ boxes. These lines are organized so that the line for the first part of $\boldsymbol{\lambda}$ is the uppermost, and the first box of the line of the part $\boldsymbol{\lambda}_{i+1}$ is glued below the last box of the line of the part $\boldsymbol{\lambda}_i$, for all $i \in [\ell(\boldsymbol{\lambda}) - 1]$. For instance, the ribbon diagram of the composition 4131 is



. (1.3.9)

1.3.4. *Integer partitions.* Again by regarding the set $\mathbb{N}$ as a graded collection as considered in Section 1.3.1, let $\mathfrak{Par}$ be the graded combinatorial collection $\mathbf{MSet}^+(\mathbb{N}_{\geqslant 1})$. Since $\#\mathbb{N}_{\geqslant 1}(n) = 1$ for all $n \geqslant 1$, it follows from (1.2.18) that the generating series of $\mathfrak{Par}$ is

$$\mathbb{G}_{\mathfrak{Par}}(t) = \prod_{n \in \mathbb{N}_{\geqslant 1}} \frac{1}{1 - t^n}. \tag{1.3.10}$$

Hence, the integer sequence of $\mathfrak{Par}$ begins by

$$1, 1, 2, 3, 5, 7, 11, 15, 22 \tag{1.3.11}$$

and is Sequence **A000041** of [**Slo**]. By definition, the objects of $\mathfrak{Par}$ are finite multisets of positive integers. We call *integer partitions* (or, for short, *partitions*) these objects. As a consequence of the definition of $\mathfrak{Par}$, the size $|\lambda|$ of any partition λ is the sum of the integers appearing in the multiset λ. Due to the definition of partitions as multisets, we can present a partition as an ordered sequence of positive integers with respect to any total order on $\mathbb{N}_{\geqslant 1}$. For this reason, we denote any partition λ by a nonincreasing sequence $(\lambda_1, \dots, \lambda_k)$ of positive integers (that is, $\lambda_i \geqslant \lambda_{i+1}$ for all $i \in [k - 1]$). Under this convention, the *length* $\ell(\lambda)$ of λ is k, and for any $i \in [\ell(\lambda)]$, the *ith part* of λ is λ_i.

1.3.5. *Permutations.* A *permutation* of *size* n is a bijection σ from $[n]$ to $[n]$. The combinatorial graded collection of all permutations is, in accordance with the notations of Section 1.1.6, denoted by $\mathfrak{S}$. The generating series of $\mathfrak{S}$ is

$$\mathbb{G}_{\mathfrak{S}}(t) = \sum_{n \in \mathbb{N}} n!\, t^n. \tag{1.3.12}$$

Hence, the integer sequence of $\mathfrak{S}$ begins by

$$1, 1, 2, 6, 24, 120, 720, 5040, 40320 \tag{1.3.13}$$

and is Sequence **A000142** of [**Slo**]. Any permutation σ of $\mathfrak{S}(n)$ is denoted as a word $\sigma(1) \dots \sigma(n)$ on $\mathbb{N}_{\geqslant 1}$. Under this convention, a permutation of size n is a word on the alphabet $[n]$ with exactly one occurrence of each letter of $[n]$. The composition operation $\circ$ of maps is a concentrated product on $\mathfrak{S}$ and the valid inputs of $\circ$ are the ordered pairs (σ_1, σ_2) such that $|\sigma_1| = |\sigma_2|$.

A *descent* of $\sigma \in \mathfrak{S}(n)$ is a position $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$. The set of all descents of σ is denoted by $\operatorname{Des}(\sigma)$. For any word u defined on an alphabet A equipped with a total order $\preccurlyeq$, the *standardized* $\operatorname{std}(u)$ of u is the permutation of size $|u|$ having the same inversions as the ones of u. In other terms $\operatorname{std}(u)$ has its letters in the same relative order as those of u, with respect to $\preccurlyeq$, where equal letters of u are ordered from left to right as the smallest to the greatest. For example, by considering the alphabet $\mathbb{N}$ equipped with the natural order of integers, $\operatorname{std}(211241) = 412563$. This map std is a surjective collection morphism from $\mathbb{N}^*$ to $\mathfrak{S}$.

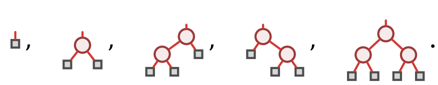
1.3.6. *Binary trees.* Let $\mathfrak{BT}_\bullet$ be the combinatorial graded collection satisfying the relation

$$\mathfrak{BT}_\bullet = \{\perp\} + \left[\, \{\bullet\}, [\mathfrak{BT}_\bullet, \mathfrak{BT}_\bullet]^{+}_{\times} \right]^{+}_{\times}, \tag{1.3.14}$$

where $\perp$ is an object of size 0 called *leaf* and $\bullet$ is an atomic object called *internal node*. We call *binary tree* each object of $\mathfrak{BT}_\bullet$. By definition, a binary tree $\mathfrak{t}$ is either the leaf $\perp$ or an ordered pair $(\bullet, (\mathfrak{t}_1, \mathfrak{t}_2))$ where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are binary trees. Observe that this description of binary trees is recursive. For instance,

$$\perp, \quad (\bullet, (\perp, \perp)), \quad (\bullet, ((\bullet, (\perp, \perp)), \perp)), \quad (\bullet, (\perp, (\bullet, (\perp, \perp)))), \quad (\bullet, ((\bullet, (\perp, \perp)), (\bullet, (\perp, \perp)))), \tag{1.3.15}$$

are binary trees. If $\mathfrak{t}$ is a binary tree different from the leaf, by definition, $\mathfrak{t}$ can be expressed as $\mathfrak{t} = (\bullet, (\mathfrak{t}_1, \mathfrak{t}_2))$ where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two binary trees. In this case, $\mathfrak{t}_1$ (resp. $\mathfrak{t}_2$) is the *left subtree* (resp. *right subtree*) of $\mathfrak{t}$. By drawing each leaf by ⊥ and each binary tree with at least one internal node by an internal node ○ attached below it, from left to right, to its left and right subtrees by means of edges —, the binary trees of (1.3.15) are depicted by



(1.3.16)

By definition of the sum and the Cartesian product operations over graded collections, the size of a binary tree $\mathfrak{t}$ satisfies

$$|\mathfrak{t}| = \begin{cases} 0 & \text{if } \mathfrak{t} = \perp, \\ 1 + |\mathfrak{t}_1| + |\mathfrak{t}_2| & \text{otherwise } (\mathfrak{t} = (\bullet, (\mathfrak{t}_1, \mathfrak{t}_2))). \end{cases} \tag{1.3.17}$$

In other words, the size of $\mathfrak{t}$ is the number of occurrences of $\bullet$ it contains. Since $\mathbb{G}_{\{\perp\}}(t) = 1$ and $\mathbb{G}_{\{\bullet\}}(t) = t$, it follows from (1.2.2) and (1.2.7) that the generating series of $\mathfrak{BT}_\bullet$ satisfies the quadratic algebraic equation

$$1 - \mathbb{G}_{\mathfrak{BT}_\bullet}(t) + t\mathbb{G}_{\mathfrak{BT}_\bullet}(t)^2 = 0. \tag{1.3.18}$$

The unique solution having a combinatorial meaning of (1.3.18) is

$$\mathbb{G}_{\mathfrak{BT}_\bullet}(t) = \frac{1 - \sqrt{1 - 4t}}{2t} = \sum_{n \in \mathbb{N}} \frac{1}{n+1} \binom{2n}{n} t^n \tag{1.3.19}$$

The integer sequence of $\mathfrak{BT}_\bullet$ begins by

$$1, 1, 2, 5, 14, 42, 132, 429, 1430 \tag{1.3.20}$$

and is Sequence **A000108** of [**Slo**]. These numbers are known as *Catalan numbers*.

2. Posets

We consider now collections endowed with partial order relations compatible with their indexations. Such structures are important in combinatorics since they lead, for instance, to the construction of alternative bases of combinatorial spaces (see forthcoming Section 1.3 of Chapter 3). We provide general definitions about posets and consider as examples three important ones: the cube, Tamari, and right weak order posets.

2.1. Posets on collections. Let us provide the main definitions about collections endowed with the structure of a poset.

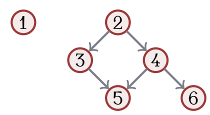
2.1.1. *Elementary definitions.* An *I-poset* is a pair $(\mathcal{Q}, \preccurlyeq_\mathcal{Q})$ where $\mathcal{Q}$ is an I-collection and $\preccurlyeq_\mathcal{Q}$ is both a relation on $\mathcal{Q}$ (recall that relations on collections preserve the indexes) and a partial order relation. The *strict order relation* of $\preccurlyeq$ is the relation $\prec$ on $\mathcal{Q}$ satisfying, for all $x, y \in \mathcal{Q}$, $x \prec y$ if $x \preccurlyeq y$ and $x \neq y$.

The *interval* between two objects x and z of $\mathcal{Q}$ is the set $[x, z] := \{y \in \mathcal{Q} : x \preccurlyeq_\mathcal{Q} y \preccurlyeq_\mathcal{Q} z\}$. When all intervals of $\mathcal{Q}$ are finite, $\mathcal{Q}$ is *locally finite*. Observe that when $\mathcal{Q}$ is combinatorial, $\mathcal{Q}$ is locally finite. For any $i \in I$, an object x of $\mathcal{Q}(i)$ is a *greatest* (resp. *least*) *element* if for all $y \in \mathcal{Q}(i)$, $y \preccurlyeq_\mathcal{Q} x$ (resp. $x \preccurlyeq_\mathcal{Q} y$). Moreover, for any $i \in I$, an object x of $\mathcal{Q}(i)$ is a *maximal* (resp. *minimal*) *element* if for all $y \in \mathcal{Q}(i)$, $x \preccurlyeq_\mathcal{Q} y$ (resp. $y \preccurlyeq_\mathcal{Q} x$) implies $x = y$. If x and y are two different objects of $\mathcal{Q}$, y *covers* x if $[x, y] = \{x, y\}$. Two objects x and y are *comparable* (resp. *incomparable*) in $\mathcal{Q}$ if $x \preccurlyeq_\mathcal{Q} y$ or $y \preccurlyeq_\mathcal{Q} x$ (resp. neither $x \preccurlyeq_\mathcal{Q} y$ nor $y \preccurlyeq_\mathcal{Q} x$ holds). If for any $i \in I$ and any i-objects x and y of $\mathcal{Q}$, x and y are comparable, $\mathcal{Q}$ is a *total order*. A *chain* of $\mathcal{Q}$ is a sequence $(x_1, \dots, x_k)$ such that $x_j \preccurlyeq_\mathcal{Q} x_{j+1}$

for all $j \in [k-1]$. An *antichain* of $\mathbb{Q}$ is a subset of pairwise incomparable elements of $\mathbb{Q}$. The *Hasse diagram* of $(\mathbb{Q}, \preccurlyeq_{\mathbb{Q}})$ is the directed graph having $\mathbb{Q}$ as set of vertices and all the pairs (x, y) where y covers x as set of arcs.

Besides, if $(\mathbb{Q}_1, \preccurlyeq_{\mathbb{Q}_1})$ and $(\mathbb{Q}_2, \preccurlyeq_{\mathbb{Q}_2})$ are two I-posets, a map $\phi : \mathbb{Q}_1 \to \mathbb{Q}_2$ is a *poset morphism* if ϕ is a collection morphism and for all $x, y \in \mathbb{Q}_1$ such that $x \preccurlyeq_{\mathbb{Q}_1} y$, $\phi(x) \preccurlyeq_{\mathbb{Q}_2} \phi(y)$. Besides, $\mathbb{Q}_2$ is a *subposet* of $\mathbb{Q}_1$ if $\mathbb{Q}_2$ is a subcollection of $\mathbb{Q}_1$ and $\preccurlyeq_{\mathbb{Q}_2}$ is the restriction of $\preccurlyeq_{\mathbb{Q}_1}$ on $\mathbb{Q}_2$. For any $i \in I$, we call *i-subposet* of $\mathbb{Q}$ the subposet of $\mathbb{Q}$ obtained by restricting $\preccurlyeq_{\mathbb{Q}}$ on $\mathbb{Q}(i)$.

We shall define posets $\mathbb{Q}$ by drawing Hasse diagrams, where minimal elements are drawn uppermost and vertices are labeled by the elements of $\mathbb{Q}$. For instance, the Hasse diagram



(2.1.1)

denotes the simple ("simple" here means the property of collections stated in Section 1.1.1) poset $([6], \preccurlyeq)$ satisfying among others $3 \preccurlyeq 5$ and $2 \preccurlyeq 6$.

2.1.2. *Operations over posets.* If $(\mathbb{Q}_1, \preccurlyeq_{\mathbb{Q}_1})$ and $(\mathbb{Q}_2, \preccurlyeq_{\mathbb{Q}_2})$ are two I-posets, the sum $\mathbb{Q}_1 + \mathbb{Q}_2$ is endowed with the relation $\preccurlyeq$ satisfying, $x \preccurlyeq y$ whenever $x, y \in \mathbb{Q}_1$ and $x \preccurlyeq_{\mathbb{Q}_1} y$, or $x, y \in \mathbb{Q}_2$ and $x \preccurlyeq_{\mathbb{Q}_2} y$. Since $\preccurlyeq$ is an order relation, $\mathbb{Q}_1 + \mathbb{Q}_2$ is an I-poset, called *sum* of $(\mathbb{Q}_1, \preccurlyeq_{\mathbb{Q}_1})$ and $(\mathbb{Q}_2, \preccurlyeq_{\mathbb{Q}_2})$. For any $p \in \mathbb{N}$ and any I-posets $(\mathbb{Q}_1, \preccurlyeq_{\mathbb{Q}_1})$, ..., $(\mathbb{Q}_p, \preccurlyeq_{\mathbb{Q}_p})$, the Hadamard product $\left[\mathbb{Q}_1, \dots, \mathbb{Q}_p\right]_{\boxtimes}$ is endowed with the relation $\preccurlyeq$ satisfying $(x_1, \dots, x_p) \preccurlyeq (y_1, \dots, y_p)$ for any $(x_1, \dots, x_p), (y_1, \dots, y_p) \in \left[\mathbb{Q}_1, \dots, \mathbb{Q}_p\right]_{\boxtimes}$ such that $x_k \preccurlyeq_{\mathbb{Q}_k} y_k$ for all $k \in [p]$. Since $\preccurlyeq$ is an order relation, $\left[\mathbb{Q}_1, \dots, \mathbb{Q}_p\right]_{\boxtimes}$ is an I-poset, called *Hadamard product* of $(\mathbb{Q}_1, \preccurlyeq_{\mathbb{Q}_1})$, ..., $(\mathbb{Q}_p, \preccurlyeq_{\mathbb{Q}_p})$. Let $(\mathbb{Q}, \preccurlyeq_{\mathbb{Q}})$ be an I-poset. Let $\mathbf{Com}(\mathbb{Q})$ be the subcollection of $[\mathbb{Q}, \mathbb{Q}]_{\boxtimes}$ restrained on the ordered pairs (x, y) such that $x \preccurlyeq_{\mathbb{Q}} y$, called *pairs of comparable objects*. By definition, $\mathbf{Com}(\mathbb{Q})$ is endowed with the restriction of the order relation of $[\mathbb{Q}, \mathbb{Q}]_{\boxtimes}$ on $\mathbf{Com}(\mathbb{Q})$. We call $\mathbf{Com}(\mathbb{Q})$ the *poset of pairs of comparable objects* of $\mathbb{Q}$. Finally, the *dual* of $\mathbb{Q}$ is the I-poset $(\mathbb{Q}, \bar{\preccurlyeq}_{\mathbb{Q}})$ such that $x \bar{\preccurlyeq}_{\mathbb{Q}} y$ holds whenever $y \preccurlyeq_{\mathbb{Q}} x$ for any $x, y \in \mathbb{Q}$.

2.2. Examples. We consider here three well-known combinatorial posets.

2.2.1. *The cube poset.* Let $\preccurlyeq$ be the partial order relation on the combinatorial collection $\mathfrak{Com}$ of compositions generated by the covering relation $\lessdot$ defined, for any composition $\boldsymbol{\lambda}$ of length k, by

$$(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{i-1}, \boldsymbol{\lambda}_i, \boldsymbol{\lambda}_{i+1}, \boldsymbol{\lambda}_{i+2}, \dots, \boldsymbol{\lambda}_k) \lessdot (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{i-1}, \boldsymbol{\lambda}_i + \boldsymbol{\lambda}_{i+1}, \boldsymbol{\lambda}_{i+2}, \dots, \boldsymbol{\lambda}_k). \qquad (2.2.1)$$

For instance, $2123 \lessdot 215$ and $2123 \preccurlyeq 8$. This order is the *refinement order* of compositions. The Hasse diagram of $(\mathfrak{Com}, \preccurlyeq)$ restricted on $\mathfrak{Com}(4)$ is shown in Figure 1.1.

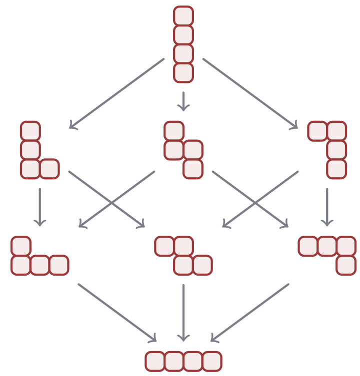


FIGURE 1.1. The Hasse diagram of the refinement order of compositions of size 4, where each composition is represented through its ribbon diagram.

Observe that for all compositions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, $\boldsymbol{\lambda} \preccurlyeq \boldsymbol{\mu}$ if and only if $\mathrm{Des}(\boldsymbol{\mu}) \subseteq \mathrm{Des}(\boldsymbol{\lambda})$. Each n-subposet of the refinement order of compositions is known as the *cube poset* of dimension $n-1$. Moreover, the cube poset of dimension $n-1$ is isomorphic to the dual of the poset of all subsets of $[n-1]$ ordered by set inclusion. An isomorphism is provided by the map Des sending a composition of size n to a subset of $[n-1]$.

2.2.2. *The Tamari order on binary trees.* Let $\preccurlyeq$ be the partial order relation on the combinatorial collection $\mathfrak{BT}_\bullet$ of binary trees generated by the covering relation $\lessdot$ defined by

$$(\dots(\bullet, ((\bullet, (\mathfrak{r}_1, \mathfrak{r}_2)), \mathfrak{r}_3)) \dots) \lessdot (\dots(\bullet, (\mathfrak{r}_1, (\bullet, (\mathfrak{r}_2, \mathfrak{r}_3)))) \dots), \tag{2.2.2}$$

where $\mathfrak{r}_1$, $\mathfrak{r}_2$, and $\mathfrak{r}_3$ are any binary trees. We call $\lessdot$ the *right rotation* relation. At this moment, the definition of this relation on binary trees is informal but, in Section 2.3 of Chapter 2, we shall develop precise tools to define and handle such operations on binary trees and more generally on syntax trees. The order $\preccurlyeq$ is the *Tamari order* on binary trees. The Hasse diagram of $(\mathfrak{BT}_\bullet, \preccurlyeq)$ restricted on $\mathfrak{BT}_\bullet(4)$ is shown in Figure 1.2.

2.2.3. *The right weak order on permutations.* Let $\preccurlyeq$ be the partial order relation on the combinatorial collection $\mathfrak{S}$ of permutations generated by the covering relation $\lessdot$ defined by

$$u \,\mathrm{ab}\, v \lessdot u \,\mathrm{ba}\, v, \tag{2.2.3}$$

where u and v are words on $\mathbb{N}_{\geqslant 1}$, and a and b are letters such that $\mathrm{a} < \mathrm{b}$. This order is the *right weak order* of permutations. The Hasse diagram of $(\mathfrak{S}, \preccurlyeq)$ restricted on $\mathfrak{S}(4)$ is shown in Figure 1.3.

3. Rewrite systems

A rewrite system describes a process whose goal is to transform iteratively an object into another one. We consider rewrite systems on I-collections, so that an i-object, $i \in I$,

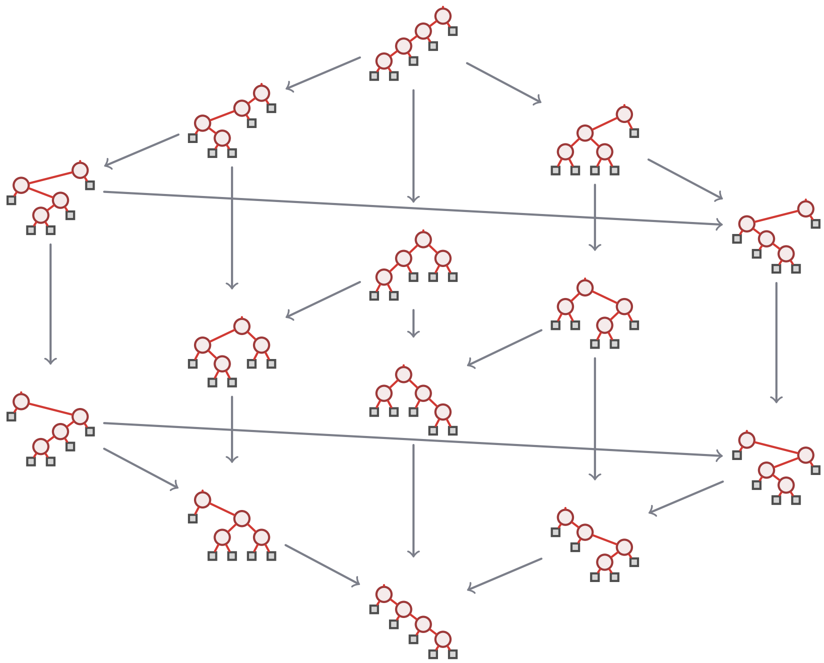


FIGURE 1.2. The Hasse diagram of the Tamari poset of binary trees of size 4.

can be transformed only into i-objects. As we shall see, rewrite systems and posets have some close connections because it is possible, in some cases, to construct posets from rewrite systems.

3.1. Rewrite systems on collections. Let us provide the main definitions about collections endowed with the structure of a rewrite system. Two properties of rewrite systems are fundamental: the termination and the confluence. We provide strategies to prove that a given rewrite system satisfies one or the other.

3.1.1. *Elementary definitions.* Let C be an I-collection. An *I-rewrite system* is a pair $(C, \to)$ where C is an I-collection and $\to$ is a relation on C. We call $\to$ a *rewrite rule*. When $x_0, x_1, \dots, x_k$ are objects of C such that $k \in \mathbb{N}$ and

$$x_0 \to x_1 \to \dots \to x_k, \tag{3.1.1}$$

we say that x_0 is *rewritable* by $\to$ into x_k in k *steps*. The reflexive and transitive closure of $\to$ is denoted by $\overset{*}{\to}$. The directed graph $(C, \to)$ consisting in C as set of vertices and $\to$ as set of arcs is the *rewriting graph* of $(C, \to)$.

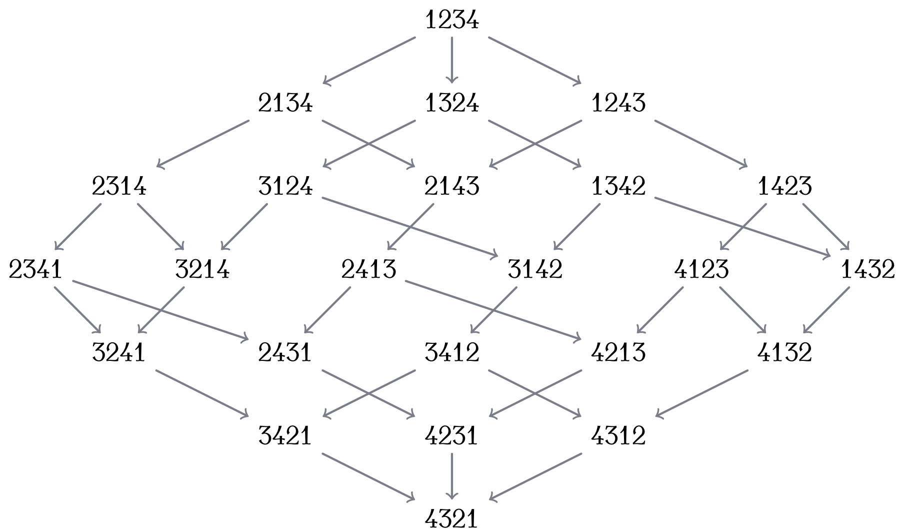


FIGURE 1.3. The Hasse diagram of the right weak poset of permutations of size 4.

3.1.2. *Termination.* When there is no infinite chain

$$x_0 \to x_1 \to x_2 \to \cdots \tag{3.1.2}$$

where all $x_j \in C$, $j \in \mathbb{N}$, $(C, \to)$ is *terminating*. Observe that, if C is combinatorial, due to the fact that for any $i \in I$, each set $C(i)$ is finite and the fact that the rewriting relation preserves the indexes, if such an infinite chain (3.1.2) exists, then it is of the form

$$x_0 \to \cdots \to x_r \to \cdots \to x_r \to \cdots , \tag{3.1.3}$$

for a certain $r \in \mathbb{N}$. A *normal form* of $(C, \to)$ is an object x of C such that for all $x' \in C$, $x \xrightarrow{*} x'$ imply $x' = x$. In other words, a normal form of $(C, \to)$ is an object which is not rewritable by $\to$. This set of objects, which is a subcollection of C, is denoted by $\mathcal{N}_{(C,\to)}$. The following result provides a tool in the aim to show that a combinatorial rewrite system is terminating.

THEOREM 3.1.1. *Let $(C, \to)$ be a combinatorial rewrite system. Then, $(C, \to)$ is terminating if and only if the binary relation $\xrightarrow{*}$ is an order relation and endows C with a structure of a combinatorial poset.*

When C is combinatorial and $(C, \to)$ is terminating, by Theorem 3.1.1, $\left(C, \xrightarrow{*}\right)$ is a combinatorial poset and we call it the *poset induced* by $\to$.

In practice, Theorem 3.1.1 is used as follows. To show that a combinatorial I-rewrite system $(C, \to)$ is terminating, we construct a map $\theta : C \to \mathbb{Q}$ where $(\mathbb{Q}, \preccurlyeq)$ is an I-poset such that for any $x, x' \in C$, $x \to x'$ implies $\theta(x) \prec \theta(x')$. Such a map θ is a *termination invariant*. Indeed, since each $C(i)$, $i \in I$, is finite, this property leads to the fact that there is no infinite chain of the form (3.1.3). In most cases, $\mathbb{Q}$ is a set of tuples of integers of a fixed length, and $\preccurlyeq$ is the lexicographic order on these tuples.

3.1.3. *Confluence.* When for any objects x, y_1, and y_2 of C such that $x \xrightarrow{*} y_1$ and $x \xrightarrow{*} y_2$, there exists an object x' of C such that $y_1 \xrightarrow{*} x'$ and $y_2 \xrightarrow{*} x'$, the rewrite system $(C, \to)$ is *confluent*. When $\to$ is both terminating and confluent, $\to$ is *convergent*.

An object x of C is a *branching* object if there exist two different objects y_1 and y_2 satisfying $x \to y_1$ and $x \to y_2$. In this case, the pair $\{y_1, y_2\}$ is a *branching pair* for x. We say that a branching pair $\{y_1, y_2\}$ is *joinable* if there exists an object z of C such that $y_1 \xrightarrow{*} z$ and $y_2 \xrightarrow{*} z$. In practice, showing that a terminating rewrite system is confluent is made simple, thank to the following result.

Theorem 3.1.2. *Let $(C, \to)$ be a rewrite system. If $(C, \to)$ is terminating and all its branching pairs are joinable, $(C, \to)$ is confluent.*

3.1.4. *Closures.* Let $(C, \to)$ be an I-rewrite system such that C is endowed with a set $\mathcal{P}$ of concentrated products. Then, let $(C, \to_{\mathcal{P}})$ be the rewrite system such that $\to_{\mathcal{P}}$ contains $\to$ (as a binary relation) and satisfies moreover

$$\star\left(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_p\right) \quad \to_{\mathcal{P}} \quad \star\left(x_1, \dots, x_{j-1}, y', x_{j+1}, \dots, x_p\right) \tag{3.1.4}$$

for any product $\star$ of arity p of $\mathcal{P}$, such that the left and right members of (3.1.4) are valid inputs for $\star$ and $y \to y'$. The fact that all products $\star$ of $\mathcal{P}$ are concentrated ensures that the left and the right members of (3.1.4) have the same index, so that $(C, \to_{\mathcal{P}})$ is an I-rewrite system. We call $(C, \to_{\mathcal{P}})$ the *$\mathcal{P}$-closure* of $(C, \to)$. Such closures provide convenient and concise ways to define rewrite systems.

3.2. Examples. Let us review some examples of rewrite systems on various combinatorial collections.

3.2.1. *A first rewrite system on words.* Let $A := \{\mathrm{a}, \mathrm{b}\}$ be an alphabet, and consider the graded rewrite system $(A^*, \to)$ defined by

$$ux \to xu \tag{3.2.1}$$

for any $u \in A^*$ and $x \in A$. We have, for instance,

$$\mathrm{aaba} \to \mathrm{aaab} \to \mathrm{baaa} \to \mathrm{abaa} \to \mathrm{aaba}. \tag{3.2.2}$$

This rewrite system is not terminating but, since for each word $u \in A^*$ there is at most a word $v \in A^*$ satisfying $u \to v$, $(A^*, \to)$ is confluent.

3.2.2. *A second rewrite system on words.* Let us now study the graded rewrite system $(A^*, \to)$ defined by $\mathrm{aba} \to \mathrm{bab}$ where A is the alphabet of the previous example. Consider the graded complete ternary product $\star : A^* \times A^* \times A^* \to A^*$ on A^* defined for any $u, v, w \in A^*$ by $\star(u, v, w) := u \cdot v \cdot w$ where $\cdot$ is the concatenation product of words. Let $\mathcal{P} := \{\star\}$ and $(A^*, \to_{\mathcal{P}})$ be the $\mathcal{P}$-closure of $(A^*, \to)$. By definition of closures, $\to_{\mathcal{P}}$ satisfies

$$\mathrm{aba} \to_{\mathcal{P}} \mathrm{bab} \tag{3.2.3}$$

and

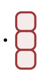
$$\mathrm{aba} \cdot v \cdot w \to_{\mathcal{P}} \mathrm{bab} \cdot v \cdot w, \qquad u \cdot \mathrm{aba} \cdot w \to_{\mathcal{P}} u \cdot \mathrm{bab} \cdot w, \qquad u \cdot v \cdot \mathrm{aba} \to_{\mathcal{P}} u \cdot v \cdot \mathrm{bab}, \tag{3.2.4}$$

for any words u, v, and w on A. All this is equivalent to the fact that $\to_{\mathcal{P}}$ is the rewrite rule satisfying

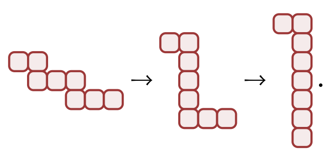
$$u \cdot \mathrm{aba} \cdot w \to_{\mathcal{P}} u \cdot \mathrm{bab} \cdot w, \tag{3.2.5}$$

for any words u and w on A. The rewrite system $(A^*, \to_{\mathcal{P}})$ is terminating since, for any words u and v on A, if $u \to_{\mathcal{P}} v$, then $|v|_{\mathrm{b}} = |u|_{\mathrm{b}} + 1$. Hence, the map $\theta : A^n \to [0, n]$ defined for any $n \in \mathbb{N}$ and $u \in A^n$ by $\theta(u) := |u|_{\mathrm{b}}$ is a termination invariant. The normal forms of $(A^*, \to_{\mathcal{P}})$ are the words that do not admit aba as factor. Moreover, $(A^*, \to_{\mathcal{P}})$ is not confluent since $\mathrm{ababa} \to_{\mathcal{P}} \mathrm{babba}$ and $\mathrm{ababa} \to_{\mathcal{P}} \mathrm{abbab}$, and $\{\mathrm{babba}, \mathrm{abbab}\}$ is a non-joinable branching pair for ababa (because these two elements are normal forms).

3.2.3. *A rewrite system on compositions.* Let the graded rewrite system $(\mathfrak{Com}, \to)$ defined, by seeing compositions through their ribbon diagrams, by

$\boldsymbol{\lambda} \cdot$  $\cdot\, \boldsymbol{\mu} \to \boldsymbol{\lambda} \cdot$  $\cdot\, \boldsymbol{\mu}$, (3.2.6)

where $\cdot$ is the concatenation of the compositions (seen as words of integers) and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are any compositions. We have, for instance,

 (3.2.7)

The rewrite system $(\mathfrak{Com}, \to)$ is terminating since, for any compositions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, if $\boldsymbol{\lambda} \to \boldsymbol{\mu}$, then $\boldsymbol{\mu} \prec \boldsymbol{\lambda}$ where $\preccurlyeq$ is the refinement order of compositions (see Section 2.2.1).

Bibliographic notes

About collections. Our exposition about combinatorial objects through combinatorial collections is very elementary in the sense that it requires a very small amount of mathematical knowledge. However, combinatorial collections form a general and powerful tool to work with algebraic structures involving combinatorial objects. For instance, graded collections appear in the context of operads (see Section 1.2 of Chapter 4) or graded associative algebras (see Section 3.1 of Chapter 3), colored collections appear in the context of colored operads (see Section 2.1 of Chapter 5), cyclic collections appear in the context of cyclic operads (see Section 2.2 of Chapter 5), symmetric collections appear in the context of symmetric operads (see Section 2.3 of Chapter 5), and 2-graded collections appear in the context of pros (see Section 3 of Chapter 5). There are other sensible tools to encode combinatorial sets. Flajolet and Sedgewick provided a complete description of what we call combinatorial graded collections under the name of combinatorial classes in [**FS09**], as a prelude for a conspectus of the field of analytic combinatorics. The proofs of most of the properties about generating series of Section 1.2 can be found here. The nice translations of most of the combinatorial operations involving combinatorial sets as algebraic operations on their generating series, together with its simplicity, are one of the main pros of this theory. By shifting in the world of labeled objects, it is relevant to work with species of structures, that are roughly speaking combinatorial graded collections C with an action of the symmetric group $\mathfrak{S}(n)$ on each $C(n)$ which can be thought as a relabeling action. In this context, it is more accurate to work with exponential generating series, instead of ordinary ones when we consider such combinatorial graded collections. This theory has been introduced by Joyal [**Joy81**] and developed afterwards by the Quebec school of combinatorics [**BLL98**,**BLL13**]. Species of structures are very good candidates to work with symmetric operads [**Mén15**] since the action of the symmetric group of a symmetric operad is encapsulated into the action of the symmetric group of an underlying species of structure. In this book, to work with symmetric operads, we shall consider symmetric collections. An other interesting way to describe combinatorial objects passes through polynomial functors [**Koc09**].

About the Tamari poset. The Tamari poset is a combinatorial poset on binary trees introduced in the study of nonassociative operations [**Tam62**]. Indeed, the covering relation generating this poset can be thought as a way to move brackets in expressions where a nonassociative product intervenes. Moreover, seen on binary trees, this operation translates as a right rotation, a fundamental operation on binary search trees, used in an algorithmic context [**Knu98**]. This operation is used to maintain binary trees with a small height in order to access efficiently, from the roots, to their internal nodes. Some of these trees are known as balanced binary trees [**AVL62**] and form efficient structures to represent dynamic sets (sets supporting the addition and the suppression of elements). A lot of properties of the Tamari poset are known, like the number of intervals of each of its n-subposets [**Cha06**] (equivalently, this is the number of pairs of comparable trees enumerated by their size), and the fact that these posets are lattices [**HT72**], for all $n \in \mathbb{N}$. Generalizations of this poset have been introduced by Bergeron and Préville-Ratelle [**BPR12**] under the name of m-Tamari poset. This poset is defined on the combinatorial collection of all $m+1$-ary trees (see Section 1.2.2 of Chapter 2). The number of intervals of each of its n-subposets, and the fact that these posets are lattices are known from [**BMFPR11**], for all $n \in \mathbb{N}$.

About the right weak poset. The right weak poset of permutations is, like the Tamari poset, also a lattice [GR63,YO69]. In a surprising way, despite its apparent simplicity, there is no known description of the number of intervals of each n-subposet, $n \in \mathbb{N}$, of the right weak poset. Some other combinatorial poset structures exist on $\mathfrak{S}$ like the Bruhat order, whose generating relation is similar to the one of the right weak poset. The definition of the Bruhat order on permutations comes from the general notion of Bruhat order [Bjö84] in Coxeter groups [Cox34]. As a last noteworthy fact, the cube, the Tamari, and the right weak posets are linked through surjective morphisms of combinatorial posets [LR02]. Indeed, a map between the right weak poset to the Tamari poset is based upon the binary search tree insertion algorithm [Knu98,HNT05]. This algorithm consists in inserting the letters of a permutation to form step by step a binary tree. Moreover, a map between the Tamari poset to the cube poset uses the canopies [LR98] of the binary trees. The canopy of a binary tree is a binary word encoding the orientations (to the left or to the right) of its leaves.

About rewrite systems. A general reference about rewrite rules and rewrite systems is [BN98]. In this text, a general method using maps called measure functions to show that (not necessarily combinatorial) rewrite systems are terminating is presented. Besides, Theorem 3.1.2 is a highly important result in the theory of rewrite systems, known as the diamond lemma, and is due to Newman [New42]. There are some additional useful tools in this theory like the Knuth-Bendix completion algorithm [KB70]. This semi-algorithm takes as input a non-confluent rewrite system and outputs, if possible, a confluent one having the same reflexive, symmetric, and transitive closures. In an algebraic context, the Knuth-Bendix completion algorithm leads to the Buchberger algorithm [Buc76]. This algorithm computes Gröbner bases from polynomial ideals.

CHAPTER 2

Treelike structures

This second chapter is devoted to present general notions about treelike structures. We present more precisely the ones appearing in the algebraic and combinatorial context of nonsymmetric operads. Rewrite systems of syntax trees are exposed, as well as methods to prove their termination and their confluence.

1. Planar rooted trees

Let us start with our prototypical treelike structures, the planar rooted trees. Most of the treelike structures we shall consider in this book are variants or enrichments of planar rooted trees.

1.1. Collection of planar rooted trees.

The combinatorial graded collection of the planar rooted trees can be defined concisely in a recursive way by using some operations over combinatorial graded collections (see Section 1.2 of Chapter 1). However, to define rigorously the usual notions of internal node, leaf, child, father, path, subtree, *etc.*, we need the notion of language associated with a tree. Indeed, a planar rooted tree is in fact a finite language satisfying some properties. Therefore, in this section, we shall adopt the point of view of defining most of the properties of a planar rooted tree through its language.

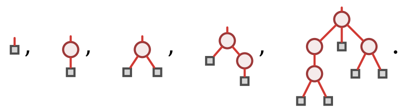
1.1.1. *First definitions.* Let $\mathfrak{PRT}$ be the graded collection satisfying the relation

$$\mathfrak{PRT} = \left[\{\bullet\}, \mathbf{List}^{+} (\mathfrak{PRT}) \right]_{\times}^{+} \tag{1.1.1}$$

where $\bullet$ is an atomic object called *node*. Since $\mathfrak{PRT}(0) = \emptyset$, this collection is combinatorial. We call *planar rooted tree* each object of $\mathfrak{PRT}$. By definition, a planar rooted tree $\mathfrak{t}$ is an ordered pair $(\bullet, (\mathfrak{t}_1, \dots, \mathfrak{t}_k))$, $k \in \mathbb{N}$, where $(\mathfrak{t}_1, \dots, \mathfrak{t}_k)$ is a (possibly empty) tuple of planar rooted trees. This definition is recursive. By convention, the planar rooted tree $(\bullet, ())$ is denoted by $\perp$ and is called the *leaf*. Observe that the leaf is of size 1. For instance,

$$\perp, \quad (\bullet, (\perp)), \quad (\bullet, (\perp, \perp)), \quad (\bullet, (\perp, (\bullet, (\perp)))), \quad (\bullet, ((\bullet, ((\bullet, (\perp, \perp)))), \perp, (\bullet, (\perp, \perp)))) \tag{1.1.2}$$

are planar rooted trees. The *root arity* of a planar rooted tree $\mathfrak{t} := (\bullet, (\mathfrak{t}_1, \dots, \mathfrak{t}_k))$ is k. If $\mathfrak{t}$ is a planar rooted tree different from the leaf, by definition, $\mathfrak{t}$ can be expressed as $\mathfrak{t} = (\bullet, (\mathfrak{t}_1, \dots, \mathfrak{t}_k))$ where $k \in \mathbb{N}_{\geqslant 1}$ and all $\mathfrak{t}_i$, $i \in [k]$, are planar rooted trees. In this case, for any $i \in [k]$, $\mathfrak{t}_i$ is the *ith suffix subtree* of $\mathfrak{t}$. Planar rooted trees are depicted by drawing each leaf by and each planar rooted tree different from the leaf by a node attached below it, from left to right, to its suffix subtrees $\mathfrak{t}_1, \dots, \mathfrak{t}_k$ by means of edges. For instance, the planar rooted trees of (1.1.2) are depicted by



(1.1.3)

By definition of the Cartesian product and the list collection operations over graded collections (see Sections 1.2.3 and 1.2.5 of Chapter 1), the size of a planar rooted tree $\mathfrak{t}$ having a root arity of k ssatisfies

$$|\mathfrak{t}| = 1 + \sum_{i \in [k]} |\mathfrak{t}_i|. \tag{1.1.4}$$

In other words, the size of $\mathfrak{t}$ is the number of occurrences of $\bullet$ it contains. We also deduce from (1.1.1) that the generating series of $\mathfrak{PRT}$ satisfies

$$\mathbb{G}_{\mathfrak{PRT}}(t) = \frac{t}{1 - \mathbb{G}_{\mathfrak{PRT}}(t)} \tag{1.1.5}$$

so that it satisfies the quadratic algebraic equation

$$t - \mathbb{G}_{\mathfrak{PRT}}(t) + \mathbb{G}_{\mathfrak{PRT}}(t)^2 = 0. \tag{1.1.6}$$

1.1.2. *Induction and structural induction.* One among the most obvious techniques to prove that all the planar rooted trees of a subcollection C of $\mathfrak{PRT}$ satisfy a predicate $P(\mathfrak{t})$ (that is, a statement involving a variable $\mathfrak{t}$ taking value in C) consists in performing a proof by induction on the size of the trees of C.

There is another method which is in some cases much more elegant than this approach, called *structural induction* on trees. A subcollection C of $\mathfrak{PRT}$ is *inductive* if C is nonempty and, if $\mathfrak{t} \in C$, all suffix subtrees $\mathfrak{t}_i$ of $\mathfrak{t}$ belong to C. Observe in particular that $\perp$ belongs to any inductive subcollection of $\mathfrak{PRT}$.

Theorem 1.1.1. *Let C be an inductive subcollection of $\mathfrak{PRT}$ and $P(\mathfrak{t})$ be a predicate on C. If*

(i) the statement $P(\perp)$ holds;
(ii) for any $\mathfrak{t}_1, \dots, \mathfrak{t}_k \in C$ such that $\mathfrak{t} := (\bullet, (\mathfrak{t}_1, \dots, \mathfrak{t}_k))$ belongs to C, the fact that all $P(\mathfrak{t}_i)$, $i \in [k]$, hold implies that $P(\mathfrak{t})$ holds;

then, all the planar rooted trees $\mathfrak{s}$ of C satisfy $P(\mathfrak{s})$.

Theorem 1.1.1 provides a powerful tool to prove properties $P(\mathfrak{t})$ of planar rooted trees belonging to inductive combinatorial subsets C. In practice, to perform a structural induction in order to show that all the objects $\mathfrak{t}$ of C satisfy $P(\mathfrak{t})$, we check that C is inductive and that Properties *(i)* and *(ii)* of Theorem 1.1.1 hold.

1.1.3. *Tree languages.* To rigorously specify nodes in planar rooted trees, we shall use a useful interpretation of planar rooted trees as special languages on the alphabet $\mathbb{N}_{\geqslant 1}$. Recall that a partial right monoid action of a monoid A^* of words (endowed with the concatenation product $\cdot$) on a set S is a map $\bullet : S \times A^* \to S$ satisfying $x \bullet \epsilon = x$, and for any $x \in S$, $u \in A^*$, and $a \in A$, $x \bullet ua$ is defined if and only if $(x \bullet u) \bullet a$ is defined, and these two elements are the same when they are defined. Let

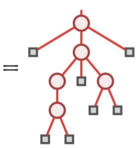
$$\bullet : \mathfrak{PRT} \times \mathbb{N}^*_{\geqslant 1} \to \mathfrak{PRT} \tag{1.1.7}$$

be the right partial monoid action defined recursively by

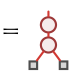
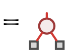
$$(\bullet, (\mathfrak{t}_1, \dots, \mathfrak{t}_k)) \bullet u := \begin{cases} (\bullet, (\mathfrak{t}_1, \dots, \mathfrak{t}_k)) & \text{if } u = \epsilon, \\ \mathfrak{t}_i \bullet v & \text{otherwise } (u = iv \text{ where } v \in \mathbb{N}^*_{\geqslant 1} \text{ and } i \in \mathbb{N}_{\geqslant 1}), \end{cases} \tag{1.1.8}$$

for any $(\bullet, (\mathfrak{t}_1, \dots, \mathfrak{t}_k)) \in \mathfrak{PRT}$ and $u \in \mathbb{N}^*_{\geqslant 1}$. Observe that this action is partial since each $\mathfrak{t}_i$ in (1.1.8) is well-defined only if i is no greater than the root arity of $\mathfrak{t}$. The *tree language* $\mathcal{N}(\mathfrak{t})$ of $\mathfrak{t}$ is the finite language on $\mathbb{N}_{\geqslant 1}$ of all the words u such that $\mathfrak{t} \bullet u$ is a well-defined planar rooted tree.

For instance, by setting

$\mathfrak{t} :=$  , (1.1.9)

we have

$\mathfrak{t} \bullet 1 =$ , $\mathfrak{t} \bullet 231 =$ , $\mathfrak{t} \bullet 3 =$ , $\mathfrak{t} \bullet 21 =$ , $\mathfrak{t} \bullet 23 =$ , (1.1.10)

and, among others, the actions of the words 11, 24, and 2321 on $\mathfrak{t}$ are all undefined. Moreover, the tree language of $\mathfrak{t}$ is

$$\mathcal{N}(\mathfrak{t}) = \{\epsilon, 1, 2, 21, 211, 2111, 2112, 22, 23, 231, 232, 3\}. \tag{1.1.11}$$

Let $\mathcal{L}_{\mathfrak{PRT}}$ be the combinatorial graded collection of all finite and nonempty prefix languages $\mathcal{L}$ on $\mathbb{N}_{\geqslant 1}$ such that if $ui \in \mathcal{L}$ where $u \in \mathbb{N}^*_{\geqslant 1}$ and $i \in \mathbb{N}_{\geqslant 2}$, $ui' \in \mathcal{L}$ where $i' := i - 1$. The size of such a language is its cardinality. For instance, the set $\mathcal{N}(\mathfrak{t})$ of (1.1.11) is an object of size 12 of $\mathcal{L}_{\mathfrak{PRT}}$, and $\{\epsilon, 1, 11, 12, 2\}$ is an object of size 5.

Proposition 1.1.2. *The combinatorial graded collections $\mathfrak{PRT}$ and $\mathcal{L}_{\mathfrak{PRT}}$ are isomorphic. Seen as a morphism of combinatorial collections $\mathcal{N} : \mathfrak{PRT} \to \mathcal{L}_{\mathfrak{PRT}}$, $\mathcal{N}$ is an isomorphism between these two collections.*

Proposition 1.1.2 is used in practice to define planar rooted trees through their languages. This will be useful later when operations on planar rooted trees will be described.

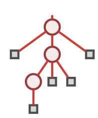
1.1.4. *Additional definitions.* Let $\mathfrak{t}$ be a planar rooted tree. We say that each word of $\mathcal{N}(\mathfrak{t})$ is a *node* of $\mathfrak{t}$. A node u of $\mathfrak{t}$ is an *internal node* if there is an $i \in \mathbb{N}_{\geqslant 1}$ such that ui is a node of $\mathfrak{t}$. A node u of $\mathfrak{t}$ which is not an internal node is a *leaf*. The set of all internal nodes (resp. leaves) of $\mathfrak{t}$ is denoted by $\mathcal{N}_{\bullet}(\mathfrak{t})$ (resp. $\mathcal{N}_{\perp}(\mathfrak{t})$). The *root* of $\mathfrak{t}$ is the node ϵ (which can be either an internal node or a leaf). The *degree* $\mathrm{deg}(\mathfrak{t})$ of $\mathfrak{t}$ is $\# \mathcal{N}_{\bullet}(\mathfrak{t})$ and the *arity* $\mathrm{ari}(\mathfrak{t})$ of $\mathfrak{t}$ is $\# \mathcal{N}_{\perp}(\mathfrak{t})$. A node u of $\mathfrak{t}$ is an *ancestor* of a node v of $\mathfrak{t}$ if $u \neq v$ and $u \preccurlyeq_{\mathrm{p}} v$. Moreover, for any $i \in \mathbb{N}_{\geqslant 1}$, a node v is the *ith child* of a node u if $v = ui$. In this case, u is the (unique) *father* of v. The *arity* of a node is the number of children it has. The lexicographic order on the words of $\mathcal{N}(\mathfrak{t})$ induces a total order on the nodes of $\mathfrak{t}$ called *depth-first order*. The *ith leaf* of $\mathfrak{t}$ is the ith leaf encountered by considering the nodes of $\mathfrak{t}$ according to the depth-first order. A *path* in $\mathfrak{t}$ is a sequence $(u_1, \dots, u_k)$ of nodes of $\mathfrak{t}$ such that for any $j \in [k-1]$, u_j is the father of u_{j+1}. Such a path is *maximal* if u_1 is the root of $\mathfrak{t}$ and u_k is a leaf. The *length* of a path is the number of nodes it contains. The *height* $\mathrm{ht}(\mathfrak{t})$ of $\mathfrak{t}$ is the maximal length of its maximal paths minus 1. This is also the length of a longest word of $\mathcal{N}(\mathfrak{t})$ minus 1. For any node u of $\mathfrak{t}$, the planar rooted tree $\mathfrak{t} \bullet u$ is the *suffix subtree* of $\mathfrak{t}$ rooted at u. By extension, the *ith suffix subtree* of u is the planar rooted tree $\mathfrak{t} \bullet ui$ when i is no greater than the arity of u. A planar rooted tree $\mathfrak{s}$ is a *prefix subtree* of $\mathfrak{t}$ if $\mathcal{N}(\mathfrak{s}) \subseteq \mathcal{N}(\mathfrak{t})$. A planar rooted tree $\mathfrak{s}$ is a *factor subtree* of $\mathfrak{t}$ rooted at a node u if $\mathfrak{s}$ is a prefix subtree of a suffix subtree of $\mathfrak{t}$ rooted at u.

Let us provide some examples for these notions. Consider the planar rooted tree $\mathfrak{t}$ of (1.1.9). Then,

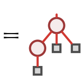
$$\mathcal{N}_{\bullet}(\mathfrak{t}) = \{\epsilon, 2, 21, 211, 23\}, \tag{1.1.12a}$$

$$\mathcal{N}_{\perp}(\mathfrak{t}) = \{1, 2111, 2112, 22, 231, 232, 3\}, \tag{1.1.12b}$$

so that $\mathrm{deg}(\mathfrak{t}) = 5$ and $\mathrm{ari}(\mathfrak{t}) = 7$. The 3rd leaf of $\mathfrak{t}$ is 2112, and the 2nd child of the internal node 23 of $\mathfrak{t}$ is 232 and is a leaf. Besides, the sequences $(\epsilon, 2, 21)$ and $(\epsilon, 2, 23)$ are non-maximal paths in $\mathfrak{t}$, and on the contrary, the paths $(\epsilon, 1)$, $(\epsilon, 2, 21, 211, 2112)$, and $(\epsilon, 2, 22)$ are maximal. The maximal path $(\epsilon, 2, 21, 211, 2112)$ have a maximal length among all maximal paths of $\mathfrak{t}$ and thus, the height of $\mathfrak{t}$ is 4. Finally, the planar rooted tree

$\mathfrak{s} :=$  (1.1.13)

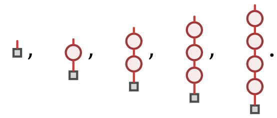
is a prefix subtree of $\mathfrak{t}$, and, the planar rooted tree

$\mathfrak{r} :=$  , (1.1.14)

being a suffix subtree of $\mathfrak{s}$ rooted at the node 2, is a factor subtree of $\mathfrak{t}$ rooted at the node 2.

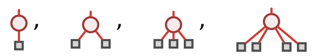
1.2. Subcollections of planar rooted trees. By basically restraining the possible arities of the internal nodes of planar rooted trees, we obtain several subcollections of $\mathfrak{PRT}$. We review here the families formed by ladders, corollas, k-ary trees, and Schröder trees. Besides, among these families, some admit alternative size functions (and form therefore different combinatorial graded collections).

1.2.1. *Ladders and corollas.* A *ladder* is a planar rooted tree of arity 1. The first ladders are



(1.2.1)

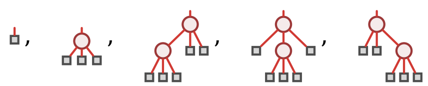
This set of ladders forms a subcollection $\mathfrak{Lad}$ of $\mathfrak{PRT}$. Besides, a *corolla* is a planar rooted tree of degree 1. The first corollas are



(1.2.2)

This set of corollas forms a subcollection $\mathfrak{Cor}$ of $\mathfrak{PRT}$. Observe that $(\bullet, (\perp))$ is the only planar rooted that is both a ladder and a corolla.

1.2.2. *k-ary trees.* Let $k \in \mathbb{N}_{\geqslant 1}$. A *$k$-ary* tree is a planar rooted tree $\mathfrak{t}$ such that all internal nodes are of arity k. For instance, the first 3-ary trees are



(1.2.3)

This set of k-ary trees forms a subcollection $\mathfrak{Ary}^{(k)}$ of $\mathfrak{PRT}$ expressing recursively as

$$\mathfrak{Ary}^{(k)} = \{\perp\} + \left[\{\bullet\}, \mathbf{List}^{+}_{\{k\}}\left(\mathfrak{Ary}^{(k)}\right)\right]^{+}_{\times}, \tag{1.2.4}$$

where $\perp$ and $\bullet$ are both atomic. One can immediately observe that $\mathfrak{Ary}^{(1)} = \mathfrak{Lad}$.

By structural induction (see Theorem 1.1.1) on $\mathfrak{Ary}^{(k)}$ (which is an inductive subcollection of $\mathfrak{PRT}$), it follows that for any k-ary tree $\mathfrak{t}$, the arity and the degree of $\mathfrak{t}$ are related by

$$\mathrm{ari}(\mathfrak{t}) - \mathrm{deg}(\mathfrak{t})(k-1) = 1. \tag{1.2.5}$$

This implies that a k-ary tree of a given arity has an imposed degree and conversely, a k-ary tree of a given degree has an imposed arity. Hence, since the size of a k-ary tree $\mathfrak{t}$ is $\mathrm{ari}(\mathfrak{t}) + \mathrm{deg}(\mathfrak{t})$ and there are finitely many planar rooted trees of a fixed size, there are finitely many k-ary trees of a fixed arity, and there are finitely many k-ary trees of a fixed degree. As a consequence, the graded collections $\mathfrak{Ary}^{(k)}_{\perp}$ and $\mathfrak{Ary}^{(k)}_{\bullet}$ of all k-ary trees such that the size of a tree of $\mathfrak{Ary}^{(k)}_{\perp}$ is its arity and the size of a tree of $\mathfrak{Ary}^{(k)}_{\bullet}$ is its degree are combinatorial. On the one hand, the generating series of $\mathfrak{Ary}^{(k)}_{\perp}$ satisfies the algebraic equation

$$t - \mathbb{G}_{\mathfrak{Ary}^{(k)}_{\perp}}(t) + \mathbb{G}_{\mathfrak{Ary}^{(k)}_{\perp}}(t)^{k} = 0. \tag{1.2.6}$$

On the other hand, the generating series of $\mathfrak{Ary}_\bullet^{(k)}$ satisfies the algebraic equation

$$1 - \mathbb{G}_{\mathfrak{Ary}_\bullet^{(k)}}(t) + t\mathbb{G}_{\mathfrak{Ary}_\bullet^{(k)}}(t)^k = 0 \tag{1.2.7}$$

and one can deduce that

$$\#\mathfrak{Ary}_\bullet^{(k)}(n) = \frac{1}{(k-1)n+1}\binom{kn}{n}. \tag{1.2.8}$$

For instance, the integer sequences of $\mathfrak{Ary}_\bullet^{(k)}$ begin with

$$1, 1, 1, 1, 1, 1, 1, 1, \qquad k = 1, \tag{1.2.9a}$$

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \qquad k = 2, \tag{1.2.9b}$$

$$1, 1, 3, 12, 55, 273, 1428, 7752, 43263, \qquad k = 3, \tag{1.2.9c}$$

$$1, 1, 4, 22, 140, 969, 7084, 53820, 420732, \qquad k = 4. \tag{1.2.9d}$$

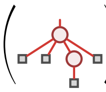
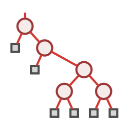
The second, third, and fourth sequences above are, respectively, Sequences **A000108**, **A001764**, and **A002293** of [**Slo**]. These are known as the *Fuss-Catalan numbers*.

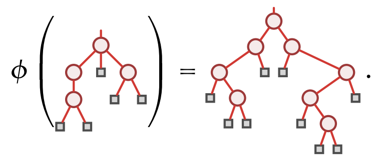
From now on, we call *binary tree* any 2-ary tree. Recall that these objects have been introduced in Section 1.3.6 of Chapter 1. If $\mathfrak{t}$ is a binary tree and u is an internal node of $\mathfrak{t}$, $u1$ and $u2$ are nodes of $\mathfrak{t}$. We call $u1$ (resp. $u2$) the *left* (resp. *right*) *child* of u, and $\mathfrak{t} \bullet u1$ (resp. $\mathfrak{t} \bullet u2$) the *left* (resp. *right*) *subtree* of u in $\mathfrak{t}$. The left (resp. right) subtree of $\mathfrak{t}$ is the *left* (resp. *right*) *subtree* of the root of $\mathfrak{t}$. Besides, a *left* (resp. *right*) *comb tree* is a binary tree $\mathfrak{t}$ such that for all internal nodes u of $\mathfrak{t}$, all right (resp. left) subtrees of u are leaves. The *infix order* induced by $\mathfrak{t}$ is the total order on the set of its internal nodes defined recursively by setting that all the internal nodes of $\mathfrak{t} \bullet 1$ are smaller than the root of $\mathfrak{t}$, and that the root of $\mathfrak{t}$ is smaller than all the internal nodes of $\mathfrak{t} \bullet 2$.

Let us denote by $\mathfrak{BT}_\perp$ the combinatorial graded collection $\mathfrak{Ary}_\perp^{(2)}$ of binary trees where the size of a tree is its arity. As a consequence of (1.1.6) and (1.2.6), we observe that the generating series of $\mathfrak{PRT}$ satisfies the same algebraic relation as the one of $\mathfrak{BT}_\perp$. Therefore, $\mathfrak{PRT}$ and $\mathfrak{BT}_\perp$ are isomorphic as graded collections. Let us describe an explicit isomorphism between these two collections. Let $\phi : \mathfrak{PRT} \to \mathfrak{BT}_\perp$ be the map recursively defined, for any planar rooted tree $\mathfrak{t}$, by

$$\phi(\mathfrak{t}) := \begin{cases} \perp \in \mathfrak{BT}_\perp & \text{if } \mathfrak{t} = \perp, \\ (\bullet, (\phi(\mathfrak{t}_1), \phi((\bullet, (\mathfrak{t}_2, \dots, \mathfrak{t}_k))))) & \text{otherwise } (\mathfrak{t} = (\bullet, (\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_k)) \text{ with } k \in \mathbb{N}_{\geqslant 1}). \end{cases} \tag{1.2.10}$$

One has, for instance,

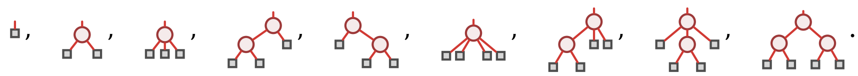
$\phi\Bigg($  $\Bigg) =$  , (1.2.11a)



(1.2.11b)

Proposition 1.2.1. *The combinatorial graded collections $\mathfrak{PRT}$ and $\mathfrak{BT}_\perp$ are isomorphic. The map ϕ defined by* (1.2.10) *is an isomorphism between these two collections.*

1.2.3. *Schröder trees.* A *Schröder tree* is a planar rooted tree such that all internal nodes are of arities 2 or more. Some among the first Schröder trees are



(1.2.12)

This set of Schröder trees forms a subcollection $\mathfrak{Sch}$ of $\mathfrak{PRT}$ expressing recursively as

$$\mathfrak{Sch} = \{\perp\} + \left[\{\bullet\}, \mathbf{List}^{+}_{\mathbb{N}_{\geqslant 2}}(\mathfrak{Sch}) \right]^{+}_{\times}, \tag{1.2.13}$$

where $\perp$ and $\bullet$ are both atomic.

By structural induction on $\mathfrak{Sch}$ (which is an inductive subcollection of $\mathfrak{PRT}$), it follows that there are finitely many Schröder trees of a given arity n. For this reason, the graded collection $\mathfrak{Sch}_\perp$ of all the Schröder trees such that the size of a tree of $\mathfrak{Sch}_\perp$ is its arity is combinatorial. Conversely, considering the degrees of the trees for their sizes does not form a combinatorial graded collection since there are infinitely many Schröder trees of degree 1 (the corollas). The generating series of $\mathfrak{Sch}_\perp$ satisfies the algebraic quadratic equation

$$t - (1+t)\mathbb{G}_{\mathfrak{Sch}_\perp}(t) + 2\mathbb{G}_{\mathfrak{Sch}_\perp}(t)^2 = 0. \tag{1.2.14}$$

Let $\mathrm{nar}(n, k)$ be the number of binary trees of arity n having exactly k internal nodes having an internal node as a left child. Then, for all $0 \leqslant k \leqslant n-2$, it is known that

$$\mathrm{nar}(n, k) = \frac{1}{k+1} \binom{n-2}{k} \binom{n-1}{k}. \tag{1.2.15}$$

These are *Narayana numbers*. The cardinalities of the sets $\mathfrak{Sch}_\perp(n)$ express by

$$\#\mathfrak{Sch}_\perp(n) = \sum_{k \in [0, n-2]} 2^k \, \mathrm{nar}(n, k), \tag{1.2.16}$$

for all $n \in \mathbb{N}_{\geqslant 2}$. The integer sequence of $\mathfrak{Sch}_\perp$ begins by

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793 \tag{1.2.17}$$

and forms Sequence A001003 of [Slo].

2. Syntax trees

We are now in position to introduce syntax trees and rewrite systems on syntax trees. These objects are central in the theory of operads since the elements of free nonsymmetric operads can be seen as syntax trees. Rewrite systems on syntax trees provide tools to establish presentations by generators and relations of operads.

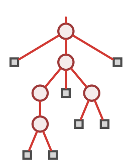
2.1. Collections of syntax trees. Syntax trees are, roughly speaking, planar rooted trees where internal nodes are labeled by objects of a fixed graded collection. These trees can be endowed with two size functions (where the size is the degree or the arity), leading to the definition of two distinct graded collections of syntax trees.

2.1.1. *Main definitions.* Let C be an augmented graded collection. A *syntax tree* on C (or, for short, a *C-syntax tree*) is a planar rooted tree $\mathfrak{t}$ endowed with a map

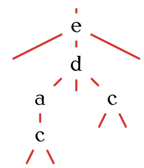
$$\omega_{\mathfrak{t}} : \mathcal{N}_{\bullet}(\mathfrak{t}) \to C \tag{2.1.1}$$

sending each internal node u of $\mathfrak{t}$ of arity k to an object of size k of C. This map $\omega_{\mathfrak{t}}$ is the *labeling map* of $\mathfrak{t}$. We say that an internal node u of $\mathfrak{t}$ is *labeled* by $x \in C$ if $\omega_{\mathfrak{t}}(u) = x$. The collection C is the *labeling collection* of $\mathfrak{t}$. The *underlying planar rooted tree* of $\mathfrak{t}$ is the planar rooted tree obtained by forgetting the map $\omega_{\mathfrak{t}}$. For any $x \in C$, the *corolla* labeled by x is the C-syntax tree $\mathfrak{c}(x)$ having exactly one internal node labeled by x and with $|x|$ leaves as children. All the notions about planar rooted trees defined in Section 1 apply to C-syntax trees as well. More precisely, for any property $P(\mathfrak{s})$ on a planar rooted tree $\mathfrak{s}$, we say that $P(\mathfrak{t})$ holds if the underlying planar rooted tree $\mathfrak{s}$ of $\mathfrak{t}$ is such that $P(\mathfrak{s})$ holds. Moreover, the notions of suffix, prefix, and factor subtrees of planar rooted trees naturally extend on C-syntax trees by taking into account the labeling maps. In graphical representations of a C-syntax tree $\mathfrak{t}$, instead of drawing each internal node u of $\mathfrak{t}$ by $\circ$, we draw u by its label $\omega_{\mathfrak{t}}(u)$.

For instance, consider the labeling collection $C := C(1) \sqcup C(2) \sqcup C(3)$ where $C(1) := \{\mathrm{a}, \mathrm{b}\}$, $C(2) := \{\mathrm{c}\}$, and $C(3) := \{\mathrm{d}, \mathrm{e}\}$, and the planar rooted tree

$\mathfrak{t} :=$  . (2.1.2)

By endowing $\mathfrak{t}$ with the labeling map defined by $\omega_{\mathfrak{t}}(\epsilon) := \mathrm{e}$, $\omega_{\mathfrak{t}}(2) := \mathrm{d}$, $\omega_{\mathfrak{t}}(21) := \mathrm{a}$, $\omega_{\mathfrak{t}}(211) := \mathrm{c}$, and $\omega_{\mathfrak{t}}(23) := \mathrm{c}$, $\mathfrak{t}$ is a C-syntax tree. This C-syntax tree is depicted more concisely as



. (2.1.3)

We denote by $\mathfrak{S}\mathfrak{T}^C$ the graded collection of all the C-syntax trees, where the size of a C-syntax tree $\mathfrak{t}$ is the size of its underlying planar rooted tree in $\mathfrak{P}\mathfrak{R}\mathfrak{T}$. When C is additionally combinatorial, by structural induction on planar rooted trees, it follows that for any $\mathfrak{t} \in \mathfrak{P}\mathfrak{R}\mathfrak{T}$, there are finitely many labeling maps $\omega_{\mathfrak{t}}$ for $\mathfrak{t}$. For this reason, $\mathfrak{S}\mathfrak{T}^C$ is in this case combinatorial. Besides, let $\mathfrak{Lad}^C$, $\mathfrak{Cor}^C$, $\mathfrak{Ary}^{(k),C}$, and $\mathfrak{Sch}^C$ be, respectively, the subcollections of $\mathfrak{S}\mathfrak{T}^C$ consisting in the C-syntax trees whose underlying planar rooted trees are ladders, corollas, k-ary trees, and Schröder trees. The concepts of inductive subcollections of $\mathfrak{S}\mathfrak{T}^C$ and of structural induction presented in Section 1.1.2 extend obviously on C-syntax trees.

2.1.2. *Alternative definition and generating series.* The graded collection $\mathfrak{S}\mathfrak{T}^C$ can be described as follows. Let $\mathcal{S}^C$ be the graded collection satisfying the relation

$$\mathcal{S}^C = \{\perp\} + \left[\{\bullet\}, C \odot \mathcal{S}^C\right]^{+}_{\times} \tag{2.1.4}$$

where both $\perp$ and $\bullet$ are atomic, and $\odot$ is the composition product over graded collections defined in Section 1.2.9 of Chapter 1. Then, the combinatorial collections $\mathfrak{S}\mathfrak{T}^C$ and $\mathcal{S}^C$ are isomorphic through the morphism $\phi : \mathfrak{S}\mathfrak{T}^C \to \mathcal{S}^C$ of combinatorial collections recursively defined, for any $\mathfrak{t} \in \mathfrak{S}\mathfrak{T}^C$ of root arity k, by

$$\phi(\mathfrak{t}) := \begin{cases} \perp \in \mathcal{S}^C & \text{if } \mathfrak{t} = \perp, \\ (\bullet, (\omega_{\mathfrak{t}}(\epsilon), (\phi\,(\mathfrak{t}_1)\,, \dots, (\mathfrak{t}_k)))) & \text{otherwise.} \end{cases} \tag{2.1.5}$$

From this equivalence and (2.1.4), we obtain, when C is combinatorial, that the generating series of $\mathfrak{S}\mathfrak{T}^C$ satisfies

$$\mathbb{G}_{\mathfrak{S}\mathfrak{T}^C}(t) = t + t\,\mathbb{G}_C\left(\mathbb{G}_{\mathfrak{S}\mathfrak{T}^C}(t)\right), \tag{2.1.6}$$

where $\mathbb{G}_C(t)$ is the generating series of C. For instance, by considering the combinatorial collection C defined above, we have $\mathbb{G}_C(t) = 2t + t^2 + 2t^3$, so that

$$t + (2t-1)\,\mathbb{G}_{\mathfrak{S}\mathfrak{T}^C}(t) + t\,\mathbb{G}_{\mathfrak{S}\mathfrak{T}^C}(t)^2 + 2t\,\mathbb{G}_{\mathfrak{S}\mathfrak{T}^C}(t)^3 = 0. \tag{2.1.7}$$

2.1.3. *Subcollections of syntax trees.* For well-chosen combinatorial augmented graded collections C, it is possible to recover a large part of the families of planar rooted trees described in Section 1.2. Indeed, one has $\mathfrak{S}\mathfrak{T}^{\mathbb{N}_{\geqslant 1}} \simeq \mathfrak{P}\mathfrak{R}\mathfrak{T}$, $\mathfrak{S}\mathfrak{T}^{\mathbb{N}_{\geqslant 2}} \simeq \mathfrak{Sch}$, and, when $\bullet_k$ is an object of size $k \in \mathbb{N}_{\geqslant 1}$, $\mathfrak{S}\mathfrak{T}^{\{\bullet_k\}} \simeq \mathfrak{Ary}^{(k)}$.

2.1.4. *Alternative sizes.* Let $\mathfrak{S}\mathfrak{T}^C_\perp$ be the graded collection of all the C-syntax trees such that the size of a tree is its arity. One has $\mathfrak{S}\mathfrak{T}^C_\perp \simeq \mathcal{S}^C$ where $\mathcal{S}^C$ is the graded collection defined in (2.1.4) wherein $\perp$ is atomic and $\bullet$ is of size 0. When C is graded, combinatorial, augmented, and has no object of size 1, we can show by structural induction on $\mathfrak{S}\mathfrak{T}^C_\perp$ that there are finitely many C-syntax trees of a given arity $n \in \mathbb{N}_{\geqslant 1}$. For this reason, $\mathfrak{S}\mathfrak{T}^C_\perp$ is combinatorial. In this case, the generating series of $\mathfrak{S}\mathfrak{T}^C_\perp$ satisfies

$$\mathbb{G}_{\mathfrak{S}\mathfrak{T}^C_\perp}(t) = t + \mathbb{G}_C\left(\mathbb{G}_{\mathfrak{S}\mathfrak{T}^C_\perp}(t)\right). \tag{2.1.8}$$

Let also $\mathfrak{S}\mathfrak{T}^C_\bullet$ be the graded collection of all the C-syntax trees such that the size of a tree is its degree. One has $\mathfrak{S}\mathfrak{T}^C_\bullet \simeq \mathcal{S}^C$ where $\mathcal{S}^C$ is the graded collection defined in (2.1.4) wherein $\bullet$ is atomic and $\perp$ is of size 0. When C is graded, augmented, and finite, we can show by structural induction on $\mathfrak{S}\mathfrak{T}^C_\bullet$ that there are finitely many C-syntax trees of a given degree n. For this reason, $\mathfrak{S}\mathfrak{T}^C_\bullet$ is combinatorial. In this case, the generating series of $\mathfrak{S}\mathfrak{T}^C_\bullet$ satisfies

$$\mathbb{G}_{\mathfrak{S}\mathfrak{T}^C_\bullet}(t) = 1 + t\,\mathbb{G}_C\left(\mathbb{G}_{\mathfrak{S}\mathfrak{T}^C_\bullet}(t)\right). \tag{2.1.9}$$

Observe that $\mathfrak{S}\mathfrak{T}^C_\bullet$ is not an augmented graded collection.

2.2. Grafting operations. Three fundamental grafting operations on syntax trees are presented here. These operations turn $\mathfrak{S}\mathfrak{T}^C_\perp$ into a collection with concentrated products in the sense of Section 1.1.7 of Chapter 1.

2.2.1. *Partial grafting.* Let for any $n, m \in \mathbb{N}_{\geqslant 1}$ and $i \in [n]$ the product

$$\circ_i^{(n,m)} : \mathfrak{S}\mathfrak{T}^C_\perp(n) \times \mathfrak{S}\mathfrak{T}^C_\perp(m) \to \mathfrak{S}\mathfrak{T}^C_\perp \tag{2.2.1}$$

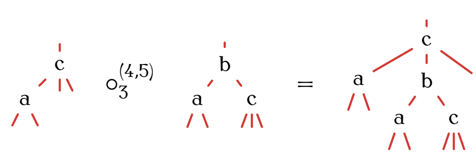
where for any $\mathfrak{t} \in \mathfrak{S}\mathfrak{T}^C_\perp(n)$, $\mathfrak{s} \in \mathfrak{S}\mathfrak{T}^C_\perp(m)$, and $i \in [n]$, the syntax tree $\mathfrak{r} := \mathfrak{t} \circ_i^{(n,m)} \mathfrak{s}$ is defined as follows. The underlying planar rooted tree of $\mathfrak{r}$ admits the tree language

$$\mathcal{N}(\mathfrak{r})s := (\mathcal{N}(\mathfrak{t}) \setminus \{u\}) \cup \{uv : v \in \mathcal{N}(\mathfrak{s})\}, \tag{2.2.2}$$

and the labeling map of $\mathfrak{r}$ satisfies, for any $w \in \mathcal{N}_\bullet(\mathfrak{r})$,

$$\omega_{\mathfrak{r}}(w) := \begin{cases} \omega_{\mathfrak{t}}(w) & \text{if } w \in \mathcal{N}_\bullet(\mathfrak{t}), \\ \omega_{\mathfrak{s}}(v) & \text{otherwise } (w = uv \text{ and } v \in \mathcal{N}_\bullet(\mathfrak{s})). \end{cases} \tag{2.2.3}$$

Observe that by Proposition 1.1.2, $\mathfrak{r}$ is wholly specified by its tree language $\mathcal{N}(\mathfrak{r})$ defined in (2.2.2). In more intuitive terms, the tree $\mathfrak{r}$ is obtained by connecting the root of $\mathfrak{s}$ onto the ith leaf of $\mathfrak{t}$. For instance, by considering the same labeling collection C as above,



. (2.2.4)

We call each $\circ_i^{(n,m)}$ a *partial grafting operation*.

Observe that since

$$\mathrm{ari}\left(\mathfrak{t} \circ_i^{(n,m)} \mathfrak{s}\right) = \mathrm{ari}(\mathfrak{t}) + \mathrm{ari}(\mathfrak{s}) - 1, \tag{2.2.5}$$

the product $\circ_i^{(n,m)}$ is concentrated and is not graded. Besides, by a slight abuse of notation, we shall sometimes omit the (n, m) in the notation of $\circ_i^{(n,m)}$ in order to denote it in a more concise way by $\circ_i$.

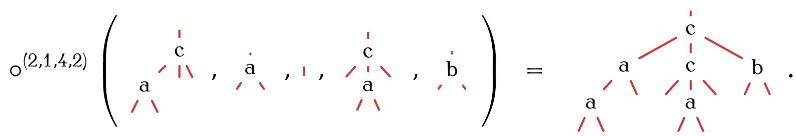
2.2.2. *Complete grafting.* Let for any $n, m_1, \dots, m_n \in \mathbb{N}_{\geqslant 1}$ the product

$$\circ^{(m_1,\dots,m_n)} : \mathfrak{ST}_\perp^C(n) \times \mathfrak{ST}_\perp^C(m_1) \times \dots \times \mathfrak{ST}_\perp^C(m_n) \to \mathfrak{ST}_\perp^C \tag{2.2.6}$$

where for any $\mathfrak{t} \in \mathfrak{ST}_\perp^C(n)$, $\mathfrak{s}_1 \in \mathfrak{ST}_\perp^C(m_1)$, …, $\mathfrak{s}_n \in \mathfrak{ST}_\perp^C(m_n)$,

$$\circ^{(m_1,\dots,m_n)}(\mathfrak{t}, \mathfrak{s}_1, \dots, \mathfrak{s}_n) := (\dots((\mathfrak{t} \circ_n \mathfrak{s}_n) \circ_{n-1} \mathfrak{s}_{n-1}) \dots) \circ_1 \mathfrak{s}_1. \tag{2.2.7}$$

In more intuitive terms, the syntax tree expressed by (2.2.7) is obtained by connecting the root of each $\mathfrak{s}_i$ onto the ith leaf of $\mathfrak{t}$. For instance, by considering the same labeling collection C as before,


(2.2.8)

We call each $\circ^{(m_1,\dots,m_n)}$ a *complete grafting operation*.

Observe that since

$$\mathrm{ari}\left(\circ^{(m_1,\dots,m_n)}(\mathfrak{t}, \mathfrak{s}_1, \dots, \mathfrak{s}_n)\right) = \mathrm{ari}(\mathfrak{s}_1) + \dots + \mathrm{ari}(\mathfrak{s}_n), \tag{2.2.9}$$

the product $\circ^{(m_1,\dots,m_n)}$ is concentrated and is not graded (because the size of the first operand $\mathfrak{t}$ does not intervene in the size of the result). Besides, by a slight abuse of notation, we shall sometimes omit the $(m_1, \dots, m_n)$ in the notation of $\circ^{(m_1,\dots,m_n)}$ in order to denote it in a more concise way by $\circ$. Moreover, we shall denote by $\mathfrak{t} \circ [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$ the C-syntax tree $\circ(\mathfrak{t}, \mathfrak{s}_1, \dots, \mathfrak{s}_n)$.

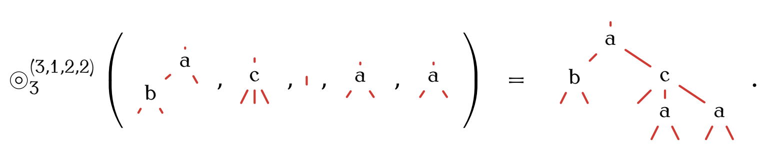
2.2.3. *Context grafting.* Let for any $n, m, k_1, \dots, k_m \in \mathbb{N}_{\geqslant 1}$ and $i \in [n]$ the product

$$\circledcirc_i^{(n,k_1,\dots,k_m)} : \mathfrak{ST}_\perp^C(n) \times \mathfrak{ST}_\perp^C(m) \times \mathfrak{ST}_\perp^C(k_1) \times \dots \times \mathfrak{ST}_\perp^C(k_m) \to \mathfrak{ST}_\perp^C \tag{2.2.10}$$

where for any $\mathfrak{t} \in \mathfrak{ST}_\perp^C(n)$, $\mathfrak{s} \in \mathfrak{ST}_\perp^C(m)$, $\mathfrak{r}_1 \in \mathfrak{ST}_\perp^C(k_1)$, …, $\mathfrak{r}_m \in \mathfrak{ST}_\perp^C(k_m)$,

$$\circledcirc_i(\mathfrak{t}, \mathfrak{s}, \mathfrak{r}_1, \dots, \mathfrak{r}_m) := \mathfrak{t} \circ_i (\mathfrak{s} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_m]). \tag{2.2.11}$$

For instance, by considering the same labeling collection C as before,


(2.2.12)

We call each $\circledcirc_i^{(n,k_1,\dots,k_m)}$ a *context grafting operation*.

Observe that since

$$\operatorname{ari}\left(\circledcirc_i^{(n,k_1,\dots,k_m)}(\mathfrak{t},\mathfrak{s},\mathfrak{r}_1,\dots,\mathfrak{r}_m)\right)=\operatorname{ari}(\mathfrak{t})-1+\operatorname{ari}(\mathfrak{r}_1)+\dots+\operatorname{ari}(\mathfrak{r}_m), \tag{2.2.13}$$

the product $\circledcirc_i^{(n,k_1,\dots,k_m)}$ is concentrated and is not graded. Besides, by a slight abuse of notation, we shall sometimes omit the $(n,k_1,\dots,k_m)$ in the notation of $\circledcirc_i^{(n,k_1,\dots,k_m)}$ in order to denote it in a more concise way by $\circledcirc_i$.

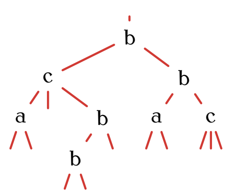
2.3. Patterns and rewrite systems. We focus now on the theory of rewrite systems exposed in Section 3 of Chapter 1 on the particular case of syntax trees. Intuitively, a rewrite rule on syntax trees works by replacing factor subtrees in a syntax tree by other ones. We explain techniques to prove termination and confluence of these particular rewrite systems.

2.3.1. *Occurrence and avoidance of patterns.* Let C be an augmented graded collection, and $\mathfrak{s}$ and $\mathfrak{t}$ be two C-syntax trees. For any node u of $\mathfrak{t}$, $\mathfrak{s}$ *occurs* at position u in $\mathfrak{t}$ if $\mathfrak{s}$ is a factor subtree of $\mathfrak{t}$ rooted at u. In this case, we say that $\mathfrak{t}$ *admits an occurrence* of the *pattern* $\mathfrak{s}$. Conversely, $\mathfrak{t}$ *avoids* $\mathfrak{s}$ if there is no occurrence of $\mathfrak{s}$ in $\mathfrak{t}$.

This property can be rephrased as follows by using the context grafting operations. A syntax tree $\mathfrak{t}$ admits an occurrence of $\mathfrak{s}$ if there exists syntax trees $\mathfrak{t}'$, $\mathfrak{r}_1$, ..., $\mathfrak{r}_{|\mathfrak{s}|}$ and $i \in [|\mathfrak{t}'|]$ such that

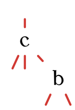
$$\mathfrak{t} = \circledcirc_i\left(\mathfrak{t}',\mathfrak{s},\mathfrak{r}_1,\dots,\mathfrak{r}_{|\mathfrak{s}|}\right). \tag{2.3.1}$$

By extension, $\mathfrak{t}$ avoids a set P of C-syntax trees if $\mathfrak{t}$ avoids all the patterns of P. For instance, consider the graded collection $C := C(2) \sqcup C(3)$ where $C(2) := \{\mathrm{a},\mathrm{b}\}$ and $C(3) := \{\mathrm{c}\}$, and the C-syntax tree

$\mathfrak{t} :=$ 

. (2.3.2)

Then, $\mathfrak{t}$ admits an occurrence of



(2.3.3)

at position 1 and two occurrences of

(2.3.4)

at positions 11 and 21.

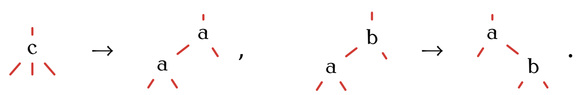
2.3.2. *Rewrite systems.* Let $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \to\right)$ be a rewrite system on syntax trees and

$$\mathscr{P} := \left\{ \circledcirc_i^{(n,k_1,\dots,k_m)} : n, m, k_1, \dots, k_m \in \mathbb{N}_{\geqslant 1}, i \in [n] \right\} \tag{2.3.5}$$

the set of all the context grafting operations. Since, as we observed in Section 2.2.3, all products of $\mathscr{P}$ are concentrated, we can consider the $\mathscr{P}$-closure of $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \to\right)$. Therefore, let us denote by $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \Rightarrow\right)$ the $\mathscr{P}$-closure of $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \to\right)$, called simply *closure* of $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \to\right)$. In other terms, $\Rightarrow$ is the rewrite rule satisfying

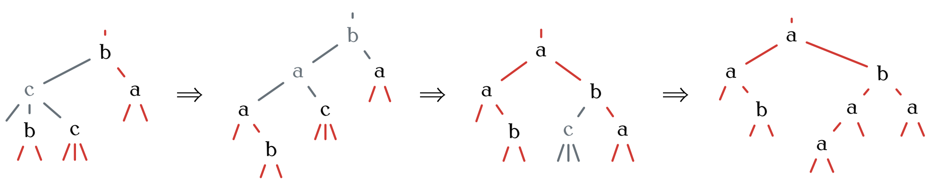
$$\circledcirc_i \left(\mathfrak{t}, \mathfrak{s}, \mathfrak{r}_1, \dots, \mathfrak{r}_m\right) \Rightarrow \circledcirc_i \left(\mathfrak{t}, \mathfrak{s}', \mathfrak{r}_1, \dots, \mathfrak{r}_m\right) \tag{2.3.6}$$

for any C-syntax trees $\mathfrak{t}$, $\mathfrak{s}$, $\mathfrak{s}'$, $\mathfrak{r}_1$, …, $\mathfrak{r}_m$ where $\mathfrak{t}$ is of arity n, $\mathfrak{s}$ is of arity m, $i \in [n]$, and $\mathfrak{s} \to \mathfrak{s}'$. In intuitive terms, one has $\mathfrak{q} \Rightarrow \mathfrak{q}'$ for two C-syntax trees $\mathfrak{q}$ and $\mathfrak{q}'$ if there are two C-syntax trees $\mathfrak{s}$ and $\mathfrak{s}'$ such that $\mathfrak{s} \to \mathfrak{s}'$ and, by replacing an occurrence of $\mathfrak{s}$ by $\mathfrak{s}'$ in $\mathfrak{q}$, we obtain $\mathfrak{q}'$. For instance, by considering the same labeling set C as before, let $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \to\right)$ be the rewrite system defined by



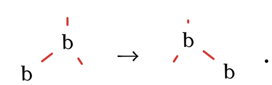
(2.3.7)

One has the following chain of rewritings



(2.3.8)

Observe by the way that the right rotation operation on binary trees considered in Section 2.2.2 of Chapter 1 can be expressed as the closure of the rewrite system $(\mathfrak{S}\mathfrak{T}_\perp^B, \to)$ such that $B := B(2) := \{\mathtt{b}\}$ defined by



(2.3.9)

In this text, we shall mainly consider rewrite systems $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \Rightarrow\right)$ defined as closures of rewrite systems $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \to\right)$ such that the number of pairs $(\mathfrak{t}, \mathfrak{t}')$ satisfying $\mathfrak{t} \to \mathfrak{t}'$ is finite. We say in this case that $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \Rightarrow\right)$ is of *finite type*. In this context, the *degree* of $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \Rightarrow\right)$ is the maximal degree among the C-syntax trees appearing as left members of $\to$. The *arity* of $\left(\mathfrak{S}\mathfrak{T}_\perp^C, \Rightarrow\right)$ is the maximal arity among the C-syntax trees appearing as left (or, equivalently, as right) members of $\to$.

2.3.3. *Proving termination.* We have observed in Section 3.1.2 of Chapter 1 that termination invariants provide tools to show that a combinatorial rewrite system is terminating. This idea extends on rewrite systems on syntax trees defined as closures of other ones in the following way.

Let $\left(\mathfrak{ST}_\perp^C, \to\right)$ be a combinatorial rewrite system and $\left(\mathfrak{ST}_\perp^C, \Rightarrow\right)$ be its closure. Assume that $\theta : \mathfrak{ST}_\perp^C \to \mathbb{Q}$ is a termination invariant for $\left(\mathfrak{ST}_\perp^C, \to\right)$, where $(\mathbb{Q}, \preccurlyeq)$ is a poset. We say that θ is *compatible with the closure* if, for any C-syntax trees $\mathfrak{s}$ and $\mathfrak{s}'$ such that $\mathfrak{s} \to \mathfrak{s}'$, the inequality

$$\theta\left(\circledcirc_i\left(\mathfrak{t}, \mathfrak{s}, \mathfrak{r}_1, \dots, \mathfrak{r}_m\right)\right) \prec \theta\left(\circledcirc_i\left(\mathfrak{t}, \mathfrak{s}', \mathfrak{r}_1, \dots, \mathfrak{r}_m\right)\right) \tag{2.3.10}$$

holds for all C-syntax trees $\mathfrak{t}$, $\mathfrak{r}_1$, ..., $\mathfrak{r}_m$, and all $i \in [\mathrm{ari}(\mathfrak{t})]$ where $m := \mathrm{ari}(\mathfrak{s}) = \mathrm{ari}(\mathfrak{s}')$. Now, as a consequence of (2.3.6) and Theorem 3.1.1 of Chapter 1, one has the following result.

Proposition 2.3.1. *Let C be a combinatorial augmented graded collection without object of size 1, $\left(\mathfrak{ST}_\perp^C, \to\right)$ be a rewrite system, and $\left(\mathfrak{ST}_\perp^C, \Rightarrow\right)$ be the closure of $\left(\mathfrak{ST}_\perp^C, \to\right)$. If*

(i) there exists a poset $\mathbb{Q}$ and a termination invariant $\theta : \mathfrak{ST}_\perp^C \to \mathbb{Q}$ for $\left(\mathfrak{ST}_\perp^C, \to\right)$;

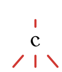
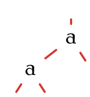
(ii) the map θ is compatible with the closure;

then, $\left(\mathfrak{ST}_\perp^C, \Rightarrow\right)$ is terminating.

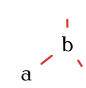
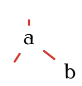
Consider, for instance, the rewrite system $\left(\mathfrak{ST}_\perp^C, \to\right)$ defined by (2.3.7). By setting $\mathbb{Q} := \mathbb{N}^2$ and $\preccurlyeq$ as the lexicographic order on $\mathbb{N}^2$, let us define the map $\theta : \mathfrak{ST}_\perp^C \to \mathbb{Q}$, for any C-syntax tree $\mathfrak{t}$, by $\theta(\mathfrak{t}) := (\deg(\mathfrak{t}), \mathrm{tam}(\mathfrak{t}))$, where

$$\mathrm{tam}(\mathfrak{t}) := \sum_{\substack{u \in \mathcal{N}_\bullet(\mathfrak{t}) \\ u \text{ of arity } 2}} \deg(\mathfrak{t} \bullet u2). \tag{2.3.11}$$

In other words, $\mathrm{tam}(\mathfrak{t})$ is the sum, for all binary nodes u of $\mathfrak{t}$, of the number of internal nodes appearing in the 2nd suffix subtrees of u. One can check that $\theta(\mathfrak{t}) \prec \theta(\mathfrak{t}')$ for all the C-syntax trees $\mathfrak{r}$ and $\mathfrak{r}'$ such that $\mathfrak{r} \to \mathfrak{r}'$. Indeed,

θ () $= (1, 0) \prec (2, 0) = \theta$ (), (2.3.12a)

and

θ () $= (2, 0) \prec (2, 1) = \theta$ (). (2.3.12b)

Moreover, the fact that θ is compatible with the closure is a straightforward verification. Therefore, the closure $\left(\mathfrak{ST}_\perp^C, \Rightarrow\right)$ of $\left(\mathfrak{ST}_\perp^C, \to\right)$ is terminating.

2.3.4. *Proving confluence.* In the same way as the tool to show that a rewrite system on C-syntax trees is terminating presented in Section 2.3.3, we present here a tool to prove that rewrite systems on syntax trees defined as closures of other ones are confluent. This criterion requires now some precise properties.

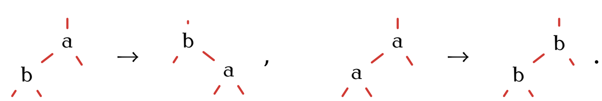
Proposition 2.3.2. *Let C be a combinatorial augmented graded collection without object of size 1, $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \to\right)$ be a rewrite system, and $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \Rightarrow\right)$ be the closure of $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \to\right)$. If $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \Rightarrow\right)$ is*

(i) of finite type;
(ii) terminating;
(iii) such that all its branching pairs consisting in trees with $2\ell - 1$ internal nodes or less are joinable, where ℓ is its degree;

then, $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \Rightarrow\right)$ is confluent.

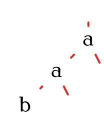
Proposition 2.3.2 yields an algorithmic way to check if a terminating rewrite system $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \Rightarrow\right)$ defined as the closure of an other one $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \to\right)$ is confluent by enumerating all the C-syntax trees $\mathfrak{t}$ of degrees at most $2\ell - 1$ (where ℓ is the degree of $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \Rightarrow\right)$) and by computing the parts $G_\mathfrak{t}$ of the rewriting graphs of $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \Rightarrow\right)$ consisting in the trees reachable from $\mathfrak{t}$. If each $G_\mathfrak{t}$ contains exactly one normal form (which correspond to a vertex with no outgoing edge in $G_\mathfrak{t}$), $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \Rightarrow\right)$ is confluent.

For instance, by considering the same labeling set C as above, let $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \to\right)$ be the rewrite system defined by



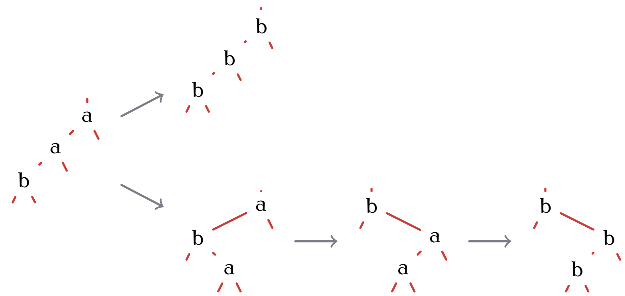
(2.3.13)

The degree of the closure $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \Rightarrow\right)$ of $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \to\right)$ is $\ell := 2$ and it is possible to show that $\left(\mathfrak{S}\mathfrak{T}^C_\perp, \Rightarrow\right)$ is terminating. Consider

$\mathfrak{t} :=$  ,

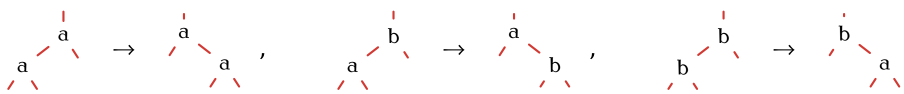
(2.3.14)

which is a C-syntax tree of degree $2\ell - 1 = 3$. The graph $G_{\mathfrak{t}}$ associated with $\mathfrak{t}$ is of the form



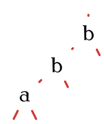
, (2.3.15)

and shows that $\left(\mathfrak{S}\mathfrak{T}^{C}_{\perp}, \Rightarrow\right)$ is not confluent. Indeed, $\mathfrak{t}$ is a non-joinable branching tree. On the other hand, consider the rewrite system $\left(\mathfrak{S}\mathfrak{T}^{C}_{\perp}, \to\right)$ defined by

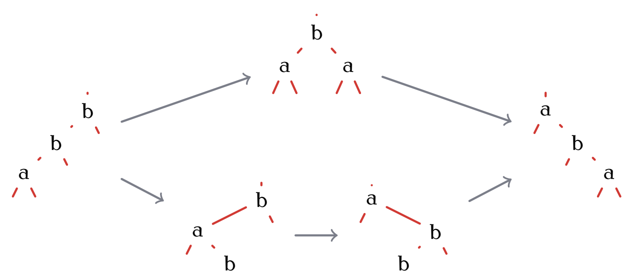


. (2.3.16)

The degree of the closure $\left(\mathfrak{S}\mathfrak{T}^{C}_{\perp}, \Rightarrow\right)$ of $\left(\mathfrak{S}\mathfrak{T}^{C}_{\perp}, \to\right)$ is $\ell := 2$ and here also, it is possible to show that $\left(\mathfrak{S}\mathfrak{T}^{C}_{\perp}, \Rightarrow\right)$ is terminating. Consider

$\mathfrak{t} :=$  , (2.3.17)

a C-syntax tree of degree $2\ell - 1 = 3$. The graph $G_{\mathfrak{t}}$ associated with $\mathfrak{t}$ is of the form



, (2.3.18)

This graph satisfies the required property stated above, and, as a systematic study of cases shows, all other graphs $G_{\mathfrak{s}}$ where $\mathfrak{s}$ is a C-syntax tree of degree 3 or less, also. For this reason, $\left(\mathfrak{S}\mathfrak{T}^{C}_{\perp}, \Rightarrow\right)$ is confluent.

3. Treelike structures

We expose here two additional variants of trees. The first one are rooted trees and are structures intervening in the study of free pre-Lie algebras (see forthcoming

Section 3.3 of Chapter 3). The second ones are colored syntax trees and can be seen as objects of free colored operads (see forthcoming Section 2.1 of Chapter 5).

3.1. Rooted trees. Let $\mathfrak{RT}$ be the graded collection satisfying the relation

$$\mathfrak{RT} = \left[\, \{\bullet\}, \mathbf{MSet}^{+}(\mathfrak{RT}) \,\right]^{+}_{\times}. \tag{3.1.1}$$

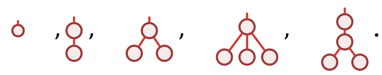
where $\bullet$ is an atomic object called *node* and **MSet** is the multiset collection operation over graded collections defined in Section 1.2.6 of Chapter 1. We call *rooted tree* each object of $\mathfrak{RT}$. By definition, a rooted tree $\mathfrak{t}$ is an ordered pair $(\bullet, \wr \mathfrak{t}_1, \dots, \mathfrak{t}_k \int)$ where $\wr \mathfrak{t}_1, \dots, \mathfrak{t}_k \int$ is a multiset of rooted trees. Like the case of planar rooted trees, this definition is recursive. For instance,

$$(\bullet, \emptyset), \quad (\bullet, \wr(\bullet, \emptyset)\int), \quad (\bullet, \wr(\bullet, \emptyset), (\bullet, \emptyset)\int), \quad (\bullet, \wr(\bullet, \emptyset), (\bullet, \emptyset), (\bullet, \emptyset)\int), \quad (\bullet, \wr(\bullet, \wr(\bullet, \emptyset), (\bullet, \emptyset)\int)\int), \tag{3.1.2}$$

are rooted trees. If $\mathfrak{t} = (\bullet, \wr \mathfrak{t}_1, \dots, \mathfrak{t}_k \int)$ is a rooted tree, each $\mathfrak{t}_i$, $i \in [k]$, is a *suffix subtree* of $\mathfrak{t}$.

Rooted trees are different kinds of trees than planar rooted trees presented in Section 1. The difference is due to the fact that rooted trees are defined by using multisets of rooted trees, while planar rooted trees are defined by using tuples of planar rooted trees. Hence, the order of the suffix subtrees of a rooted tree is not significant.

By drawing each rooted tree by a node ⦽ attached below it to its subtrees by means of edges —, the rooted trees of (3.1.2) are depicted by



(3.1.3)

By definition of the product and multiset operations over combinatorial collections, the size of a rooted tree $\mathfrak{t}$ satisfies

$$|\mathfrak{t}| := 1 + \sum_{i \in [k]} |\mathfrak{t}_i|. \tag{3.1.4}$$

The integer sequence of $\mathfrak{RT}$ begins by

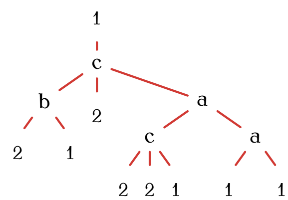
$$1, 1, 2, 4, 9, 20, 48, 115, 286 \tag{3.1.5}$$

and forms Sequence A000081 of [**Slo**].

3.2. Colored syntax trees. Let $\mathfrak{C}$ be a set of colors and C be a $\mathfrak{C}$-colored collection (see Section 1.1.4 of Chapter 1) such that the graduation of C is augmented. A *$\mathfrak{C}$-colored C-syntax tree* is a triple $(a, \mathfrak{t}, u)$ where $\mathfrak{t}$ is a C-syntax tree of arity $n \in \mathbb{N}_{\geqslant 1}$, $a \in \mathfrak{C}$, $u \in \mathfrak{C}^n$, and for any internal nodes u and v of $\mathfrak{t}$ such that v is the ith child of u, $\mathrm{out}(y) = \mathrm{in}_i(x)$ where x (resp. y) is the label of u (resp. v). The set of all $\mathfrak{C}$-colored C-syntax trees is denoted by $\mathfrak{CST}^C$. This set is a $\mathfrak{C}$-colored collection by setting that $\mathrm{out}((a, \mathfrak{t}, u)) := a$ and $\mathrm{in}((a, \mathfrak{t}, u)) := u$ for all $(a, \mathfrak{t}, u) \in \mathfrak{CST}^C$. By a slight abuse of notation, if u is an internal node of $\mathfrak{t}$, we denote by $\mathrm{out}(u)$ (resp. $\mathrm{in}(u)$) the color $\mathrm{out}(x)$ (resp. word of colors $\mathrm{in}(x)$)

where x is the label of u. We say that a $\mathfrak{C}$-colored C-syntax tree $\mathfrak{t}$ is *monochrome* if C is a monochrome colored collection. In graphical representations of a $\mathfrak{C}$-colored C-syntax tree $(a, \mathfrak{t}, u)$, we draw $\mathfrak{t}$ together with its output color above its root and its input color $u(i)$ below its ith leaf for any $i \in [|u|]$.

For instance, consider the set of colors $\mathfrak{C} := \{1, 2\}$ and the $\mathfrak{C}$-colored collection C defined by $C := C(2) \sqcup C(3)$ with $C(2) := \{\mathsf{a}, \mathsf{b}\}$, $C(3) := \{\mathsf{c}\}$, $\mathrm{out}(\mathsf{a}) := 1$, $\mathrm{out}(\mathsf{b}) := 2$, $\mathrm{out}(\mathsf{c}) := 1$, $\mathrm{in}(\mathsf{a}) := 11$, $\mathrm{in}(\mathsf{b}) := 21$, and $\mathrm{in}(\mathsf{c}) := 221$. The tree



(3.2.1)

is a $\mathfrak{C}$-colored C-syntax tree. Its output color is 1 and its word of input colors is 21222111. Besides, $(1, \perp, 1)$ and $(1, \perp, 2)$ are two $\mathfrak{C}$-colored C-syntax trees of degree 0 and arity 1.

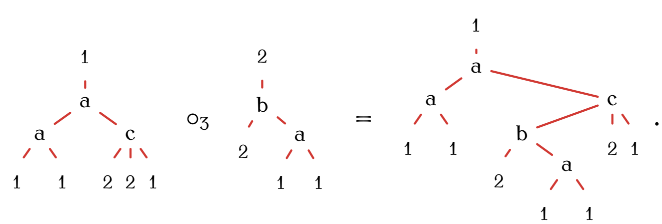
The partial grafting operation of syntax trees (see Section 2.2.1) admits a generalization on colored syntax trees. Let for any $(a, u) \in \mathfrak{C} \times \mathfrak{C}^n$, $(b, v) \in \mathfrak{C} \times \mathfrak{C}^m$, and $i \in [n]$ such that $b = u(i)$ the product

$$\circ_i^{((a,u),(b,v))} : \mathfrak{CST}^C(a, u) \times \mathfrak{CST}^C(b, v) \to \mathfrak{CST}^C \tag{3.2.2}$$

defined, for any $(a, \mathfrak{t}, u) \in \mathfrak{CST}^C(a, u)$ and $(b, \mathfrak{s}, v) \in \mathfrak{CST}^C(b, v)$ by

$$(a, \mathfrak{t}, u) \circ_i^{((a,u),(b,v))} (b, \mathfrak{s}, v) := \left(a, \mathfrak{t} \circ_i^{(n,m)} \mathfrak{s}, u \hookleftarrow_i v\right), \tag{3.2.3}$$

where $u \hookleftarrow_i v$ is the word obtained by replacing the ith letter of u by v, and $\circ_i^{(n,m)}$ is the partial grafting of syntax trees. For instance, by considering the same labeling $\mathfrak{C}$-colored collection as above,



(3.2.4)

We call each $\circ_i^{(n,m)}$ a *partial grafting operation*.

Observe that since

$$\mathrm{ind}\left((a, \mathfrak{t}, u) \circ_i^{(n,m)} (b, \mathfrak{s}, v)\right) = (a, u \hookleftarrow_i v) \tag{3.2.5}$$

where ind denotes the index of an object of a colored collection (see Section 1.1.1 of Chapter 1), the product $\circ_i^{(n,m)}$ is concentrated. Besides, by a slight abuse of notation, we shall sometimes omit the (n, m) in the notation of $\circ_i^{(n,m)}$ in order to denote it in a more concise way by $\circ_i$.

The generalizations of the complete and context grafting products of syntax trees (see Sections 2.2.2 and 2.2.3) on colored syntax trees follow from the definition of the partial grafting operation of colored syntax trees just given. These two products are also concentrated.

Most of the notions exposed in Section 2.3 about syntax trees and rewrite systems on syntax trees naturally extend on colored syntax trees like, among others, the notions of occurrences of patterns, the complete grafting operations, and the criteria offered by Propositions 2.3.1 and 2.3.2 to, respectively, prove the termination and the confluence of rewrite system on syntax trees.

Bibliographic notes

About trees. The concept of tree encompasses a large range of quite different combinatorial objects. For instance, in graph theory, trees are connected acyclic graphs while in combinatorics, one encounters mostly rooted trees. Among rooted trees, some of these can be planar (the order of the children of a node is relevant) or not. In addition to this, the internal nodes, the leaves, or the edges of the trees can be labeled, and some conditions for the arities of their nodes can be imposed. One of the first occurrences of the concept of tree came from the work of Cayley [**Cay57**]. Nowadays, trees appear among others in computer science as data structures [**Knu98**, **CLRS09**], in combinatorics in relation with enumerating questions and Lagrange inversion [**Lab81**, **FS09**], and in algebraic combinatorics, where several families of trees are endowed with algebraic structures [**LR98**, **HNT05**, **Cha08**]. Besides, the bijection between the combinatorial collections of the planar rooted trees and the one of binary trees appearing in Proposition 1.2.1 is known as the rotation correspondence and is due to Knuth [**Knu97**]. This bijection, offering a means of encoding a planar rooted tree by a binary tree, admits applications in algebraic combinatorics [**NT13**, **EFM14**].

About enumerating properties. Formula (1.2.8) for the Fuss-Catalan numbers, enumerating the combinatorial collection of the k-ary trees with respect to their number of internal nodes has been established in [**DM47**]. Besides, Formula (1.2.16) enumerating the combinatorial set of the Schröder trees with respect to their number of leaves uses the Narayana numbers [**Nar55**]. These numbers admit the following combinatorial interpretation: the 2-graded collection C of binary trees, where the index of a binary tree $\mathfrak{t}$ is the pair (n, k) where n is the arity of $\mathfrak{t}$ and k is the number of internal nodes of $\mathfrak{t}$ having an internal node as a left child satisfies $\#C(n, k) = \mathrm{nar}(n, k)$.

About rewrite rules on trees. The Buchberger algorithm, which is a completion algorithm (see the end of Chapter 1), admits adaptations in the context of rewrite systems of trees and operads [**DK10**, **BD16**]. Several works use rewrite systems on trees to provide presentations of operads (see, for instance, [**Hof10**, **LV12**, **Gir16b**, **CCG18**]).

CHAPTER 3

Algebraic structures

This chapter deals with vector spaces obtained from graded collections. A general framework for algebraic structures having products and coproducts is presented. Most of the algebraic structures encountered in algebraic combinatorics like associative, dendriform, pre-Lie algebras, and Hopf bialgebras fit into this framework. This chapter contains classical examples of such structures.

1. Polynomials spaces

We introduce here the notion of polynomial spaces. All the algebraic structures considered in this book are polynomial spaces endowed with some operations or co-operations. A set of operations, analogous to the operations on graded collections of Section 1.2 of Chapter 1, over graded polynomial spaces are considered. We also review some links between changes of bases of polynomial spaces, posets, and incidence algebras.

1.1. Polynomials on collections. Intuitively, a polynomial on an I-collection C is a finite formal sum of objects of C with coefficients in a field $\mathbb{K}$. In what follows, $\mathbb{K}$ can be any field of characteristic 0.

1.1.1. *Polynomials.* Let C be an I-collection. A *polynomial on C* (or, for short, a *C-polynomial*) is a map

$$f : C \to \mathbb{K} \tag{1.1.1}$$

such that the set

$$\mathrm{Supp}(f) := \{x \in C : f(x) \neq 0\} \tag{1.1.2}$$

is finite, where the symbol 0 appearing in (1.1.2) is the zero of $\mathbb{K}$. We call $\mathrm{Supp}(f)$ the *support* of f. The *coefficient* $f(x)$ of $x \in C$ in f is denoted by $\langle x, f\rangle$. An object x of C *appears* in f if $\langle x, f\rangle \neq 0$. A C-polynomial f is a *C-monomial* if $\mathrm{Supp}(f)$ is a singleton. We say that f is *homogeneous* if there is an index $i \in I$ such that $\mathrm{Supp}(f) \subseteq C(i)$. For any finite subcollection X of C, the *characteristic polynomial* of X is the C-polynomial $\mathrm{ch}(X)$ defined, for any $x \in C$, by

$$\langle x, \mathrm{ch}(X)\rangle := \begin{cases} 1 \in \mathbb{K} & \text{if } x \in X, \\ 0 \in \mathbb{K} & \text{otherwise.} \end{cases} \tag{1.1.3}$$

Given two C-polynomials f_1 and f_2, the *scalar product* of f_1 and f_2 is the scalar

$$\langle f_1, f_2 \rangle := \sum_{x \in C} \langle x, f_1 \rangle \langle x, f_2 \rangle \tag{1.1.4}$$

of $\mathbb{K}$. This notation for the scalar product of C-polynomials is consistent with the notation $\langle x, f \rangle$ for the coefficient of x in f because by (1.1.4), the coefficient $\langle x, f \rangle$ and the scalar product $\langle \mathrm{ch}(\{x\}), f \rangle$ are equal.

In the particular case where C is a graded collection, the *degree* $\deg(f)$ of f is undefined if $\mathrm{Supp}(f) = \emptyset$ and is otherwise the greatest size of an object appearing in $\mathrm{Supp}(f)$.

1.1.2. *Polynomial spaces.* The set of all C-polynomials is denoted by $\mathbb{K}\langle C \rangle$. The *underlying collection* of $\mathbb{K}\langle C \rangle$ is C. For any property P of collections (see Section 1 of Chapter 1), we say by extension that $\mathbb{K}\langle C \rangle$ *satisfies the property P* if C satisfies P.

This set $\mathbb{K}\langle C \rangle$ is endowed with the following two operations. First, the *addition*

$$+ : \mathbb{K}\langle C \rangle \times \mathbb{K}\langle C \rangle \to \mathbb{K}\langle C \rangle \tag{1.1.5}$$

is defined, for any $f_1, f_2 \in \mathbb{K}\langle C \rangle$ and $x \in C$, by

$$\langle x, f_1 + f_2 \rangle := \langle x, f_1 \rangle + \langle x, f_2 \rangle . \tag{1.1.6}$$

Second, the *scalar multiplication*

$$\cdot : \mathbb{K} \times \mathbb{K}\langle C \rangle \to \mathbb{K}\langle C \rangle \tag{1.1.7}$$

is defined, for any $f \in \mathbb{K}\langle C \rangle$, $\lambda \in \mathbb{K}$, and $x \in C$, by

$$\langle x, \lambda \cdot f \rangle := \lambda \langle x, f \rangle . \tag{1.1.8}$$

Endowed with these two operations, $\mathbb{K}\langle C \rangle$ is a $\mathbb{K}$-vector space, named *polynomial space on C* (or, for short, *C-polynomial space*). Moreover, $\mathbb{K}\langle C \rangle$ decomposes as a direct sum

$$\mathbb{K}\langle C \rangle = \bigoplus_{i \in I} \mathbb{K}\langle C(i) \rangle . \tag{1.1.9}$$

We call each $\mathbb{K}\langle C(i) \rangle$ the *i-homogeneous component* of $\mathbb{K}\langle C \rangle$. In the sequel, we shall also write $\mathbb{K}\langle C \rangle (i)$ for $\mathbb{K}\langle C(i) \rangle$.

By using now the linear structure of $\mathbb{K}\langle C \rangle$, any C-polynomial f can be expressed as the finite sum of C-monomials

$$f = \sum_{x \in C} \langle x, f \rangle \cdot \mathrm{ch}(\{x\}), \tag{1.1.10}$$

which is denoted, by a slight abuse of notation, by

$$f = \sum_{x \in C} \langle x, f \rangle \, x. \tag{1.1.11}$$

The notation (1.1.11) for f as a linear combination of objects of C is the *sum notation* of C-polynomials.

Since for any C-polynomial f, there is unique way to express f as a finite sum of the form (1.1.11), the set

$$\{\mathrm{ch}(\{x\}) : x \in C\} \tag{1.1.12}$$

forms a basis of $\mathbb{K}\langle C\rangle$. This basis is called *fundamental basis* of $\mathbb{K}\langle C\rangle$, and, by a slight but convenient abuse of notation, each basis element $\mathrm{ch}(\{x\})$, $x \in C$, is simply denoted by x. Observe that each basis of $\mathbb{K}\langle C\rangle$ is indexed by C. Moreover, Let us emphasize the fact any polynomial space $\mathbb{K}\langle C\rangle$ is always seen through its explicit basis C (contrarily when working abstractly with a vector space $\mathcal{V}$ without explicit basis). In the sequel, we shall define products on C which extend by linearity on $\mathbb{K}\langle C\rangle$. Properties of such products (like associativity or commutativity) can be defined and checked only on C.

Besides, we are sometimes led to consider several bases of $\mathbb{K}\langle C\rangle$ and work with many of them at the same time. In this case, to distinguish elements expressed on different bases, we denote them by putting elements of C as indexes of a letter naming the basis. For instance, the elements of the B-basis of $\mathbb{K}\langle C\rangle$ are denoted by B_x, $x \in C$.

Let C_1 and C_2 be two I-collections. A *morphism* between $\mathbb{K}\langle C_1\rangle$ and $\mathbb{K}\langle C_2\rangle$ is a linear map

$$\phi : \mathbb{K}\langle C_1\rangle \to \mathbb{K}\langle C_2\rangle \tag{1.1.13}$$

such that for any $x \in C_1$, $\phi(x) \in \mathbb{K}\langle C_2\rangle(\mathrm{ind}(x))$. Observe that any combinatorial collection morphism $\psi : C_1 \to C_2$ gives rise to a polynomial space morphism $\bar{\psi} : \mathbb{K}\langle C_1\rangle \to \mathbb{K}\langle C_2\rangle$ obtained by extending ψ linearly. Besides, $\mathbb{K}\langle C_2\rangle$ is a *subspace* of $\mathbb{K}\langle C_1\rangle$ if there exists an injective morphism from $\mathbb{K}\langle C_2\rangle$ to $\mathbb{K}\langle C_1\rangle$. For any subset J of I, we denote by $\mathbb{K}\langle C\rangle(J)$ the polynomial space $\mathbb{K}\langle C(J)\rangle$. Since $C(J)$ is by definition a subcollection of C, $\mathbb{K}\langle C\rangle(J)$ is a subspace of $\mathbb{K}\langle C\rangle$.

1.1.3. *Combinatorial graded polynomial spaces.* When C is a combinatorial graded collection, as a particular case of (1.1.9), $\mathbb{K}\langle C\rangle$ decomposes as a direct sum

$$\mathbb{K}\langle C\rangle = \bigoplus_{n\in\mathbb{N}} \mathbb{K}\langle C\rangle(n). \tag{1.1.14}$$

Moreover, since C is combinatorial, each $\mathbb{K}\langle C(n)\rangle$, $n \in \mathbb{N}$, is finite dimensional. For this reason, the *Hilbert series* of $\mathbb{K}\langle C\rangle$, defined by

$$\mathbb{H}_{\mathbb{K}\langle C\rangle}(t) = \sum_{n\in\mathbb{N}} \dim \mathbb{K}\langle C\rangle(n)\, t^n, \tag{1.1.15}$$

is a well-defined series. We can observe that the Hilbert series $\mathbb{H}_{\mathbb{K}\langle C\rangle}(t)$ of $\mathbb{K}\langle C\rangle$ and the generating series $\mathbb{G}_C(t)$ of C are the same power series.

1.1.4. *Rewrite systems and quotient spaces.* For any I-collection C, any rewrite system $(C, \to)$ gives rise to a subspace $\mathcal{R}_{(C,\to)}$ of $\mathbb{K}\langle C\rangle$ generated by all the homogeneous C-polynomials $x' - x$ whenever x and x' are two objects of C such that $x \to x'$. We call $\mathcal{R}_{(C,\to)}$ the *space induced* by $(C, \to)$. Conversely, when $\mathcal{R}$ is a subspace of $\mathbb{K}\langle C\rangle$ such that there exists a rewrite system $(C, \to)$ such that $\mathcal{R}$ and $\mathcal{R}_{(C,\to)}$ are isomorphic, we say that $(C, \to)$ is an *orientation* of $\mathcal{R}$. When $(C, \to)$ is convergent, one has a concrete description of the quotient space $\mathbb{K}\langle C\rangle /_{\mathcal{R}_{(C,\to)}}$ involving the normal forms $\mathcal{N}_{(C,\to)}$ of $(C, \to)$ provided by the following result.

Proposition 1.1.1. *Let $(C, \to)$ be a convergent rewrite system. Then, as spaces*

$$\mathbb{K}\langle C\rangle /_{\mathcal{R}_{(C,\to)}} \simeq \mathbb{K}\left\langle \mathcal{N}_{(C,\to)}\right\rangle. \tag{1.1.16}$$

1.2. Operations over polynomial spaces. In the same way as operations over collections allow to create new collections from already existing ones (see Section 1.2 of Chapter 1), there exist analogous operations over polynomial spaces. Some of these are consequences of the definitions of operations over collections. We present here the main ones. Alternatively, one of the aims of this section is to show that the usual operations over spaces (direct sum, quotient, and tensor product) produce polynomial spaces.

1.2.1. *Direct sum.* The sum of two collections translates as the direct sum of the associated polynomial spaces. Indeed, for any I-collections C_1 and C_2,

$$\mathbb{K}\langle C_1 + C_2\rangle \simeq \mathbb{K}\langle C_1\rangle \oplus \mathbb{K}\langle C_2\rangle. \tag{1.2.1}$$

An isomorphism between the two spaces of (1.2.1) is provided by the map

$$\phi : \mathbb{K}\langle C_1 + C_2\rangle \to \mathbb{K}\langle C_1\rangle \oplus \mathbb{K}\langle C_2\rangle, \tag{1.2.2}$$

linearly defined for any $x \in C_1 + C_2$ by

$$\phi(x) := \begin{cases} x \in \mathbb{K}\langle C_1\rangle & \text{if } x \in C_1, \\ x \in \mathbb{K}\langle C_2\rangle & \text{otherwise } (x \in C_2). \end{cases} \tag{1.2.3}$$

For this reason, we shall identify the two spaces of (1.2.1).

1.2.2. *Quotient space.* If C is an I-collection and $\mathcal{V}$ is a space included in $\mathbb{K}\langle C\rangle$, the *quotient space* of $\mathbb{K}\langle C\rangle$ by $\mathcal{V}$ is the space $\mathbb{K}\langle C\rangle /_{\mathcal{V}}$ of all the equivalence classes

$$[f]_{\mathcal{V}} := \{f + g : g \in \mathcal{V}\}, \tag{1.2.4}$$

for all $f \in \mathbb{K}\langle C\rangle$, endowed with its natural vector space structure. We call *canonical surjection map* the linear map $\theta : \mathbb{K}\langle C\rangle \to \mathbb{K}\langle C\rangle /_{\mathcal{V}}$ defined linearly by $\theta(x) := [x]_{\mathcal{V}}$ for any object x of C. It is always possible to see $\mathbb{K}\langle C\rangle /_{\mathcal{V}}$ as a D-polynomial space by providing an adequate I-collection D so that

$$\mathbb{K}\langle C\rangle /_{\mathcal{V}} \simeq \mathbb{K}\langle D\rangle. \tag{1.2.5}$$

For this reason, we shall identify any quotient space with a polynomial space.

1.2.3. *Tensor product.* The Cartesian product of collections translates as the tensor product of the associated polynomial spaces. Indeed, for any $p \in \mathbb{N}$, any index sets $I_1, \dots, I_p$, and any I_1-collection $C_1, \dots,$ any I_p-collection C_p,

$$\mathbb{K}\left\langle \left[C_1, \dots, C_p \right]_\times \right\rangle \simeq \mathbb{K}\langle C_1 \rangle \otimes \dots \otimes \mathbb{K}\langle C_p \rangle. \tag{1.2.6}$$

An isomorphism between the two spaces of (1.2.6) is provided by the map

$$\phi : \mathbb{K}\left\langle \left[C_1, \dots, C_p \right]_\times \right\rangle \to \mathbb{K}\langle C_1 \rangle \otimes \dots \otimes \mathbb{K}\langle C_p \rangle, \tag{1.2.7}$$

linearly defined for any $(x_1, \dots, x_p) \in \left[C_1, \dots, C_p \right]_\times$ by

$$\phi\left((x_1, \dots, x_p)\right) := x_1 \otimes \dots \otimes x_p. \tag{1.2.8}$$

For this reason, we shall identify the two spaces of (1.2.6). Moreover, as a consequence, the tuple notation for tensors is linear. That is, for any $f_1 \in \mathbb{K}\langle C_1 \rangle, \dots, f_p \in \mathbb{K}\langle C_p \rangle$,

$$(f_1, \dots, f_p) = \sum_{(x_1, \dots, x_p) \in \left[C_1, \dots, C_p \right]_\times} \left(\prod_{k \in [p]} \langle x_k, f_k \rangle \right) (x_1, \dots, x_p). \tag{1.2.9}$$

1.2.4. *Tensor algebras.* If $\mathcal{V}$ is a $\mathbb{K}$-vector space, the *tensor algebra* of $\mathcal{V}$ is the space $\mathrm{T}\,\mathcal{V}$ defined by

$$\mathrm{T}\,\mathcal{V} := \bigoplus_{p \in \mathbb{N}} \mathcal{V}^{\otimes p} \tag{1.2.10}$$

where $\mathcal{V}^{\otimes p}$, $p \in \mathbb{N}$, denotes the space of all tensors on $\mathcal{V}$ of order $p \in \mathbb{N}$. A basis of $\mathrm{T}\,\mathcal{V}$ is formed by all tensors on any basis of $\mathcal{V}$. As a special case of the one of tensor products discussed in the above section, the list collection operation applied to a graded collection translates as the tensor algebra of the associated graded polynomial space. Indeed, for any $p \in \mathbb{N}$ and I-collection C,

$$\mathbb{K}\left\langle \mathbf{List}_{\{p\}}(C) \right\rangle \simeq \mathbb{K}\langle C \rangle^{\otimes p}. \tag{1.2.11}$$

so that, by using the correspondence between direct sums of spaces and sums of collections, we obtain

$$\mathbb{K}\langle \mathbf{List}(C) \rangle \simeq \mathrm{T}\,\mathbb{K}\langle C \rangle. \tag{1.2.12}$$

An isomorphism between the two spaces of (1.2.12) is provided by the map

$$\phi : \mathbb{K}\langle \mathbf{List}(C) \rangle \to \mathrm{T}\,\mathbb{K}\langle C \rangle, \tag{1.2.13}$$

linearly defined for any $(x_1, \dots, x_p) \in \mathbf{List}(C)$ by

$$\phi\left((x_1, \dots, x_p)\right) := x_1 \otimes \dots \otimes x_p. \tag{1.2.14}$$

For this reason, we shall identify the two spaces of (1.2.12).

1.2.5. *Symmetric algebras.* If $\mathcal{V}$ is a $\mathbb{K}$-vector space, the *symmetric algebra* of $\mathcal{V}$ is the space $\mathrm{S}\mathcal{V}$ defined by

$$\mathrm{S}\mathcal{V} := \mathrm{T}\,\mathcal{V}/_{\mathcal{V}_{\mathrm{S}}}, \tag{1.2.15}$$

where $\mathcal{V}_{\mathrm{S}}$ is the subspace of $\mathrm{T}\,\mathbb{K}\langle C\rangle$ consisting in all the tensors

$$u \otimes x_1 \otimes x_2 \otimes v - u \otimes x_2 \otimes x_1 \otimes v, \tag{1.2.16}$$

where $u, v \in \mathrm{T}\,\mathcal{V}$ and $x_1, x_2 \in \mathcal{V}$. A basis of $\mathrm{S}\mathcal{V}$ is formed by all monomials on any basis of $\mathcal{V}$. The multiset collection operation applied to a graded collection translates as the symmetric algebra of the associated graded polynomial space. Indeed, for any I-collection C,

$$\mathbb{K}\langle \mathbf{MSet}(C)\rangle \simeq \mathrm{S}\mathbb{K}\langle C\rangle. \tag{1.2.17}$$

An isomorphism between the two spaces of (1.2.17) is provided by the map

$$\phi : \mathbb{K}\langle \mathbf{MSet}(C)\rangle \to \mathrm{S}\mathbb{K}\langle C\rangle, \tag{1.2.18}$$

linearly defined for any $\wr x_1, \dots, x_p \int \in \mathbf{MSet}(C)$ by

$$\phi\left(\wr x_1, \dots, x_p \int\right) := y_1^{\alpha_1} \dots y_\ell^{\alpha_\ell}, \tag{1.2.19}$$

where ℓ is the number of distinct elements of $\wr x_1, \dots, x_p \int$ and each α_i, $i \in [\ell]$, denotes the multiplicity of y_i in $\wr x_1, \dots, x_p \int$. For this reason, we shall identify the two spaces of (1.2.17).

1.2.6. *Exterior algebras.* If $\mathcal{V}$ is a $\mathbb{K}$-vector space, the *exterior algebra* of $\mathcal{V}$ is the space $\mathrm{E}\mathcal{V}$ defined by

$$\mathrm{E}\mathcal{V} := \mathrm{T}\,\mathcal{V}/_{\mathcal{V}_{\mathrm{E}}}, \tag{1.2.20}$$

where $\mathcal{V}_{\mathrm{E}}$ is the subspace of $\mathrm{T}\,\mathcal{V}$ consisting in all the tensors

$$u \otimes x_1 \otimes x_2 \otimes v + u \otimes x_2 \otimes x_1 \otimes v, \tag{1.2.21}$$

where $u, v \in \mathrm{T}\,\mathcal{V}$ and $x_1, x_2 \in \mathcal{V}$. A basis of $\mathrm{E}\mathcal{V}$ is formed by all monomials on a basis of $\mathcal{V}$ without repeated letters. The set collection operation applied to a graded collection translates as the exterior algebra of the associated graded polynomial space. Indeed, for any I-collection C,

$$\mathbb{K}\langle \mathbf{Set}(C)\rangle \simeq \mathrm{E}\mathbb{K}\langle C\rangle. \tag{1.2.22}$$

An isomorphism between the two spaces of (1.2.22) is provided by the map

$$\phi : \mathbb{K}\langle \mathbf{Set}(C)\rangle \to \mathrm{E}\mathbb{K}\langle C\rangle, \tag{1.2.23}$$

linearly defined for any $\{x_1, \dots, x_p\} \in \mathbf{Set}(C)$ by

$$\phi\left(\{x_1, \dots, x_p\}\right) := x_1 \dots x_p. \tag{1.2.24}$$

For this reason, we shall identify the two spaces of (1.2.22).

1.2.7. *Duality for combinatorial polynomial spaces.* Assume in this section that C is combinatorial. The *dual* of $\mathbb{K}\langle C\rangle$ is the $\mathbb{K}$-vector space $\mathbb{K}\langle C\rangle^\star$ defined by

$$\mathbb{K}\langle C\rangle^\star := \bigoplus_{i\in I} \mathbb{K}\langle C\rangle(i)^\star, \tag{1.2.25}$$

where for any $i \in I$, $\mathbb{K}\langle C\rangle(i)^\star$ is the dual space of $\mathbb{K}\langle C\rangle(i)$. Since C is combinatorial, all the $\mathbb{K}\langle C\rangle(i)$ are finite dimensional spaces, so that $\mathbb{K}\langle C\rangle(i)^\star \simeq \mathbb{K}\langle C\rangle(i)$, and thus,

$$\mathbb{K}\langle C\rangle^\star \simeq \mathbb{K}\langle C\rangle. \tag{1.2.26}$$

For this reason, we shall identify $\mathbb{K}\langle C\rangle$ and $\mathbb{K}\langle C\rangle^\star$ in this book once C is combinatorial.

The *duality bracket* between $\mathbb{K}\langle C\rangle$ and $\mathbb{K}\langle C\rangle^\star$ is the linear map

$$\langle - \rangle : \mathbb{K}\langle C\rangle \otimes \mathbb{K}\langle C\rangle^\star \simeq \mathbb{K}\langle [C, C]_\times\rangle \to \mathbb{K} \tag{1.2.27}$$

defined linearly, for all $(x, x') \in [C, C]_\times$, by

$$\left\langle (x, x') \right\rangle := \begin{cases} 1 & \text{if } x = x', \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.28}$$

To not overload the notation, we write $\langle x, x'\rangle$ instead of $\langle (x, x')\rangle$. Observe that for any $f_1 \in \mathbb{K}\langle C\rangle$ and $f_2 \in \mathbb{K}\langle C\rangle^\star$, $\langle f_1, f_2\rangle$ is equal to the scalar product (1.1.4) of f_1 and f_2. Moreover, the duality bracket extends for any $p \in \mathbb{N}$ on $\mathbb{K}\left\langle \left[\mathbf{List}_{\{p\}}(C), \mathbf{List}_{\{p\}}(C) \right]_\times \right\rangle$ linearly by

$$\left\langle (x_1, \dots, x_p), (x'_1, \dots, x'_p) \right\rangle := \prod_{k\in[p]} \langle x_k, x'_k\rangle \tag{1.2.29}$$

for any objects $(x_1, \dots, x_p)$ and $(x'_1, \dots, x'_p)$ of $\mathbf{List}_{\{p\}}(C)$.

1.3. Changes of basis and posets. It is very usual, given a polynomial space $\mathbb{K}\langle C\rangle$, to consider a poset structure on C to define new bases of $\mathbb{K}\langle C\rangle$. Indeed, such new bases are defined by considering sums of elements greater (or smaller) than other ones. In this context, incidence algebras of posets and their Möbius functions play an important role. We expose here these concepts.

1.3.1. *Incidence algebras.* Let $(\mathcal{Q}, \preccurlyeq)$ be a locally finite I-poset. The *incidence algebra* of $(\mathcal{Q}, \preccurlyeq)$ is the polynomial space $\mathbb{K}\langle \mathbf{Com}(\mathcal{Q})\rangle$ ($\mathbf{Com}(\mathcal{Q})$ is defined in Section 2.1.2 of Chapter 1) endowed with the linear binary product $\star$ (the notion of products in polynomial spaces is presented in the following Section 2 but here, only elementary notions about these are needed) defined, for any objects (x, y) and (x', y') of $\mathbf{Com}(\mathcal{Q})$ by

$$(x, y) \star (x', y') := \begin{cases} (x, y') & \text{if } y = x', \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.1}$$

This product is obviously associative. Moreover, for each $i \in I$, on the i-homogeneous component of $\mathbb{K}\langle \mathbf{Com}(\mathcal{Q})\rangle$, the $\mathbf{Com}(\mathcal{Q})$-polynomial

$$\mathbb{1}_i := \sum_{x \in C(i)} (x, x) \tag{1.3.2}$$

plays the role of a unit, that is, $f \star \mathbb{1}_i = f = \mathbb{1}_i \star f$ for all $f \in \mathbb{K}\langle \mathbf{Com}(C)\rangle(i)$. Let for any $i \in I$ the $\mathbf{Com}(\mathcal{Q})$-polynomial ζ_i, called *i-zeta polynomial* of $(\mathcal{Q}, \preccurlyeq)$, defined by

$$\zeta_i := \sum_{\substack{x, y \in C(i) \\ x \preccurlyeq y}} (x, y). \tag{1.3.3}$$

This $\mathbf{Com}(\mathcal{Q})$-polynomial encodes some properties of the order $\preccurlyeq$. For instance, the coefficient in $\zeta_i \star \zeta_i$ of each $(x, y) \in \mathbf{Com}(\mathcal{Q}(i))$ is the cardinality of the interval $[x, y]$ in $(\mathcal{Q}, \preccurlyeq)$. The *i-Möbius polynomial* of $(\mathcal{Q}, \preccurlyeq)$ is the $\mathbf{Com}(\mathcal{Q})$-polynomial μ_i satisfying

$$\mu_i \star \zeta_i = \mathbb{1}_i = \zeta_i \star \mu_i. \tag{1.3.4}$$

In other words, μ_i is the inverse of ζ_i with respect to the product $\star$. Recall that, as exposed in Section 1.1.1, polynomials on collections are functions associating a coefficient with any object. For this reason, ζ_i and μ_i are functions associating a coefficient with any pair of comparable objects of $\mathcal{Q}$.

THEOREM 1.3.1. *Let $(\mathcal{Q}, \preccurlyeq)$ be a locally finite I-poset. Then, the i-Möbius polynomial μ_i, $i \in I$, of $(\mathcal{Q}, \preccurlyeq)$ is a well-defined element of $\mathbb{K}\langle \mathbf{Com}(\mathcal{Q})\rangle$ and its coefficients satisfy $\langle (x, x), \mu_i\rangle = 1$ for all $x \in \mathcal{Q}(i)$, and*

$$\langle (x, z), \mu_i\rangle = -\sum_{\substack{y \in \mathcal{Q}(i) \\ x \preccurlyeq y \prec z}} \langle (x, y), \mu_i\rangle \tag{1.3.5}$$

for all $x, z \in C(i)$ such that $x \neq z$.

Theorem 1.3.1 provides a recursive way to compute the coefficients of μ_i, $i \in I$, as a consequence of the finiteness of each interval of $\mathcal{Q}(i)$.

1.3.2. *Changes of basis.* Let C be a combinatorial I-collection and $\preccurlyeq$ be a partial order relation on C such that $(C, \preccurlyeq)$ is an I-poset. Consider the family

$$\left\{\mathrm{B}_x^{\preccurlyeq}, x \in C\right\} \tag{1.3.6}$$

of elements of $\mathbb{K}\langle C\rangle$ defined, from the fundamental basis of $\mathbb{K}\langle C\rangle$, by

$$\mathrm{B}_x^{\preccurlyeq} := \sum_{\substack{y \in C \\ x \preccurlyeq y}} y. \tag{1.3.7}$$

Observe that since C is combinatorial and $\preccurlyeq$ preserves the indexes of the objects of C, each $\mathrm{B}_x^{\preccurlyeq}$ is a homogeneous C-polynomial. We call the family (1.3.6) the *$\mathrm{B}^{\preccurlyeq}$-family* of $\mathbb{K}\langle C\rangle$.

Proposition 1.3.2. *Let $(C, \preccurlyeq)$ be a combinatorial I-poset. The $\mathsf{B}^{\preccurlyeq}$-family forms a basis of $\mathbb{K}\langle C\rangle$ and*

$$x = \sum_{\substack{y \in C \\ x \preccurlyeq y}} \langle (x, y), \mu_i \rangle \, \mathsf{B}_y^{\preccurlyeq} \tag{1.3.8}$$

for all $x \in C(i)$, $i \in I$, where μ_i is the i-Möbius polynomial of $(C, \preccurlyeq)$.

2. Bialgebras

Bialgebras are polynomial spaces endowed with operations. These operations are very general in the sense that they can have several inputs and outputs. These structures encompass all the algebraic structures seen in this work.

2.1. Biproducts on polynomial spaces.

Polynomial spaces are rather poor algebraic structures. It is usual in combinatorics to handle spaces endowed with several products. When a polynomial space is graded and its products are compatible with the sizes of the underlying combinatorial objects, all this form a graded algebra. This notion is detailed here, as well as the concepts of coproduct, duality, and coalgebras and bialgebras.

2.1.1. *Biproducts.* Let C be an I-collection and $\mathbb{K}\langle C\rangle$ be a polynomial space. A *biproduct* on $\mathbb{K}\langle C\rangle$ is a linear map

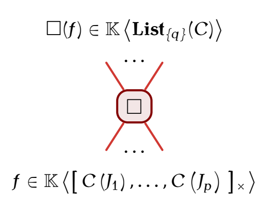
$$\square : \mathbb{K}\left\langle \left[\, C\left(J_1\right), \dots, C\left(J_p\right) \,\right]_{\times} \right\rangle \to \mathbb{K}\left\langle \mathbf{List}_{\{q\}}(C) \right\rangle \tag{2.1.1}$$

where $p, q \in \mathbb{N}$, and J_1, ..., J_p are nonempty subsets of I. Equivalently, by using the interpretation of the tensor product and of tensor algebras shown in Section 1.2, (2.1.1) is equivalent to

$$\square : \mathbb{K}\langle C\rangle\left(J_1\right) \otimes \cdots \otimes \mathbb{K}\langle C\rangle\left(J_p\right) \to \mathbb{K}\langle C\rangle^{\otimes q}. \tag{2.1.2}$$

The *arity* (resp. *coarity*) of $\square$ is p (resp. q) and the *index domain* of $\square$ is the set $J_1 \times \cdots \times J_p$. A tuple $\left(x_1, \dots, x_p\right)$ is a *valid input* for $\square$ if $\square\left(\left(x_1, \dots, x_p\right)\right)$ is defined, that is, $\left(\mathrm{ind}\left(x_1\right), \dots, \mathrm{ind}\left(x_p\right)\right)$ belongs to the index domain of $\square$. The *image* $\mathrm{Im}(\square)$ of $\square$ is the usual image of $\square$ as a linear map. To not overload the notation, we shall write $\square\left(x_1, \dots, x_p\right)$ instead of $\square\left(\left(x_1, \dots, x_p\right)\right)$ for any valid input $\left(x_1, \dots, x_p\right)$ for $\square$.

The biproduct $\square$ can be seen as an operation taking a valid input consisting in a bunch of p objects of C and outputting bunches of q objects of C. This biproduct is depicted by a rectangle labeled by its name, with p incoming edges (below the rectangle) and q outgoing edges (above the rectangle) as



. (2.1.3)

2.1.2. *Completion.* Let $\square$ a biproduct on $\mathbb{K}\langle C\rangle$ of the form (2.1.1). In the case where the index domain of $\square$ is I^p, we say that $\square$ is *complete*. Otherwise, the *completion* of $\square$ is the complete biproduct

$$\dot{\square} : \mathbb{K}\left\langle \mathbf{List}_{\{p\}}(C)\right\rangle \to \mathbb{K}\left\langle \mathbf{List}_{\{q\}}(C)\right\rangle \tag{2.1.4}$$

defined linearly, for any object $\left(x_1, \dots, x_p\right)$ of $\mathbf{List}_{\{p\}}(C)$, by

$$\dot{\square}\left(x_1, \dots, x_p\right) := \begin{cases} \square\left(x_1, \dots, x_p\right) & \text{if } \left(x_1, \dots, x_p\right) \text{ is a valid input for } \square, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.5}$$

In the sequel, we shall provide properties and constructions involving complete biproducts. Nevertheless, all these apply also on general biproducts since one can always work with the completion of a noncomplete biproduct.

2.1.3. *Spaces of complete biproducts.* The set of all the complete biproducts of arity p and coarity q on $\mathbb{K}\langle C\rangle$ has a structure of a $\mathbb{K}$-vector space. Indeed, if $\square_1$ and $\square_2$ are two such biproducts, the *addition* of $\square_1$ and $\square_2$ is the biproduct $\square_1 + \square_2$ defined by

$$\left(\square_1 + \square_2\right)\left(x_1, \dots, x_p\right) := \square_1\left(x_1, \dots, x_p\right) + \square_2\left(x_1, \dots, x_p\right) \tag{2.1.6}$$

for any object $\left(x_1, \dots, x_p\right)$ of $\mathbf{List}_{\{p\}}(C)$. Moreover, for any coefficient $\lambda \in \mathbb{K}$, if $\square$ is such a biproduct, the *scalar multiplication* of $\square$ by λ is the biproduct $\lambda\square$ defined by

$$(\lambda\square)\left(x_1, \dots, x_p\right) := \lambda\square\left(x_1, \dots, x_p\right) \tag{2.1.7}$$

for any object $\left(x_1, \dots, x_p\right)$ of $\mathbf{List}_{\{p\}}(C)$.

2.1.4. *Structure coefficient maps.* Let

$$\xi : \mathbf{List}_{\{p\}}(C) \times \mathbf{List}_{\{q\}}(C) \to \mathbb{K} \tag{2.1.8}$$

be a map such for any object $\left(x_1, \dots, x_p\right)$ of $\mathbf{List}_{\{p\}}(C)$, there are finitely many objects $\left(y_1, \dots y_q\right)$ of $\mathbf{List}_{\{q\}}(C)$ such that $\xi\left(\left(x_1, \dots, x_p\right), \left(y_1, \dots, y_q\right)\right) \neq 0$. From this map ξ, let the complete biproduct

$$\square : \mathbb{K}\left\langle \mathbf{List}_{\{p\}}(C)\right\rangle \to \mathbb{K}\left\langle \mathbf{List}_{\{q\}}(C)\right\rangle \tag{2.1.9}$$

satisfying, for any objects x_1, ..., x_p of C,

$$\square\left(x_1, \dots, x_p\right) = \sum_{\left(y_1, \dots, y_q\right) \in \mathbf{List}_{\{q\}}(C)} \xi\left(\left(x_1, \dots, x_p\right), \left(y_1, \dots, y_q\right)\right) \left(y_1, \dots, y_q\right). \tag{2.1.10}$$

Due to the condition satisfied by ξ, there is a finite number of tuples $\left(y_1, \dots, y_q\right)$ appearing in the right member of (2.1.10). Hence, all the $\square\left(x_1, \dots, x_p\right)$ are C-polynomials, so that $\square$ is a well-defined complete biproduct on $\mathbb{K}\langle C\rangle$.

Conversely, from any complete biproduct $\square$ on $\mathbb{K}\langle C\rangle$ of arity p and coarity q, one can recover a map ξ of the form (2.1.8) such that (2.1.10) holds. We call ξ the *structure coefficient map* of $\square$. Beside, we say that $\square$ is *degenerate* if all its structure coefficients are zero.

2.1.5. *Dual biproducts.* Assume here that C is combinatorial so that we can identify $\mathbb{K}\langle C\rangle$ with its dual $\mathbb{K}\langle C\rangle^\star$ as mentioned in Section 1.2.7. Given a complete biproduct $\square$ on $\mathbb{K}\langle C\rangle$ of arity p and coarity q, let

$$\square^\star : \mathbb{K}\left\langle \mathbf{List}_{\{q\}}(C)\right\rangle^\star \to \mathbb{K}\left\langle \mathbf{List}_{\{p\}}(C)\right\rangle^\star \tag{2.1.11}$$

be the map linearly defined, for all objects $(y_1, \dots, y_q)$ of $\mathbf{List}_{\{q\}}(C)$, by

$$\square^\star\left(y_1, \dots, y_q\right) := \sum_{\left(x_1, \dots, x_p\right) \in \mathbf{List}_{\{p\}}(C)} \left\langle \square\left(x_1, \dots, x_p\right), \left(y_1, \dots, y_q\right)\right\rangle \left(x_1, \dots, x_p\right). \tag{2.1.12}$$

In the case where (2.1.12) is a finite sum for any object $(y_1, \dots, y_q)$ of $\mathbf{List}_{\{q\}}(C)$, its right member is a C-polynomial so that $\square^\star$ is a biproduct on $\mathbb{K}\langle C\rangle^\star$, called *dual biproduct* of $\square$.

Observe that $\square^\star$ is of arity q and coarity p, and is complete. Observe also that in (2.1.12), the coefficient $\left\langle \square\left(x_1, \dots, x_p\right), \left(y_1, \dots, y_q\right)\right\rangle$ is in fact equal to $\xi\left(\left(x_1, \dots, x_p\right), \left(y_1, \dots, y_q\right)\right)$ where ξ is the structure coefficient map of $\square$. Hence, if one sees the map ξ as a matrix whose rows are indexed by the $(x_1, \dots, x_p)$ and the columns by the $(y_1, \dots, y_q)$, the structure coefficient map of $\square^\star$ is the transpose of this matrix.

2.2. Products on polynomial spaces. We focus here on products, that are particular biproducts on polynomial spaces. In all this section, $\mathbb{K}\langle C\rangle$ is a polynomial space.

2.2.1. *Products.* A *product* is a biproduct of coarity 1. Let

$$\star : \mathbb{K}\left\langle \left[C(J_1), \dots, C\left(J_p\right)\right]_\times \right\rangle \to \mathbb{K}\langle C\rangle \tag{2.2.1}$$

be a product of arity $p \in \mathbb{N}$, where J_1, ..., J_p are nonempty subsets of I. When there is a map $\omega : J_1 \times \cdots \times J_p \to I$ satisfying, for any valid input $(x_1, \dots, x_p)$ for $\star$,

$$\star\left(x_1, \dots, x_p\right) \in \mathbb{K}\langle C\rangle\left(\omega\left(\operatorname{ind}(x_1), \dots, \operatorname{ind}\left(x_p\right)\right)\right), \tag{2.2.2}$$

we say that $\star$ is *ω-concentrated* (or simply *concentrated* when it is not useful specify ω). In intuitive terms, this means that the indexes of the monomials appearing in a product depend only on the indexes of their operands.

There is a close connection between products on collections (see Section 1.1.7 of Chapter 1) and products on polynomial spaces. Indeed, when C is an I-collection with a product

$$\star : C(J_1) \times \cdots \times C\left(J_p\right) \to C \tag{2.2.3}$$

of arity p, $\star$ gives rise to a product

$$\bar{\star} : \mathbb{K}\left\langle \left[C(J_1), \dots, C\left(J_p\right)\right]_\times \right\rangle \to \mathbb{K}\langle C\rangle \tag{2.2.4}$$

on $\mathbb{K}\langle C\rangle$ defined by extending $\star$ by linearity. This product $\bar{\star}$ is called the *linearization* of $\star$. When $\star$ is a ω-concentrated product on C (see the aforementioned section), its linearization $\bar{\star}$ is an ω-concentrated product on $\mathbb{K}\langle C\rangle$.

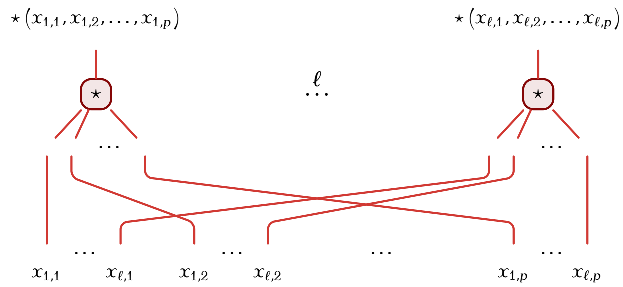
2.2.2. *Tensor powers.* By considering that $\star$ is a product on $\mathbb{K}\langle C\rangle$ of the form (2.2.1), let us introduce for any $\ell \in \mathbb{N}_{\geqslant 1}$, the biproduct

$$\mathrm{T}_\ell(\star) : \mathbb{K}\left\langle \left[\, \mathbf{List}_{\{\ell\}}(C(J_1)), \dots, \mathbf{List}_{\{\ell\}}(C(J_p))\right]_\times \right\rangle \to \mathbb{K}\left\langle \mathbf{List}_{\{\ell\}}(C)\right\rangle \tag{2.2.5}$$

defined linearly by

$$\begin{aligned} &\mathrm{T}_\ell(\star)\left((x_{1,1}, \dots, x_{\ell,1}), (x_{1,2}, \dots, x_{\ell,2}), \dots, (x_{1,p}, \dots, x_{\ell,p})\right) \\ &\qquad := \left(\star\left(x_{1,1}, \dots, x_{1,p}\right), \star\left(x_{2,1}, \dots, x_{2,p}\right), \dots, \star\left(x_{\ell,1}, \dots, x_{\ell,p}\right)\right), \end{aligned} \tag{2.2.6}$$

for all $(x_{1,k}, \dots, x_{\ell,k}) \in \mathbf{List}_{\{\ell\}}(C(J_k))$, $k \in [p]$. Graphically, $\mathrm{T}_\ell(\star)$ is the biproduct



. (2.2.7)

This product $\mathrm{T}_\ell(\star)$ can be seen as the ℓth-tensor power of $\star$ seen as a linear map. For this reason, $\mathrm{T}_\ell(\star)$ is called the *ℓth tensor power* of $\star$.

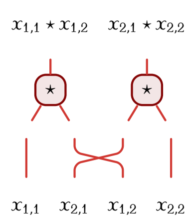
Let us provide an example. When $\star$ is a complete binary product on $\mathbb{K}\langle C\rangle$, $\mathrm{T}_2(\star)$ is of the form

$$\mathrm{T}_2(\star) : \mathbb{K}\left\langle \left[\, \mathbf{List}_{\{2\}}(C), \mathbf{List}_{\{2\}}(C)\right]_\times \right\rangle \to \mathbb{K}\left\langle \mathbf{List}_{\{2\}}(C)\right\rangle \tag{2.2.8}$$

and it satisfies

$$(x_{1,1}, x_{2,1})\ \mathrm{T}_2(\star)\ (x_{1,2}, x_{2,2}) = (x_{1,1} \star x_{1,2}, x_{2,1} \star x_{2,2}) \tag{2.2.9}$$

for all objects $(x_{1,1}, x_{2,1})$ and $(x_{1,2}, x_{2,2})$ of $\mathbf{List}_{\{2\}}(C)$. In (2.2.9), since $\star$ and $\mathrm{T}_2(\star)$ are binary products, we denote them in infix way. We follow this convention in all this text. Graphically, $\mathrm{T}_2(\star)$ is the biproduct



(2.2.10)

2.2.3. *Products of arity zero.* A product η of arity 0 on $\mathbb{K}\langle C\rangle$ is of the form

$$\eta : \mathbb{K}\langle [\,]_\times \rangle \simeq \mathbb{K} \to \mathbb{K}\langle C\rangle \tag{2.2.11}$$

where, as explained in Section 1.2.3 of Chapter 1, the empty Cartesian product $[\,]_\times$ of collections contains exactly one element, namely the empty tuple. Hence, η is totally determined by the image $\eta(1) \in \mathbb{K}\langle C\rangle$ where $1 \in \mathbb{K}$. In this way, there is a correspondence between products of arity zero and elements of $\mathbb{K}\langle C\rangle$. By a slight abuse of notation, we shall write sometimes η instead of $\eta(1)$. In this way, η is no longer a map but an element of $\mathbb{K}\langle C\rangle$.

2.2.4. *Product properties.* We now list some properties a product $\star$ on $\mathbb{K}\langle C\rangle$ of the form (2.2.1) can satisfy.

In the particular case where $\mathbb{K}\langle C\rangle$ is a graded polynomial space, $\star$ is *graded* if $\star$ is ω-concentrated for the map $\omega : \mathbb{N}^p \to \mathbb{N}$ defined by $\omega\left((n_1, \dots, n_p)\right) := n_1 + \dots + n_p$. This notion is analogous to the one of the same name for collections with products exposed in Section 1.1.7 of Chapter 1. Observe that when C is a graded collection with a graded product $\star$, its linearization $\bar{\star}$ is a graded product on $\mathbb{K}\langle C\rangle$.

We now assume that $\mathbb{K}\langle C\rangle$ is any polynomial space. If $\{\mathrm{B}_x : x \in C\}$ is a basis of $\mathbb{K}\langle C\rangle$ such that, for any valid input $\left(x_1, \dots, x_p\right)$ for $\star$ there is an object x of C satisfying

$$\star\left(\mathrm{B}_{x_1}, \dots, \mathrm{B}_{x_p}\right) = \mathrm{B}_x, \tag{2.2.12}$$

we say that the B-basis of $\mathbb{K}\langle C\rangle$ is a *set-basis* with respect to $\star$.

Assume now that $\star$ is of arity 2 so that $\star$ is of the form

$$\star : \mathbb{K}\langle [C(J_1), C(J_2)]_\times\rangle \to \mathbb{K}\langle C\rangle \tag{2.2.13}$$

where J_1 and J_2 are two nonempty subsets of I. In the case where $\mathrm{Im}(\star)$ is contained in $\mathbb{K}\langle C(J_1 \cap J_2)\rangle$, the *associator* of $\star$ is the ternary product

$$(-,-,-)_\star : \mathbb{K}\langle [C(J_1), C(J_1 \cap J_2), C(J_2)]_\times\rangle \to \mathbb{K}\langle C\rangle \tag{2.2.14}$$

defined linearly for all valid inputs (x_1, x_2, x_3) for $(-,-,-)_\star$ by

$$(x_1, x_2, x_3)_\star := (x_1 \star x_2) \star x_3 - x_1 \star (x_2 \star x_3). \tag{2.2.15}$$

When, for all valid inputs (x_1, x_2, x_3) for $(-,-,-)_\star$, one has

$$(x_1, x_2, x_3)_\star = 0, \tag{2.2.16}$$

we say that $\star$ is *associative*. The *commutator* of $\star$ is the binary product

$$[-,-]_\star : \mathbb{K}\langle [C(J_1 \cap J_2), C(J_2 \cap J_1)]_\times\rangle \to \mathbb{K}\langle C\rangle \tag{2.2.17}$$

defined linearly for all valid inputs (x_1, x_2) for $[-,-]_\star$ by

$$[x_1, x_2]_\star := x_1 \star x_2 - x_2 \star x_1. \tag{2.2.18}$$

When, for all valid inputs (x_1, x_2) for $[x_1, x_2]_\star$, one has

$$[x_1, x_2]_\star = 0, \tag{2.2.19}$$

the product $\star$ is *commutative*. When there is a product $\mathbb{1}_\star$ of arity 0 such that, for all $x \in C(J_1 \cap J_2)$,

$$x \star \mathbb{1}_\star(1) = x = \mathbb{1}_\star(1) \star x, \tag{2.2.20}$$

we say that $\star$ is *unitary* and that $\mathbb{1}_\star$ is the *unit* of $\star$. Observe that if $\mathbb{K}\langle C\rangle$ is graded and $\star$ is a graded product, $\mathbb{1}_\star(1)$ is necessarily of degree 0.

2.2.5. *Coproducts.* A *coproduct* is a biproduct of arity 1. Observe that when $\mathbb{K}\langle C\rangle$ is combinatorial and that $\star$ is a concentrated complete product, its dual $\star^\star$ is a coproduct. This is not true in general when $\mathbb{K}\langle C\rangle$ is not combinatorial or not concentrated since the conditions exposed in Section 2.1.5 for the well definition of $\star^\star$ could not be satisfied.

A coproduct υ of coarity 0 on $\mathbb{K}\langle C\rangle$ is of the form

$$\upsilon : \mathbb{K}\langle C\rangle \to \mathbb{K}\langle [\,]_\times\rangle \simeq \mathbb{K} \tag{2.2.21}$$

and can therefore be seen as a linear form on $\mathbb{K}\langle C\rangle$.

All the properties of products defined in Sections 2.2.1 and 2.2.4 hold for coproducts which admit dual products in the following way. For any property P on products, we say that a coproduct Δ admitting a product $\Delta^\star$ as dual *satisfies the property "coP"* if $\Delta^\star$ satisfies P. For instance, Δ is *cograded* if $\Delta^\star$ is graded, and Δ is *coassociative* if $\Delta^\star$ is associative. Moreover, Δ is *counitary* if there exists a normal form $\mathbb{1}_\Delta$ on $\mathbb{K}\langle C\rangle$ called *counit* such that its dual $\mathbb{1}_\Delta^\star$ is the unit of $\Delta^\star$.

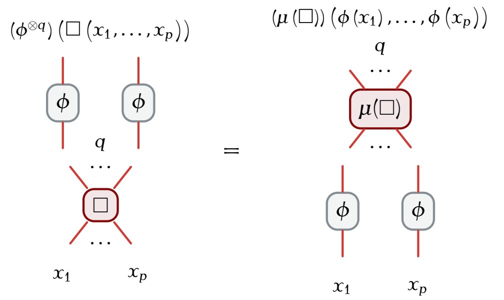
2.3. Polynomial bialgebras. We now consider polynomial spaces endowed with a set of biproducts. The main definitions and properties of these structures are listed.

2.3.1. *Elementary definitions.* A *polynomial bialgebra* is a pair $(\mathbb{K}\langle C\rangle, \mathscr{B})$ where $\mathbb{K}\langle C\rangle$ is a polynomial space endowed with a (possibly infinite) set $\mathscr{B}$ of biproducts. When $\mathscr{B}$ contains only products (resp. coproducts), $(\mathbb{K}\langle C\rangle, \mathscr{B})$ is a *polynomial algebra* (resp. *polynomial coalgebra*). To not overload the notation, we shall simply write $\mathbb{K}\langle C\rangle$ instead of $(\mathbb{K}\langle C\rangle, \mathscr{B})$ when the context is clear.

Let $(\mathbb{K}\langle C_1\rangle, \mathscr{B}_1)$ and $(\mathbb{K}\langle C_2\rangle, \mathscr{B}_2)$ be polynomial bialgebras. These bialgebras are *μ-compatible* if there exists a bijective map $\mu : \mathscr{B}_1 \to \mathscr{B}_2$ that sends any biproduct of $\mathscr{B}_1$ to a biproduct of $\mathscr{B}_2$ of the same arity, the same coarity, and the same index domain. When $(\mathbb{K}\langle C_1\rangle, \mathscr{B}_1)$ and $(\mathbb{K}\langle C_2\rangle, \mathscr{B}_2)$ are μ-compatible, a *μ-polynomial bialgebra morphism* (or simply a *polynomial bialgebra morphism* when there is no ambiguity) from $\mathbb{K}\langle C_1\rangle$ to $\mathbb{K}\langle C_2\rangle$ is a polynomial space morphism $\phi : \mathbb{K}\langle C_1\rangle \to \mathbb{K}\langle C_2\rangle$ such that

$$\left(\phi^{\otimes q}\right)\left(\square\left(x_1, \dots, x_p\right)\right) = \left(\mu\left(\square\right)\right)\left(\phi\left(x_1\right), \dots, \phi\left(x_p\right)\right) \tag{2.3.1}$$

for all biproducts $\square$ or arity p and coarity q of $\mathcal{B}_1$, and valid inputs $(x_1, \dots, x_p)$ for $\square$, where $\phi^{\otimes q}$ is the qth tensor power $\mathrm{T}_q(\phi)$ of ϕ. Graphically, (2.3.1) reads as



(2.3.2)

In the special case where $\square$ is a product η of arity 0 (see Section 2.2.3) of $\mathcal{B}_1$, (2.3.1) implies that the morphism ϕ satisfies $\phi(\eta) = \mu(\eta)$.

Besides, when $(\mathbb{K}\langle C_1\rangle, \mathcal{B}_1)$ and $(\mathbb{K}\langle C_2\rangle, \mathcal{B}_2)$ are μ-compatible, $(\mathbb{K}\langle C_2\rangle, \mathcal{B}_2)$ is a *sub-bialgebra* of $(\mathbb{K}\langle C_1\rangle, \mathcal{B}_1)$ if there is an injective μ-polynomial bialgebra morphism from $\mathbb{K}\langle C_2\rangle$ to $\mathbb{K}\langle C_1\rangle$. Let $(\mathbb{K}\langle C\rangle, \mathcal{B})$ be a polynomial bialgebra. For any subset $\mathfrak{G}$ of $\mathbb{K}\langle C\rangle$, the *bialgebra generated* by $\mathfrak{G}$ is the smallest sub-bialgebra $\mathbb{K}\langle C\rangle^{\mathfrak{G}}$ of $\mathbb{K}\langle C\rangle$ containing $\mathfrak{G}$. When $\mathbb{K}\langle C\rangle^{\mathfrak{G}} = \mathbb{K}\langle C\rangle$ and $\mathfrak{G}$ is minimal with respect to the inclusion among the subsets of $\mathfrak{G}$ satisfying this property, $\mathfrak{G}$ is a *minimal generating set* of $\mathbb{K}\langle C\rangle$. A bialgebra $\mathbb{K}\langle C\rangle$ can have several minimal generating sets.

A *polynomial bialgebra ideal* of $\mathbb{K}\langle C\rangle$ is a space $\mathcal{V}$ included in $\mathbb{K}\langle C\rangle$ such that

$$\square(x_1, \dots, x_{i-1}, f, x_{i+1}, \dots, x_p) \in \bigoplus_{j \in [q]} \mathbb{K}\left\langle \mathbf{List}_{\{j-1\}}(C)\right\rangle \otimes \mathcal{V} \otimes \mathbb{K}\left\langle \mathbf{List}_{\{q-j\}}(C)\right\rangle \tag{2.3.3}$$

for all biproducts $\square$ on $\mathbb{K}\langle C\rangle$ of $\mathcal{B}$ of arity p and coarity q, and all $i \in [p]$, $f \in \mathcal{V}$, $x_k \in C$, $k \in [p] \setminus \{i\}$ such that the left member of (2.3.3) is well-defined. Note that in (2.3.3), we use the identification (1.2.11) between the space of all the tensors on $\mathbb{K}\langle C\rangle$ of a given order $\ell \in \mathbb{N}$ and the polynomial space on $\mathbf{List}_{\{\ell\}}(C)$. Given a polynomial bialgebra ideal $\mathcal{V}$ of $\mathbb{K}\langle C\rangle$, the *quotient bialgebra* $\mathbb{K}\langle C\rangle/_{\mathcal{V}}$ of $\mathbb{K}\langle C\rangle$ by $\mathcal{V}$ is defined as follows. Let $\theta : \mathbb{K}\langle C\rangle \to \mathbb{K}\langle C\rangle/_{\mathcal{V}}$ be the canonical surjection map from $\mathbb{K}\langle C\rangle$ to its quotient space $\mathbb{K}\langle C\rangle/_{\mathcal{V}}$. Any biproduct $\square$ on $\mathbb{K}\langle C\rangle$ of $\mathcal{B}$ of arity p and coarity q gives rise to a biproduct $\square_{\mathcal{V}}$ on $\mathbb{K}\langle C\rangle/_{\mathcal{V}}$ of arity p and coarity q, and with the same index domain as the one of $\square$. This biproduct $\square_{\mathcal{V}}$ is defined linearly, for any valid input $(x_1, \dots, x_p)$ for $\square$, by

$$\square_{\mathcal{V}}(\theta(x_1), \dots, \theta(x_p)) := \mathrm{T}_q(\theta)\left(\square(x_1, \dots, x_p)\right). \tag{2.3.4}$$

Condition (2.3.3) ensures that (2.3.4) is well-defined.

2.3.2. *Combinatorial polynomial bialgebras.* In practice, and even more so in this book, most of the encountered polynomial bialgebras are of the form $(\mathbb{K}\langle C\rangle, \mathcal{B})$ where $\mathcal{B}$ contains only products and coproducts. We say in this case that $\mathbb{K}\langle C\rangle$ is *uniform*. Moreover, in most practical cases, C is a graded, a bigraded, or a colored combinatorial collection. When $(\mathbb{K}\langle C\rangle, \mathcal{B})$ is combinatorial, uniform, and all the products and coproducts of $\mathcal{B}$ are complete and concentrated, the *dual bialgebra* of $\mathbb{K}\langle C\rangle$ is the bialgebra $\left(\mathbb{K}\langle C\rangle^{\star}, \mathcal{B}^{\star}\right)$ where $\mathcal{B}^{\star}$ is the set of the dual biproducts of the biproducts of $\mathcal{B}$. Note that this bialgebra is also uniform.

It is very common, given a uniform combinatorial bialgebra $(\mathbb{K}\langle C\rangle, \mathcal{B})$, to endow C with a structure of a combinatorial poset $(C, \preccurlyeq)$ in order to construct $\mathrm{B}^{\preccurlyeq}$-families (see Section 1.3.2). For instance, when a biproduct $\square$ has a complicated structure coefficient map, considering an adequate partial order relation $\preccurlyeq$ on C such that the $\mathrm{B}^{\preccurlyeq}$-family is a set-basis with respect to $\square$ allows to infer properties of $\square$ (such as minimal generating sets of $\mathbb{K}\langle C\rangle$, a description of the nontrivial relations satisfied by these generators, or even freeness properties).

2.3.3. *Set-theoretic algebras.* Let $(\mathbb{K}\langle C\rangle, \mathcal{P})$ be a polynomial algebra and $\{\mathrm{B}_x : x \in C\}$ be one of its bases which is additionally a set-basis for all products of $\mathcal{P}$ at the same time. In this case, it is possible to forget the linear structure of $\mathbb{K}\langle C\rangle$. Indeed, each product $\bar{\star}$ of arity p on $\mathbb{K}\langle C\rangle$ gives rise to a product $\star$ on C defined, for any valid input $\left(x_1, \dots, x_p\right)$ for $\bar{\star}$, by

$$\star\left(x_1, \dots, x_p\right) := y \tag{2.3.5}$$

whenever

$$\bar{\star}\left(\mathrm{B}_{x_1}, \dots, \mathrm{B}_{x_p}\right) = \mathrm{B}_y \tag{2.3.6}$$

for an $y \in C$. This endows the collection C with products in the sense of Section 1.1.7 of Chapter 1. We say in this case that C is a *set-theoretic algebra*.

A large part of the concepts presented above about bialgebras work for the particular case of set-theoretic algebras with some adjustments. For instance, to define quotients of a set-theoretic algebra $(C, \mathcal{P}')$, we do not work with polynomial algebra ideals but with congruences of set-theoretic algebras. To be a little more precise, a *set-theoretic algebra congruence* is a relation $\equiv$ on C which is an equivalence relation satisfying

$$\star\left(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_p\right) \equiv \star\left(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_p\right) \tag{2.3.7}$$

for all products $\star$ of arities p, all $i \in [p]$ such that x_i and x_i' are objects of C satisfying $x_i \equiv x_i'$, and all valid inputs $\left(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_p\right)$ and $\left(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_p\right)$ for $\star$.

In the sequel, if "N" is the name of an algebraic structure, we call *"set-N"* the corresponding set-theoretic structure. For instance, a set-theoretic unitary associative algebra is a monoid. We shall further encounter in this way set-operads, colored set-operads, and set-pros.

3. Types of polynomial bialgebras

A *type of polynomial bialgebra* is specified by biproduct symbols together with their arities and coarities, and the possible relations between them (like, for instance, associativity, commutativity, cocommutativity, or distributivity). In this section, we list some of the very ordinary types of uniform polynomial bialgebras in combinatorics like associative, dendriform and pre-Lie algebras, and Hopf bialgebras. We give concrete examples for each of these.

3.1. Associative and coassociative algebras. An *associative algebra* is a polynomial space endowed with a complete associative binary product. An associative algebra is *unitary* if its product is unitary. Besides, an associative algebra is *commutative* if its product is commutative. To perfectly fit to the definition of types of bialgebras given above, the type of unitary associative and commutative algebras is made of a product symbol $\star$ of arity 2 and a product symbol $\mathbb{1}$ of arity 0 together with the relations $(f_1, f_2, f_3)_\star = 0$, $[f_1, f_2]_\star = 0$, $f \star \mathbb{1}(1) = f = \mathbb{1}(1) \star f$, where and f_1, f_2, and f_3 are any elements of the space. Moreover, by following the definitions of Section 2.3.1, a polynomial space morphism $\phi : \mathbb{K}\langle C_1 \rangle \to \mathbb{K}\langle C_2 \rangle$ is a unitary commutative algebra morphism between two unitary commutative algebras $\mathbb{K}\langle C_1 \rangle$ and $\mathbb{K}\langle C_2 \rangle$ if for any $f_1, f_2 \in \mathbb{K}\langle C_1 \rangle$, $\phi(f_1 \star_1 f_2) = \phi(f_1) \star_2 \phi(f_2)$ and $\phi(\mathbb{1}_1) = \mathbb{1}_2$, where $\star_1$ (resp. $\star_2$) is the binary product of $\mathbb{K}\langle C_1 \rangle$ (resp. $\mathbb{K}\langle C_2 \rangle$) and $\mathbb{1}_1$ (resp. $\mathbb{1}_2$) is the unit of $\mathbb{K}\langle C_1 \rangle$ (resp. $\mathbb{K}\langle C_2 \rangle$).

A *coassociative coalgebra* is a polynomial space endowed with a coassociative coproduct. A coassociative coalgebra is *counitary* if its coproduct is counitary. Besides, a coassociative coalgebra is *cocommutative* if its coproduct is cocommutative.

In all this section, $A := \{\mathrm{a}_1, \dots, \mathrm{a}_\ell\}$ is a finite alphabet.

3.1.1. *Concatenation algebra.* The *concatenation product* is the complete binary product $\cdot$ on $\mathbb{K}\langle A^* \rangle$ defined as the concatenation product of A^* extended by linearity. Since $\cdot$ is graded and all $\mathbb{K}\langle A^n \rangle$ are finite dimensional for all $n \in \mathbb{N}$, $(\mathbb{K}\langle A^* \rangle, \cdot)$ is a combinatorial graded algebra. Moreover, $\cdot$ is associative, noncommutative, and admits the product of arity zero ϵ, where ϵ is the empty word, as unit so that $(\mathbb{K}\langle A^* \rangle, \cdot, \epsilon)$ is a unitary noncommutative associative algebra called *concatenation algebra* on A.

3.1.2. *Shuffle algebra.* The *shuffle product* is the binary product $\sqcup\!\sqcup$ on $\mathbb{K}\langle A^* \rangle$ linearly and recursively defined by

$$u \sqcup\!\sqcup \epsilon := u =: \epsilon \sqcup\!\sqcup u, \tag{3.1.1a}$$

$$ua \sqcup\!\sqcup vb := (u \sqcup\!\sqcup vb) \cdot a + (ua \sqcup\!\sqcup v) \cdot b \tag{3.1.1b}$$

for any $u, v \in A^*$ and $a, b \in A$, where $\cdot$ is the concatenation product of the concatenation algebra on A. Intuitively, $\sqcup\!\sqcup$ consists in summing in all the ways of interlacing the letters of the two words as input. For instance,

$$\begin{aligned}
\mathtt{a}_1\mathtt{a}_2 \sqcup\!\sqcup \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 &= \mathtt{a}_1\mathtt{a}_2\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2 \\
&\quad + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1 \\
&\quad + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2 \qquad (3.1.2) \\
&= 2\mathtt{a}_1\mathtt{a}_2\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 \\
&\quad + 2\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1 + 3\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2.
\end{aligned}$$

Since $\sqcup\!\sqcup$ is graded and all $\mathbb{K}\langle A^n\rangle$ are finite dimensional for all $n \in \mathbb{N}$, $(\mathbb{K}\langle A^*\rangle, \sqcup\!\sqcup)$ is a combinatorial graded algebra. Moreover, $\sqcup\!\sqcup$ is associative, commutative, and admits ϵ as unit so that $(\mathbb{K}\langle A^*\rangle, \sqcup\!\sqcup, \epsilon)$ is a unitary commutative associative algebra called *shuffle algebra* on A.

3.1.3. *Deconcatenation coalgebra.* Let $\Delta_\cdot$ be the dual coproduct of the concatenation product $\cdot$ of $\mathbb{K}\langle A^*\rangle$ considered in Section 3.1.1. By (2.1.12), for all $u \in A^*$,

$$\Delta_\cdot(u) = \sum_{v,w \in A^*} \langle v \cdot w, u\rangle\, (v, w) = \sum_{\substack{v,w \in A^* \\ v \cdot w = u}} (v, w). \qquad (3.1.3)$$

This coproduct is known as the *deconcatenation coproduct*. For instance,

$$\Delta_\cdot(\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2) = (\epsilon, \mathtt{a}_1\mathtt{a}_1\mathtt{a}_2) + (\mathtt{a}_1, \mathtt{a}_1\mathtt{a}_2) + (\mathtt{a}_1\mathtt{a}_1, \mathtt{a}_2) + (\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2, \epsilon). \qquad (3.1.4)$$

Let also υ be the dual coproduct of the unit ϵ for the concatenation product considered in Section 3.1.1. This coproduct υ satisfies $\upsilon(\epsilon) = 1$ and $\upsilon(u) = 0$ for all nonempty words u. The coalgebra $(\mathbb{K}\langle A^*\rangle, \Delta_\cdot, \upsilon)$ is a counitary noncocommutative coassociative coalgebra called *deconcatenation coalgebra* on A.

3.1.4. *Unshuffle coalgebra.* Let $\Delta_{\sqcup\!\sqcup}$ be the dual coproduct of the shuffle product $\sqcup\!\sqcup$ of $\mathbb{K}\langle A^*\rangle$. Again by (2.1.12), for all $u \in A^*$,

$$\Delta_{\sqcup\!\sqcup}(u) = \sum_{v,w \in A^*} \langle v \sqcup\!\sqcup w, u\rangle\, (v, w). \qquad (3.1.5)$$

The coefficient $\langle v \sqcup\!\sqcup w, u\rangle$ counts the number of ways to decompose u as two disjoint subwords v and w, and thus,

$$\Delta_{\sqcup\!\sqcup}(u) = \sum_{\substack{J_1, J_2 \subseteq [|u|] \\ J_1 \sqcup J_2 = [|u|]}} \left(u_{|J_1}, u_{|J_2}\right). \qquad (3.1.6)$$

This coproduct can alternatively be expressed by

$$\Delta_{\sqcup\!\sqcup}(\mathtt{a}) = (\epsilon, \mathtt{a}) + (\mathtt{a}, \epsilon) \qquad (3.1.7)$$

for any $\mathtt{a} \in A$, and

$$\Delta_{\sqcup\!\sqcup}(u) = \prod_{i \in [|u|]} \Delta(u(i)) \qquad (3.1.8)$$

for any $u \in A^*$, where the product of (3.1.8) denotes the iterated version of the 2nd tensor power $\mathrm{T}_2(\cdot)$ of the concatenation product $\cdot$ (see Section 2.2.2). This product $\mathrm{T}_2(\cdot)$ is associative due to the fact that $\cdot$ is associative, and thus, its iterated version is well-defined. This coproduct is known as the *unshuffling coproduct*. For instance,

$$\begin{aligned}\Delta_{\sqcup\!\sqcup}(\mathrm{a}_1\mathrm{a}_1\mathrm{a}_2) &= ((\epsilon,\mathrm{a}_1)+(\mathrm{a}_1,\epsilon))\ \mathrm{T}_2(\cdot)\ ((\epsilon,\mathrm{a}_1)+(\mathrm{a}_1,\epsilon))\ \mathrm{T}_2(\cdot)\ ((\epsilon,\mathrm{a}_2)+(\mathrm{a}_2,\epsilon))\\ &= (\epsilon,\mathrm{a}_1\mathrm{a}_1\mathrm{a}_2)+(\mathrm{a}_2,\mathrm{a}_1\mathrm{a}_1)+(\mathrm{a}_1,\mathrm{a}_1\mathrm{a}_2)+(\mathrm{a}_1\mathrm{a}_2,\mathrm{a}_1)\\ &\quad+(\mathrm{a}_1,\mathrm{a}_1\mathrm{a}_2)+(\mathrm{a}_1\mathrm{a}_2,\mathrm{a}_1)+(\mathrm{a}_1\mathrm{a}_1,\mathrm{a}_2)+(\mathrm{a}_1\mathrm{a}_1\mathrm{a}_2,\epsilon)\\ &= (\epsilon,\mathrm{a}_1\mathrm{a}_1\mathrm{a}_2)+(\mathrm{a}_2,\mathrm{a}_1\mathrm{a}_1)+2\,(\mathrm{a}_1,\mathrm{a}_1\mathrm{a}_2)\\ &\quad+2\,(\mathrm{a}_1\mathrm{a}_2,\mathrm{a}_1)+(\mathrm{a}_1\mathrm{a}_1,\mathrm{a}_2)+(\mathrm{a}_1\mathrm{a}_1\mathrm{a}_2,\epsilon)\,.\end{aligned} \tag{3.1.9}$$

The coalgebra $(\mathbb{K}\langle A^*\rangle, \sqcup\!\sqcup, v)$, where v is the counit considered in Section 3.1.3, is a counitary cocommutative coassociative coalgebra called *unshuffle coalgebra* on A.

3.2. Dendriform algebras. A *dendriform algebra* is a polynomial space $\mathbb{K}\langle C\rangle$ endowed with two complete binary products $\prec$ and $\succ$ satisfying

$$(x_1 \prec x_2) \prec x_3 = x_1 \prec (x_2 \prec x_3) + x_1 \prec (x_2 \succ x_3)\,, \tag{3.2.1a}$$

$$(x_1 \succ x_2) \prec x_3 = x_1 \succ (x_2 \prec x_3)\,, \tag{3.2.1b}$$

$$(x_1 \prec x_2) \succ x_3 + (x_1 \succ x_2) \succ x_3 = x_1 \succ (x_2 \succ x_3)\,, \tag{3.2.1c}$$

for all objects x_1, x_2, and x_3 of C.

A polynomial algebra $(\mathbb{K}\langle C\rangle, \star)$, where $\star$ is a binary product, admits a *dendriform algebra structure* if its product can be split into two operations

$$\star = \prec + \succ, \tag{3.2.2}$$

where $\prec$ and $\succ$ are two non-degenerate binary products such that $(\mathbb{K}\langle C\rangle, \prec, \succ)$ is a dendriform algebra. Expression (3.2.2) uses the addition of biproducts exposed in Section 2.1.3. Observe that if $(\mathbb{K}\langle C\rangle, \star)$ admits a dendriform algebra structure, $\star$ is associative. The associativity of $\prec + \succ$ is a consequence of Relations (3.2.1a), (3.2.1b), and (3.2.1c) of dendriform algebras.

In all this section, $A := \{\mathrm{a}_1, \dots, \mathrm{a}_\ell\}$ is a finite alphabet.

3.2.1. *Shuffle dendriform algebra.* Consider on $\mathbb{K}\langle A^*\rangle$ the binary products $\prec$ and $\succ$ defined linearly and recursively by

$$u \prec \epsilon := u =: \epsilon \succ u, \tag{3.2.3a}$$

$$w \succ \epsilon =: 0 := \epsilon \prec w, \tag{3.2.3b}$$

$$ua \prec v := (u \sqcup\!\sqcup v) \cdot a, \tag{3.2.3c}$$

$$u \succ vb := (u \sqcup\!\sqcup v) \cdot b \tag{3.2.3d}$$

for any $u, v \in A^*$, $w \in A^+$, and $a, b \in A$, where $\cdot$ is the concatenation product of words. In other words, $u \prec v$ (resp. $u \succ v$) is the sum of all the words w obtained by shuffling u and v such that the last letter of w comes from u (resp. v). For example,

$$\begin{aligned} \mathtt{a}_1\mathtt{a}_2 \prec \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 &= \mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2 \\ &= \mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2 + 3\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2, \end{aligned} \tag{3.2.4a}$$

$$\begin{aligned} \mathtt{a}_1\mathtt{a}_2 \succ \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 &= \mathtt{a}_1\mathtt{a}_2\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 \\ &\quad + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1 \\ &= 2\mathtt{a}_1\mathtt{a}_2\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1 + \mathtt{a}_2\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1 + 2\mathtt{a}_2\mathtt{a}_1\mathtt{a}_1\mathtt{a}_2\mathtt{a}_1. \end{aligned} \tag{3.2.4b}$$

These two products endow $\mathbb{K}\langle A^*\rangle$ with a structure of a dendriform algebra called *shuffle dendriform algebra* on A. This shows moreover that the shuffle algebra $(\mathbb{K}\langle A^*\rangle, \sqcup\!\sqcup)$ admits a dendriform algebra structure since

$$u \sqcup\!\sqcup v = u \prec v + u \succ v \tag{3.2.5}$$

for all $u, v \in A^*$. This offers also a way to recover the recursive definition (see (3.1.1a) and (3.1.1b)) of $\sqcup\!\sqcup$.

3.2.2. *Max dendriform algebra.* Assume here that A is a totally ordered alphabet by $\mathtt{a}_i \leqslant \mathtt{a}_j$ if $i \leqslant j$. Consider on $\mathbb{K}\langle A^+\rangle$ the binary products $\prec$ and $\succ$ defined linearly by

$$u \prec v := \begin{cases} u \cdot v & \text{if } \max_\leqslant(u) \geqslant \max_\leqslant(v) \\ 0 & \text{otherwise,} \end{cases} \tag{3.2.6a}$$

$$u \succ v := \begin{cases} u \cdot v & \text{if } \max_\leqslant(u) < \max_\leqslant(v) \\ 0 & \text{otherwise,} \end{cases} \tag{3.2.6b}$$

for all $u, v \in A^+$, where $\cdot$ is the concatenation product of words. These two products endow $\mathbb{K}\langle A^+\rangle$ with a structure of a dendriform algebra called *max dendriform algebra*. Moreover, we have here $\cdot = \prec + \succ$ where $\cdot$ is the associative algebra product of concatenation of $\mathbb{K}\langle A^+\rangle$ so that $(\mathbb{K}\langle A^+\rangle, \cdot)$ admits a dendriform algebra structure.

3.3. Pre-Lie algebras. A *pre-Lie algebra* is a polynomial space $\mathbb{K}\langle C\rangle$ endowed with a binary product $\curvearrowleft$ satisfying

$$(x_1 \curvearrowleft x_2) \curvearrowleft x_3 - x_1 \curvearrowleft (x_2 \curvearrowleft x_3) = (x_1 \curvearrowleft x_3) \curvearrowleft x_2 - x_1 \curvearrowleft (x_3 \curvearrowleft x_2) \tag{3.3.1}$$

for all objects x_1, x_2, and x_3 of C. This relation (3.3.1) of pre-Lie algebras says that the associator $(-,-,-)_\curvearrowleft$ is symmetric in its two last entries, that is $(x_1, x_2, x_3)_\curvearrowleft = (x_1, x_3, x_2)_\curvearrowleft$.

3.3.1. *Pre-Lie algebras from associative algebras.* When $(\mathbb{K}\langle C\rangle, \star)$ is an associative algebra, $\star$ satisfies in particular (3.3.1) since both left and right members are equal to zero. For this reason, $(\mathbb{K}\langle C\rangle, \star)$ is a pre-Lie algebra.

3.3.2. *Pre-Lie algebra of rooted trees.* Recall that $\mathfrak{RT}$ is the combinatorial graded collection of all rooted trees (see Section 3.1 of Chapter 2). Consider now on $\mathbb{K}\langle\mathfrak{RT}\rangle$ the products

$$\curlywedge^{(p)} : \mathbb{K}\left\langle \mathbf{List}_{\{p\}}(\mathfrak{RT})\right\rangle \to \mathbb{K}\langle\mathfrak{RT}\rangle \tag{3.3.2}$$

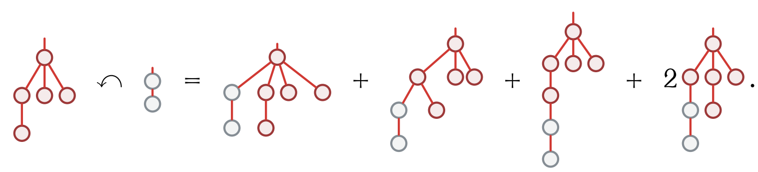
defined linearly for all $p \in \mathbb{N}_{\geqslant 1}$ and all rooted trees $\mathfrak{t}_1, \dots, \mathfrak{t}_p$ by

$$\curlywedge^{(p)}\left(\mathfrak{t}_1, \dots, \mathfrak{t}_p\right) := \left(\bullet, \wr \mathfrak{t}_1, \dots, \mathfrak{t}_p \int\right). \tag{3.3.3}$$

Intuitively, $\curlywedge^{(p)}$ consists in grafting all the trees $\mathfrak{t}_1, \dots, \mathfrak{t}_p$ onto a common root. This product is symmetric with respect to all its inputs. Now, let $\curvearrowleft$ be the binary product on $\mathbb{K}\langle\mathfrak{RT}\rangle$ defined linearly and recursively by

$$\mathfrak{s} \curvearrowleft \mathfrak{t} := \curlywedge^{(p+1)}\left(\mathfrak{s}_1, \dots, \mathfrak{s}_p, \mathfrak{t}\right) + \sum_{i \in [p]} \curlywedge^{(p)}(\mathfrak{s}_1, \dots, \mathfrak{s}_{i-1}, (\mathfrak{s}_i \curvearrowleft \mathfrak{t}), \mathfrak{s}_{i+1}, \dots, \mathfrak{s}_p) \tag{3.3.4}$$

for any $\mathfrak{s}, \mathfrak{t} \in \mathfrak{RT}$ where $\mathfrak{s} = \left(\bullet, \wr \mathfrak{s}_1, \dots, \mathfrak{s}_p \int\right)$. Intuitively, $\curvearrowleft$ consists in summing all the ways of connecting the root of the second operand on a node of the first. For example,


(3.3.5)

This product endows $\mathbb{K}\langle\mathfrak{RT}\rangle$ with a structure of a pre-Lie algebra, called the *pre-Lie algebra of rooted trees*.

3.4. Hopf bialgebras. A *Hopf bialgebra* is a polynomial space $\mathbb{K}\langle C\rangle$ endowed with a complete binary product $\star$ and a complete binary coproduct Δ such that $(\mathbb{K}\langle C\rangle, \star, \mathbb{1})$ is a unitary associative algebra, $(\mathbb{K}\langle C\rangle, \Delta, \upsilon)$ is a counitary coassociative coalgebra, and

$$\Delta(x_1 \star x_2) = \Delta(x_1)\ \mathrm{T}_2(\star)\, \Delta(x_2), \tag{3.4.1a}$$

$$\upsilon(x_1 \star x_2) = \upsilon(x_1)\, \upsilon(x_2), \tag{3.4.1b}$$

$$\Delta(\mathbb{1}) = (\mathbb{1}, \mathbb{1}), \tag{3.4.1c}$$

$$\upsilon(\mathbb{1}) = 1, \tag{3.4.1d}$$

for all objects x_1 and x_2 of C. When $(\mathbb{K}\langle C\rangle, \star, \mathbb{1}, \Delta, \upsilon)$ is combinatorial and all its (co)-products are concentrated, its dual is well-defined and is still a Hopf bialgebra.

Let us now provide some classical definitions about Hopf bialgebras.

3.4.1. *Primitive and group-like elements.* Let $(\mathbb{K}\langle C\rangle, \star, \mathbb{1}, \Delta, \upsilon)$ be a Hopf bialgebra. An element f of $\mathbb{K}\langle C\rangle$ is *primitive* if $\Delta(f) = (\mathbb{1}, f) + (f, \mathbb{1})$. The set $\mathcal{P}_{\mathbb{K}\langle C\rangle}$ of all primitive elements of $\mathbb{K}\langle C\rangle$ forms a subspace of $\mathbb{K}\langle C\rangle$ and the commutator $[-,-]_\star$ endows $\mathcal{P}_{\mathbb{K}\langle C\rangle}$ with a structure of a Lie algebra. Besides, an element f of $\mathbb{K}\langle C\rangle$ is *group-like* if $\Delta(f) = (f, f)$.

3.4.2. *Convolution product and antipode.* Given two Hopf bialgebras $(\mathbb{K}\langle C_1\rangle, \star_1, \mathbb{1}_1, \Delta_1, \upsilon_1)$ and $(\mathbb{K}\langle C_2\rangle, \star_2, \mathbb{1}_2, \Delta_2, \upsilon_2)$, if ω and ω' are two Hopf bialgebra morphisms from $\mathbb{K}\langle C_1\rangle$ to $\mathbb{K}\langle C_2\rangle$, the *convolution* of ω and ω' is the map

$$\omega * \omega' : \mathbb{K}\langle C_1\rangle \to \mathbb{K}\langle C_2\rangle \tag{3.4.2}$$

defined linearly, for any object x of C_1, by

$$\left(\omega * \omega'\right)(x) := \sum_{y_1, y_2 \in C_1} \xi_{\Delta_1}\left((x), (y_1, y_2)\right) \omega(y_1) \star_2 \omega'(y_2), \tag{3.4.3}$$

where ξ_{Δ_1} is the structure coefficient map of Δ_1. This convolution product is associative, as a consequence of the fact that Δ_1 is coassociative and $\star_2$ is associative.

Now, let $(\mathbb{K}\langle C\rangle, \star, \mathbb{1}, \Delta, \upsilon)$ be a Hopf bialgebra. Let $\nu : \mathbb{K}\langle C\rangle \to \mathbb{K}\langle C\rangle$ be the linear map defined as the inverse of the identity map $\mathrm{Id}_{\mathbb{K}\langle C\rangle}$ on $\mathbb{K}\langle C\rangle$. This map ν is the *antipode* of $\mathbb{K}\langle C\rangle$ and it can be undefined in certain cases.

3.4.3. *Combinatorial connected graded Hopf bialgebras.* In algebraic combinatorics, one encounters very particular Hopf bialgebras. Most of these are combinatorial connected graded Hopf bialgebras $(\mathbb{K}\langle C\rangle, \star, \mathbb{1}, \Delta, \upsilon)$. These structures have hence a graded product (that is, $x_1 \star x_2$ is homogeneous and of degree $|x_1| + |x_2|$ for all objects x_1 and x_2 of C), a cograded coproduct (that is, the sum of the sizes of each (y_1, y_2) appearing in $\Delta(x)$ is $|x|$ for all objects x of C). Moreover, each $\mathbb{K}\langle C\rangle(n)$, $n \in \mathbb{N}$, is finite dimensional and, since $\#C(0) = 1$, $\mathbb{K}\langle C\rangle(0)$ can be identified with $\mathbb{K}$. Additionally, since $\star$ is graded, this implies that the unit $\mathbb{1}$ is the unique element of $C(0)$. Finally, the counit υ is the linear map behaving as the identity map on $\mathbb{K}\langle C\rangle(0)$ and sending all the elements of $\mathbb{K}\langle C\rangle(\mathbb{N}_{\geqslant 1})$ to 0.

Proposition 3.4.1. *Let $(\mathbb{K}\langle C\rangle, \star, \mathbb{1}, \Delta, \upsilon)$ be a combinatorial connected graded Hopf bialgebra. Then, $\mathbb{K}\langle C\rangle$ admits a unique antipode and it satisfies, for any $x \in C$, the recurrence*

$$\nu(x) = \delta_{\mathbb{1},x} - \sum_{\substack{y_1, y_2 \in C \\ y_2 \neq \mathbb{1}}} \xi_\Delta\left((x), (y_1, y_2)\right) \nu(y_1) \star y_2, \tag{3.4.4}$$

where $\delta_{-,-}$ is the Kronecker symbol and ξ_Δ is the structure coefficient map of Δ.

Therefore, (3.4.4) implies that the antipode of $\mathbb{K}\langle C\rangle$ can be computed by induction on the sizes of the objects.

3.4.4. *Shuffle deconcatenation Hopf bialgebra.* Let $A := \{\mathrm{a}_1, \dots, \mathrm{a}_\ell\}$ be a finite alphabet. The concatenation product $\cdot$ (see Section 3.1.1), the unit ϵ (see Section 3.1.1), the unshuffling coproduct $\Delta_{\sqcup\!\sqcup}$ (see Section 3.1.4), and the counit υ (see Section 3.1.3) endow $\mathbb{K}\langle A^*\rangle$ with a structure of a combinatorial connected graded Hopf bialgebra $(\mathbb{K}\langle A^*\rangle, \cdot, \epsilon, \Delta_{\sqcup\!\sqcup}, \upsilon)$. Its dual bialgebra is the Hopf bialgebra $(\mathbb{K}\langle A^*\rangle, \sqcup\!\sqcup, \epsilon, \Delta_\cdot, \upsilon)$ where $\sqcup\!\sqcup$ is the shuffle product (see Section 3.1.2) and $\Delta_\cdot$ is the deconcatenation coproduct (see again Section 3.1.3).

3.4.5. *Noncommutative symmetric functions.* Consider the combinatorial graded polynomial space **NCSym** := $\mathbb{K}\langle\mathfrak{Com}\rangle$ of the compositions (see Section 1.3.3 of Chapter 1). Let $\{\mathsf{S}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathfrak{Com}\}$ be the basis of the *complete noncommutative symmetric functions* of **NCSym** and $\star$ be the binary product defined linearly, for any $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathfrak{Com}$, by

$$\mathsf{S}_{\boldsymbol{\lambda}} \star \mathsf{S}_{\boldsymbol{\mu}} := \mathsf{S}_{\boldsymbol{\lambda}\cdot\boldsymbol{\mu}}, \tag{3.4.5}$$

where $\boldsymbol{\lambda} \cdot \boldsymbol{\mu}$ is the concatenation of the compositions (seen as words of integers). Moreover, let Δ be the binary coproduct defined linearly, for any $\boldsymbol{\lambda} \in \mathfrak{Com}$, by

$$\Delta\left(\mathsf{S}_{\boldsymbol{\lambda}}\right) := \prod_{j \in [\ell(\boldsymbol{\lambda})]} \left(\sum_{\substack{n,m \in \mathbb{N} \\ n+m=\boldsymbol{\lambda}_j}} \left(\mathsf{S}_{(n)}, \mathsf{S}_{(m)}\right) \right), \tag{3.4.6}$$

where the product of (3.4.6) denotes the iterated version of 2nd tensor power $\mathrm{T}_2(\star)$ of $\star$, and for any $n \in \mathbb{N}_{\geqslant 1}$, $\mathsf{S}_{(n)}$ is the basis element indexed by the composition of length 1 whose only part is n, and $\mathsf{S}_{(0)}$ is identified with the unit 1 of $\mathbb{K}$. For instance,

$$\begin{aligned}
\Delta\left(\mathsf{S}_{121}\right) &= \left(\left(1, \mathsf{S}_1\right) + \left(\mathsf{S}_1, 1\right)\right) \mathrm{T}_2(\star) \left(\left(1, \mathsf{S}_2\right) + \left(\mathsf{S}_1, \mathsf{S}_1\right) + \left(\mathsf{S}_2, 1\right)\right) \mathrm{T}_2(\star) \left(\left(1, \mathsf{S}_1\right) + \left(\mathsf{S}_1, 1\right)\right) \\
&= \left(1, \mathsf{S}_{121}\right) + \left(\mathsf{S}_1, \mathsf{S}_{111}\right) + \left(\mathsf{S}_1, \mathsf{S}_{12}\right) + \left(\mathsf{S}_1, \mathsf{S}_{21}\right) + 2\left(\mathsf{S}_{11}, \mathsf{S}_{11}\right) + \left(\mathsf{S}_{11}, \mathsf{S}_2\right) \\
&\quad + \left(\mathsf{S}_2, \mathsf{S}_{11}\right) + \left(\mathsf{S}_{111}, \mathsf{S}_1\right) + \left(\mathsf{S}_{12}, \mathsf{S}_1\right) + \left(\mathsf{S}_{21}, \mathsf{S}_1\right) + \left(\mathsf{S}_{121}, 1\right).
\end{aligned} \tag{3.4.7}$$

The product $\star$ and the coproduct Δ endow **NCSym** with a structure of a combinatorial connected graded Hopf bialgebra.

Moreover, let $\{\mathsf{R}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathfrak{Com}\}$ be the family defined by

$$\mathsf{R}_{\boldsymbol{\lambda}} := \sum_{\substack{\boldsymbol{\mu} \in \mathfrak{Com} \\ \boldsymbol{\lambda} \preccurlyeq \boldsymbol{\mu}}} (-1)^{\ell(\boldsymbol{\lambda}) - \ell(\boldsymbol{\mu})} \mathsf{S}_{\boldsymbol{\mu}}, \tag{3.4.8}$$

where $\preccurlyeq$ is the refinement order of compositions (see Section 2.2.1 of Chapter 1). For instance,

$$\mathsf{R}_{212} = \mathsf{S}_{212} - \mathsf{S}_{23} - \mathsf{S}_{32} + \mathsf{S}_5. \tag{3.4.9}$$

By triangularity, this family forms a basis of **NCSym** and is known as the basis of *ribbon noncommutative symmetric functions*. On this basis, one has, for any $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathfrak{Com}$,

$$\mathsf{R}_{\boldsymbol{\lambda}} \star \mathsf{R}_{\boldsymbol{\mu}} := \mathsf{R}_{\boldsymbol{\lambda}\cdot\boldsymbol{\mu}} + \mathsf{R}_{\boldsymbol{\lambda} \triangleright \boldsymbol{\mu}}, \tag{3.4.10}$$

for any $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathfrak{Com}$, where $\boldsymbol{\lambda} \cdot \boldsymbol{\mu}$ is the concatenation of the compositions and

$$\boldsymbol{\lambda} \triangleright \boldsymbol{\mu} := \left(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{\ell(\boldsymbol{\lambda})-1}, \boldsymbol{\lambda}_{\ell(\boldsymbol{\lambda})} + \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_{\ell(\boldsymbol{\mu})}\right). \tag{3.4.11}$$

For instance,

$$\mathsf{R}_{3112} \star \mathsf{R}_{142} = \mathsf{R}_{3112142} + \mathsf{R}_{311342}. \tag{3.4.12}$$

This Hopf bialgebra **NCSym** is usually known as the *Hopf bialgebra of noncommutative symmetric functions*. To explain this name, consider an alphabet $A := \{a_1, a_2, \dots\}$ equipped with a total order $\preccurlyeq$ where $1 \leqslant i \leqslant j$ implies $a_i \preccurlyeq a_j$. Now, let the noncommutative series

$$\mathrm{R}_{\boldsymbol{\lambda}}(A) := \sum_{\substack{u \in A^* \\ \mathrm{cmp}(u) = \boldsymbol{\lambda}}} u, \tag{3.4.13}$$

of $\mathbb{K}\langle\langle A^* \rangle\rangle$ defined for all $\boldsymbol{\lambda} \in \mathfrak{Com}$, where cmp is defined in Section 1.3.3 of Chapter 1. Observe that all $\mathrm{R}_{\boldsymbol{\lambda}}(A)$ are polynomials when A is finite, but are series in the other case. For instance,

$$\mathrm{R}_{31}(\{a_1, a_2\}) = a_1a_1a_2a_1 + a_1a_2a_2a_1 + a_2a_2a_2a_1, \tag{3.4.14a}$$

$$\begin{aligned} \mathrm{R}_{21}(\{a_1, a_2, a_3\}) = {} & a_1a_2a_1 + a_1a_3a_1 + a_1a_3a_2 + a_2a_2a_1 \\ & + a_2a_3a_1 + a_2a_3a_2 + a_3a_3a_1 + a_3a_3a_2, \end{aligned} \tag{3.4.14b}$$

$$\begin{aligned} \mathrm{R}_{121}(\{a_1, a_2, a_3\}) = {} & a_2a_1a_2a_1 + a_2a_1a_3a_1 + a_2a_1a_3a_2 + a_3a_1a_2a_1 + a_3a_1a_3a_1 \\ & + a_3a_1a_3a_2 + a_3a_2a_2a_1 + a_3a_2a_3a_1 + a_3a_2a_3a_2. \end{aligned} \tag{3.4.14c}$$

The linear span of all the $\mathrm{R}_{\boldsymbol{\lambda}}(A)$, $\boldsymbol{\lambda} \in \mathfrak{Com}$, is the space of the noncommutative symmetric functions on A. The associative algebra structure of **NCSym** is compatible with these series in the sense that

$$\mathrm{R}_{\boldsymbol{\lambda}}(A) \cdot \mathrm{R}_{\boldsymbol{\mu}}(A) = (\mathrm{R}_{\boldsymbol{\lambda}} \star \mathrm{R}_{\boldsymbol{\mu}})(A) \tag{3.4.15}$$

for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathfrak{Com}$, where the product $\cdot$ of the left member of (3.4.15) is the usual product of noncommutative series of $\mathbb{K}\langle\langle A^* \rangle\rangle$.

3.4.6. *Free quasi-symmetric noncommutative symmetric functions.* Let the graded combinatorial polynomial space $\mathbf{FQSym} := \mathbb{K}\langle\mathfrak{S}\rangle$ of the permutations. Let $\{\mathrm{F}_\sigma : \sigma \in \mathfrak{S}\}$ be the basis of the *fundamental free quasi-symmetric functions* of **FQSym** and $\star$ be the binary product defined linearly, for any $\sigma, \nu \in \mathfrak{S}$, by

$$\mathrm{F}_\sigma \star \mathrm{F}_\nu := \sum_{\pi \in \mathfrak{S}} \langle \pi, \sigma ⧢ \bar{\nu} \rangle \mathrm{F}_\pi, \tag{3.4.16}$$

where $\bar{\nu}$ is the word obtained by incrementing each letter of ν by $|\sigma|$, and $⧢$ is the shuffle product of words defined in Section 3.1.2. For instance

$$\mathrm{F}_{21} \star \mathrm{F}_{12} = \mathrm{F}_{2134} + \mathrm{F}_{2314} + \mathrm{F}_{2341} + \mathrm{F}_{3214} + \mathrm{F}_{3241} + \mathrm{F}_{3421}. \tag{3.4.17}$$

This product is known as the *shifted shuffle product* and is sometimes denoted also as $\overline{⧢}$. Let moreover Δ be the binary coproduct defined linearly, for any $\pi \in \mathfrak{S}$, by

$$\Delta(\mathrm{F}_\pi) := \sum_{0 \leqslant i \leqslant |\pi|} \left(\mathrm{F}_{\mathrm{std}(\pi(1)\dots\pi(i))}, \mathrm{F}_{\mathrm{std}(\pi(i+1)\dots\pi(|\pi|))}\right), \tag{3.4.18}$$

where std is defined in Section 1.3.5 of Chapter 1. For instance

$$\Delta(\mathrm{F}_{42513}) = (1, \mathrm{F}_{42513}) + (\mathrm{F}_1, \mathrm{F}_{2413}) + (\mathrm{F}_{21}, \mathrm{F}_{312}) + (\mathrm{F}_{213}, \mathrm{F}_{12}) + (\mathrm{F}_{3241}, \mathrm{F}_1) + (\mathrm{F}_{42513}, 1). \tag{3.4.19}$$

The product $\star$ and the coproduct Δ endow **FQSym** with a structure of a combinatorial connected graded Hopf bialgebra.

This Hopf bialgebra **FQSym** is usually known as the *Hopf bialgebra of free quasi-symmetric functions*. Indeed, as for **NCSym**, there is a way to see the elements of **FQSym** as noncommutative series. For this, consider an alphabet $A := \{a_1, a_2, \dots\}$ equipped with a total order $\preccurlyeq$ where $1 \leqslant i \leqslant j$ implies $a_i \preccurlyeq a_j$. Let the noncommutative series

$$F_\sigma(A) := \sum_{\substack{u \in A^* \\ \operatorname{std}(u) = \sigma^{-1}}} u, \tag{3.4.20}$$

of $\mathbb{K}\langle\langle A^* \rangle\rangle$ defined for all $\sigma \in \mathfrak{S}$. For instance

$$F_{312}(\{a_1, a_2, a_3\}) = a_2a_2a_1 + a_2a_3a_1 + a_3a_3a_1 + a_3a_3a_2, \tag{3.4.21a}$$

$$F_{132}(\{a_1, a_2, a_3\}) = a_1a_2a_1 + a_1a_3a_1 + a_1a_3a_2 + a_2a_3a_2. \tag{3.4.21b}$$

The linear span of all the $F_\sigma(A)$, $\sigma \in \mathfrak{S}$, is the space of the free quasi-symmetric functions on A. The associative algebra structure on **FQSym** is compatible with these series in the sense that

$$F_\sigma(A) \cdot F_\nu(A) = (F_\sigma \star F_\nu)(A) \tag{3.4.22}$$

for all $\sigma, \nu \in \mathfrak{S}$, where the product $\cdot$ of the left member of (3.4.22) is the usual product of noncommutative series of $\mathbb{K}\langle\langle A^* \rangle\rangle$.

Furthermore, the Hopf bialgebras **FQSym** and **NCSym** are related through the injective morphism of Hopf bialgebras $\phi : \mathbf{NCSym} \to \mathbf{FQSym}$ defined linearly by

$$\phi(R_\lambda) := \sum_{\substack{\sigma \in \mathfrak{S} \\ \operatorname{Des}(\sigma^{-1}) = \operatorname{Des}(\lambda)}} F_\sigma \tag{3.4.23}$$

for all $\lambda \in \mathfrak{Com}$. For instance,

$$\phi(R_{21}) = F_{312} + F_{132}. \tag{3.4.24}$$

Observe, with the help of (3.4.14b), (3.4.21a), and (3.4.21b), in particular that (3.4.24) holds on the noncommutative series associated with the elements of **NCSym** and **FQSym**, that is, $R_{21}(A) = F_{312}(A) + F_{132}(A)$.

Bibliographic notes

About Incidence algebras. One of the first apparitions of incidence algebras in combinatorics is due to Rota [**Rot64**]. These structures, associated with any locally finite poset, provide an abstraction of the principle of inclusion-exclusion [**Sta11**] through their Möbius functions. Indeed, the usual inclusion-exclusion principle comes from the Möbius function of the cube poset. Besides, in our exposition, we have presented the elements of incidence algebras as polynomials of pairs of comparable elements, but in the literature [**Sta11**], it is most common to see these elements as maps associating a coefficient with each pair of comparable elements. These two points of view are therefore equivalent but the definition of the product of incidence algebras in terms of polynomials is simpler.

Dendriform algebras. Dendriform algebras are types of polynomial algebras introduced by Loday [Lod01]. These structures can be used as devices to split the product of an associative algebra into two parts by putting a dendriform algebra structure onto it. For instance, the dendriform algebra structure put onto the shuffle algebra (see Section 3.1.2 and 3.2.1) leads to a recursive expression for the shuffle product (see (3.1.1a) and (3.1.1b)) known since Ree [Ree58]. We invite the reader to take a look at [LR98, Agu00, Lod02, Foi07, EFMP08, EFM09, LV12] for a supplementary review of properties of dendriform algebras. Besides, in the recent years, a lot of generalizations of dendriform algebras and their dual notions were introduced, each of them splitting an associative product in different ways and in more than two pieces. Tridendriform algebras [LR04], quadri-algebras [AL04], ennea-algebras [Ler04], m-dendriform algebras of Leroux [Ler07], m-dendriform algebras of Novelli [Nov14], and polydendriform algebras [Gir16c, Gir16d] are examples of such structures.

About pre-Lie algebras. Pre-Lie algebras were introduced by Vinberg [Vin63] and Gerstenhaber [Ger63] independently. These structures appear under different names in the literature, for instance as Vinberg algebras, left-symmetric algebras, or chronological algebras. The appellation pre-Lie algebra is now very natural since, given a pre-Lie algebra $(\mathbb{K}\langle C\rangle, \curvearrowleft)$, the commutator of $\curvearrowleft$ endows $\mathbb{K}\langle C\rangle$ with a structure of a Lie algebra. In the context of combinatorics, several pre-Lie products are defined on combinatorial spaces by summing over all the ways to compose (in a certain sense) two combinatorial objects. For this reason, in an intuitive way, pre-Lie algebras encode the combinatorics of the composition of combinatorial objects in all possible ways [Cha08]. Besides, the free objects in the category of pre-Lie algebras have been described by Chapoton and Livernet [CL01]. They have shown that the free pre-Lie algebra generated by a set $\mathfrak{G}$ is the combinatorial space of all rooted trees whose nodes are labeled on $\mathfrak{G}$, and the product of two such rooted trees is the sum of all the ways to connect the root of the second tree to a node of the first. Thereby, the pre-Lie algebra $(\mathbb{K}\langle\mathfrak{RT}\rangle, \curvearrowleft)$ of rooted trees (see Section 3.3.2) is the free pre-Lie algebra generated by a singleton. For more details on pre-Lie algebras, see [Man11].

About bialgebras. In the field of algebraic combinatorics, many types of bialgebras have emerged recently. In [Lod08], Loday defined the notion of triples of operads, leading to the constructions of various kinds of bialgebras. This leads also to the discovery of analogs of the Poincaré-Birkhoff-Witt and Cartier-Milnor-Moore theorems and rigidity theorems (see as well [Cha02] and [BDO18]). Loday defined among others infinitesimal bialgebras, forming an example of bialgebras having an associative binary product and a coassociative binary coproduct satisfying a compatibility relation. Let us describe some other types of bialgebras that play a role in combinatorics. Bidendriform bialgebras, introduced by Foissy [Foi07] are one of these. These bialgebras have two products satisfying the dendriform relations and two coproducts satisfying the dual relations of dendriform products, and all of these together satisfy some compatibility relations. There is a notion of bidendriform bialgebra structure onto a Hopf bialgebra which leads to a rigidity theorem in the sense that a Hopf bialgebra admitting a bidendriform bialgebra structure is self-dual, free as an associative algebra, and free as a coassociative coalgebra. Moreover, in [Foi12], Foissy considered algebraic structures, named **Dup**-**Dendr** bialgebras, having two binary products satisfying the duplicial relations [BF03, Lod08], two binary coproducts such that their dual products satisfy the dendriform relations, and such that these four (co)products satisfy several compatibility relations. These structures lead to rigidity theorems in the sense that any **Dup**-**Dendr** bialgebra is free as a duplicial algebra. In the same way, Foissy introduced also in [Foi15] structures named **Com**-**PLie** bialgebras, that are spaces with an associative and commutative binary product, a pre-Lie product, and a binary coproduct that satisfy compatibility relations. Another interesting example has been brought by Livernet [Liv06] wherein bialgebra structures having a pre-Lie product and a coproduct satisfying the dual relation of the so-called nonassociative permutative relation have been considered to construct here again a rigidity theorem.

About Hopf bialgebras. The Hopf bialgebra **NCSym** of noncommutative symmetric functions has been introduced in [GKL+95] as a generalization of the usual symmetric functions [Mac15]. This generalization is a consequence of the fact that there is a surjective morphism from **NCSym** to the algebra of symmetric functions. The Hopf bialgebra **FQSym** of free quasi-symmetric functions has been introduced by Malvenuto and Reutenauer [MR95] and is sometimes called the Malvenuto-Reutenauer algebra. Due to its interpretation [DHT02] as an algebra of noncommutative series $\mathrm{F}_\sigma(A)$, each element of **FQSym** can be seen as a particular function, whence its name. Other classical examples of Hopf bialgebras include the Poirier-Reutenauer Hopf bialgebra of tableaux [PR95], also known as the Hopf bialgebra of free symmetric functions **FSym** [DHT02, HNT05]. This Hopf bialgebra is defined on the combinatorial space of all standard Young tableaux. The Loday-Ronco Hopf bialgebra [LR98], also known as the Hopf bialgebra of binary search trees **PBT** [HNT05] is defined on the combinatorial space of all binary trees. As other modern examples of combinatorial spaces endowed with a Hopf bialgebra structure, one can cite **WQSym** [Hiv99] involving packed words, **PQSym** [NT07] involving parking functions, **Bell** [Rey07] involving set partitions, **Baxter** [LR12, Gir12] involving ordered pairs of twin binary trees, and **Camb** [CP17] involving Cambrian trees. The study of all these structures uses a large set of tools. Indeed, it relies on algorithms transforming words into combinatorial objects, congruences of free monoids, partials orders structures and lattices, and polytopes and their geometric realizations. Besides, a polynomial realization of a combinatorial Hopf bialgebra $\mathbb{K}\langle C\rangle$ consists in seeing $\mathbb{K}\langle C\rangle$ as an algebra of noncommutative series so that its product is the usual product of series and its coproduct is obtained by alphabet doubling (see, for instance, [Hiv03]). In this text, only the polynomial realizations of **NCSym** and **FQSym** have been detailed, but all the Hopf bialgebras discussed here have polynomial realizations.

CHAPTER 4

Nonsymmetric operads

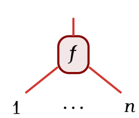
This chapter introduces nonsymmetric operads. Our presentation relies on the framework of graded collections and graded spaces introduced in the previous chapters. We consider here also set-operads, algebras over operads, free operads, presentations by generators and relations, Koszul duality and Koszulity of operads. At the end of the chapter, several examples of operads on a large family of combinatorial collections are provided.

1. Operads as polynomial spaces

Let us start by posing the main definitions about operads. We first present the notion of partial composition maps on abstract operators and then focus on operads and algebras over operads.

1.1. Composition maps. Intuitively, the elements of an operad are operations with several inputs and one output that can be composed. We introduce here the notion of abstract operator and two ways to compose them by the so-called partial or full compositions. To consider operads, we shall expect that partial and full compositions satisfy some relations. One of the aims of this section is to give an intuition about these relations. In all this section, $\mathbb{K}\langle C\rangle$ is an augmented graded polynomial space.

1.1.1. *Abstract operators.* From now, we shall see any homogeneous element f of $\mathbb{K}\langle C\rangle$ of degree n as an operator with n inputs and a single output, called *abstract operator*. We will use in the context the term *arity* instead of size or degree, so that the arity of f is n. Any abstract operator is depicted by following the drawing conventions of biproducts exposed in Section 2.1.1 of Chapter 3. Therefore, f is depicted by



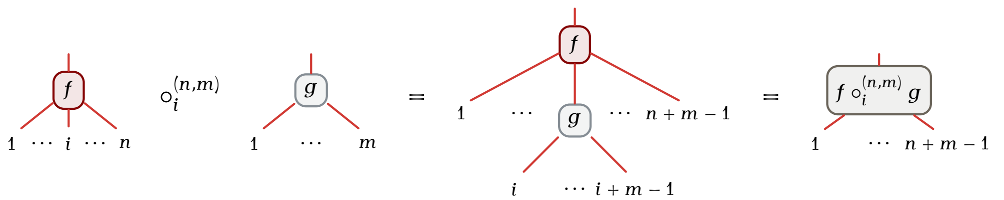
. (1.1.1)

The reason to see the elements of $\mathbb{K}\langle C\rangle$ in this way is based on the fact that we shall consider compositions operations on $\mathbb{K}\langle C\rangle$ subjected to relations that are easy to understand through this formalism.

1.1.2. *Partial composition maps.* Let for all $n, m \in \mathbb{N}_{\geqslant 1}$ and $i \in [n]$ binary products of the form

$$\circ_i^{(n,m)} : \mathbb{K}\langle [C(n), C(m)]_\times \rangle \to \mathbb{K}\langle C \rangle (n + m - 1). \tag{1.1.2}$$

On abstract operators, these products $\circ_i^{(n,m)}$ behave in the following way. For any $f \in \mathbb{K}\langle C \rangle (n)$ and $g \in \mathbb{K}\langle C \rangle (m)$, $f \circ_i^{(n,m)} g$ is the abstract operator



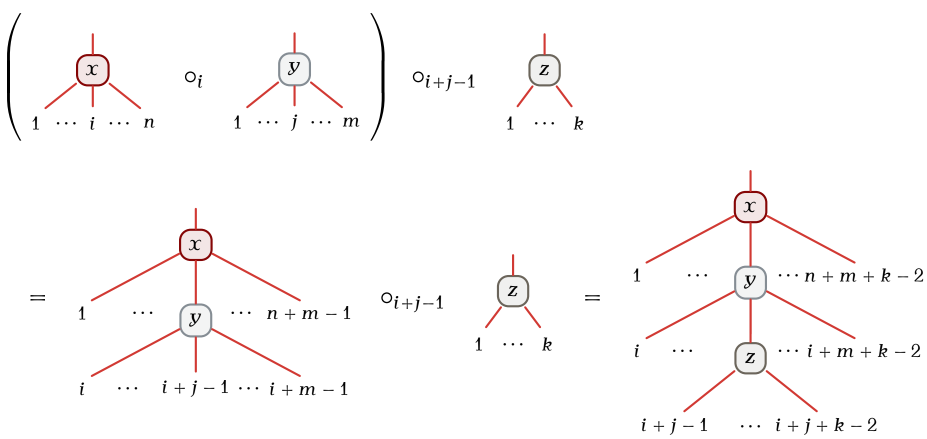
(1.1.3)

In words, $f \circ_i^{(n,m)} g$ is obtained by plugging the output of g onto the ith input of f. Observe that since one input of f is used to make the connection with the output of g, the right member of (1.1.3) is of arity $n + m - 1$. Moreover, observe also that the products $\circ_i^{(n,m)}$ are concentrated. By a slight abuse of notation, we shall sometimes omit the (n, m) in the notation of $\circ_i^{(n,m)}$ in order to denote it in a more concise way by $\circ_i$.

When for any objects $x \in C(n)$, $y \in C(m)$, $z \in C(k)$, and any integers $i \in [n]$ and $j \in [m]$, the relations

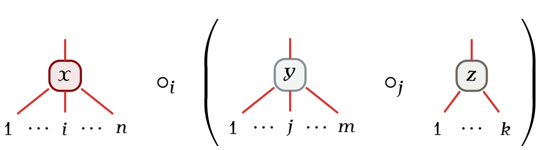
$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z) \tag{1.1.4}$$

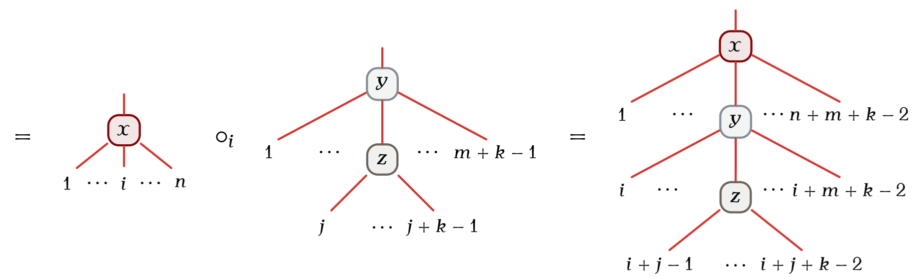
hold, we say that the products $\circ_i$ are *series associative*. To understand this relation, let us consider the abstract operators expressed by the left and right members of (1.1.4). On the one hand, we have



, (1.1.5)

and on the other,





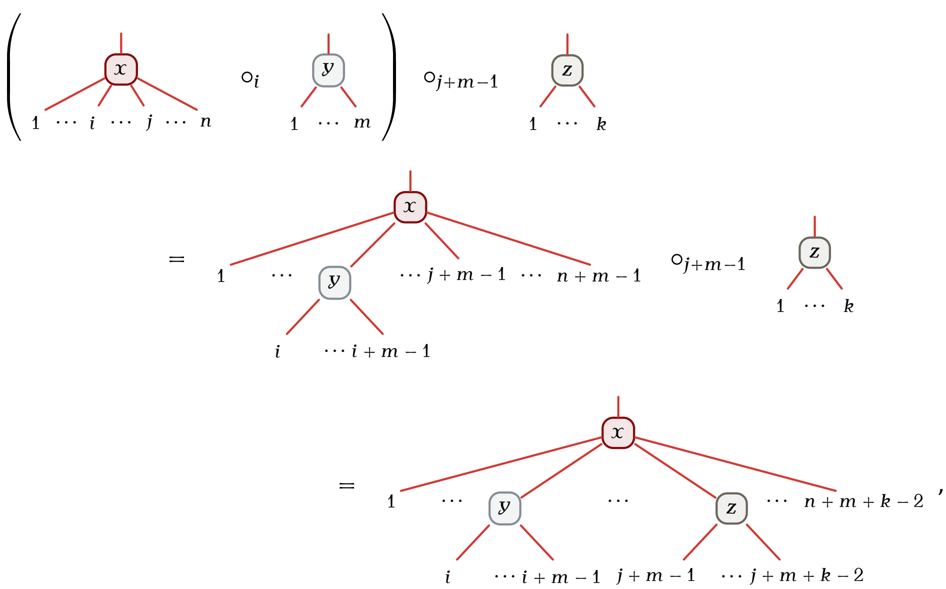
(1.1.6)

We observe that the two obtained abstract operators are the same, as expressed by (1.1.4).

Besides, when for any objects $x \in C(n)$, $y \in C(m)$, $z \in C(k)$, and any integers $i, j \in [n]$ such that $i < j$, the relations

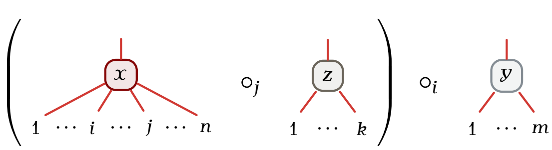
$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y \tag{1.1.7}$$

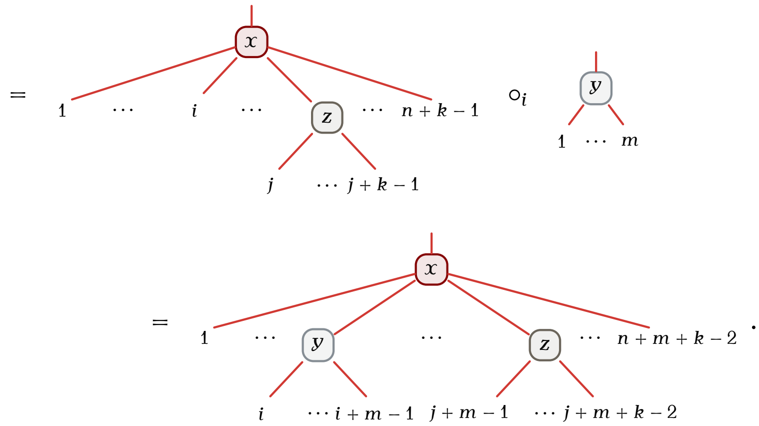
hold, we say that the products $\circ_i$ are *parallel associative*. To understand this relation, let us consider the abstract operators expressed by the left and right members of (1.1.7). On the one hand, we have



(1.1.8)

and on the other,





(1.1.9)

We observe that the two obtained abstract operators are the same, as expressed by (1.1.7).

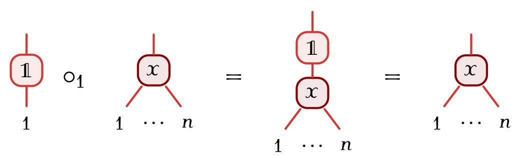
Finally, when there exists a product of arity 0 (see Section 2.2.3 of Chapter 3) of the form

$$\mathbb{1} : \mathbb{K}\langle [\,]_{\times} \rangle \to \mathbb{K}\langle C \rangle (1), \tag{1.1.10}$$

such that for any object $x \in C(n)$ and any integer $i \in [n]$ the relations

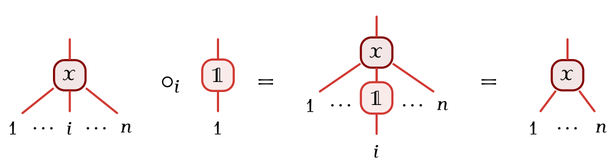
$$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1} \tag{1.1.11}$$

hold, we say that the products $\circ_i$ are *unital* and that $\mathbb{1}$ is the *unit*. To understand this relation, let us consider the abstract operators associated with each member of (1.1.11). This leads to the relation



(1.1.12)

for the left part of (1.1.11) and



(1.1.13)

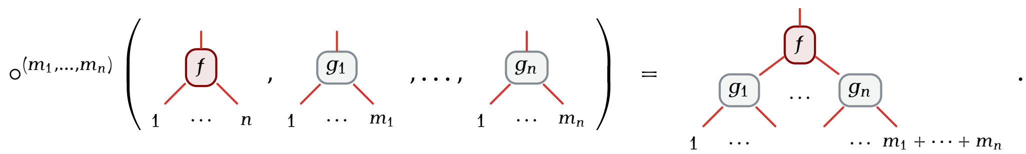
for its right part, saying that $\mathbb{1}$ is an operator of arity 1 behaving as the identity map.

When the products $\circ_i$ are series associative, parallel associative, and unital, the $\circ_i$ are called *partial composition maps*.

1.1.3. *Full composition maps.* Let for all $n, m_1, \dots, m_n \in \mathbb{N}_{\geqslant 1}$ products of the form

$$\circ^{(m_1,\dots,m_n)} : \mathbb{K}\langle [C(n), C(m_1), \dots, C(m_n)]_\times \rangle \to \mathbb{K}\langle C \rangle (m_1 + \dots + m_n). \tag{1.1.14}$$

On abstract operators, these products $\circ^{(m_1,\dots,m_n)}$ behave in the following way. For any $f \in \mathbb{K}\langle C\rangle(n)$ and $g_i \in \mathbb{K}\langle C\rangle(m_i)$, $i \in [n]$, $\circ^{(m_1,\dots,m_n)}(f, g_1, \dots, g_n)$ is the abstract operator



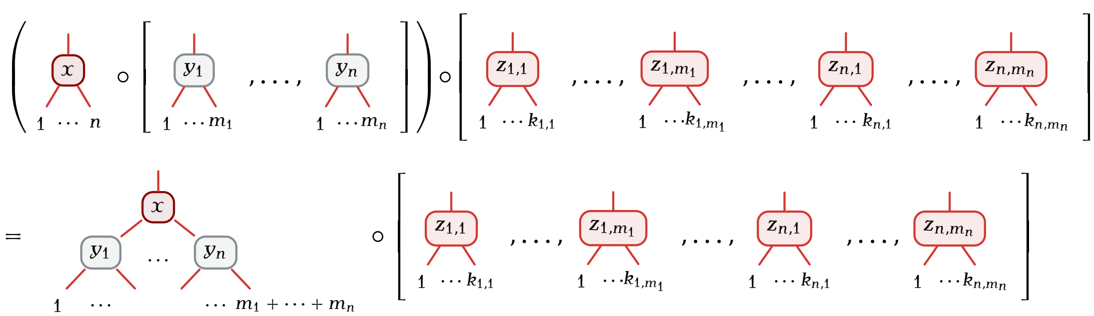
(1.1.15)

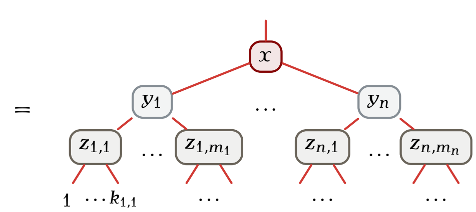
In words, $\circ^{(m_1,\dots,m_n)}(f, g_1, \dots, g_n)$ is obtained by plugging the outputs of the g_i, $i \in [n]$, onto the ith inputs of f simultaneously. Observe that since each input of f is connected to a g_i, the inputs of the right member of (1.1.15) are the ones of the g_i, $i \in [n]$, so that its arity is $m_1 + \dots + m_n$. Moreover, observe also that the products $\circ^{(m_1,\dots,m_n)}$ are concentrated. By a slight abuse of notation, we shall sometimes omit the $(m_1, \dots, m_n)$ in the notation of $\circ^{(m_1,\dots,m_n)}$ in order to denote it in a more concise way by $\circ$. Moreover, we shall write $f \circ [g_1, \dots, g_n]$ instead of $\circ(f, g_1, \dots, g_n)$.

When for any objects $x \in C(n)$, $y_i \in C(m_i)$, $i \in [n]$, $z_{i,j} \in C(k_{i,j})$, $i \in [n]$, $j \in [m_i]$, the relations

$$(x \circ [y_1, \dots, y_n]) \circ [z_{1,1}, \dots, z_{1,m_1}, \dots, z_{n,1}, \dots, z_{n,m_n}]$$
$$= x \circ [y_1 \circ [z_{1,1}, \dots, z_{1,m_1}], \dots y_n \circ [z_{n,1}, \dots, z_{n,m_n}]] \tag{1.1.16}$$

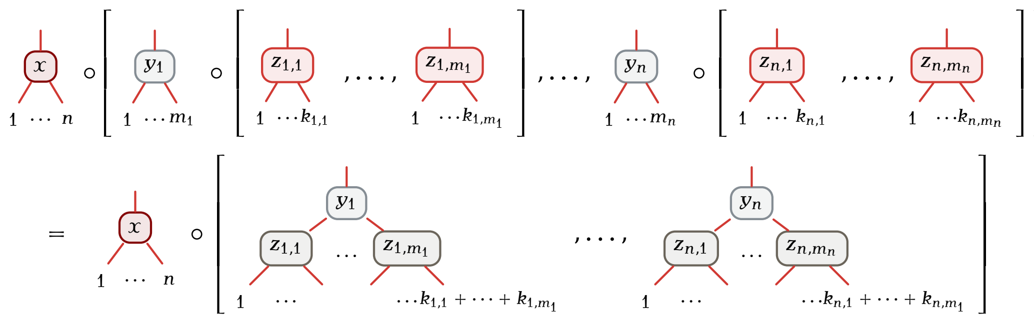
hold, we say that the product $\circ$ is *associative*. To understand this relation, let us consider the abstract operators expressed by the left and right members of (1.1.16). On the one hand, we have

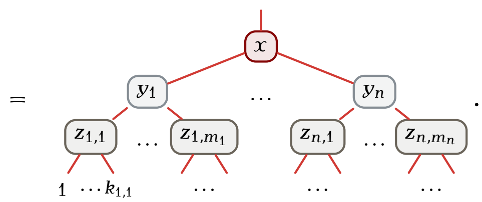




, (1.1.17)

and on the other,





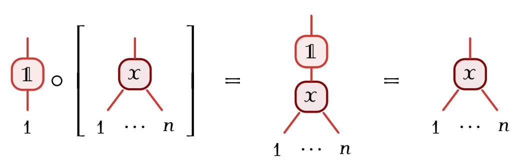
. (1.1.18)

We observe that the two obtained abstract operators are the same, as expressed by (1.1.16).

Besides, when there exists a product $\mathbb{1}$ of arity 0 on $\mathbb{K}\langle C\rangle$ satisfying $\mathbb{1} \in \mathbb{K}\langle C\rangle(1)$ and such that for any object $x \in C(n)$ the relations

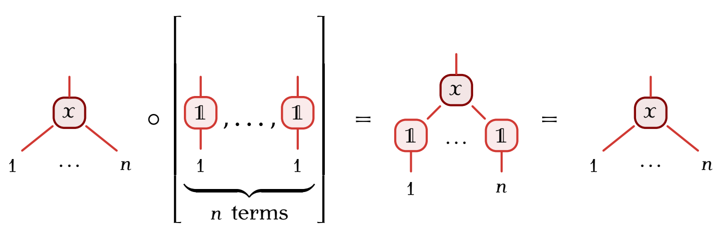
$$\mathbb{1} \circ [x] = x = x \circ \Big[\underbrace{\mathbb{1}, \dots, \mathbb{1}}_{n \text{ terms}}\Big] \qquad (1.1.19)$$

hold, we say that the products $\circ$ are *unital* and that $\mathbb{1}$ is the *unit*. To understand this relation, let us consider the abstract operators associated with each member of (1.1.19). This leads to the relation



(1.1.20)

for the left part of (1.1.19) and



(1.1.21)

for its right part, saying that $\mathbb{1}$ is an operator of arity 1 behaving as the identity map.

When the products $\circ$ are associative and unital, the $\circ$ are called *full composition maps*.

1.1.4. *Equivalence between partial and full composition maps.* Let $\circ_i$ be partial composition maps on $\mathbb{K}\langle C\rangle$. We construct from the $\circ_i$ the products $\circ^{(m_1,\dots,m_n)}$, $n, m_1, \dots, m_n \in \mathbb{N}_{\geqslant 1}$, on $\mathbb{K}\langle C\rangle$ defined linearly in the following way. For any $x \in C(n)$, $y_i \in C(m_i)$, $i \in [n]$, let us set

$$\circ^{(m_1,\dots,m_n)}(x, y_1, \dots, y_n) := (\dots((x \circ_n y_n) \circ_{n-1} y_{n-1}) \dots) \circ_1 y_1. \tag{1.1.22}$$

Proposition 1.1.1. *Let $\mathbb{K}\langle C\rangle$ be an augmented graded polynomial space endowed with partial composition maps $\circ_i$. Then, the products $\circ$ on $\mathbb{K}\langle C\rangle$ defined by* (1.1.22) *are full composition maps.*

Conversely, let $\circ$ be full composition maps on $\mathbb{K}\langle C\rangle$ and $\mathbb{1}$ their unit. We construct from the $\circ$ and $\mathbb{1}$ the products $\circ_i^{(n,m)}$, $n, m \in \mathbb{N}_{\geqslant 1}$, $i \in [n]$, on $\mathbb{K}\langle C\rangle$ defined linearly in the following way. For any $x \in C(n)$ and $y \in C(m)$, let us set

$$x \circ_i^{(n,m)} y := x \circ \left[\underbrace{\mathbb{1}, \dots, \mathbb{1}}_{i-1 \text{ terms}}, y, \underbrace{\mathbb{1}, \dots, \mathbb{1}}_{n-i \text{ terms}}\right], \tag{1.1.23}$$

where $\mathbb{1}$ is the unit of the $\circ$.

Proposition 1.1.2. *Let $\mathbb{K}\langle C\rangle$ be an augmented graded polynomial space endowed with full composition maps $\circ$ and their unit $\mathbb{1}$. Then, the products $\circ_i$ on $\mathbb{K}\langle C\rangle$ defined by* (1.1.23) *are partial composition maps.*

1.2. Operads. Operads are algebraic structures furnishing a formalization of the notion of abstract operators and their partial and full compositions. They allow, for instance, to mimic the composition of abstract operators for various collections of combinatorial objects (words, trees, graphs, *etc.*) and make them behave like operators. We provide here definitions about these algebraic structures and present set-operads.

1.2.1. *First definitions.* A *nonsymmetric operad* (or an *operad* for short) is a triple

$$\left(\mathbb{K}\langle C\rangle, \left\{\circ_i^{(n,m)} : n, m \in \mathbb{N}_{\geqslant 1}, i \in [n]\right\}, \mathbb{1}\right) \tag{1.2.1}$$

where $\mathbb{K}\langle C\rangle$ is an augmented graded polynomial space, the $\circ_i$ are partial composition maps, and $\mathbb{1}$ is their unit. Equivalently, by Propositions 1.1.1 and 1.1.2, an operad is a triple

$$\left(\mathbb{K}\langle C\rangle, \left\{\circ^{(m_1,\dots,m_n)} : n \in \mathbb{N}_{\geqslant 1}, m_i \in \mathbb{N}_{\geqslant 1}, i \in [n]\right\}, \mathbb{1}\right) \tag{1.2.2}$$

where $\mathbb{K}\langle C\rangle$ is an augmented graded polynomial space, the $\circ$ are full composition maps, and $\mathbb{1}$ is their unit. For this reason, in the sequel, we shall consider operads through partial or full composition maps indifferently. Moreover, given an operad $\mathbb{K}\langle C\rangle$ defined through partial composition maps $\circ_i$, we call *full composition maps* of $\mathbb{K}\langle C\rangle$ the full composition maps $\circ$ defined in (1.1.22). Conversely, if $\mathbb{K}\langle C\rangle$ is defined through full composition maps $\circ$, we call *partial composition maps* of $\mathbb{K}\langle C\rangle$ the partial composition maps $\circ_i$ defined in (1.1.23).

Since an operad is a particular polynomial algebra, all the properties, definitions, and notations about polynomial algebras exposed in Section 2 of Chapter 3 remain valid for operads (like operad morphisms, suboperads, generating sets, operad ideals and quotients, *etc.*). In particular, to be more precise, if $\mathbb{K}\langle C_1\rangle$ and $\mathbb{K}\langle C_2\rangle$ are two operads, a map $\phi : \mathbb{K}\langle C_1\rangle \to \mathbb{K}\langle C_2\rangle$ is an *operad morphism* if ϕ is a graded polynomial space morphism, it sends the unit of $\mathbb{K}\langle C_1\rangle$ to the unit of $\mathbb{K}\langle C_2\rangle$, and

$$\phi(x \circ_i y) = \phi(x) \circ_i \phi(y) \tag{1.2.3}$$

for all $x \in C_1(n)$, $y \in C_1$, and $i \in [n]$. If $\mathbb{K}\langle C\rangle$ is an operad and $\mathfrak{G}$ is a subset of $\mathbb{K}\langle C\rangle$, the *operad generated* by $\mathfrak{G}$ is the smallest suboperad $\mathbb{K}\langle C\rangle^{\mathfrak{G}}$ of $\mathbb{K}\langle C\rangle$ containing $\mathfrak{G}$. A space $\mathcal{V}$ included in $\mathbb{K}\langle C\rangle$ is an *operad ideal* of $\mathbb{K}\langle C\rangle$ if $x \circ_i f \in \mathcal{V}$ and $f \circ_j y \in \mathcal{V}$ for all homogeneous element f of $\mathcal{V}$ of degree m, $x \in C(n)$, $y \in C$, $i \in [n]$, and $j \in [m]$. The *quotient operad* $\mathbb{K}\langle C\rangle/_{\mathcal{V}}$ of $\mathbb{K}\langle C\rangle$ by $\mathcal{V}$ is defined as follows. Let $\theta : \mathbb{K}\langle C\rangle \to \mathbb{K}\langle C\rangle/_{\mathcal{V}}$ be the canonical surjection map from $\mathbb{K}\langle C\rangle$ to $\mathbb{K}\langle C\rangle/_{\mathcal{V}}$. The space $\mathbb{K}\langle C\rangle/_{\mathcal{V}}$ is endowed with the structure of an operad through the partial composition maps defined by

$$\theta(x) \circ_i \theta(y) := \theta(x \circ_i y) \tag{1.2.4}$$

for any $x \in C(n)$, $y \in C$, and $i \in [n]$, where the second occurrence of $\circ_i$ in (1.2.4) is the partial composition map of $\mathbb{K}\langle C\rangle$.

1.2.2. *Additional definitions.* Let $\mathbb{K}\langle C\rangle$ be an operad. An element f of arity 2 of $\mathbb{K}\langle C\rangle$ is *associative* if $f \circ_1 f - f \circ_2 f = 0$. If $\mathbb{K}\langle C_1\rangle$ and $\mathbb{K}\langle C_2\rangle$ are two operads, an *operad antimorphism* is a graded polynomial space morphism $\phi : \mathbb{K}\langle C_1\rangle \to \mathbb{K}\langle C_2\rangle$ sending the unit of $\mathbb{K}\langle C_1\rangle$ to the unit of $\mathbb{K}\langle C_2\rangle$ and satisfying

$$\phi(x \circ_i y) = \phi(x) \circ_{n-i+1} \phi(y) \tag{1.2.5}$$

for any $x \in C_1(n)$, $y \in C_1$, and $i \in [n]$. A *symmetry* of $\mathbb{K}\langle C\rangle$ is either an operad automorphism or an operad antiautomorphism of $\mathbb{K}\langle C\rangle$. The set of all symmetries of $\mathbb{K}\langle C\rangle$ forms a group for the map composition, called *group of symmetries* of $\mathbb{K}\langle C\rangle$.

The *Hadamard product* of a sequence $\mathbb{K}\langle C_1\rangle$, ..., $\mathbb{K}\langle C_p\rangle$, $p \in \mathbb{N}$, of operads is the operad on the graded polynomial space $\mathbb{K}\langle [C_1, \dots, C_p]_{\boxtimes}\rangle$ where $[C_1, \dots, C_p]_{\boxtimes}$ is the Hadamard product on collections (see Section 1.2.4 of Chapter 1). The partial composition maps $\circ_i$ of this operad are defined linearly by

$$(x_1, \dots, x_p) \circ_i (y_1, \dots, y_p) := (x_1 \circ_i y_1, \dots, y_p \circ_i y_p), \tag{1.2.6}$$

for any objects $(x_1, \dots, x_p)$ and $(y_1, \dots, y_p)$ of $[C_1, \dots, C_p]_{\boxtimes}$, where the occurrences of $\circ_i$ in (1.2.6) are, from left to right, the partial composition maps of $\mathbb{K}\langle C_1\rangle$, ..., $\mathbb{K}\langle C_p\rangle$. The unit of the operad $\mathbb{K}\langle [C_1, \dots, C_p]_{\boxtimes}\rangle$ is $(\mathbb{1}_1, \dots, \mathbb{1}_p)$ where $\mathbb{1}_k$ is the unit of $\mathbb{K}\langle C_k\rangle$ for any $k \in [p]$.

1.2.3. *Set-operads.* An operad $\mathbb{K}\langle C\rangle$ is a *set-operad* if C is a set-basis (see Section 2.2.4 of Chapter 3) with respect to all the partial composition maps of $\mathbb{K}\langle C\rangle$, and the unit $\mathbb{1}$ is an object of C. This implies in particular that for all $x \in C(n)$, $y \in C(m)$, and $i \in [n]$, $x \circ_i y$ is an object of C. To study a set-operad $\mathbb{K}\langle C\rangle$, it is in some cases convenient to forget about its linear structure and see its partial composition maps $\circ_i$ as set-theoretic maps (see Section 2.3.3 of Chapter 3). Let us consider now that $\mathbb{K}\langle C\rangle$ is a set-operad and let us review some properties that operads of this kind of operad can satisfy.

A collection of maps

$$\tau_n : C(n) \to [n] \tag{1.2.7}$$

where $n \in \mathbb{N}_{\geqslant 1}$ are *root maps* of $\mathbb{K}\langle C\rangle$ if, for any $x \in C(n)$, $y \in C(m)$, and $i \in [n]$,

$$\tau_{n+m-1}(x \circ_i y) = \begin{cases} \tau_n(x) + m - 1 & \text{if } i \leqslant \tau_n(x) - 1, \\ \tau_n(x) + \tau_m(y) - 1 & \text{if } i = \tau_n(x), \\ \tau_n(x) & \text{otherwise } (i \geqslant \tau_n(x)). \end{cases} \tag{1.2.8}$$

In this case, we say that $\mathbb{K}\langle C\rangle$ is a *rooted operad* with respect to the maps τ_n, $n \in \mathbb{N}_{\geqslant 1}$. More intuitively, this property says that in a rooted operad, each object x of $C(n)$ has a particular input $\tau_n(x)$ which is preserved by the partial composition maps.

Besides, let for any $y \in C(m)$, $n \in \mathbb{N}_{\geqslant 1}$, and $i \in [n]$ the maps

$$\bullet_i^{(n,y)} : C(n) \to C(n + m - 1), \tag{1.2.9}$$

defined for any $x \in C(n)$ by

$$\bullet_i^{(n,y)}(x) := x \circ_i y. \tag{1.2.10}$$

When for all $y \in C(m)$, $n \in \mathbb{N}_{\geqslant 1}$, and $i \in [n]$, all the maps $\bullet_i^{(n,y)}$ are injective, $\mathbb{K}\langle C\rangle$ is a *basic operad*. More intuitively, this property says that in a basic operad, one can recover the object x from $x \circ_i y$ with the knowledge of i and y.

1.3. Algebras over operads. One of the main interests of the theory of operads is that each operad encodes a category of type of algebras. In this way, by studying a single operad, it is possible to get general results about all the algebras of the encoded category. Moreover, morphisms between operads offer general constructions to, given an algebra of one type, obtain an algebra of another type. We explain here all these notions and expose also the concept of free algebras over operads.

1.3.1. *From operads to types of algebras.* Any operad $\mathbb{K}\langle C\rangle$ encodes a type of polynomial algebras (see Section 3 of Chapter 3) called *algebras over* $\mathbb{K}\langle C\rangle$ (or, for short, $\mathbb{K}\langle C\rangle$*-algebras*). A $\mathbb{K}\langle C\rangle$-algebra is a (not necessarily graded) polynomial space $\mathbb{K}\langle D\rangle$, where D is a collection, which is endowed for all $n \in \mathbb{N}_{\geqslant 1}$ with linear maps

$$\bullet_n : \mathbb{K}\left\langle \left[C(n), \mathbf{List}_{\{n\}}(D) \right]_\times \right\rangle \to \mathbb{K}\langle D\rangle \tag{1.3.1}$$

satisfying the relations imposed by the operad structure of $\mathbb{K}\langle C\rangle$, that are, for all $x \in C(n)$, $y \in C(m)$, $i \in [n]$, and $(a_1, \dots, a_{n+m-1}) \in \mathbf{List}_{\{n+m-1\}}(D)$,

$$\bullet_{n+m-1}(x \circ_i y, (a_1, \dots, a_{n+m-1})) = \bullet_n(x, (a_1, \dots, a_{i-1}, \bullet_m(y, (a_i, \dots, a_{i+m-1})), a_{i+m}, \dots, a_{n+m-1})), \quad (1.3.2a)$$

and for all $a_1 \in D$,

$$\bullet_1(\mathbb{1}, (a_1)) = a_1. \quad (1.3.2b)$$

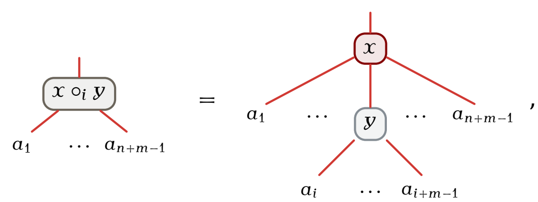
In other words, any object x of C of arity n plays the role of a complete product (in the sense of Section 2.1.1 of Chapter 3) of the form

$$x : \mathbb{K}\left\langle \mathbf{List}_{\{n\}}(D)\right\rangle \to \mathbb{K}\langle D\rangle, \quad (1.3.3)$$

defined, for any $(a_1, \dots, a_n) \in \mathbf{List}_{\{n\}}(D)$ by

$$x(a_1, \dots, a_n) := \bullet_n(x, (a_1, \dots, a_n)). \quad (1.3.4)$$

Under this point of view, Relation (1.3.2a) reads as



(1.3.5)

and Relation (1.3.2b) says that $\mathbb{1}$ is the identity map on $\mathbb{K}\langle D\rangle$. From now, to define an algebra $\mathbb{K}\langle D\rangle$ over an operad $\mathbb{K}\langle C\rangle$, we shall simply describe how the objects x of C behave as linear products on $\mathbb{K}\langle D\rangle$.

Observe that by (1.3.2a), any associative element of $\mathbb{K}\langle C\rangle$ gives rise to an associative operation on $\mathbb{K}\langle D\rangle$ (details are given in further Section 3.1.1).

1.3.2. *Categories of algebras.* The class of all the $\mathbb{K}\langle C\rangle$-algebras forms a category, called *category of $\mathbb{K}\langle C\rangle$-algebras*, wherein morphisms between $\mathbb{K}\langle C\rangle$-algebras are polynomial algebra morphisms (see Section 2.3.1 of Chapter 3). More concretely, if $\mathbb{K}\langle D_1\rangle$ and $\mathbb{K}\langle D_2\rangle$ are two $\mathbb{K}\langle C\rangle$-algebras such that D_1 and D_2 are two collections on a same set of indexes, a map

$$\phi : \mathbb{K}\langle D_1\rangle \to \mathbb{K}\langle D_2\rangle \quad (1.3.6)$$

is a $\mathbb{K}\langle C\rangle$-algebra morphism if ϕ is a polynomial space morphism and satisfies

$$\phi(x(a_1, \dots, a_n)) = x(\phi(a_1), \dots, \phi(a_n)) \quad (1.3.7)$$

for all $n \in \mathbb{N}_{\geqslant 1}$, $x \in C(n)$, and $a_1, \dots, a_n \in D_1$.

PROPOSITION 1.3.1. *Let $\mathbb{K}\langle C_1\rangle$ and $\mathbb{K}\langle C_2\rangle$ be two operads and $\phi : \mathbb{K}\langle C_1\rangle \to \mathbb{K}\langle C_2\rangle$ be an operad morphism. Then, if $\mathbb{K}\langle D\rangle$ is a $\mathbb{K}\langle C_2\rangle$-algebra, by setting for any $n \in \mathbb{N}_{\geqslant 1}$, $x \in C_1(n)$, and $a_1, \dots, a_n \in D$,*

$$x\,(a_1, \dots, a_n) := (\phi(x))\,(a_1, \dots, a_n)\,, \tag{1.3.8}$$

the space $\mathbb{K}\langle D\rangle$ becomes a $\mathbb{K}\langle C_1\rangle$-algebra.

Proposition 1.3.1 brings a way to construct $\mathbb{K}\langle C_1\rangle$-algebras from both a $\mathbb{K}\langle C_2\rangle$-algebra and an operad morphism between $\mathbb{K}\langle C_1\rangle$ and $\mathbb{K}\langle C_2\rangle$. Some classical constructions of algebras come within this framework. For instance, it is well-known that any dendriform algebra leads to an associative algebra by considering the product obtained by summing the two dendriform products (see Section 3.2 of Chapter 3). This construction is in fact the consequence of an operad morphism from the associative operad to the dendriform operad (see the forthcoming Sections 3.1.1 and 3.2.3).

1.3.3. *Free algebras over operads.* Let us now describe particular algebras over operads. Let S be a graded collection and let us consider the graded space

$$\mathbb{K}\langle C\rangle^{(S)} := \mathbb{K}\langle C \odot S\rangle \tag{1.3.9}$$

where $\odot$ is the composition product of graded collections (see Section 1.2.9 of Chapter 1). Let us endow $\mathbb{K}\langle C\rangle^{(S)}$ with the products $x \in C(n)$, $n \in \mathbb{N}_{\geqslant 1}$, defined linearly, for all objects $\left(y_i, \left(s_{i,1}, \dots, s_{i,|y_i|}\right)\right)$ of $(C \odot S)\,(m_i)$, $m_i \in \mathbb{N}_{\geqslant 1}$, $i \in [n]$, by

$$\begin{aligned} x\left(\left(y_1, \left(s_{1,1}, \dots, s_{1,|y_1|}\right)\right), \dots, \left(y_n, \left(s_{n,1}, \dots, s_{n,|y_n|}\right)\right)\right) \\ := \left(x \circ [y_1, \dots, y_n]\,, \left(s_{1,1}, \dots, s_{1,|y_1|}, \dots, s_{n,1}, \dots, s_{n,|y_n|}\right)\right). \end{aligned} \tag{1.3.10}$$

PROPOSITION 1.3.2. *Let $\mathbb{K}\langle C\rangle$ be an operad and S be a graded collection. Then, the space $\mathbb{K}\langle C\rangle^{(S)}$ endowed with the linear products $x \in C$ defined by (1.3.10) is a $\mathbb{K}\langle C\rangle$-algebra.*

Let now

$$\iota : S \to \mathbb{K}\langle C\rangle^{(S)} \tag{1.3.11}$$

be the map defined for any $s \in S$ by $\iota(s) := (\mathbb{1}, (s))$, where as usual, $\mathbb{1}$ denotes the unit of $\mathbb{K}\langle C\rangle$. This map can be seen as an inclusion of S into $\mathbb{K}\langle C\rangle^{(S)}$.

THEOREM 1.3.3. *Let $\mathbb{K}\langle C\rangle$ be an operad and S be a graded collection. Then, $\mathbb{K}\langle C\rangle^{(S)}$ is the unique $\mathbb{K}\langle C\rangle$-algebra (up to isomorphism) such that for any graded $\mathbb{K}\langle C\rangle$-algebra $\mathbb{K}\langle D\rangle$ and any map $f : S \to \mathbb{K}\langle D\rangle$ respecting the sizes, there exists a unique $\mathbb{K}\langle C\rangle$-algebra morphism $\phi : \mathbb{K}\langle C\rangle^{(S)} \to \mathbb{K}\langle D\rangle$ such that $f = \phi \circ \iota$.*

Theorem 1.3.3 provides the fact that $\mathbb{K}\langle C\rangle^{(S)}$ satisfies a universality property saying (with the notations of the statement of the theorem) that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\ f\ } & \mathbb{K}\langle D\rangle \\ {\scriptstyle\iota}\big\downarrow & \nearrow_{\phi} & \\ \mathbb{K}\langle C\rangle^{(S)} & & \end{array} \tag{1.3.12}$$

commutes and therefore, that $\mathbb{K}\langle C\rangle^{(S)}$ is a free object in the category of the $\mathbb{K}\langle C\rangle$-algebras. For this reason, we call $\mathbb{K}\langle C\rangle^{(S)}$ the *free $\mathbb{K}\langle C\rangle$-algebra over S*.

When $\bullet$ is an atom, the free $\mathbb{K}\langle C\rangle$-algebra $\mathbb{K}\langle C\rangle^{(\{\bullet\})}$ admits the following description. First, since $\bullet$ is of size 1, by Relation (1.2.26a) of Chapter 1, $\mathbb{K}\langle C\rangle^{(\{\bullet\})}$ is isomorphic (as a polynomial space) to $\mathbb{K}\langle C\rangle$ and each basis element $\left(x, \bullet^{|x|}\right)$ of $\mathbb{K}\langle C\rangle^{(\{\bullet\})}$ can be identified with the basis element $x \in C$ of $\mathbb{K}\langle C\rangle$. Moreover, by (1.3.10), the operations $x \in C(n)$, $n \in \mathbb{N}_{\geqslant 1}$, of $\mathbb{K}\langle C\rangle$ satisfy, for any $y_1, \dots, y_n \in C$,

$$x\,(y_1, \dots y_n) := x \circ [y_1, \dots y_n]\,. \tag{1.3.13}$$

2. Free operads, presentations, and Koszulity

Free operads are intuitively operads wherein partial composition maps satisfy only the required relations. These operads can be realized as spaces of syntax trees. We present here some general notions for operads related to free operads: presentations by generators and relations, Koszul duality, and Koszulity for binary and quadratic operads.

2.1. Free operads. Let us start by defining free operads, exposing the universality property they satisfy, and a notion of factorization of the elements of a operad relying on free operads.

2.1.1. *Operads of syntax trees.* Let $\mathfrak{G}$ be an augmented graded collection. The *free operad* over $\mathfrak{G}$ is the operad

$$\mathbf{FO}(\mathfrak{G}) := \mathbb{K}\left\langle \mathfrak{S}\mathfrak{T}_{\perp}^{\mathfrak{G}} \right\rangle, \tag{2.1.1}$$

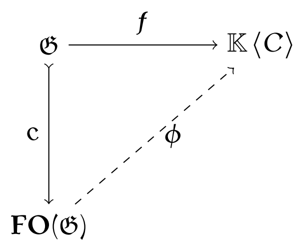
where $\mathfrak{S}\mathfrak{T}_{\perp}^{\mathfrak{G}}$ is the graded collection of all the $\mathfrak{G}$-syntax trees (see Section 2.1 of Chapter 2). The space $\mathbf{FO}(\mathfrak{G})$ is endowed with the linearizations of the partial grafting operations $\circ_i$, $i \in \mathbb{N}_{\geqslant 1}$, defined in Section 2.2.1 of Chapter 2. The unit of $\mathbf{FO}(\mathfrak{G})$ is the only $\mathfrak{G}$-syntax tree $\perp$ of arity 1 and degree 0.

Recall, as defined in Section 2.1 of Chapter 2, that for any $x \in \mathfrak{G}$, $\mathrm{c}(x)$ is the corolla labeled by x. We shall from now see c as a map

$$\mathrm{c} : \mathfrak{G} \to \mathbf{FO}(\mathfrak{G}) \tag{2.1.2}$$

called *inclusion map*. In the sequel, if required by the context, we shall implicitly see any element x of $\mathfrak{G}$ as the corolla $\mathrm{c}(x)$ of $\mathbf{FO}(\mathfrak{G})$. For instance, when $x \in \mathfrak{G}(n)$ and $y \in \mathfrak{G}$, we shall simply denote by $x \circ_i y$ the syntax tree $\mathrm{c}(x) \circ_i \mathrm{c}(y)$ for any $i \in [n]$.

Free operads satisfy the following universality property. The free operad $\mathbf{FO}(\mathfrak{G})$ is the unique operad (up to isomorphism) such that for any operad $\mathbb{K}\langle C\rangle$ and any map $f : \mathfrak{G} \to \mathbb{K}\langle C\rangle$ respecting the arities, there exists a unique operad morphism $\phi : \mathbf{FO}(\mathfrak{G}) \to \mathbb{K}\langle C\rangle$ such that $f = \phi \circ \mathrm{c}$. In other terms, the diagram



(2.1.3)

commutes.

2.1.2. *Evaluations and treelike expressions.* Let $\mathbb{K}\langle C\rangle$ be an operad. Since C is an augmented graded collection, one can consider the free operad $\mathbf{FO}(C)$ of the C-syntax trees. By definition, the fundamental basis of $\mathbf{FO}(C)$ is the set of the syntax trees on C. The *evaluation map* of $\mathbb{K}\langle C\rangle$ is the map

$$\mathrm{ev} : \mathbf{FO}(C) \to \mathbb{K}\langle C\rangle \tag{2.1.4}$$

defined linearly by induction, for any C-syntax tree $\mathfrak{t}$, by

$$\mathrm{ev}(\mathfrak{t}) := \begin{cases} \mathbb{1} \in \mathbb{K}\langle C\rangle & \text{if } \mathfrak{t} = \perp, \\ \omega_{\mathfrak{t}}(\epsilon) \circ [\mathrm{ev}(\mathfrak{t}_1), \dots, \mathrm{ev}(\mathfrak{t}_k)] & \text{otherwise}, \end{cases} \tag{2.1.5}$$

where the $\circ$ are the full composition maps of $\mathbb{K}\langle C\rangle$, $\omega_{\mathfrak{t}}(\epsilon)$ is the label of the root of $\mathfrak{t}$, and k is the root arity of $\mathfrak{t}$. This map ev is the unique surjective operad morphism from $\mathbf{FO}(C)$ to $\mathbb{K}\langle C\rangle$ satisfying $\mathrm{ev}(\mathrm{c}(x)) = x$ for all $x \in C$.

For any element f of $\mathbb{K}\langle C\rangle$, a *treelike expression* of f is an element f' of $\mathbf{FO}(C)$ such that $\mathrm{ev}(f') = f$. A treelike expression can therefore be thought as a factorization in an operad.

2.2. Presentations by generators and relations. To understand the structure of an operad, it is in most of the cases fruitful to see it as a quotient of a free operad, leading to the notion of presentation by generators and relations. Indeed, by comparing the presentations of two operads, it is most of the time easy to construct injective or surjective morphisms between them. Moreover, knowing a presentation of an operad facilitates the description of the category of the algebras it encodes. We present here a tool coming from the theory of rewrite systems on syntax trees to establish presentations.

2.2.1. *Presentations.* A *presentation* of an operad $\mathbb{K}\langle C\rangle$ consists in a pair $(\mathfrak{G}, \mathfrak{R})$ such that $\mathfrak{G}$ is an augmented graded collection, $\mathfrak{R}$ is a subspace of $\mathbf{FO}(\mathfrak{G})$ and

$$\mathbb{K}\langle C\rangle \simeq \mathbf{FO}(\mathfrak{G})/_{\langle\mathfrak{R}\rangle} \tag{2.2.1}$$

where $\langle\mathfrak{R}\rangle$ is the operad ideal of $\mathbf{FO}(\mathfrak{G})$ generated by $\mathfrak{R}$. We call $\mathfrak{G}$ the *set of generators* and $\mathfrak{R}$ the *space of relations* of $\mathbb{K}\langle C\rangle$.

We say that a presentation $(\mathfrak{G}, \mathfrak{R})$ of $\mathbb{K}\langle C\rangle$ is *quadratic* if $\mathfrak{R}$ is a homogeneous subspace of $\mathbf{FO}(\mathfrak{G})$ consisting in syntax trees of degree 2. Besides, we say that $(\mathfrak{G}, \mathfrak{R})$ is *binary* if $\mathfrak{G}$ has only elements of size (arity) 2. By extension, we say also that $\mathbb{K}\langle C\rangle$ is *quadratic* (resp. *binary*) if it admits a quadratic (resp. binary) presentation.

There is a close link between operad ideals, closures of rewrite rules of syntax trees (see Section 2.3.2 of Chapter 2), and spaces induced by rewrite rules (see Section 1.1.4 of Chapter 3) brought by the following statement.

Proposition 2.2.1. *Let $\mathfrak{G}$ be an augmented graded collection and $\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \to\right)$ be a rewrite system. Then,*

$$\left\langle \mathfrak{R}_{\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \to\right)} \right\rangle = \mathfrak{R}_{\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \Rightarrow\right)}. \tag{2.2.2}$$

In the statement of Proposition 2.2.1, recall that $\mathfrak{R}_{\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \to\right)}$ denote the space induced by $\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \to\right)$ and $\mathfrak{R}_{\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \Rightarrow\right)}$ denotes the space induced by the closure of $\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \to\right)$.

2.2.2. *Proving presentations through rewrite systems.* Rewrite systems on syntax trees (see Section 3 of Chapter 1 and Section 2.3 of Chapter 2) are powerful tools to prove that a given operad admits a conjectured presentation. The following result provides a way to establish presentations of operads.

Theorem 2.2.2. *Let $\mathbb{K}\langle C\rangle$ be an operad, $\mathfrak{G}$ be a subcollection of C, and $\mathfrak{R}$ be a subspace of $\mathbf{FO}(\mathfrak{G})$ of syntax trees of degrees 2 or more. If*

(i) the collection $\mathfrak{G}$ is a generating set of $\mathbb{K}\langle C\rangle$ as an operad;
(ii) for any $f \in \mathfrak{R}$, $\mathrm{ev}(f) = 0$;
(iii) there exists a rewrite system $\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \to\right)$ being an orientation of $\mathfrak{R}$, such that its closure $\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \Rightarrow\right)$ is convergent, and its set of normal forms $\mathcal{N}_{\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \Rightarrow\right)}$ is isomorphic (as a graded collection) to C,

then $(\mathfrak{G}, \mathfrak{R})$ is a presentation of $\mathbb{K}\langle C\rangle$.

In practice, there are at least two ways to use Theorem 2.2.2 to establish a presentation of an operad $\mathbb{K}\langle C\rangle$. The first one is the most obvious: it consists first in finding a generating set $\mathfrak{G}$ of $\mathbb{K}\langle C\rangle$, then conjecturing (likely with the help of the computer) a space of relations $\mathfrak{R}$ and a rewrite system $\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \to\right)$ such that all conditions *(i)*, *(ii)*, and *(iii)* are satisfied. This can be technical (especially to prove that the closure $\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \Rightarrow\right)$ is convergent), and relies heavily on computer exploration. The second way requires as a prerequisite that $\mathbb{K}\langle C\rangle$ is combinatorial (and thus, all its homogeneous components are finite dimensional). In this case, we need here also to find a generating set $\mathfrak{G}$ of $\mathbb{K}\langle C\rangle$, a space of relations $\mathfrak{R}$ and a rewrite system $\left(\mathfrak{S}\mathfrak{T}_{\perp}^{\mathfrak{G}}, \to\right)$ such that *(i)*, and *(ii)* hold, and that C and $\mathcal{N}_{\left(\mathfrak{S}\mathfrak{T}_{\perp}^{\mathfrak{G}}, \Rightarrow\right)}$ are isomorphic as graded combinatorial collections. The difference with the first way occurs for *(iii)*: it is now sufficient to prove that $\left(\mathfrak{S}\mathfrak{T}_{\perp}^{\mathfrak{G}}, \Rightarrow\right)$ is terminating (and not necessarily convergent). Indeed, if $\left(\mathfrak{S}\mathfrak{T}_{\perp}^{\mathfrak{G}}, \Rightarrow\right)$ is terminating, since $\mathbb{K}\langle C\rangle$ is combinatorial,

$$\dim \mathbb{K}\langle C\rangle(n) = \#\mathcal{N}_{\left(\mathfrak{S}\mathfrak{T}_{\perp}^{\mathfrak{G}}, \Rightarrow\right)}(n) \geqslant \dim \mathbf{FO}(\mathfrak{G})/_{\langle\mathfrak{R}\rangle}(n) \tag{2.2.3}$$

for all $n \in \mathbb{N}_{\geqslant 1}$. The inequality of (2.2.3) comes from the fact that, since we do not know if $\left(\mathfrak{S}\mathfrak{T}_{\perp}^{\mathfrak{G}}, \Rightarrow\right)$ is confluent, it can have more normal forms of arity n than the dimension of $\mathbf{FO}(\mathfrak{G})/_{\langle\mathfrak{R}\rangle}$ in arity n. It follows from (2.2.3), by using straightforward arguments, that there is an operad isomorphism from $\mathbf{FO}(\mathfrak{G})/_{\langle\mathfrak{R}\rangle}$ to $\mathbb{K}\langle C\rangle$.

2.2.3. *Realizations and presentations.* Defining an operad can be done in at least two different ways. The first way consists in describing explicitly an augmented graded polynomial space $\mathbb{K}\langle C\rangle$ together with algorithms for the computation of the partial composition maps $\circ_i$ involving objects of C. This concrete manner provides a *realization* of an operad. The second way consists in defining an operad through its presentation $(\mathfrak{G}, \mathfrak{R})$, that is, an operad which is by definition isomorphic to $\mathbf{FO}(\mathfrak{G})/_{\langle\mathfrak{R}\rangle}$. This manner provides only an abstract definition of an operad since nor the underlying space neither the partial composition maps of the operad are known at this stage. In practice, to fully understand an operad, it is most of the time useful to know one of its realizations and one of its presentations.

2.2.4. *From presentations to types of algebras.* The knowledge of a presentation $(\mathfrak{G}, \mathfrak{R})$ of an operad $\mathbb{K}\langle C\rangle$ leads to a simple description of the category of $\mathbb{K}\langle C\rangle$-algebras. Indeed, the symbols of $\mathfrak{G}$ specify the products of the algebras of the category, and the relations of $\mathfrak{R}$ specify the relations between these products. This relies on the fact that since $\mathfrak{G}$ is a generating set of $\mathbb{K}\langle C\rangle$, any $f \in \mathbb{K}\langle C\rangle(n)$ writes as an expression involving the linear structure of $\mathbb{K}\langle C\rangle$, its partial composition maps, and elements of $\mathfrak{G}$. Now, for any $\mathbb{K}\langle C\rangle$-algebra $\mathbb{K}\langle D\rangle$, Relation (1.3.2a) implies that one can write any $f(a_1, \dots, a_n)$, $a_1, \dots, a_n \in \mathbb{K}\langle D\rangle$, in terms of a linear combination of compositions of products of $\mathfrak{G}$. Hence, the knowledge of the behavior of each product $x \in \mathfrak{G}$ on $\mathbb{K}\langle D\rangle$ is enough to know the behavior of any product $f \in \mathbb{K}\langle C\rangle(n)$, $n \in \mathbb{N}_{\geqslant 1}$, on $\mathbb{K}\langle D\rangle$. Moreover, the relations between the products of $\mathfrak{G}$ satisfied by any $\mathbb{K}\langle C\rangle$-algebra are encoded by the elements of $\mathfrak{R}$. Indeed, each element f of $\mathfrak{R}$ is a formal sum of $\mathfrak{G}$-syntax trees which is, by definition, equated with 0 (that is, $\mathrm{ev}(f) = 0$).

2.3. Koszulity. Given a presentation of a quadratic and binary operad, one can compute a presentation of another operad, namely of its Koszul dual. This kind of duality has a close connection with the concept of Koszulity of operads which is defined originally in an algebraic way. This property on operads can be rephrased in terms of properties of orientations of spaces of relations and rewrite systems. As a concrete consequence of Koszulity, given a combinatorial Koszul operad, its Hilbert series and the one of its Koszul dual are inverse (in a certain sense) one of the other.

2.3.1. *Koszul duality.* Let $\mathbb{K}\langle C\rangle$ be an operad admitting a binary and quadratic presentation $(\mathfrak{G}, \mathcal{R})$ where $\mathfrak{G}$ is finite, the *Koszul dual* of $\mathbb{K}\langle C\rangle$ is the operad $\mathbb{K}\langle C\rangle^!$, isomorphic to the operad admitting the presentation $\left(\mathfrak{G}, \mathcal{R}^\perp\right)$ where $\mathcal{R}^\perp$ is the annihilator of $\mathcal{R}$ in $\mathbf{FO}(\mathfrak{G})$ with respect to the linear map

$$\langle - \rangle : \mathbb{K}\left\langle \left[\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}(3), \mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}(3) \right]_\times \right\rangle \to \mathbb{K} \tag{2.3.1}$$

linearly defined, for all $x, x', y, y' \in \mathfrak{G}(2)$, by

$$\left\langle \left(x \circ_i y, x' \circ_{i'} y'\right) \right\rangle := \begin{cases} 1 & \text{if } x = x', y = y', \text{ and } i = i' = 1, \\ -1 & \text{if } x = x', y = y', \text{ and } i = i' = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.2}$$

To not overload the notation, we write $\langle \mathfrak{t}, \mathfrak{s}\rangle$ instead of $\langle(\mathfrak{t}, \mathfrak{s})\rangle$ for any pair $(\mathfrak{t}, \mathfrak{s})$ of $\mathfrak{G}$-syntax trees of arity 3 and degree 2.

Then, with knowledge of a presentation of $\mathbb{K}\langle C\rangle$, one can compute a presentation of $\mathbb{K}\langle C\rangle^!$.

2.3.2. *Koszulity.* An operad $\mathbb{K}\langle C\rangle$ admitting a quadratic presentation is *Koszul* if its Koszul complex is acyclic. Furthermore, when $\mathbb{K}\langle C\rangle$ is Koszul, combinatorial, and admits a binary and quadratic presentation, the Hilbert series of $\mathbb{K}\langle C\rangle$ and of its Koszul dual $\mathbb{K}\langle C\rangle^!$ are related by

$$\mathbb{H}_{\mathbb{K}\langle C\rangle}\left(-\mathbb{H}_{\mathbb{K}\langle C\rangle^!}(-t)\right) = t = \mathbb{H}_{\mathbb{K}\langle C\rangle^!}\left(-\mathbb{H}_{\mathbb{K}\langle C\rangle}(-t)\right). \tag{2.3.3}$$

Relation (2.3.3) can be used either to prove that an operad is not Koszul (it is the case when the coefficients of the hypothetical Hilbert series of the Koszul dual admits coefficients that are not nonnegative integers) or to compute the Hilbert series of the Koszul dual of a Koszul operad.

The Koszulity of an operad $\mathbb{K}\langle C\rangle$ can be proved by using rewrite systems on syntax trees, in the following way.

PROPOSITION 2.3.1. *Let $\mathbb{K}\langle C\rangle$ be an operad admitting a quadratic presentation $(\mathfrak{G}, \mathcal{R})$. If there exists an orientation $\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \to\right)$ of $\mathcal{R}$ such that its closure $\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \Rightarrow\right)$ is a convergent rewrite system, then $\mathbb{K}\langle C\rangle$ is Koszul.*

When $\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \Rightarrow\right)$ satisfies the conditions contained in the statement of Proposition 2.3.1, the set of $\mathfrak{G}$-syntax trees that are normal forms $\mathcal{N}_{\left(\mathfrak{S}\mathfrak{T}_\perp^{\mathfrak{G}}, \Rightarrow\right)}$ forms a basis of $\mathbf{FO}(\mathfrak{G})/_{\langle\mathcal{R}\rangle}$, called *Poincaré-Birkhoff-Witt basis*.

One of the main merits of Koszul operads with Poincaré-Birkhoff-Witt bases is that they come with a generic way to build an associated realization. Assume that $(\mathfrak{G}, \mathfrak{R})$ is a quadratic presentation and let us set the goal to find a realization of $\mathbf{FO}(\mathfrak{G})/_{\langle \mathfrak{R} \rangle}$. If one can construct an orientation $\left(\mathfrak{ST}_\perp^{\mathfrak{G}}, \to\right)$ of $\mathfrak{R}$ such that $\left(\mathfrak{ST}_\perp^{\mathfrak{G}}, \Rightarrow\right)$ is a convergent rewrite system, by Proposition 2.3.1, the set of all normal forms of $\left(\mathfrak{ST}_\perp^{\mathfrak{G}}, \Rightarrow\right)$ forms a basis of $\mathbf{FO}(\mathfrak{G})/_{\langle \mathfrak{R} \rangle}$. Moreover, to compute the partial composition $\mathfrak{t} \circ_i \mathfrak{s}$ of two such normal forms $\mathfrak{t}$ and $\mathfrak{s}$, start with the syntax tree $\mathfrak{r} := \mathfrak{t} \circ_i \mathfrak{s}$ obtained by using the partial composition map $\circ_i$ of $\mathbf{FO}(\mathfrak{G})$, and then rewrite $\mathfrak{r}$ using $\Rightarrow$ as much as possible in order to obtain a normal form $\mathfrak{r}'$. This process is well-defined since $\left(\mathfrak{ST}_\perp^{\mathfrak{G}}, \Rightarrow\right)$ is convergent. We have established the fact that the space $\mathbb{K}\left\langle \mathcal{N}_{\left(\mathfrak{ST}_\perp^{\mathfrak{G}}, \Rightarrow\right)} \right\rangle$ is isomorphic to $\mathbf{FO}(\mathfrak{G})/_{\langle \mathfrak{R} \rangle}$ and that the partial composition maps just described endow this first space with an operad structure having $(\mathfrak{G}, \mathfrak{R})$ as presentation.

3. Main operads

We provide here classical examples of operads. These examples are divided into three categories depending on the general families of the involved combinatorial objects: words, trees, or graphs. We also present two general constructions to obtain, respectively, operads on words and operads on graphs. Table 4.1 contains an overview of these.

3.1. Operads of words. Five examples of operads are provided here. Their common point is that they are defined on graded spaces of families of words. The associative and diassociative operads seem not, at first glance, operads of words. We shall explain how to provide a realization of these two operads as operads of words through a general construction of operads from monoids.

3.1.1. *Associative operad.* Let $A := \{\mathfrak{a}_n : n \in \mathbb{N}_{\geqslant 1}\}$ be the graded collection where $|\mathfrak{a}_n| := n$ for any $n \in \mathbb{N}_{\geqslant 1}$. The *associative operad* $\mathbf{As}$ is the space $\mathbb{K}\langle A \rangle$ endowed with the partial composition maps $\circ_i$ defined linearly, for any $\mathfrak{a}_n \in A(n)$, $\mathfrak{a}_m \in A(m)$, and $i \in [n]$, by

$$\mathfrak{a}_n \circ_i \mathfrak{a}_m := \mathfrak{a}_{n+m-1}. \tag{3.1.1}$$

The unit of $\mathbf{As}$ is $\mathfrak{a}_1$. This operad is a set-operad, is combinatorial, and its Hilbert series satisfies

$$\mathbb{H}_{\mathbf{As}}(t) = \frac{t}{1-t}. \tag{3.1.2}$$

Moreover, $\mathbf{As}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \{\mathfrak{a}_2\}$ and $\mathfrak{R}$ is the space generated by

$$\mathfrak{c}(\mathfrak{a}_2) \circ_1 \mathfrak{c}(\mathfrak{a}_2) - \mathfrak{c}(\mathfrak{a}_2) \circ_2 \mathfrak{c}(\mathfrak{a}_2). \tag{3.1.3}$$

Operad	Objects	Arity	Set-operad	Binary	Quadratic
As	Integers	Value	Yes	Yes	Yes
Per	Permutations	Length	Yes	No	Yes
Dias	Word on $\{0, 1\}$ with one 0	Length	Yes	Yes	Yes
T$\mathcal{M}$	Words on $\mathcal{M}$	Length	Yes	No	No
Motz	Motzkin paths	Points	Yes	No	Yes
Mag	Binary trees	Leaves	Yes	Yes	Yes
Dup	Binary trees	Int. nodes	Yes	Yes	Yes
Dendr	Binary trees	Int. nodes	No	Yes	Yes
BS	Schröder trees	Leaves	Yes	Yes	Yes
PLie	Standard rooted trees	Nodes	No	No	Yes
NAP	Standard rooted trees	Nodes	Yes	No	?
NCT	Noncrossing trees	Sides	Yes	Yes	Yes
BNC	Bicolored noncross. config.	Sides	Yes	Yes	Yes
Grav	Gravity chord config.	Sides	Yes	No	No
C$\mathcal{M}$	$\bar{\mathcal{M}}$-config.	Sides	Yes	No	No
NC$\mathcal{M}$	Noncross. $\bar{\mathcal{M}}$-config.	Sides	Yes	No	No

TABLE 4.1. Main properties of some operads. Here, $\mathcal{M}$ is a monoid.

Since $\mathfrak{G}$ contains only $\mathtt{a}_2$, any algebra over **As** is a space $\mathbb{K}\langle D\rangle$ endowed with a binary product $\mathtt{a}_2$. Moreover, since $\mathfrak{R}$ contains the element (3.1.3), we have for any $f_1, f_2, f_3 \in \mathbb{K}\langle D\rangle$,

$$\begin{aligned} 0 &= (\mathtt{a}_2 \circ_1 \mathtt{a}_2 - \mathtt{a}_2 \circ_2 \mathtt{a}_2)(f_1, f_2, f_3) \\ &= (\mathtt{a}_2 \circ_1 \mathtt{a}_2)(f_1, f_2, f_3) - (\mathtt{a}_2 \circ_2 \mathtt{a}_2)(f_1, f_2, f_3) \\ &= \mathtt{a}_2(\mathtt{a}_2(f_1, f_2), f_3) - \mathtt{a}_2(f_1, \mathtt{a}_2(f_2, f_3)). \end{aligned} \tag{3.1.4}$$

This is equivalent to the relation

$$(f_1\, \mathtt{a}_2\, f_2)\, \mathtt{a}_2\, f_3 - f_1\, \mathtt{a}_2\, (f_2\, \mathtt{a}_2\, f_3) = 0 \tag{3.1.5}$$

written in infix way, implying that $\mathtt{a}_2$ is associative. Hence, any **As**-algebra is an associative algebra.

3.1.2. *Operad of permutations.* For any permutation σ of $\mathfrak{S}(n)$, $i \in [n]$, and $k \in \mathbb{N}$, let $\uparrow_i^k(\sigma)$ be the word on $\mathbb{N}$ obtained by incrementing by k the letters of σ greater than i. The *operad of permutations* **Per** is the space $\mathbb{K}\langle\mathfrak{S}\rangle$ endowed with the partial composition maps $\circ_i$ defined linearly, for any $\sigma \in \mathfrak{S}(n)$, $\nu \in \mathfrak{S}(m)$, and $i \in [n]$ in the following way. First, let $\sigma' := \uparrow_{\sigma(i)}^{m-1}(\sigma)$ and $\nu' := \uparrow_0^{\sigma(i)-1}(\nu)$. The partial composition of σ and ν is defined as

$$\sigma \circ_i \nu := \sigma'_{|[1,i-1]}\, \nu'\, \sigma'_{|[i+1,n]}. \tag{3.1.6}$$

For instance,

$$123 \circ_2 12 = 1234, \tag{3.1.7a}$$

$$7415623 \circ_4 231 = 941675823 \tag{3.1.7b}$$

are two partial compositions in **Per**. The unit of **Per** is the permutation $1 \in \mathfrak{S}(1)$. This operad is a set-operad, is combinatorial, and its Hilbert series satisfies

$$\mathcal{H}_{\mathbf{Per}}(t) = \sum_{n \in \mathbb{N}_{\geqslant 1}} n!\, t^n. \tag{3.1.8}$$

A *simple permutation* is a permutation σ such that for all factors u of σ, if the letters of u form an interval of $\mathbb{N}$ then $|u| = 1$ or $|u| = |\sigma|$. For instance, the permutation 6**2413**57 is not simple since the letters of the factor $u := 2413$ form an interval of $\mathbb{N}$. On the other hand, the permutation 5137462 is simple.

The operad **Per** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G}$ is the set of all simple permutations of sizes 2 or more and $\mathfrak{R}$ is the space generated by

$$\mathrm{c}(12) \circ_1 \mathrm{c}(12) - \mathrm{c}(12) \circ_2 \mathrm{c}(12), \tag{3.1.9a}$$

$$\mathrm{c}(21) \circ_1 \mathrm{c}(21) - \mathrm{c}(21) \circ_2 \mathrm{c}(21). \tag{3.1.9b}$$

3.1.3. *Diassociative operad.* Let $E := \{\mathfrak{e}_{n,k} : n \in \mathbb{N}_{\geqslant 1}, k \in [n]\}$ be the graded collection where $|\mathfrak{e}_{n,k}| := n$ for any $n \in \mathbb{N}_{\geqslant 1}$ and $k \in [n]$. The *diassociative operad* **Dias** is the space $\mathbb{K}\langle E\rangle$ endowed with the partial composition maps $\circ_i$ defined linearly, for any $\mathfrak{e}_{n,k} \in E(n)$, $\mathfrak{e}_{m,\ell} \in E(m)$, and $i \in [n]$, by

$$\mathfrak{e}_{n,k} \circ_i \mathfrak{e}_{m,\ell} = \begin{cases} \mathfrak{e}_{n+m-1,k+m-1} & \text{if } i < k, \\ \mathfrak{e}_{n+m-1,k+\ell-1} & \text{if } i = k, \\ \mathfrak{e}_{n+m-1,k} & \text{otherwise } (i > k). \end{cases} \tag{3.1.10}$$

The unit of **Dias** is $\mathfrak{e}_{1,1}$. This operad is a set-operad, is combinatorial, and its Hilbert series satisfies

$$\mathcal{H}_{\mathbf{Dias}}(t) = \frac{t}{(1-t)^2} = \sum_{n \in \mathbb{N}_{\geqslant 1}} n\, t^n. \tag{3.1.11}$$

Moreover, **Dias** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \{\mathfrak{e}_{2,1}, \mathfrak{e}_{2,2}\}$ and $\mathfrak{R}$ is the space generated by, by denoting by $\dashv$ (resp. $\vdash$) the elements $\mathfrak{e}_{2,1}$ (resp. $\mathfrak{e}_{2,2}$),

$$\mathrm{c}(\dashv) \circ_1 \mathrm{c}(\dashv) - \mathrm{c}(\dashv) \circ_2 \mathrm{c}(\dashv), \quad \mathrm{c}(\dashv) \circ_1 \mathrm{c}(\dashv) - \mathrm{c}(\dashv) \circ_2 \mathrm{c}(\vdash), \tag{3.1.12a}$$

$$\mathsf{c}(\dashv) \circ_1 \mathsf{c}(\vdash) - \mathsf{c}(\vdash) \circ_2 \mathsf{c}(\dashv), \tag{3.1.12b}$$

$$\mathsf{c}(\vdash) \circ_1 \mathsf{c}(\dashv) - \mathsf{c}(\vdash) \circ_2 \mathsf{c}(\vdash), \quad \mathsf{c}(\vdash) \circ_1 \mathsf{c}(\vdash) - \mathsf{c}(\vdash) \circ_2 \mathsf{c}(\vdash). \tag{3.1.12c}$$

It is possible to show that the closure $\left(\mathfrak{S}\mathfrak{T}^{\mathfrak{G}}_{\perp}, \Rightarrow\right)$ of the orientation $\left(\mathfrak{S}\mathfrak{T}^{\mathfrak{G}}_{\perp}, \to\right)$ of $\mathcal{R}$ defined by

$$\mathsf{c}(\dashv) \circ_2 \mathsf{c}(\dashv) \to \mathsf{c}(\dashv) \circ_1 \mathsf{c}(\dashv), \quad \mathsf{c}(\dashv) \circ_2 \mathsf{c}(\vdash) \to \mathsf{c}(\dashv) \circ_1 \mathsf{c}(\dashv), \tag{3.1.13a}$$

$$\mathsf{c}(\vdash) \circ_2 \mathsf{c}(\dashv) \to \mathsf{c}(\dashv) \circ_1 \mathsf{c}(\vdash), \tag{3.1.13b}$$

$$\mathsf{c}(\vdash) \circ_1 \mathsf{c}(\dashv) \to \mathsf{c}(\vdash) \circ_2 \mathsf{c}(\vdash), \quad \mathsf{c}(\vdash) \circ_1 \mathsf{c}(\vdash) \to \mathsf{c}(\vdash) \circ_2 \mathsf{c}(\vdash). \tag{3.1.13c}$$

is convergent. Its normal forms are the syntax trees that avoid the trees appearing in the left members of (3.1.13a), (3.1.13b), and (3.1.13c). All this implies, by Proposition 2.3.1, that **Dias** is Koszul.

Besides, any algebra over **Dias** is space $\mathbb{K}\langle D\rangle$ endowed with two binary products $\dashv$ and $\vdash$ such that both $\dashv$ and $\vdash$ are associative (as consequences of (3.1.12a) and (3.1.12c)), and, for any $x, y, z \in D$,

$$x \dashv y \dashv z = x \dashv (y \vdash z), \tag{3.1.14a}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{3.1.14b}$$

$$(x \dashv y) \vdash z = x \vdash y \vdash z. \tag{3.1.14c}$$

These structures are called *diassociative algebras*.

3.1.4. *From monoids to operads.* We describe here a general way for constructing operads of words. Let $\mathcal{M}$ be a monoid with an associative product $\star$ admitting $\mathbb{1}$ as unit. We denote by $\mathbf{T}\mathcal{M}$ the space $\mathbb{K}\langle\mathcal{M}^+\rangle$ where $\mathcal{M}^+$ is the graded collection of all nonempty words on $\mathcal{M}$ seen as an alphabet. The space $\mathbf{T}\mathcal{M}$ is endowed with the partial composition maps $\circ_i$ defined linearly, for any $u \in \mathcal{M}(n)$, $v \in \mathcal{M}(m)$, and $i \in [n]$, by

$$u \circ_i v := u(1) \dots u(i-1)\,(u(i) \star v(1)) \,\dots\, (u(i) \star v(m))\, u(i+1) \dots u(n). \tag{3.1.15}$$

Proposition 3.1.1. *For any monoid $\mathcal{M}$, $\mathbf{T}\mathcal{M}$ is an operad.*

The unit of $\mathbf{T}\mathcal{M}$ is the unit $\mathbb{1}$ of the monoid $\mathcal{M}$, seen as a word of length 1. The operad $\mathbf{T}\mathcal{M}$ is a set-operad. Moreover, when $\mathcal{M}$ is finite, $\mathbf{T}\mathcal{M}$ is combinatorial and its Hilbert series satisfies

$$\mathbb{H}_{\mathbf{T}\mathcal{M}}(t) = \frac{t}{1 - mt} = \sum_{n \in \mathbb{N}_{\geqslant 1}} m^n\, t^n \tag{3.1.16}$$

where $m := \#\mathcal{M}$.

Let us consider an example. Let $\mathcal{M} := \{\mathsf{a}, \mathsf{b}\}^*$ be a free monoid of words. Then, $\mathbf{T}\mathcal{M}$ is the space of all words whose letters are words on $\{\mathsf{a}, \mathsf{b}\}$. We call such element *multiwords*. For instance, $(\mathsf{aa}, \mathsf{ba}, \mathsf{b}, \epsilon, \mathsf{a})$ is a multiword of arity 5 of $\mathbf{T}\mathcal{M}$ and

$$(\mathsf{aa}, \mathsf{ba}, \mathsf{b}, \epsilon, \mathsf{a}) \circ_3 (\mathsf{ab}, \epsilon, \mathsf{a}) = (\mathsf{aa}, \mathsf{ba}, \mathsf{bab}, \mathsf{b}, \mathsf{ba}, \epsilon, \mathsf{a}) \tag{3.1.17}$$

is a partial composition in $\mathbf{T}\mathcal{M}$.

Proposition 3.1.2. *Let $\mathcal{M}$ be a monoid. Then, the operad $\mathbf{T}\mathcal{M}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathcal{M} \sqcup \{\mathbb{1}\mathbb{1}\}$ and $\mathfrak{R}$ is the space generated by*

$$\mathsf{c}(\mathbb{1}\mathbb{1}) \circ_1 \mathsf{c}(\mathbb{1}\mathbb{1}) - \mathsf{c}(\mathbb{1}\mathbb{1}) \circ_2 \mathsf{c}(\mathbb{1}\mathbb{1}), \tag{3.1.18a}$$

$$\mathsf{c}(x) \circ_1 \mathsf{c}(y) - \mathsf{c}(x \star y), \qquad x, y \in \mathcal{M}, \tag{3.1.18b}$$

$$\mathsf{c}(\mathbb{1}\mathbb{1}) \circ [\mathsf{c}(x), \mathsf{c}(x)] - \mathsf{c}(x) \circ_1 \mathsf{c}(\mathbb{1}\mathbb{1}), \qquad x \in \mathcal{M}. \tag{3.1.18c}$$

Observe that the presentation of $\mathbf{T}\mathcal{M}$ provided by Proposition 3.1.2 is not minimal in the sense that the exhibited generating set $\mathfrak{G}$ may be not minimal.

The operads **As** and **Dias** can be obtained through this construction $\mathbf{T}$. First, one can check that $\mathbf{As} \simeq \mathbf{T}\{\mathbb{1}\}$ where $\{\mathbb{1}\}$ is the trivial monoid. An isomorphism between **As** and $\mathbf{T}\{\mathbb{1}\}$ is provided by the linear map $\phi : \mathbf{As} \to \mathbf{T}\{\mathbb{1}\}$ satisfying $\phi(\mathfrak{a}_n) = \mathbb{1}^n$ for all $n \in \mathbb{N}_{\geqslant 1}$. For instance,

$$\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1} \circ_3 \mathbb{1}\mathbb{1} = \mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1} \tag{3.1.19}$$

is a partial composition in this realization of **As**. Besides, **Dias** is isomorphic to the suboperad of $\mathbf{T}(\mathbb{N}, \max)$ generated by the words 01 and 10. An isomorphism between **Dias** and $\mathbf{T}(\mathbb{N}, \max)^{\{01,10\}}$ is provided by the linear map $\phi : \mathbf{Dias} \to \mathbf{T}(\mathbb{N}, \max)^{\{01,10\}}$ satisfying $\phi\left(1^k 0 1^\ell\right) = \mathfrak{e}_{k+1+\ell, k+1}$ for all $k, \ell \in \mathbb{N}$. For instance,

$$11011 \circ_3 01 = 110111, \tag{3.1.20a}$$

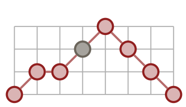
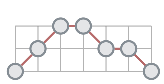
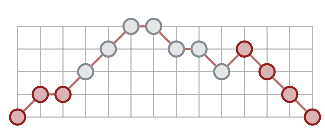
$$11011 \circ_4 01 = 110111 \tag{3.1.20b}$$

are two partial compositions in this realization of **Dias**.

3.1.5. *Operad of Motzkin words.* A *Motzkin word* is a nonempty word u on $\mathbb{N}$ starting and finishing by 0 and such that $|u(i) - u(i+1)| \leqslant 1$ for all $i \in [|u| - 1]$. We denote here by M the graded collection of all the Motzkin words where the size of a word is its length. Let **Motz** be the suboperad of $\mathbf{T}(\mathbb{N}, +)$ generated by the set $\{00, 010\}$. It is possible to show by induction on the arities that $\mathbf{Motz} = \mathbb{K}\langle M \rangle$. From the definition of the construction $\mathbf{T}$, the partial composition maps $\circ_i$ of **Motz** behave as follows. Given two Motzkin words u and v, $u \circ_i v$ is the Motzkin word obtained by replacing the letter at position i in u by a copy of v wherein each of its letters is incremented by $u(i)$. For instance,

$$0112321010 \circ_4 0122110 = 0112344332321010 \tag{3.1.21}$$

is a partial composition in **Motz**. By representing a Motzkin word u as a path in the quarter plane (that is, by drawing points $(i-1, u(i))$ for all positions i and by connecting all pairs of adjacent points by lines), (3.1.21) becomes

 $\circ_4$  $=$  . (3.1.22)

The unit of **Motz** is , the Motzkin word 0. This operad is a set-operad, is combinatorial, and its Hilbert series satisfies

$$\mathbb{H}_{\mathbf{Motz}}(t) = \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t}. \tag{3.1.23}$$

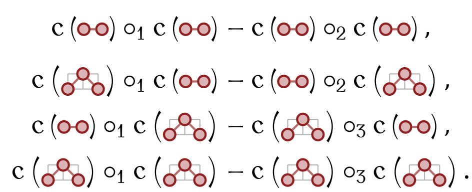
The first coefficients of its Hilbert series are

$$1, 1, 2, 4, 9, 21, 51, 127, 323 \tag{3.1.24}$$

and form Sequence A001006 of [Slo]. Moreover, **Motz** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where

(3.1.25)

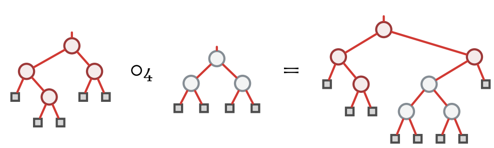
and $\mathfrak{R}$ is the space generated by



(3.1.26a)
(3.1.26b)
(3.1.26c)
(3.1.26d)

3.2. Operads of trees. Six examples of operads are provided here. Their common point is that they are defined on augmented graded spaces of families of trees: binary trees (seen endowed with several size functions), bicolored Schröder trees, and labeled rooted trees.

3.2.1. *Magmatic operad.* The *magmatic operad* **Mag** is the space $\mathbb{K}\langle \mathfrak{BT}_\perp \rangle$ (where $\mathfrak{BT}_\perp$ is the combinatorial graded collection of binary trees defined in Section 1.2.2 of Chapter 2) endowed with the partial composition maps $\circ_i$ defined as the linearizations of the partial grafting defined in Section 2.2.1 of Chapter 2. For instance,

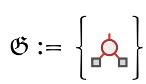


(3.2.1)

is a partial composition in **Mag**. The unit of **Mag** is . This operad is a set-operad, is combinatorial, and its Hilbert series satisfies

$$\mathbb{H}_{\mathbf{Mag}}(t) = \frac{1 - \sqrt{1 - 4t}}{2} = \sum_{n \in \mathbb{N}_{\geqslant 1}} \binom{2n-1}{n-1} \frac{1}{n} t^n. \tag{3.2.2}$$

Moreover, **Mag** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where

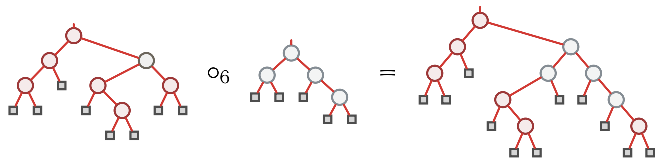


(3.2.3)

and $\mathfrak{R}$ is the trivial space.

Any algebra over **Mag** is a space $\mathbb{K}\langle D \rangle$ with a binary product which does satisfy any required relation.

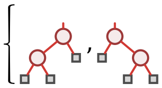
3.2.2. *Duplicial operad.* The *duplicial operad* **Dup** is the space $\mathbb{K}\langle \mathbf{Aug}(\mathfrak{BT}_\bullet)\rangle$ (where $\mathfrak{BT}_\bullet$ is the combinatorial graded collection of binary trees defined in Section 1.3.6 of Chapter 1) endowed with the partial composition maps $\circ_i$ defined linearly, for any $\mathfrak{t} \in \mathbf{Aug}(\mathfrak{BT}_\bullet)(n)$, $\mathfrak{s} \in \mathbf{Aug}(\mathfrak{BT}_\bullet)(m)$, and $i \in [n]$, by $\mathfrak{r} := \mathfrak{t} \circ_i \mathfrak{s}$ where $\mathfrak{r}$ is the binary tree obtained by replacing the ith (with respect to the infix order) internal node u of $\mathfrak{t}$ by a copy of $\mathfrak{s}$, and by grafting the left subtree of u to the first leaf of the copy, and the right subtree of u to the last leaf of the copy. For instance,


(3.2.4)

is a partial composition in **Dup**. The unit of **Dup** is . This operad is a set-operad, is combinatorial, and its Hilbert series satisfies

$$\mathbb{H}_{\mathbf{Dup}}(t) = \frac{1 - 2t - \sqrt{1-4t}}{2t} = \sum_{n \in \mathbb{N}_{\geqslant 1}} \binom{2n}{n} \frac{1}{n+1} t^n. \tag{3.2.5}$$

Moreover, **Dup** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where

$\mathfrak{G} := \Big\{$  $\Big\}$ (3.2.6)

and $\mathfrak{R}$ is the space generated by, by denoting by $\ll$ (resp. $\gg$) the first (resp. second) tree of (3.2.6),

$$\mathrm{c}(\ll) \circ_1 \mathrm{c}(\ll) - \mathrm{c}(\ll) \circ_2 \mathrm{c}(\ll), \tag{3.2.7a}$$

$$\mathrm{c}(\gg) \circ_1 \mathrm{c}(\ll) - \mathrm{c}(\ll) \circ_2 \mathrm{c}(\gg), \tag{3.2.7b}$$

$$\mathrm{c}(\gg) \circ_1 \mathrm{c}(\gg) - \mathrm{c}(\gg) \circ_2 \mathrm{c}(\gg). \tag{3.2.7c}$$

Any algebra over **Dup** is a space $\mathbb{K}\langle D\rangle$ endowed with two binary products $\ll$ and $\gg$ such that both $\ll$ and $\gg$ are associative (as consequences of (3.2.7a) and (3.2.7b)), and, for any $x, y, z \in D$,

$$(x \ll y) \gg z = x \ll (y \gg z). \tag{3.2.8}$$

These structures are called *duplicial algebras*.

3.2.3. *Dendriform operad.* The *dendriform operad* **Dendr** is defined as the operad admitting the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\prec, \succ\}$ and $\mathfrak{R}$ is the space generated by

$$\mathrm{c}(\prec) \circ_1 \mathrm{c}(\prec) - \mathrm{c}(\prec) \circ_2 \mathrm{c}(\prec) - \mathrm{c}(\prec) \circ_2 \mathrm{c}(\succ), \tag{3.2.9a}$$

$$\mathrm{c}(\prec) \circ_1 \mathrm{c}(\succ) - \mathrm{c}(\succ) \circ_2 \mathrm{c}(\prec), \tag{3.2.9b}$$

$$\mathrm{c}(\succ) \circ_1 \mathrm{c}(\prec) + \mathrm{c}(\succ) \circ_1 \mathrm{c}(\succ) - \mathrm{c}(\succ) \circ_2 \mathrm{c}(\succ). \tag{3.2.9c}$$

The dendriform operad and the diassociative operad are the Koszul duals one of the other. This can be shown by computing a basis of $\mathcal{R}^\perp$ where $\mathcal{R}$ is the space of relations of **Dendr**, and by observing that $\mathcal{R}^\perp$ and the space of relations of **Dias** shown in Section 3.1.3 are the same (by replacing, respectively, by $\dashv$ and $\vdash$ the generators $\prec$ and $\succ$ appearing in it). As a consequence of this fact and the Koszulity of **Dias**, the Hilbert series $\mathbb{H}_{\mathbf{Dendr}}(t)$ and $\mathbb{H}_{\mathbf{Dias}}(t)$ satisfy (2.3.3). It is then possible to obtain the explicit description

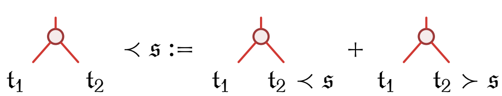
$$\mathbb{H}_{\mathbf{Dendr}}(t) = \frac{1 - 2t - \sqrt{1-4t}}{2t} = \sum_{n \in \mathbb{N}_{\geqslant 1}} \binom{2n}{n} \frac{1}{n+1} t^n \tag{3.2.10}$$

for the Hilbert series of **Dendr**. This shows that **Dendr** is, as a combinatorial polynomial space, the space $\mathbb{K}\langle \mathbf{Aug}(\mathfrak{BT}_\bullet)\rangle$.

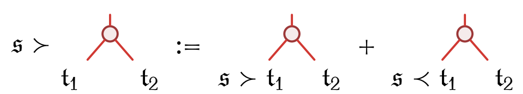
From the definition of **Dendr** by generators and relations, one can observe that any algebra over **Dendr** is a dendriform algebra (see Section 3.2 of Chapter 3). Moreover, the free dendriform algebra over one generator is the space **Dendr**, that is the linear span of all nonempty binary trees, endowed with the linear binary products $\prec$ and $\succ$ defined recursively, for any nonempty tree $\mathfrak{s}$, and binary trees $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by

$$\mathfrak{s} \prec \square := \mathfrak{s} =: \square \succ \mathfrak{s}, \tag{3.2.11a}$$

$$\square \prec \mathfrak{s} := 0 =: \mathfrak{s} \succ \square, \tag{3.2.11b}$$

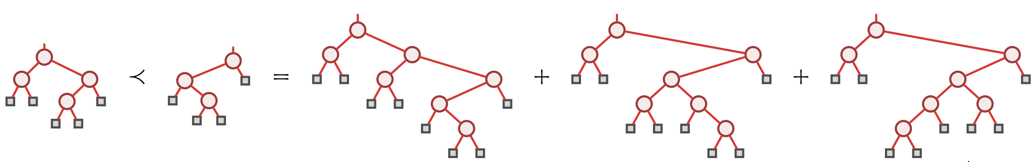


(3.2.11c)

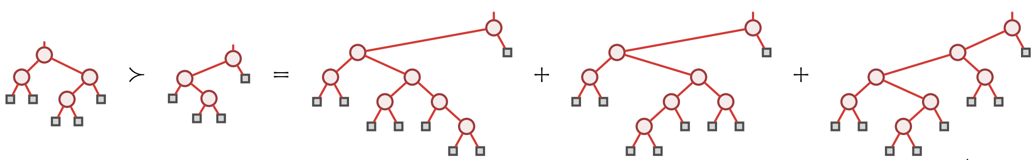


(3.2.11d)

Note that neither $\square \prec \square$ nor $\square \succ \square$ need to be defined. We have for instance,



(3.2.12a)



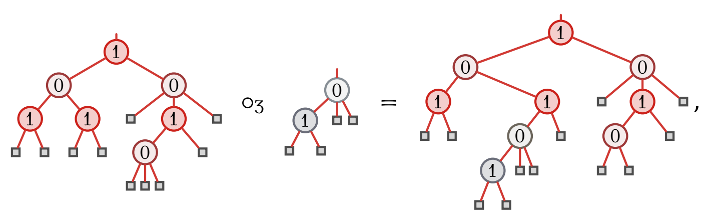
(3.2.12b)

Besides, one can check that the element $\prec + \succ$ of **Dendr** is associative. This implies that the linear map $\phi : \mathbf{As} \to \mathbf{Dendr}$ defined by $\phi(\mathfrak{a}_2) := \prec + \succ$ extends in a unique way into an operad morphism. Now, by Proposition 1.3.1, we obtain that if $\mathbb{K}\langle D\rangle$ is a dendriform algebra, the binary product $\mathfrak{a}_2$ defined for any $f_1, f_2 \in \mathbb{K}\langle D\rangle$ by

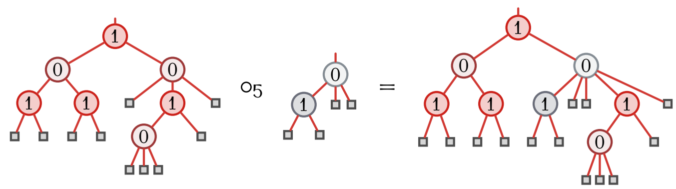
$$f_1 \,\mathfrak{a}_2\, f_2 := f_1 \,\phi(\mathfrak{a}_2)\, f_2 = f_1 \prec f_2 + f_1 \succ f_2 \tag{3.2.13}$$

is associative and endows $\mathbb{K}\langle D\rangle$ with the structure of an associative algebra.

3.2.4. *Bicolored Schröder tree operad.* A *bicolored Schröder tree* is a Schröder tree $\mathfrak{t}$ (see Section 1.2.3 of Chapter 2) such that each internal node is assigned with an element of the set $\{0, 1\}$ and all internal nodes that have a father labeled by 0 (resp. 1) are labeled by 1 (resp. 0). Let $\mathfrak{BSch}_\perp$ be the graded collection of all bicolored Schröder trees wherein the size of such trees is their number of leaves. The *bicolored Schröder tree operad* **BS** is the space $\mathbb{K}\langle \mathfrak{BSch}_\perp\rangle$ endowed with the partial composition maps $\circ_i$ defined linearly, for any $\mathfrak{t} \in \mathfrak{BSch}_\perp(n)$, $\mathfrak{s} \in \mathfrak{BSch}_\perp(m)$, and $i \in [n]$ by $\mathfrak{r} := \mathfrak{t} \circ_i \mathfrak{s}$ where $\mathfrak{r}$ is the bicolored Schröder tree obtained by grafting a copy of $\mathfrak{s}$ onto the ith leaf of $\mathfrak{t}$ and, in the case where the edge connecting this leaf and the copy of $\mathfrak{s}$ have the extremities that are internal nodes labeled by the same element $a \in \{0, 1\}$, by contracting this edge to form a single internal node labeled by a. For instance,



(3.2.14a)



(3.2.14b)

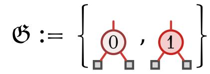
are two partial compositions in **BS**. The unit of **BS** is ⊥. This operad is a set-operad, is combinatorial, and its Hilbert series satisfies

$$\mathbb{H}_{\mathbf{BS}}(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2}. \tag{3.2.15}$$

The first coefficients of its Hilbert series are

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586 \tag{3.2.16}$$

and form Sequence A006318 of [**Slo**]. Moreover, **BS** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where



(3.2.17)

and $\mathfrak{R}$ is the space generated by, by denoting by $\star_0$ (resp. $\star_1$) the first (resp. second) tree of (3.2.17),

$$\mathrm{c}(\star_0) \circ_1 \mathrm{c}(\star_0) - \mathrm{c}(\star_0) \circ_2 \mathrm{c}(\star_0), \tag{3.2.18a}$$

$$\mathrm{c}\,(\star_1) \circ_1 \mathrm{c}\,(\star_1) - \mathrm{c}\,(\star_1) \circ_2 \mathrm{c}\,(\star_1)\,. \tag{3.2.18b}$$

Any algebra over **BS** is a space $\mathbb{K}\langle D\rangle$ endowed with two binary associative products $\star_0$ and $\star_1$. These structures are called *two-associative algebras*.

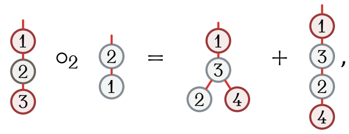
3.2.5. *Labeled rooted trees.* A *labeled rooted tree* is a rooted tree $\mathfrak{t}$ (see Section 3.1 of Chapter 2) endowed with an injective map sending each internal node of $\mathfrak{t}$ to an element of $\mathbb{N}$ called *label*. Due to the injective labeling of the nodes of any labeled rooted tree $\mathfrak{t}$, we shall identify each node of $\mathfrak{t}$ with its label. The set of all labels appearing in $\mathfrak{t}$ is denoted by $\mathcal{L}(\mathfrak{t})$. For any $i \in \mathcal{L}(\mathfrak{t})$, we denote by $\mathfrak{t}^{(i)}$ the set of the suffix subtrees rooted at the children of the node i in $\mathfrak{t}$. Moreover, for any $k \in \mathbb{N}$, we denote by $\uparrow_i^k(\mathfrak{t})$ the labeled rooted tree obtained from $\mathfrak{t}$ by incrementing by k its nodes greater than i. Let $\mathfrak{t}$ and $\mathfrak{s}$ be two labeled rooted trees such that $i \in \mathcal{L}(\mathfrak{t})$ and $(\mathcal{L}(\mathfrak{t}) \setminus \{i\}) \cap \mathcal{L}(\mathfrak{s}) = \emptyset$, and $\phi : \mathfrak{t}^{(i)} \to \mathcal{L}(\mathfrak{s})$ be a map. We denote by $\mathfrak{t} \hookleftarrow_i^{\phi} \mathfrak{s}$ the labeled rooted tree obtained by replacing the node i in $\mathfrak{t}$ by the root of a copy of $\mathfrak{s}$, and by grafting each tree $\mathfrak{r}$ of $\mathfrak{t}^{(i)}$ as a child of the node $\phi(\mathfrak{r})$ in the copy of $\mathfrak{s}$.

A *standard rooted tree* is a labeled rooted tree $\mathfrak{t}$ having all its labels in the set $[n]$ where n is the number of nodes of $\mathfrak{t}$. We denote by $\mathfrak{SRT}$ the combinatorial graded collection of the standard rooted trees wherein the size of such trees is their number of nodes. As usual, we draw standard rooted trees as rooted trees where the label of each internal node is written inside it.

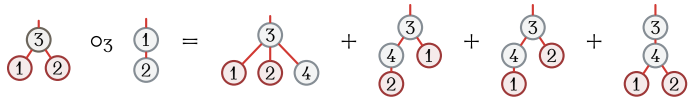
3.2.6. *Pre-Lie operad.* The *pre-Lie operad* **PLie** is the space $\mathbb{K}\langle\mathfrak{SRT}\rangle$ endowed with the partial composition maps $\circ_i$ defined linearly, for any $\mathfrak{t} \in \mathfrak{SRT}(n)$, $\mathfrak{s} \in \mathfrak{SRT}(m)$, and $i \in [n]$ in the following way. First, let $\mathfrak{t}' := \uparrow_i^{m-1}(\mathfrak{t})$ and $\mathfrak{s}' := \uparrow_0^{i-1}(\mathfrak{s})$. The partial composition of $\mathfrak{t}$ and $\mathfrak{s}$ is defined as the sum

$$\mathfrak{t} \circ_i \mathfrak{s} := \sum_{\phi : \mathfrak{t}'^{(i)} \to \mathcal{L}(\mathfrak{s}')} \mathfrak{t}' \hookleftarrow_i^{\phi} \mathfrak{s}'. \tag{3.2.19}$$

For instance,



(3.2.20a)



(3.2.20b)

are two partial compositions in **PLie**. The unit of **PLie** is . This operad is combinatorial and its Hilbert series satisfies

$$\mathcal{H}_{\mathbf{PLie}}(t) = \sum_{n \in \mathbb{N}_{\geqslant 1}} n^{n-1}\, t^n. \tag{3.2.21}$$

The first coefficients of its Hilbert series are

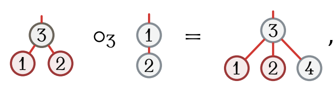
$$1, 2, 9, 64, 625, 7776, 117649, 2097152, 43046721 \tag{3.2.22}$$

and form Sequence A000169 of [Slo].

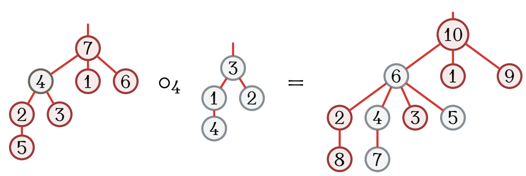
3.2.7. *Nonassociative permutative operad.* The *nonassociative permutative operad* **NAP** is the space $\mathbb{K}\langle\mathfrak{S}\mathfrak{R}\mathfrak{T}\rangle$ endowed with the partial composition maps $\circ_i$ defined linearly, for any $\mathfrak{t} \in \mathfrak{S}\mathfrak{R}\mathfrak{T}(n)$, $\mathfrak{s} \in \mathfrak{S}\mathfrak{R}\mathfrak{T}(m)$, and $i \in [n]$ in the following way. By using the notations of Section 3.2.5 about labeled rooted trees, let $\mathfrak{t}' := \uparrow_i^{m-1}(\mathfrak{t})$, $\mathfrak{s}' := \uparrow_0^{i-1}(\mathfrak{s})$, and $\phi : \mathfrak{t}'^{(i)} \to \mathcal{L}(\mathfrak{s}')$ be the map defined for any $\mathfrak{r} \in \mathfrak{t}'^{(i)}$ by $\phi(\mathfrak{r}) := j$ where j is the label of the root of $\mathfrak{s}'$. The partial composition of $\mathfrak{t}$ and $\mathfrak{s}$ is defined as

$$\mathfrak{t} \circ_i \mathfrak{s} := \mathfrak{t}' \hookleftarrow_i^{\phi} \mathfrak{s}'. \tag{3.2.23}$$

Observe that $\mathfrak{t} \circ_i \mathfrak{s}$ is a particular element appearing in the partial composition $\mathfrak{t} \circ_i \mathfrak{s}$ of the operad **PLie**. For instance,



(3.2.24a)



(3.2.24b)

are two partial compositions in **NAP**. The unit of **NAP** is . This operad is a set-operad, is combinatorial, and its Hilbert series is the same as the one of **PLie**.

3.3. Operads of graphs. As last examples, we expose here operads defined on graded spaces of families of graphs. These graphs are configurations of chords in polygons having labeled arcs. We shall also provide a general construction of operads of graphs from unitary magmas.

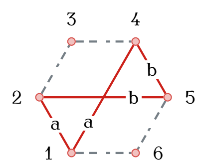
3.3.1. *Configurations of chords.* A *polygon* of *size* $n \in \mathbb{N}_{\geqslant 1}$ is a directed graph $\mathfrak{p}$ on the set of vertices $[n+1]$. An *arc* of $\mathfrak{p}$ is a pair of integers (x, y) with $1 \leqslant x < y \leqslant n+1$, a *diagonal* is an arc (x, y) different from $(x, x+1)$ and $(1, n+1)$, and an *edge* is an arc of the form $(x, x+1)$ and different from $(1, n+1)$. We denote by $\mathcal{A}_{\mathfrak{p}}$ (resp. $\mathcal{D}_{\mathfrak{p}}$, $\mathcal{E}_{\mathfrak{p}}$) the set of all arcs (resp. diagonals, edges) of $\mathfrak{p}$. For any $i \in [n]$, the *ith edge* of $\mathfrak{p}$ is the edge $(i, i+1)$, and the arc $(1, n+1)$ is the *base* of $\mathfrak{p}$.

For any set S, an *S-configuration of chords* (or simply an *S-configuration*) is a polygon $\mathfrak{c}$ endowed with a partial function

$$\phi_{\mathfrak{c}} : \mathcal{A}_{\mathfrak{c}} \to S. \tag{3.3.1}$$

When $\phi_{\mathfrak{c}}((x,y))$ is defined, we say that the arc (x,y) is *labeled* and we denote it by $\mathfrak{c}(x,y)$, otherwise, (x,y) is *unlabeled*. When the base of $\mathfrak{c}$ is labeled, we denote it by $\mathfrak{c}_0$, and when the ith edge of $\mathfrak{c}$ is labeled, we denote it by $\mathfrak{c}_i$. Two diagonals (x,y) and (x',y') of $\mathfrak{c}$ are *crossing* if $x < x' < y < y'$ or $x' < x < y' < y$. The S-configuration $\mathfrak{c}$ is *noncrossing* if it does not admit any pair of crossing labeled diagonals. The graded collection of all S-configurations (resp. noncrossing S-configurations) is denoted by $\mathfrak{CC}^S$ (resp. $\mathfrak{NCC}^S$).

In our graphical representations, each polygon is depicted so that its base is the bottommost segment, vertices are implicitly numbered from 1 to $n+1$ in the clockwise direction We shall represent any S-configuration $\mathfrak{c}$ by drawing a polygon of the same size as the one of $\mathfrak{c}$ and by labeling its arcs accordingly. For instance

$\mathfrak{c} :=$ 

(3.3.2)

is an $\{\mathrm{a},\mathrm{b}\}$-configuration of size 5. Its set of all diagonals is

$$\mathscr{D}_{\mathfrak{c}} = \{(1,3),(1,4),(1,5),(2,4),(2,5),(2,6),(3,5),(3,6),(4,6)\}, \tag{3.3.3}$$

its set of all edges is

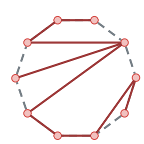
$$\mathscr{E}_{\mathfrak{c}} = \{(1,2),(2,3),(3,4),(4,5),(5,6)\}, \tag{3.3.4}$$

and its set of all arcs is

$$\mathscr{A}_{\mathfrak{c}} = \mathscr{D}_{\mathfrak{c}} \sqcup \mathscr{E}_{\mathfrak{c}} \sqcup \{(1,6)\}. \tag{3.3.5}$$

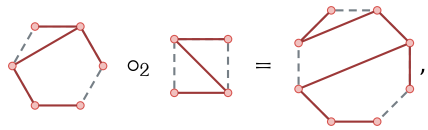
The arcs $(1,2)$ and $(1,4)$ of $\mathfrak{c}$ are labeled by a, the arcs $(2,5)$ and $(4,5)$ are labeled by b, and the other arcs are unlabeled. The labeled diagonals $(1,4)$ and $(2,5)$ are crossing so that $\mathfrak{c}$ is not noncrossing.

3.3.2. *Noncrossing tree operad.* A *noncrossing tree* is a $\{\star\}$-configuration $\mathfrak{c}$, where $\star$ is any symbol, satisfying the following conditions. First, $\mathfrak{c}$ is noncrossing and its base is labeled, and, by denoting by n the size of $\mathfrak{c}$, the graph on $[n+1]$ consisting in the edges $\{x,y\}$ if (x,y) is labeled in $\mathfrak{c}$, is connected and simply connected. For instance,

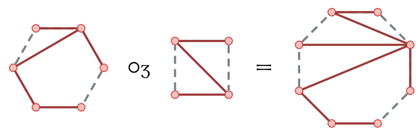

(3.3.6)

is a noncrossing tree of size 9. The graded collection of all noncrossing trees is denoted by $\mathfrak{NCT}$. The *operad of noncrossing trees* **NCT** is the space $\mathbb{K}\langle\mathfrak{NCT}\rangle$ endowed with the partial composition maps $\circ_i$ defined graphically as follows. For any $\mathfrak{c} \in \mathfrak{NCT}(n)$, $\mathfrak{d} \in \mathfrak{NCT}(m)$, and $i \in [n]$, the noncrossing tree $\mathfrak{c} \circ_i \mathfrak{d}$ is obtained by gluing the base of $\mathfrak{d}$

onto the ith edge of $\mathfrak{c}$, so that the arc $(i, i+m)$ of $\mathfrak{c} \circ_i \mathfrak{d}$ is labeled when the ith edge of $\mathfrak{c}$ is labeled and is unlabeled otherwise. For example,



(3.3.7a)



(3.3.7b)

are two partial compositions in **NCT**. The unit of **NCT** is ∘—∘. This operad is a set-operad, is combinatorial, and its Hilbert series satisfies

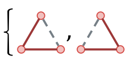
$$\mathbb{H}_{\mathbf{NCT}}(t) = \sum_{n \in \mathbb{N}_{\geqslant 1}} \binom{3n-2}{n-1} \frac{1}{n} t^n. \tag{3.3.8}$$

The first coefficients of its Hilbert series are

$$1, 2, 7, 30, 143, 728, 3876, 21318, 120175 \tag{3.3.9}$$

and form Sequence A006013 of [**Slo**]. Moreover, **NCT** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where

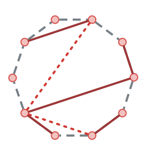
$$\mathfrak{G} := \left\{ \quad , \quad \right\} \tag{3.3.10}$$



and $\mathfrak{R}$ is the space generated by, by denoting by ↼ (resp. ⇀) the first (resp. second) noncrossing tree of (3.3.10),

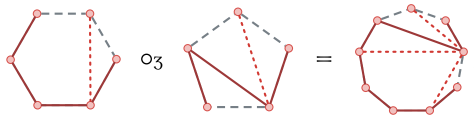
$$\mathrm{c}(⇀) \circ_1 \mathrm{c}(↼) - \mathrm{c}(↼) \circ_2 \mathrm{c}(⇀). \tag{3.3.11}$$

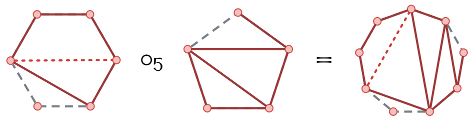
3.3.3. *Bicolored noncrossing configuration operad.* A *bicolored noncrossing configuration* is a noncrossing $\{\star, \bar{\star}\}$-configuration $\mathfrak{c}$, where $\star$ and $\bar{\star}$ are any symbols, such that all arcs labeled by $\bar{\star}$ are diagonals. We draw any arc labeled by $\star$ (resp. $\bar{\star}$) by a thick (resp. dotted) line. For instance,

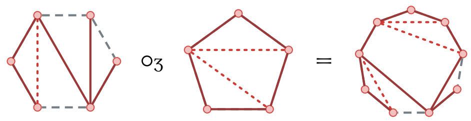


(3.3.12)

is a bicolored noncrossing configuration of size 9. By definition, we set that there is only one bicolored noncrossing configuration of size 1, having its only arc unlabeled. The graded collection of all bicolored noncrossing configurations is denoted by $\mathfrak{BNC}$. The *operad of bicolored noncrossing configurations* **BNC** is the space $\mathbb{K}\langle\mathfrak{BNC}\rangle$ endowed with the partial composition maps $\circ_i$ defined graphically as follows. For any $\mathfrak{c} \in \mathfrak{BNC}(n)$, $\mathfrak{d} \in \mathfrak{BNC}(m)$, and $i \in [n]$, the bicolored noncrossing configuration $\mathfrak{c} \circ_i \mathfrak{d}$ is obtained by gluing the base of $\mathfrak{d}$ onto the ith edge of $\mathfrak{c}$, and then, if the base of $\mathfrak{d}$ and the ith edge of $\mathfrak{c}$ are both unlabeled, the arc $(i, i+m)$ of $\mathfrak{c} \circ_i \mathfrak{d}$ becomes labeled by $\bar{\star}$; if the base of $\mathfrak{d}$ and the ith edge of $\mathfrak{c}$ are both labeled by $\star$, the arc $(i, i+m)$ of $\mathfrak{c} \circ_i \mathfrak{d}$ becomes labeled by $\star$; otherwise, the arc $(i, i+m)$ of $\mathfrak{c} \circ_i \mathfrak{d}$ becomes unlabeled. For example,


(3.3.13a)


(3.3.13b)


(3.3.13c)

are three partial compositions in **BNC**. The unit of **BNC** is . This operad is a set-operad, is combinatorial, and its Hilbert series satisfies

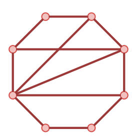
$$\mathbb{H}_{\mathbf{BNC}}(t) = \frac{1 - 4t - \sqrt{1 - 20t + 4t^2}}{6}. \tag{3.3.14}$$

The first coefficients of its Hilbert series are

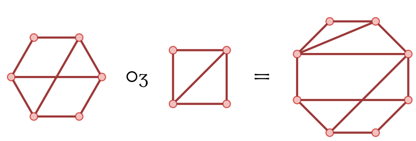
$$1, 8, 80, 992, 13760, 204416, 3180800, 51176960, 844467200 \tag{3.3.15}$$

and form Sequence A234596 of [Slo].

3.3.4. *Gravity operad.* A *gravity chord configuration* is an $\{\star\}$-configuration $\mathfrak{c}$, where $\star$ is any symbol, satisfying the following conditions. By denoting by n the size of $\mathfrak{c}$, all the edges and the base of $\mathfrak{c}$ are labeled, and if (x, y) and (x', y') are two labeled crossing diagonals of $\mathfrak{c}$ such that $x < x'$, the arc (x', y) is unlabeled. In other words, the quadrilateral formed by the vertices x, x', y, and y' of $\mathfrak{c}$ is such that its side (x', y) is unlabeled. For instance,


(3.3.16)

is a gravity chord configuration of size 7 having four labeled diagonals (observe in particular that, as required, the arc $(3, 5)$ is not labeled). By definition, we set that there is only one gravity chord configuration of size 1, having its only arc unlabeled. The graded collection of all gravity chord configurations is denoted by $\mathfrak{GCC}$. The *operad of gravity chord configurations* **Grav** is the space $\mathbb{K} \langle \mathfrak{GCC} \rangle$ endowed with the partial composition maps $\circ_i$ defined graphically as follows. For any $\mathfrak{c} \in \mathfrak{GCC}(n)$, $\mathfrak{d} \in \mathfrak{GCC}(m)$, and $i \in [n]$, the gravity chord configuration $\mathfrak{c} \circ_i \mathfrak{d}$ is obtained by gluing the base of $\mathfrak{d}$ onto the ith edge of $\mathfrak{c}$, so that the arc $(i, i+m)$ of $\mathfrak{c} \circ_i \mathfrak{d}$ is labeled. For example,


(3.3.17)

is a partial composition in **Grav**. The unit of **Grav** is ∘—∘. This operad is a set-operad, is combinatorial, and its Hilbert series satisfies

$$\mathbb{H}_{\mathbf{Grav}}(t) = t + \sum_{n \in \mathbb{N}_{\geqslant 2}} \frac{n!}{2} t^n. \tag{3.3.18}$$

The first coefficients of its Hilbert series are

$$1, 1, 3, 12, 60, 360, 2520, 20160, 181440 \tag{3.3.19}$$

and form Sequence **A001710** of [**Slo**].

3.3.5. *From unitary magmas to graph operads.* We describe here a general way for constructing operads of configurations of chords. Let $\mathcal{M}$ be a unitary magma with binary product $\star$ admitting $\mathbb{1}$ as unit. By setting $\bar{\mathcal{M}} := \mathcal{M} \setminus \{\mathbb{1}\}$, $\mathfrak{CC}_{\bar{\mathcal{M}}}$ is the graded collection of all $\mathcal{M}$-configurations where all labeled arcs have labels different from $\mathbb{1}$. We denote by $\mathbf{C}\mathcal{M}$ the space $\mathbb{K}\langle \mathfrak{CC}_{\bar{\mathcal{M}}} \rangle$. The space $\mathbf{C}\mathcal{M}$ is endowed with the partial composition maps $\circ_i$ defined linearly, for any $\mathfrak{c} \in \mathfrak{CC}_{\bar{\mathcal{M}}}(n)$, $\mathfrak{d} \in \mathfrak{CC}_{\bar{\mathcal{M}}}(m)$, and $i \in [n]$ in the following way. First, let $\mathfrak{c}'$ and $\mathfrak{d}'$ be the two $\mathcal{M}$-configurations, respectively, obtained from $\mathfrak{c}$ and $\mathfrak{d}$ by labeling by $\mathbb{1}$ the possible unlabeled edges and the possible unlabeled bases. The configuration $\mathfrak{c} \circ_i \mathfrak{d}$ is obtained by gluing the base of $\mathfrak{d}'$ onto the ith edge of $\mathfrak{c}'$, then by relabeling the arc $(i, i+m)$ of $\mathfrak{c} \circ_i \mathfrak{d}$ by $\mathfrak{c}_i \star \mathfrak{d}_0$, and finally by making unlabeled the possible arcs labeled by $\mathbb{1}$.

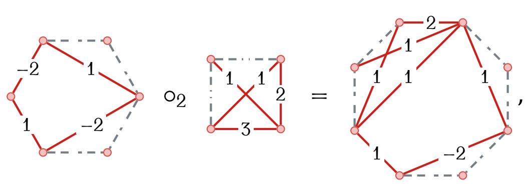
Proposition 3.3.1. *For any monoid $\mathcal{M}$, $\mathbf{C}\mathcal{M}$ is an operad.*

The unit of $\mathbf{C}\mathcal{M}$ is the $\bar{\mathcal{M}}$-configuration ∘- -∘ of size 1 having its only arc unlabeled. The operad $\mathbf{C}\mathcal{M}$ is a set-operad. Moreover, when $\mathcal{M}$ is finite, $\mathbf{C}\mathcal{M}$ is combinatorial and its Hilbert series satisfies

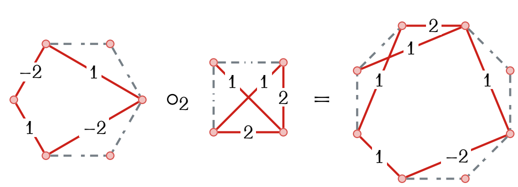
$$\mathbb{H}_{\mathbf{C}\mathcal{M}}(t) = \sum_{n \in \mathbb{N}_{\geqslant 1}} m^{\binom{n+1}{2}} t^n \tag{3.3.20}$$

where $m := \#\mathcal{M}$.

Let us consider an example. By considering the monoid $\mathbb{Z}$ for the usual integer addition, $\mathbf{C}\mathbb{Z}$ is the space of all configurations with labels in $\mathbb{Z} \setminus \{0\}$. Moreover,



(3.3.21a)



(3.3.21b)

are two partial compositions in $\mathbf{C}\mathbb{Z}$.

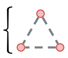
Let now the space

$$\mathbf{C}_{\mathrm{u}}\mathscr{M} := \mathbb{K}\left\langle \{\circ\text{- -}\circ\} + \bigsqcup_{n \in \mathbb{N}_{\geqslant 2}} \mathfrak{C}\mathfrak{C}_{\bar{\mathscr{M}}}(n) \right\rangle. \tag{3.3.22}$$

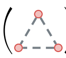
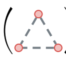
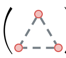
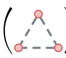
Proposition 3.3.2. *When $\mathscr{M}$ is a unitary magma, $\mathbf{C}_{\mathrm{u}}\mathscr{M}$ is an operad. Moreover, when $\mathscr{M}$ is also a monoid, $\mathbf{C}_{\mathrm{u}}\mathscr{M}$ is a suboperad of $\mathbf{C}\mathscr{M}$.*

Now, when $\mathscr{M}$ is a monoid, let us define $\mathbf{NC}\mathscr{M}$ as the space $\mathbb{K}\langle \mathfrak{N}\mathfrak{C}\mathfrak{C}_{\bar{\mathscr{M}}} \rangle$ of the noncrossing $\bar{\mathscr{M}}$-configurations. Immediately by definition of the partial composition of $\mathbf{C}\mathscr{M}$, one can observe that if $\mathfrak{c}$ and $\mathfrak{d}$ are noncrossing, any partial composition $\mathfrak{c} \circ_i \mathfrak{d}$ is also noncrossing. For this reason, $\mathbf{NC}\mathscr{M}$ is a suboperad of $\mathbf{C}\mathscr{M}$.

Proposition 3.3.3. *Let $\mathscr{M}$ be a monoid. Then, the operad $\mathbf{NC}\mathscr{M}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where*

$$\mathfrak{G} := \left\{ \circ\!\!-\!x\!-\!\!\circ : x \in \bar{\mathscr{M}} \right\} \sqcup \left\{ \quad \right\} \tag{3.3.23}$$


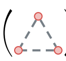
and $\mathfrak{R}$ is the space generated by

$$\mathrm{c}\left(\quad \right) \circ_1 \mathrm{c}\left(\quad \right) - \mathrm{c}\left(\quad \right) \circ_2 \mathrm{c}\left(\quad \right), \tag{3.3.24a}$$





$$\mathrm{c}\,(\circ\!\!-\!x\!-\!\!\circ) \circ_1 \mathrm{c}\,(\circ\!\!-\!y\!-\!\!\circ) - \mathrm{c}\,(\circ\!\!-\!x \star y\!-\!\!\circ), \qquad x, y \in \bar{\mathscr{M}}. \tag{3.3.24b}$$

From the presentation of $\mathbf{NC}\mathscr{M}$ provided by Proposition 3.3.3 and the one of the operad of words $\mathbf{T}\mathscr{M}$ provided by Proposition 3.1.2, we can observe that $\mathbf{T}\mathscr{M}$ is a quotient of $\mathbf{NC}\mathscr{M}$. The linear map $\phi : \mathbf{NC}\mathscr{M} \to \mathbf{T}\mathscr{M}$ satisfying, for any $x \in \bar{\mathscr{M}}$,

$$\phi\,(\circ\!\!-\!x\!-\!\!\circ) = x \in \mathbf{T}\mathscr{M}(1), \tag{3.3.25a}$$

$$\phi\left(\quad \right) = \mathbb{1}\mathbb{1} \in \mathbf{T}\mathscr{M}(2) \tag{3.3.25b}$$


extends in a unique way into an operad morphism and this morphism is surjective.

All the operads of graphs presented in Section 3.3 can be constructed directly or as suboperads of $\mathbf{C}\mathscr{M}$ or $\mathbf{C}_{\mathrm{u}}\mathscr{M}$ for suitable monoids or unitary magmas $\mathscr{M}$.

Bibliographic notes

About operads, pre-Lie systems, and combinatorics. The theory of operads arose first in the 1970s in the field of algebraic topology through the works of May [**May72**] and Boardman and Vogt [**BV73**]. The first motivation was to study loop spaces. By “operad”, most of the authors mean what we call symmetric operad, that is a nonsymmetric operad wherein each subspace consisting in the elements of arity n is endowed with the action of the symmetric group $\mathfrak{S}(n)$ (see forthcoming Section 2.3 of Chapter 5). Nonsymmetric operads appeared a little earlier in the work of Gerstenhaber under the name of pre-Lie systems [**Ger63**]. To be more precise, a pre-Lie system is a space endowed with products $\circ_i$ which are series associative (see (1.1.4)) and parallel associative (see (1.1.7)), but not necessarily unital (see (1.1.11)). The theory of operads has somewhat been neglected in the next twenty years following its discovery but it was put back on the front of the stage in the 1990s [Lod96]. At this moment, an increasing number of combinatorists began to take an interest in the subject and several works relating combinatorics and operads were performed. One can cite, for instance, [MY91] dealing with Möbius species and compositions of trees, [Lod01, CL01, Lod08, Gir15, Gir16b, Gir17] where operads on many combinatorial families are defined, and [Lod05, Cha06, Liv06, CL07, CG14] where operad structures lead to the discovery of algebraic and combinatorial properties. The main philosophy is here twofold: on the one hand, the structure thereby added on combinatorial families enables to see these in a new light, and on the other, techniques coming from combinatorics lead to establish algebraic properties of operads and the category of algebras they encode. Classical and complementary references about operads are [Mar08, Cha08, LV12, Mén15, Yau16].

About set-operads. Due to the fact that their linear structure can be forgotten, set-operads form a class of operads which is in some sense simpler than the class of general ones. Despite this apparent simplicity, set-operads remain very rich structures and, as suggested in Section 3, a lot of operads appearing in combinatorics are set-operads. Moreover, as a consequence of the lack of linear structure, there are simple techniques to establish presentations by generators and relations of set-operads by using rewrite systems on trees (as exposed in Section 2.2.2). Computer exploration is a crucial tool in this context. For instance, the works [CG14, Gir16c, Gir16b, Gir17, CCG18] use the computer to conjecture orientations of spaces of relations (used as a prerequisite of Theorem 2.2.2). Besides, we exposed in Section 1.2.3 two special notions about set-operads: the one of rooted operads is, up to a slight variation, introduced by Chapoton in [Cha14] as a tool to study series on operads, and the one of basic operads is due to Vallette [Val07] and intervenes as a prerequisite for a tool for showing that an operad is Koszul.

About Koszul duality and Koszulity. Koszul duality for binary and quadratic operads has been introduced by Ginzburg and Kapranov [GK94]. This duality is an extension of the Koszul duality of quadratic associative algebras [Pri70]. The so-called rebirth of the operads in the 1990s [Lod96] was in part due to this duality. Note that the duality exposed in Section 2.3.1 concerns only nonsymmetric operads but the theory also includes the case of symmetric operads. Besides, the definition of the Koszul property for an operad consisting in asking for the acyclicity of its Koszul complex (see, for instance, [LV12]) admits several reformulations. A first criterion is due to Vallette [Val07] (see also [Mén15]) passing by the construction of a family of posets from an operad and showing that they are Cohen-Macaulay (see, for instance, [BGS82]). The criterion using convergent rewrite systems exhibited in Proposition 2.3.1 is a consequence of the work of Dotsenko and Khoroshkin [DK10]. The concept of Poincaré-Birkhoff-Witt bases (which, as explained in Section 2.3.2, form bases of Koszul operads) arises in the work of Hoffbeck [Hof10].

About the presented examples of operads. Let us now finally give some details about the operads reviewed in Section 3. The operad of permutations **Per** is in some cases called “associative operad” [**AL07**]. Indeed, **Per** can be seen as the regularization of the associative operad **As** (see forthcoming Section 2.3 of Chapter 5). The diassociative operad **Dias** has been introduced by Loday in [**Lod01**] within its presentation by generators and relations, and its realization in terms of the elements $\mathfrak{e}_{n,k}$ is due to Chapoton [**Cha05**]. The construction **T**, associating an operad with any monoid has been brought in [**Gir15**]. As explained, it provides alternative constructions of **As** and **Dias**, and also for the triassociative operad **Trias** (see [**LR04**]). As illustrated in [**Gir15**], the construction **T** can be used in order to build operads on a large range of combinatorial graded collections (words, permutations, k-ary trees, integer compositions, directed animals, *etc.*). We have presented here in this context only the operad **Motz** of Motzkin words, obtained as a suboperad of an operad obtained from the construction **T**. Besides, the duplicial operad appeared in [**Lod08**] in the context of the study of types of bialgebras. The dendriform operad **Dendr** was defined as the Koszul dual of **Dias** in [**Lod01**]. The presentations of **Dup** and **Dendr** are very similar and they share the same graded space of binary trees. There are also two alternative realizations of **Dendr** in terms of rational functions [**Cha07**, **Lod10**]. The bicolored Schröder tree operad **BS** was considered in [**LR06**] under the name $2as$. The pre-Lie operad **PLie** has been defined by Chapoton and Livernet [**CL01**] as the operad such that algebras of the category it encodes are pre-Lie algebras (see Section 3.3 of Chapter 3). This operad is usually studied as a symmetric operad but its nonsymmetric version has the interesting property to be free [**BL10**]. The nonassociative permutative operad **NAP** has been introduced in [**Liv06**] as a symmetric operad. Unlike **PLie**, **NAP** is not free as a nonsymmetric operad (it is easy to find a nontrivial relation in degree 2 for instance). Some links between **PLie** and **NAP** have been exploited in [**Saï14**]. The operad of noncrossing trees **NCT** was defined in [**Cha07**] as a suboperad of a bigger operad **Mould**, the operad of mould. The algebras over **NCT** are sometimes called L-algebras and have been studied in [**Ler11**]. In the same text [**Cha07**], a generalization of **NCT** involving noncrossing plants was brought, that are combinatorial objects defined as configurations of chords satisfying some conditions. The operad of bicolored noncrossing configurations **BNC**, introduced in [**CG14**], is a further generalization of this latter. The operad **BNC** contains as suboperads generated by binary elements the operads of noncrossing plants, of noncrossing trees, the dipterous operad [**LR03**], and the operad of bicolored Schröder trees. The gravity operad **Grav** is, as a symmetric operad, defined by Getzler [**Get94**]. It has been studied as a nonsymmetric one in [**AP17**]. The general constructions **C**, $\mathbf{C}_{\mathrm{u}}$, and **NC**, introduced in [**Gir17**], produce operads on configurations of chords. These constructions can be used to provide alternative realizations of all the operads presented in Section 3.3, and also of some other, as the ones of multi-tildes and double multi-tildes [**LMN13**, **GLMN16**] coming from a context of formal language theory [**CCM11**]. Finally, let us mention that a general reference about some of these operads (and some others) is [**Zin12**], where a large number of morphisms between them are referenced.

CHAPTER 5

Applications and generalizations

This last chapter is devoted to review some applications of the theory of operads for enumerative prospects. To this aim, we present formal power series on operads, generalizing usual generating series. We also provide an overview on enrichments of operads: colored operads, cyclic operads, symmetric operads, and pros.

1. Series on operads

We consider here the notion of spaces of formal power series on collections, forming a generalization of usual generating series. Any product $\star$ (in the sense of Section 1.1.7 of Chapter 1) on a collection C gives rise to a product on the series on C, leading potentially to the discovery of enumerative properties on the objects of C. We shall present how associative and graded products on graded collections lead to generalizations of the usual multiplication product of generating series, and how full composition maps of set-operads lead to generalizations of the usual composition product of generating series.

1.1. Series on algebraic structures. We introduce now series on collections, which are intuitively possibly infinite formal sums of objects of a collection. Elementary definitions about these series are reviewed here.

1.1.1. *Series spaces.* Let C be an I-collection. A *series on C* (or, for short, a *C-series*) is a map $\mathbf{f} : C \to \mathbb{K}$. The *coefficient* $\mathbf{f}(x)$ of $x \in C$ in $\mathbf{f}$ is denoted by $\langle x, \mathbf{f} \rangle$. The set of all C-series is denoted by $\mathbb{K} \langle\langle C \rangle\rangle$. This set $\mathbb{K} \langle\langle C \rangle\rangle$ is endowed with the following two operations. First, the *addition* $\mathbf{f}_1 + \mathbf{f}_2$ of two C-series $\mathbf{f}_1$ and $\mathbf{f}_2$ is defined, for any $x \in C$, by $\langle x, \mathbf{f}_1 + \mathbf{f}_2 \rangle := \langle x, \mathbf{f}_1 \rangle + \langle x, \mathbf{f}_2 \rangle$. Second, the *scalar multiplication* of a C-series $\mathbf{f}$ by $\lambda \in \mathbb{K}$ is defined, for any $x \in C$, by $\langle x, \lambda \cdot \mathbf{f} \rangle := \lambda \langle x, \mathbf{f} \rangle$. Endowed with these two operations, $\mathbb{K} \langle\langle C \rangle\rangle$ is a $\mathbb{K}$-vector space, named *series space on C* (or, for short, *C-series space*).

Observe that C-polynomials (see Section 1 of Chapter 3) are particular C-series and that $\mathbb{K} \langle C \rangle$ is a subspace of $\mathbb{K} \langle\langle C \rangle\rangle$. One among the crucial differences between $\mathbb{K} \langle\langle C \rangle\rangle$ and $\mathbb{K} \langle C \rangle$ is that this last admits C as a basis while $\mathbb{K} \langle\langle C \rangle\rangle$ has no explicit basis. As a side remark, a C-series can be seen as a linear form on $\mathbb{K} \langle C \rangle$. For this reason, $\mathbb{K} \langle\langle C \rangle\rangle$ can be seen as the (usual) dual space of $\mathbb{K} \langle C \rangle$ (this has not to be confused with the dual of a combinatorial polynomial space considered in Section 1.2.7 of Chapter 3).

For any subcollection X of C, the *characteristic series* of X is the C-series $\mathrm{ch}(X)$ defined, for any $x \in C$, by $\langle x, \mathrm{ch}(X)\rangle := 1$ for all $x \in X$ and $\langle y, \mathrm{ch}(X)\rangle := 0$ for all $y \in C \setminus X$. By using now the linear structure of $\mathbb{K}\langle\langle C\rangle\rangle$, any C-series $\mathbf{f}$ can be expressed as the possibly infinite sum

$$\mathbf{f} = \sum_{x \in C} \langle x, \mathbf{f}\rangle \cdot \mathrm{ch}(\{x\}), \tag{1.1.1}$$

which is denoted, by a slight abuse of notation, by

$$\mathbf{f} = \sum_{x \in C} \langle x, \mathbf{f}\rangle\, x. \tag{1.1.2}$$

The notation (1.1.2) for $\mathbf{f}$ as a (possibly infinite) linear combination of objects of C is the *infinite sum notation* of C-series.

1.1.2. *Generating series and products.* By setting that $\{t\}$ is a graded collection wherein t is an atomic object, $\mathbb{K}\langle\langle \mathbf{MSet}(\{t\})\rangle\rangle$ is the space of the usual generating series. Indeed, by denoting by t^n each object $\wr t, \dots, t \int$ of $\mathbf{MSet}(\{t\})$ of size $n \in \mathbb{N}$, any element $\mathbf{f}$ of $\mathbb{K}\langle\langle \mathbf{MSet}(\{t\})\rangle\rangle$ is by definition of the form

$$\mathbf{f} = \sum_{n \in \mathbb{N}} \langle t^n, \mathbf{f}\rangle\, t^n. \tag{1.1.3}$$

To not overload the notation, we shall write $\mathbb{K}\langle\langle t\rangle\rangle$ for $\mathbb{K}\langle\langle \mathbf{MSet}(\{t\})\rangle\rangle$.

The usual multiplication (resp. composition) of generating series is denoted by $\cdot$ (resp. $\circ$). In this way, by setting $1 := \mathrm{ch}\left(\left\{t^0\right\}\right)$, $(\mathbb{K}\langle\langle t\rangle\rangle, \cdot, 1)$ is a unitary associative algebra. Moreover, by denoting by $t\mathbb{K}\langle\langle t\rangle\rangle$ the subspace of $\mathbb{K}\langle\langle t\rangle\rangle$ of all the series $\mathbf{f}$ such that $\left\langle t^0, \mathbf{f}\right\rangle = 0$, $(t\mathbb{K}\langle\langle t\rangle\rangle, \circ, t)$ is a unitary associative algebra.

1.1.3. *Index series.* When C is combinatorial, the *index series* of C is the I-series $\mathbf{I}(C)$, where I is seen as a simple collection, defined by

$$\mathbf{I}(C) := \sum_{x \in C} \mathrm{ind}(x) = \sum_{i \in I} \#C(i)\, i. \tag{1.1.4}$$

Since the coefficient $\langle i, \mathbf{I}(C)\rangle$ is the number of elements of index $i \in I$ of C, the series $\mathbf{I}(C)$ encodes enumerating data about C. It is then worthwhile to provide ways of expressing $\mathbf{I}(C)$ in order to compute its coefficients. We shall besides consider in the sequel index series of colored operads and of pros as analogs of usual Hilbert series.

Moreover, let $\omega : I \to \mathbb{N}$ be a map. The *ω-evaluation map* is the map

$$\mathrm{ev}^\omega : \mathbb{K}\langle\langle C\rangle\rangle \to \mathbb{K}\langle\langle t\rangle\rangle \tag{1.1.5}$$

defined, for any C-series $\mathbf{f}$, by

$$\mathrm{ev}^\omega(\mathbf{f}) := \sum_{x \in C} \langle x, \mathbf{f}\rangle\, t^{\omega(x)}. \tag{1.1.6}$$

This series is well-defined if each fiber $\omega^{-1}(n)$ is finite for any $n \in \mathbb{N}$ and is called the *ω-evaluation* of $\mathbf{f}$. When C is combinatorial and graded, one has

$$\mathrm{ev}^{\mathrm{Id}}(\mathbf{I}(C)) = \mathrm{ev}^{|-|}(\mathrm{ch}(C)) = \mathbb{G}_C(t) \tag{1.1.7}$$

where Id is the identity map on the index set $I = \mathbb{N}$ of C, and $|-|$ is the size function of C. Recall that $\mathbb{G}_C(t)$ denotes the generating series of C.

1.1.4. *Products on series.* Assume that the I-collection C is endowed with a product

$$\star : C(J_1) \times \cdots \times C(J_p) \to C \tag{1.1.8}$$

where $p \in \mathbb{N}$ and $J_1, \ldots, J_p$ are nonempty subsets of I (see Section 1.1.7 of Chapter 1). Then, let the linear map

$$\bar{\star} : \mathbb{K}\langle\langle C\rangle\rangle^{\otimes p} \to \mathbb{K}\langle\langle C\rangle\rangle \tag{1.1.9}$$

defined, for any $\mathbf{f}_1, \ldots, \mathbf{f}_p \in \mathbb{K}\langle\langle C\rangle\rangle$ and $x \in C$, by

$$\left\langle x, \bar{\star}\left(\mathbf{f}_1, \ldots, \mathbf{f}_p\right)\right\rangle := \sum_{\substack{y_1, \ldots, y_p \in C \\ \star(y_1, \ldots, y_p) = x}} \prod_{k \in [p]} \langle y_k, \mathbf{f}_k\rangle. \tag{1.1.10}$$

In other terms, by using the sum notation of series,

$$\bar{\star}\left(\mathbf{f}_1, \ldots, \mathbf{f}_p\right) = \sum_{x_k \in C(J_k), k \in [p]} \prod_{k \in [p]} \langle x_k, \mathbf{f}_k\rangle \star \left(x_1, \ldots, x_p\right). \tag{1.1.11}$$

We call $\bar{\star}$ the *series extension* of $\star$. In this way, series extensions of products on C allow to translate set-theoretic algebraic structures on C into products on series.

For instance, by considering the product $\star$ on $\mathbf{MSet}(\{t\})$ defined by $t^n \star t^m := t^{n+m}$ for any $n, m \in \mathbb{N}$, the series extension of $\star$ on $\mathbb{K}\langle\langle t\rangle\rangle$ is the multiplication $\cdot$ of generating series.

1.2. Generalizing series multiplication. We consider series extensions of binary products on graded collections satisfying some conditions and explain how they provide generalizations of the multiplication product of generating series.

1.2.1. *Series on monoids and multiplication.* Let C be a graded collection endowed with a graded complete binary associative product $\star$. In this case, its series extension $\bar{\star}$ endows $\mathbb{K}\langle\langle C\rangle\rangle$ with the structure of an associative algebra. Moreover, when $\star$ admits a unit $\mathbb{1}$, C is a monoid and the series $\mathbf{1} := \mathrm{ch}(\{\mathbb{1}\})$ of $\mathbb{K}\langle\langle C\rangle\rangle$ is the unit of $\bar{\star}$.

Proposition 1.2.1. *Let C be a graded collection endowed with a graded complete binary associative product $\star$ admitting a unit. Then, the map $\mathrm{ev}^{|-|}$ is a unitary associative algebra morphism between $(\mathbb{K}\langle\langle C\rangle\rangle, \bar{\star}, \mathbf{1})$ and $(\mathbb{K}\langle\langle t\rangle\rangle, \cdot, 1)$. Moreover, $\mathrm{ev}^{|-|}$ is surjective when $C(n) \neq \emptyset$ for all $n \in \mathbb{N}$.*

Proposition 1.2.1 implies in particular that if one obtains a nontrivial expression for the characteristic series $\mathrm{ch}(C)$ of C by using the sum of series and the product $\bar{\star}$, its $|-|$-evaluation will provide a nontrivial expression for the generating series $\mathbb{G}_C(t)$ of C. We shall present examples in the further sections.

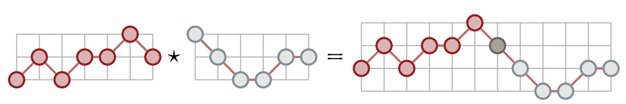
1.2.2. *A monoid of paths.* Let us consider an example of series on monoids and an application to enumeration. We call *path* any nonempty word u on $\mathbb{N}$ and we denote by $\mathfrak{Path}$ be the graded collection of all paths, where the size of a path is its length as a word minus 1. This collection is endowed with the complete binary product $\star$ defined, for any $u \in \mathfrak{Path}(n-1)$ and $v \in \mathfrak{Path}(m-1)$, by

$$u \star v := \begin{cases} \uparrow_k (u(1) \dots u(n-1)) \cdot v & \text{if } v(1) \geqslant u(n), \\ u \cdot \uparrow_k (v(2) \dots v(m)) & \text{otherwise,} \end{cases} \tag{1.2.1}$$

where $\cdot$ is the concatenation product of words, $k := |u(n) - v(1)|$, and for any path w, $\uparrow_k (w)$ is the path obtained by incrementing all the letters of w by k. For instance,

$$0101121 \star 210011 = 121223210011. \tag{1.2.2}$$

By depicting a path through its graph in the quarter plane (that is, by drawing points $(i-1, u(i))$ for all positions i and by connecting all pairs of adjacent points by lines), (1.2.2) becomes



In intuitive terms, the product $\star$ consists in concatenating the paths by superimposing the last letter of the first operand with the first letter of the second. Observe that the path 0, denoted by $\circ$, has size zero in $\mathfrak{Path}$ and is the unit of $\star$.

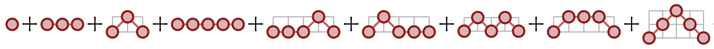
Proposition 1.2.2. *The triple $(\mathfrak{Path}, \star, \circ)$ is a graded monoid.*

By Propositions 1.2.1 and 1.2.2, we can consider series of paths and the series extension of the product of paths. To specify particular families of paths, let us introduce the following tool. For any subcollection X of $\mathfrak{Path}$, the *$\star$-Kleene star* $X^{*_\star}$ of X is the subcollection defined by

$$X^{*_\star} := \bigcup_{p \in \mathbb{N}} \underbrace{X \star \dots \star X}_{p \text{ terms}}. \tag{1.2.3}$$

In other words, $X^{*_\star}$ is the submonoid of $\mathfrak{Path}$ generated by X. As a collection, $X^{*_\star}$ contains all the paths obtained by concatenating elements of X.

1.2.3. *Enumeration of families of paths.* Let us first study the collection $\mathfrak{Path}_{\mathfrak{Sch}}$ of *Schröder paths*, that are the paths of {}$^{\star\cdot}$ starting and finishing by 0. This collection is combinatorial and its characteristic series is

$$\mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Sch}}\right) :=$$  $$+\cdots . \qquad (1.2.4)$$

By reasoning on the non-ambiguous decomposition of Schröder paths, one can establish that this series satisfies the nontrivial relation

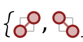
$\mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Sch}}\right) =$  $+$  $\bar{\star}\ \mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Sch}}\right) +$  $\bar{\star}\ \mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Sch}}\right) \bar{\star}$  $\bar{\star}\ \mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Sch}}\right).$ (1.2.5)

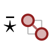
The $|-|$-evaluation of the left and right members of (1.2.5) leads, by using (1.1.7), to the algebraic relation

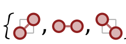
$$\mathbb{G}_{\mathfrak{Path}_{\mathfrak{Sch}}}(t) = 1 + t^2\, \mathbb{G}_{\mathfrak{Path}_{\mathfrak{Sch}}}(t) + t^2\, \mathbb{G}_{\mathfrak{Path}_{\mathfrak{Sch}}}(t)^2 \qquad (1.2.6)$$

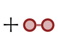
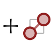
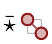
for the generating series of $\mathfrak{Path}_{\mathfrak{Sch}}$.

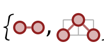
We can use similar mechanisms to obtain expressions for the generating series of other families of paths. Let us consider three of these.

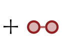
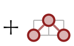
Dyck paths. The characteristic series of the collection $\mathfrak{Path}_{\mathfrak{Dyck}}$ of *Dyck paths*, that are paths of {}$^{\star\cdot}$ starting and finishing by 0 satisfies

$\mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Dyck}}\right) =$  $+$  $\bar{\star}\ \mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Dyck}}\right) \bar{\star}$  $\bar{\star}\ \mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Dyck}}\right).$ (1.2.7)

Motzkin paths. The characteristic series of the collection $\mathfrak{Path}_{\mathfrak{Motz}}$ of *Motzkin paths*, that are paths of {}$^{\star\cdot}$ starting and finishing by 0 satisfies

$\mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Motz}}\right) =$  $+$  $\bar{\star}\ \mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Motz}}\right) +$  $\bar{\star}\ \mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Motz}}\right) \bar{\star}$  $\bar{\star}\ \mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Motz}}\right).$ (1.2.8)

Fibonacci paths. The characteristic series of the collection $\mathfrak{Path}_{\mathfrak{Fib}}$ of *Fibonacci paths*, that are paths of {}$^{\star\cdot}$ satisfies

$\mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Fib}}\right) =$  $+$  $\bar{\star}\ \mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Fib}}\right) +$  $\bar{\star}\ \mathrm{ch}\left(\mathfrak{Path}_{\mathfrak{Fib}}\right).$ (1.2.9)

1.3. Generalizing series composition. We consider now series on combinatorial collections endowed with the structure of set-operads and explain how they provide generalizations of the composition product of generating series.

1.3.1. *Series on operads and composition.* Let C be a set-operad (see Section 1.2.3 of Chapter 4). In this case, the series extensions $\bar{\circ}$ of its full composition maps $\circ$ satisfy, for any $n \in \mathbb{N}_{\geqslant 1}$ and any C-series $\mathbf{f}, \mathbf{g}_1, \ldots, \mathbf{g}_n$,

$$\langle x, \mathbf{f}\ \bar{\circ}\ [\mathbf{g}_1, \ldots, \mathbf{g}_n] \rangle = \sum_{\substack{y \in C(n) \\ z_1, \ldots, z_n \in C \\ y \circ [z_1, \ldots, z_n] = x}} \langle y, \mathbf{f} \rangle \prod_{i \in [n]} \langle z_i, \mathbf{g}_i \rangle . \qquad (1.3.1)$$

Let now $\odot$ be the binary product on $\mathbb{K}\langle\langle C\rangle\rangle$ defined as the sum of all the series extensions of the full composition maps of C. More precisely, for any C-series $\mathbf{f}$ and $\mathbf{g}$,

$$\langle x, \mathbf{f} \odot \mathbf{g}\rangle := \sum_{p \in \mathbb{N}_{\geqslant 1}} \left\langle x, \mathbf{f} \mathbin{\bar{\circ}} \left[\underbrace{\mathbf{g}, \dots, \mathbf{g}}_{p \text{ terms}}\right]\right\rangle = \sum_{\substack{y \in C(n), n \in \mathbb{N} \\ z_1, \dots, z_n \in C \\ y \circ [z_1, \dots, z_n] = x}} \langle y, \mathbf{f}\rangle \prod_{i \in [n]} \langle z_i, \mathbf{g}\rangle . \tag{1.3.2}$$

As immediate observations, remark that $\odot$ is linear on left, is not linear on the right, and admits $\mathbf{1} := \mathrm{ch}(\{\mathbb{1}\})$ as a left and a right unit, where $\mathbb{1}$ is the operad unit of C.

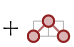
Proposition 1.3.1. *Let C be a set-operad, $x \in C(n)$, $n \in \mathbb{N}_{\geqslant 1}$, and $\mathbf{g}_1$, …, $\mathbf{g}_n$ be C-series. Then,*

$$\mathrm{ev}^{|-|}\left(x \mathbin{\bar{\circ}} [\mathbf{g}_1, \dots, \mathbf{g}_n]\right) = \prod_{i \in [n]} \mathrm{ev}^{|-|}\left(\mathbf{g}_i\right). \tag{1.3.3}$$

Proposition 1.3.2. *Let C be a set-operad. Then, the map $\mathrm{ev}^{|-|}$ is a unitary associative algebra morphism between $(\mathbb{K}\langle\langle C\rangle\rangle, \odot, \mathbf{1})$ and $(t\mathbb{K}\langle\langle t\rangle\rangle, \circ, t)$. Moreover, $\mathrm{ev}^{|-|}$ is surjective when $C(n) \neq \emptyset$ for all $n \in \mathbb{N}_{\geqslant 1}$.*

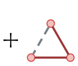
Propositions 1.3.1 and 1.3.2 imply in particular that if one obtains a nontrivial expression for the characteristic series $\mathrm{ch}(C)$ of C by using the sum of series, and the products $\bar{\circ}$ and $\odot$, its $|-|$-evaluation will provide a nontrivial expression for the generating series $\mathbb{G}_C(t)$ of C. We shall present examples in the further sections.

1.3.2. *Enumeration of Motzkin paths.* Let us consider the set-operad **Motz** of Motzkin words introduced in Section 3.1.5 of Chapter 4. By using the operad structure on the underlying graded collection $\mathfrak{Path}'_{\mathfrak{Motz}}$ of **Motz**, one obtains the nontrivial relation

$\mathrm{ch}\left(\mathfrak{Path}'_{\mathfrak{Motz}}\right) =$  $+$  $\bar{\circ}$ $\left[\right.$  $, \mathrm{ch}\left(\mathfrak{Path}'_{\mathfrak{Motz}}\right)\left.\right] +$  $\bar{\circ}$ $\left[\right.$  $, \mathrm{ch}\left(\mathfrak{Path}'_{\mathfrak{Motz}}\right), \mathrm{ch}\left(\mathfrak{Path}'_{\mathfrak{Motz}}\right)\left.\right]$ (1.3.4)

for the characteristic series of $\mathfrak{Path}'_{\mathfrak{Motz}}$. Observe that (1.3.4) and (1.2.8) are two equivalent expressions for enumerating collections of Motzkin paths (yet with different size functions). Nevertheless, (1.3.4) has the advantage of not requiring the definition of a general algebraic structure of paths (in (1.3.4), all the terms are series of Motzkin paths, while in (1.2.8),  and  are not Motzkin paths).

1.3.3. *Enumeration of noncrossing trees.* Let us consider the set-operad **NCT** of noncrossing trees introduced in Section 3.3.2 of Chapter 4. By using the operad structure on the underlying collection $\mathfrak{NCT}$ of **NCT**, one obtains the nontrivial relation

$\mathrm{ch}(\mathfrak{NCT}) =$ $+$  $\bar{\circ}\ [\mathrm{ch}(\mathfrak{NCT}), \mathrm{ch}(\mathfrak{NCT})] +$  $\bar{\circ}\ [\mathbf{f}, \mathrm{ch}(\mathfrak{NCT})]$, (1.3.5)

where $\mathbf{f}$ is the $\mathfrak{NCT}$-series satisfying

$\mathbf{f} =$ $+$  $\bar{\circ}\ [\mathbf{f}, \mathrm{ch}(\mathfrak{NCT})]$, (1.3.6)

for the characteristic series of $\mathfrak{NCT}$.

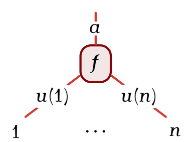
2. Enriched operads

Three enrichments of nonsymmetric operads are presented here: colored operads, cyclic operads, and symmetric operads.

2.1. Colored operads. We begin by introducing colored operads. These variations of operads involve colored collections. Intuitively, each element of a colored operad has a color for its output and colors for each of its inputs. The partial composition of two elements is defined if and only if the colors of the involved input and output coincide.

2.1.1. *Colored polynomial spaces.* Given a $\mathfrak{C}$-colored collection C (see Section 1.1.4 of Chapter 1), the polynomial space $\mathbb{K}\langle C\rangle$ is said *$\mathfrak{C}$-colored* and is endowed with the maps out and in associating with each nonzero homogeneous element f of $\mathbb{K}\langle C\rangle$, respectively, its output color $\mathrm{out}(f)$ and its word of input colors $\mathrm{in}(f)$ (see the aforementioned section). The *arity* $|f|$ of an (a, u)-homogeneous element f of $\mathbb{K}\langle C\rangle$, where (a, u) is a $\mathfrak{C}$-colored index, is the length $|u|$ of the word u. Alternatively, the arity of f is the degree of f in $\mathbb{K}\langle C'\rangle$ where C' is the graduation of C (see Section 1.2.2 of Chapter 1). Moreover, to not overload the notation, we denote by $\mathbb{K}\langle C\rangle(a, u)$ the homogeneous component $\mathbb{K}\langle C\rangle((a, u))$ of $\mathbb{K}\langle C\rangle$ for any $\mathfrak{C}$-colored index (a, u).

2.1.2. *Colored abstract operators.* We regard any homogeneous element f of an augmented $\mathfrak{C}$-colored polynomial space $\mathbb{K}\langle C\rangle$ as a *colored abstract operator*, that is, an abstract operator wherein the output and each input are associated with an element of $\mathfrak{C}$. If f is of arity n, $\mathrm{out}(f) = a$, and $\mathrm{in}(f) = u$, f is depicted as



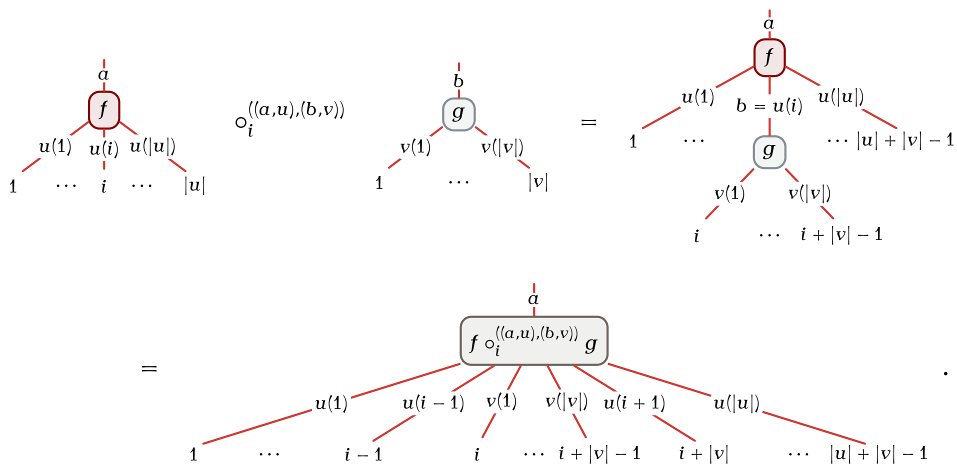
. (2.1.1)

The output and input colors of f are written onto the output and input edges.

2.1.3. *Colored operads.* Let C be an augmented $\mathfrak{C}$-colored collection. Let for all $\mathfrak{C}$-colored indexes (a, u) and (b, v), and $i \in [|u|]$ such that $b = u(i)$, binary products of the form

$$\circ_i^{((a,u),(b,v))} : \mathbb{K}\langle [C(a, u), C(b, v)]_\times\rangle \to \mathbb{K}\langle C\rangle(a, u \hookleftarrow_i v), \tag{2.1.2}$$

where $u \hookleftarrow_i v$ is the word obtained by replacing the ith letter of u by v. On abstract operators, these products $\circ_i^{((a,u),(b,v))}$ behave as the products $\circ_i^{(|u|,|v|)}$ of operads (see Section 1.1.2 of Chapter 4) but with the addition of taking into account of the output and input colors of the colored abstract operators. Indeed, for any $f \in \mathbb{K}\langle C\rangle(a, u)$ and $g \in \mathbb{K}\langle C\rangle(b, v)$, $f \circ_i^{((a,u),(b,v))} g$ is the abstract operator



(2.1.3)

Let us emphasize the fact that these products require that the output color of g is equal to the ith input color of f. By a slight abuse of notation, we shall sometimes omit the $((a, u), (b, v))$ in the notation of $\circ_i^{((a,u),(b,v))}$ in order to denote it in a more concise way by $\circ_i$.

When for any objects x, y, and z of C, the fact that the left and right members of Relation (1.1.4) (resp. Relation (1.1.7)) of Chapter 4 are well-defined implies that they are equal, $\circ_i$ is *series associative* (resp. *parallel associative*). Moreover, assume that there exists a set of elements $\{\mathbb{1}_a : a \in \mathfrak{C}\}$ of arity 1 of $\mathbb{K}\langle C\rangle$ such that for any $a \in \mathfrak{C}$, $\operatorname{out}(\mathbb{1}_a) = a = \operatorname{in}(\mathbb{1}_a)$. When for any object x of C, the fact that, by replacing by $\mathbb{1}_a$ each occurrence of $\mathbb{1}$ in Relation (1.1.11) of Chapter 4, the first or the last members of the relation are well-defined implies that they are equal to x, $\circ_i$ is *unital*. We call in this case each $\mathbb{1}_a$, $a \in \mathfrak{C}$, a *unit of color a*.

When the products $\circ_i$ are series associative, parallel associative, and unital, the $\circ_i$ are called *partial composition maps*. A *$\mathfrak{C}$-colored operad* is a $\mathfrak{C}$-colored polynomial space $\mathbb{K}\langle C\rangle$ endowed with partial composition maps. The main algebraic notions presented in Section 1.2 of Chapter 4 for operads (like full composition maps associated with partial composition maps, morphisms, quotients, group of symmetries, set-operads, *etc.*) hold straightforwardly for colored operads. When $\mathbb{K}\langle C\rangle$ is combinatorial, its *Hilbert series* is the $\mathfrak{C} \times \mathfrak{C}^+$-series

$$\mathbb{H}_{\mathbb{K}\langle C\rangle} := \mathbf{I}(C) = \sum_{(a,u)\in\mathfrak{C}\times\mathfrak{C}^+} (\dim \mathbb{K}\langle C\rangle(a, u))\,(a, u). \tag{2.1.4}$$

2.1.4. *Categorical point of view.* Recall that a monoid $\mathcal{M}$ can be seen as a category with exactly one object $\bullet$ wherein the elements of $\mathcal{M}$ are interpreted as morphisms $\phi : \bullet \to \bullet$. In the same way, an operad $\mathbb{K}\langle C\rangle$ can be seen as a multicategory with exactly one object $\bullet$. In this case, the elements of $\mathbb{K}\langle C\rangle(n)$, $n \in \mathbb{N}_{\geqslant 1}$, are interpreted as multimorphisms $\phi : \bullet^n \to \bullet$. The full composition maps of $\mathbb{K}\langle C\rangle$ translate as the composition of multimorphisms.

In a similar way, a $\mathfrak{C}$-colored operad $\mathbb{K}\langle C\rangle$ can be seen as a multicategory having $\mathfrak{C}$ as set of objects. In this case, the elements of $\mathbb{K}\langle C\rangle(a, u)$ where (a, u) is a $\mathfrak{C}$-colored index, are interpreted as multimorphisms $\phi : u(1) \times \cdots \times u(n) \to a$. The full composition maps of $\mathbb{K}\langle C\rangle$ translate as the composition of multimorphisms, where the constraints imposed by the colors in $\mathbb{K}\langle C\rangle$ become constraints imposed by the domains and codomains of multimorphisms.

2.1.5. *Free colored operads.* Let $\mathfrak{C}$ be a set of colors and $\mathfrak{G}$ be an augmented $\mathfrak{C}$-colored collection. The *free colored operad* over $\mathfrak{G}$ is the operad

$$\mathbf{FCO}(\mathfrak{G}) := \mathbb{K}\left\langle \mathfrak{CST}_{\perp}^{\mathfrak{G}} \right\rangle, \tag{2.1.5}$$

where $\mathfrak{CST}_{\perp}^{\mathfrak{G}}$ is the graded collection of all the $\mathfrak{C}$-colored $\mathfrak{G}$-syntax trees (see Section 3.2 of Chapter 2). The space $\mathbf{FCO}(\mathfrak{G})$ is endowed with the linearizations of the partial grafting operations $\circ_i$, $i \in \mathbb{N}_{\geqslant 1}$, defined in Section 3.2 of Chapter 2. The unit of color a, $a \in \mathfrak{C}$, of $\mathbf{FCO}(\mathfrak{G})$ is the only $\mathfrak{C}$-colored $\mathfrak{G}$-syntax tree $\overset{|}{\underset{|}{a}}$ of arity 1 and degree 0 and having a as output and input color.

2.1.6. *Example: bud operads.* Given an operad $\mathbb{K}\langle C\rangle$ and a set of colors $\mathfrak{C}$, there is an easy way to construct a $\mathfrak{C}$-colored operad. Let $\mathbf{Bud}_{\mathfrak{C}}(\mathbb{K}\langle C\rangle)$ be the $\mathfrak{C}$-colored space $\mathbb{K}\langle \mathbf{Col}_{\mathfrak{C}}(\mathbb{K}\langle C\rangle)\rangle$ where $\mathbf{Col}_{\mathfrak{C}}(C)$ denotes the $\mathfrak{C}$-coloration of C (see Section 1.2.10 of Chapter 1). Let us endow $\mathbf{Bud}_{\mathfrak{C}}(\mathbb{K}\langle C\rangle)$ with the partial composition maps defined linearly, for any objects (a, x, u) and (b, y, v) of $\mathbf{Col}_{\mathfrak{C}}(C)$, and $i \in [|u|]$ such that $b = u(i)$, by

$$(a, x, u) \circ_i (b, y, v) := (a, x \circ_i y, u \hookleftarrow_i v) \tag{2.1.6}$$

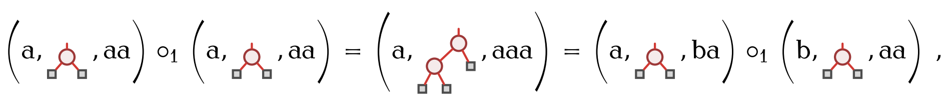
where the second occurrence of $\circ_i$ in (2.1.6) is the partial composition map of the operad $\mathbb{K}\langle C\rangle$ and $\hookleftarrow_i$ is the operation on words on $\mathfrak{C}$ defined in Section 2.1.3.

Proposition 2.1.1. *For any set of colors $\mathfrak{C}$ and any operad $\mathbb{K}\langle C\rangle$, $\mathbf{Bud}_{\mathfrak{C}}(\mathbb{K}\langle C\rangle)$ is a $\mathfrak{C}$-colored operad.*

We call $\mathbf{Bud}_{\mathfrak{C}}(\mathbb{K}\langle C\rangle)$ the *$\mathfrak{C}$-bud operad* of $\mathbb{K}\langle C\rangle$. For instance, one has in $\mathbf{Bud}_{\{\mathtt{a},\mathtt{b}\}}(\mathbf{As})$ (where $\mathbf{As}$ is the associative operad defined in Section 3.1.1 of Chapter 4) the partial composition

$$(\mathtt{a}, \mathfrak{a}_4, \mathtt{bbab}) \circ_2 (\mathtt{b}, \mathfrak{a}_3, \mathtt{aab}) = (\mathtt{a}, \mathfrak{a}_6, \mathtt{baabab}). \tag{2.1.7}$$

Let us observe that the bud operad of a free operad can be not free as a colored operad. For instance, consider the $\{\mathtt{a}, \mathtt{b}\}$-bud operad of the magmatic operad $\mathbf{Mag}$ (see Section 3.2.1 of Chapter 4). One has in $\mathbf{Bud}_{\{\mathtt{a},\mathtt{b}\}}(\mathbf{Mag})$ among others the nontrivial relation



(2.1.8)

implying that $\mathbf{Bud}_{\{\mathtt{a},\mathtt{b}\}}(\mathbf{Mag})$ is not free as a colored operad.

2.2. Cyclic operads. We focus now on cyclic operads. These variations of operads involve cyclic collections. Intuitively, in a cyclic operad, the output and the inputs of the elements play an interchangeable role. This is due to the fact that these structures are endowed with a map performing a cyclic action on the inputs and outputs of its elements.

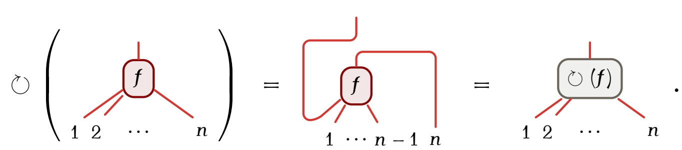
2.2.1. *Cyclic polynomial spaces.* A graded polynomial space $\mathbb{K}\langle C\rangle$ is *cyclic* if it is endowed for all $n \in \mathbb{N}$ with unary products

$$\circlearrowright_n : \mathbb{K}\langle C\rangle(n) \to \mathbb{K}\langle C\rangle(n) \tag{2.2.1}$$

such that $\circlearrowright_n^{n+1}$ is the identity map on $\mathbb{K}\langle C\rangle(n)$. We say that the $\circlearrowright_n$, $n \in \mathbb{N}$, are *cycle maps* of $\mathbb{K}\langle C\rangle$. By a slight abuse of notation, we shall sometimes omit the n in the notation of $\circlearrowright_n$ in order to denote it in a more concise way by $\circlearrowright$. As usual, if $\phi : \mathbb{K}\langle C_1\rangle \to \mathbb{K}\langle C_2\rangle$ is a morphism between two graded polynomial spaces $\mathbb{K}\langle C_1\rangle$ and $\mathbb{K}\langle C_2\rangle$, ϕ is a *cyclic polynomial space morphism* if it commutes with the cycle maps of $\mathbb{K}\langle C_1\rangle$ and $\mathbb{K}\langle C_2\rangle$.

Remark that when C is a cyclic collection (see Section 1.1.5 of Chapter 1), the linearizations of the cycle maps of C endow $\mathbb{K}\langle C\rangle$ with the structure of a cyclic polynomial space.

2.2.2. *Cyclic abstract operators.* We regard any homogeneous element f of an augmented cyclic polynomial space $\mathbb{K}\langle C\rangle$ as a *cyclic abstract operator*, that is, an abstract operator wherein the output can play the role of an input and an input can play the role of the output. In this case, $\circlearrowright$ behaves in the following way. For any $f \in \mathbb{K}\langle C\rangle(n)$,



(2.2.2)

In words, $\circlearrowright(f)$ is obtained by transforming each input of f of index $i+1$ into an input of index i for any $i \in [n-1]$, by transforming the 1st input of f into an output, and by transforming the output of f into an input of index n. It is straightforward to check that $\circlearrowright^{n+1}(f) = f$ as required due to the fact that C is a cyclic collection.

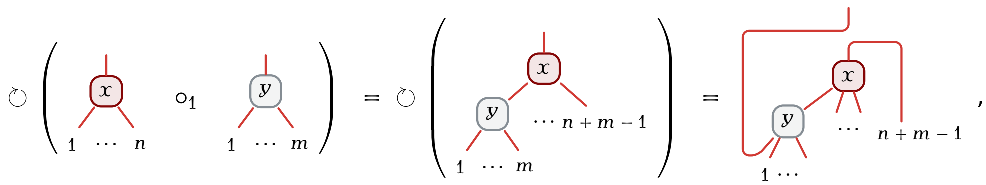
2.2.3. *Cyclic operads.* A *cyclic operad* is an operad $\mathbb{K}\langle C\rangle$ such that $\mathbb{K}\langle C\rangle$ is also cyclic as a polynomial space and satisfies, for any $x \in C(n)$, $y \in C(m)$, and $i \in [n-1]$, the compatibility relations,

$$\circlearrowright(x \circ_1 y) = \circlearrowright(y) \circ_m \circlearrowright(x), \tag{2.2.3a}$$

$$\circlearrowright(x \circ_{i+1} y) = \circlearrowright(x) \circ_i y, \tag{2.2.3b}$$

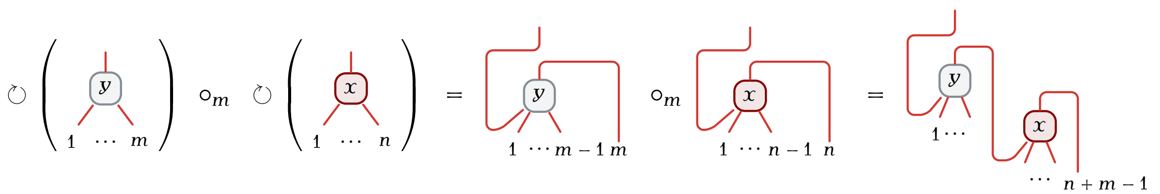
$$\circlearrowright(\mathbb{1}) = \mathbb{1}. \tag{2.2.3c}$$

To understand these relations, let us consider first the abstract operators expressed by the left and right members of (2.2.3a). On the one hand, we have



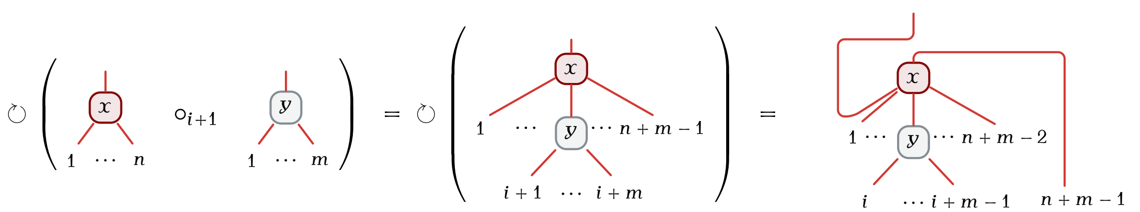
(2.2.4)

and on the other,



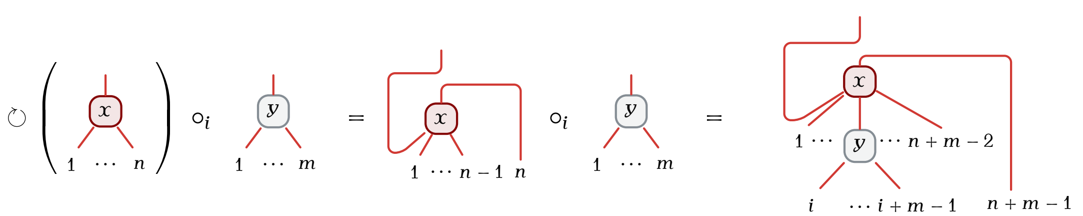
(2.2.5)

We observe that the two obtained abstract operators are the same. Indeed, for both of them, the connections between x and y are the same. This is what is expressed by (2.2.3a). Let us now consider the abstract operators expressed by the left and right members of (2.2.3b). On the one hand, we have



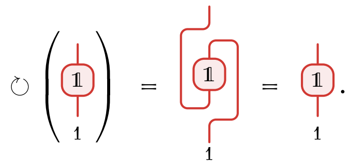
(2.2.6)

and on the other,



(2.2.7)

We observe that the two obtained abstract operators are the same. Indeed, for both of them, the connections between x and y are the same. This is what is expressed by (2.2.3b). Last relation (2.2.3c) expresses that

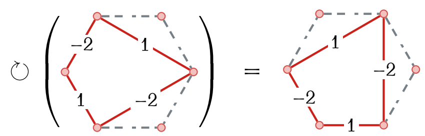


(2.2.8)

Since, in an operad, the unit $\mathbb{1}$ can be seen as the identity map (see Section 1.1.2 of Chapter 4), this map is also invertible. This is what is expressed by (2.2.3c).

2.2.4. *Example: operads of configurations of chords.* Let us consider the construction $\mathbf{C}$ associating with any monoid $\mathcal{M}$ the operad of configurations of chords $\mathbf{C}\mathcal{M}$ exposed in Section 3.3.5 of Chapter 4.

Let $\circlearrowleft$ be the cyclic map on $\mathfrak{CC}_{\bar{\mathcal{M}}}$ defined, for any $\bar{\mathcal{M}}$-configuration $\mathfrak{c}$ in the following way. The configuration $\circlearrowleft(\mathfrak{c})$ is obtained by applying a rotation of one step of $\mathfrak{c}$ in the counterclockwise direction. For instance, one has in $\mathbf{C}\mathbb{Z}$,



(2.2.9)

Proposition 2.2.1. *For any monoid $\mathcal{M}$, $\mathbf{C}\mathcal{M}$ is a cyclic operad for the cycle maps $\circlearrowleft$.*

2.3. Symmetric operads. As a last variant of operads, we consider now symmetric operads. These variations of operads involve symmetric collections. Intuitively, in a symmetric operad, the inputs of the elements can be permuted. This is due to the fact that these structures are endowed with maps letting the symmetric group of order n acting on its elements of arity n.

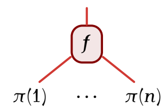
2.3.1. *Symmetric polynomial spaces.* A graded polynomial space $\mathbb{K}\langle C\rangle$ is *symmetric* if it is endowed for all $n \in \mathbb{N}$ and $\sigma \in \mathfrak{S}(n)$ with unary products

$$\circledcirc_\sigma : \mathbb{K}\langle C\rangle(n) \to \mathbb{K}\langle C\rangle(n) \tag{2.3.1}$$

such that $\circledcirc_{\mathrm{Id}_n}$ is the identity map on $\mathbb{K}\langle C\rangle(n)$, where Id_n denotes the identity map of $\mathfrak{S}_n$, and $\circledcirc_{\sigma_1} \circ \circledcirc_{\sigma_2} = \circledcirc_{\sigma_2 \circ \sigma_1}$ for any permutations σ_1 and σ_2 of $\mathfrak{S}(n)$. We say that the $\circledcirc_\sigma$, $\sigma \in \mathfrak{S}$, are *symmetric maps* of $\mathbb{K}\langle C\rangle$. As usual, if $\phi : \mathbb{K}\langle C_1\rangle \to \mathbb{K}\langle C_2\rangle$ is a morphism between two graded polynomial spaces $\mathbb{K}\langle C_1\rangle$ and $\mathbb{K}\langle C_2\rangle$, ϕ is a *symmetric polynomial space morphism* if it commutes with the symmetric maps of $\mathbb{K}\langle C_1\rangle$ and $\mathbb{K}\langle C_2\rangle$.

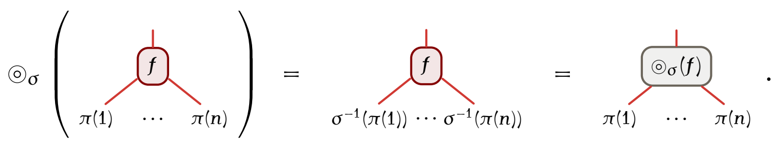
Remark that when C is a symmetric collection (see Section 1.1.6 of Chapter 1), the linearizations of the symmetric maps of C endow $\mathbb{K}\langle C\rangle$ with the structure of a symmetric polynomial space.

2.3.2. *Symmetric abstract operators.* We regard any homogeneous element f of an augmented symmetric polynomial space $\mathbb{K}\langle C\rangle$ as a *symmetric abstract operator*, that is, an abstract operator wherein the inputs are endowed with a total order. More precisely, the inputs of a symmetric abstract operator of arity n are number from 1 to n, but not necessarily from left to right in the increasing order as is the case for usual abstract operators (see Section 1.1.1 of Chapter 4). A symmetric abstract operator f of arity n is depicted as



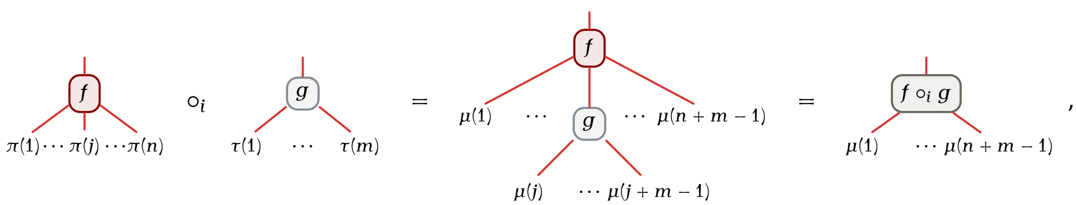
(2.3.2)

where π is a permutation of size n. Moreover, each symmetric map $\circledcirc_\sigma$, $\sigma \in \mathfrak{S}$, behaves in the following way. For any $f \in \mathbb{K}\langle C\rangle(n)$ and $\sigma \in \mathfrak{S}(n)$,



(2.3.3)

In words, $\circledcirc_\sigma(f)$ is obtained by permuting the inputs of f as specified by σ. It is straightforward to check that $(\circledcirc_{\sigma_1} \circ \circledcirc_{\sigma_2})(f) = \circledcirc_{\sigma_2 \circ \sigma_1}(f)$ for all $\sigma_1, \sigma_2 \in \mathfrak{S}(n)$ as expected since $\mathbb{K}\langle C\rangle$ is a symmetric polynomial space. On symmetric abstract operators, for any $f \in \mathbb{K}\langle C\rangle(n)$ and $g \in \mathbb{K}\langle C\rangle(m)$, the partial composition $f \circ_i g$ is the symmetric abstract operator



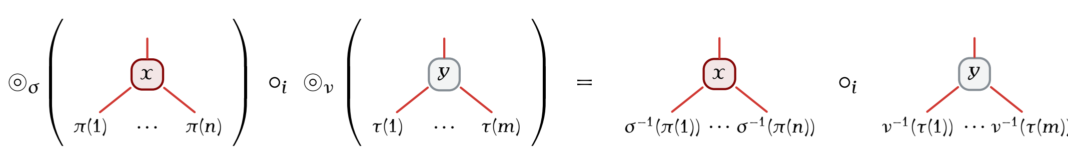
(2.3.4)

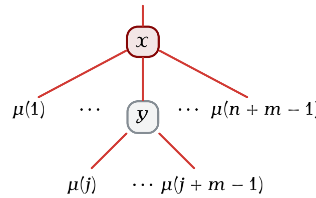
where $j \in [n]$ is such that $\pi(j) = i$ and $\mu = \pi \circ_j \tau$, where the occurrence of $\circ_i$ is the partial composition map of the operad **Per** of permutations (see Section 3.1.2 of Chapter 4).

2.3.3. *Symmetric operads.* A *symmetric operad* is an operad $\mathbb{K}\langle C\rangle$ such that $\mathbb{K}\langle C\rangle$ is also symmetric as polynomial space and satisfies, for any $x \in C(n)$, $\sigma \in \mathfrak{S}(n)$, $y \in C(m)$, $\nu \in \mathfrak{S}(m)$, and $i \in [n]$, the compatibility relation

$$\circledcirc_\sigma(x) \circ_i \circledcirc_\nu(y) = \circledcirc_{\sigma \circ_i \nu}\left(x \circ_{\sigma(i)} y\right), \tag{2.3.5}$$

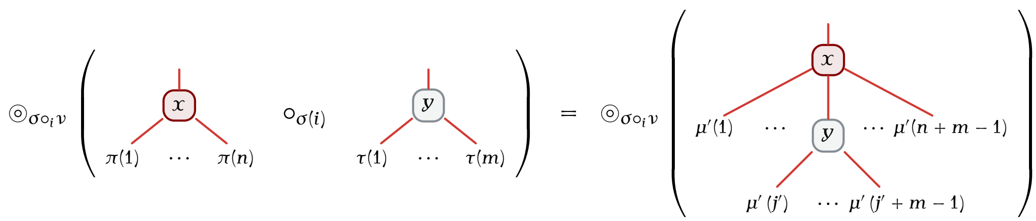
where the occurrence of $\circ_i$ in the right member of (2.3.5) refers to the partial composition maps of **Per**. To understand this relation, let us consider the abstract operators expressed by the left and right members of (2.3.5). On the one hand, we have

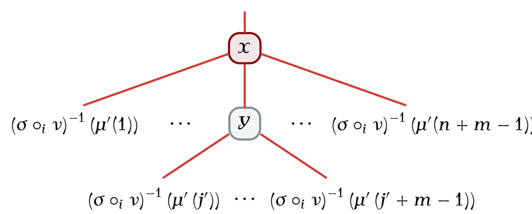


$=$ 

, (2.3.6)

where $j \in [n]$ is such that $\left(\sigma^{-1} \circ \pi\right)(j) = i$ and $\mu = \left(\sigma^{-1} \circ \pi\right) \circ_j \left(\nu^{-1} \circ \tau\right)$. On the other, we have



$=$ 

, (2.3.7)

where $j' \in [n]$ is such that $\pi(j') = \sigma(i)$ and $\mu' = \pi \circ_{j'} \tau$. Let us now explain why the last symmetric abstract operators of (2.3.6) and (2.3.7) are equal. First, from the hypothesis $\sigma^{-1}(\pi(j)) = i$ and $\pi(j') = \sigma(i)$, we deduce that $\sigma^{-1}(\pi(j')) = \sigma^{-1}(\sigma(i)) = i$, implying that $j' = j$. This provides the fact that the considered abstract operators have the same shape: the output of y is connected to the jth input of x in both cases. Now, consider the following result connecting the group theoretic composition $\circ$ of permutations and the partial composition maps $\circ_i$ of **Per**.

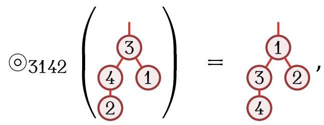
Lemma 2.3.1. *Let $n, m \in \mathbb{N}_{\geqslant 1}$, $i \in [n]$, and four permutations $\pi, \sigma \in \mathfrak{S}(n)$, $\tau, \nu \in \mathfrak{S}(m)$. Then, in the operad* **Per***,*

$$(\sigma \circ_i \nu)^{-1} \circ \left(\pi \circ_{\pi^{-1}(\sigma(i))} \tau\right) = \left(\sigma^{-1} \circ \pi\right) \circ_{\pi^{-1}(\sigma(i))} \left(\nu^{-1} \circ \tau\right). \tag{2.3.8}$$

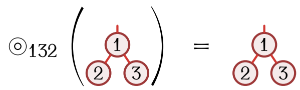
By Lemma 2.3.1, the permutations μ and $(\sigma \circ_i \nu)^{-1} \circ \mu'$ are equal. Therefore, the two obtained abstract operators are the same.

2.3.4. *Example: the symmetric operad* **NAP**. Let us consider the operad **NAP** introduced in Section 3.2.7 of Chapter 4. We endow the underlying collection $\mathfrak{SRT}$ of **NAP** with the symmetric maps $⊚_\sigma$, $\sigma \in \mathfrak{S}$, defined for any standard rooted tree $\mathfrak{t}$ of size n in the following way. The tree $⊚_\sigma(\mathfrak{t})$ is obtained by relabeling all the nodes $i \in [n]$ by $\sigma^{-1}(i)$.

For instance,



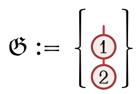
(2.3.9a)



(2.3.9b)

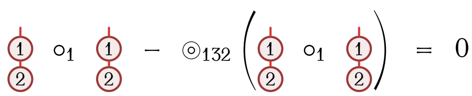
Then, the linearization of these symmetric maps endows **NAP** with the structure of a symmetric polynomial space. Together with the partial composition maps of **NAP** and its unit, **NAP** is a symmetric operad.

It is possible to show that, as a symmetric operad, the set



(2.3.10)

is a minimal generating set of **NAP**. Moreover, **NAP** contains the single nontrivial relation



(2.3.11)

involving its generator.

2.3.5. *Constructions involving operads.* If $\mathbb{K}\langle C \rangle$ is a symmetric operad, by forgetting its symmetric maps, the space $\mathbb{K}\langle C \rangle$ seen as an augmented graded polynomial space is an operad. We call this operad the *symmetric oblivion* of the symmetric operad $\mathbb{K}\langle C \rangle$.

Besides, given an operad $\mathbb{K}\langle C \rangle$, there are at least two ways to construct a symmetric operad. Let us present these. The first one consists in turning the augmented graded collection C into a symmetric collection by endowing it with the symmetric maps $\circledcirc_\sigma$, $\sigma \in \mathfrak{S}$, defined by $\circledcirc_\sigma(x) := x$ for all $x \in C$ and $\sigma \in \mathfrak{S}(|x|)$. The space $\mathbb{K}\langle C \rangle$ endowed with the linearizations of these symmetric maps on C, together with the partial composition maps and unit of the operad $\mathbb{K}\langle C \rangle$ forms a symmetric operad, called *trivial symmetric operad* of $\mathbb{K}\langle C \rangle$.

The second one consists in turning C into a symmetric collection by considering its regularization $\mathbf{Reg}(C)$ (see Section 1.2.12 of Chapter 1). Let us endow the symmetric polynomial space $\mathbb{K}\langle \mathbf{Reg}(C) \rangle$ with the partial composition maps defined linearly for any $(x, \sigma) \in (\mathbf{Reg}(C))(n)$, $(y, \nu) \in \mathbf{Reg}(C)$, and $i \in [n]$, by

$$(x, \sigma) \circ_i (y, \nu) := \left(x \circ_{\sigma^{-1}(i)} y, \sigma \circ_{\sigma^{-1}(i)} \nu\right), \tag{2.3.12}$$

where the first occurrence of a partial composition map in the right member of (2.3.12) refers to the partial composition map of the operad $\mathbb{K}\langle C \rangle$, and the second one refers to the partial composition map of the operad **Per**.

PROPOSITION 2.3.2. *For any operad $\mathbb{K}\langle C\rangle$, $\mathbb{K}\langle \mathbf{Reg}(C)\rangle$ is a symmetric operad.*

Proposition 2.3.2 is a consequence of Lemma 2.3.1. The symmetric operad $\mathbb{K}\langle \mathbf{Reg}(C)\rangle$, denoted by a slight abuse of notation by $\mathbf{Reg}(\mathbb{K}\langle C\rangle)$, is the *regularization* of $\mathbb{K}\langle C\rangle$. Let us notice that in general, the regularization of the symmetric oblivion of a symmetric operad $\mathbb{K}\langle C\rangle$ is different to $\mathbb{K}\langle C\rangle$. For instance, consider the symmetric operad **NAP** considered in Section 2.3.4. The regularization of the oblivion of **NAP** is the symmetric operad $\mathbf{Reg}(\mathbf{NAP})$, and this last is not isomorphic to **NAP** since $\dim \mathbf{NAP}(3) = 9$ and $\dim(\mathbf{Reg}(\mathbf{NAP}))(3) = 9 \times 3! = 54$.

2.3.6. *Algebras over symmetric operads.* Let $\mathbb{K}\langle C\rangle$ be a symmetric operad. An *algebra over $\mathbb{K}\langle C\rangle$* (or, for short, a *$\mathbb{K}\langle C\rangle$-algebra*) is an algebra $\mathbb{K}\langle D\rangle$ over the oblivion of $\mathbb{K}\langle C\rangle$ (see Section 1.3 of Chapter 4) satisfying the following additional condition. By denoting by $\bullet_n$, $n \in \mathbb{N}_{\geqslant 1}$, the linear maps (1.3.1) of Chapter 4 endowing the space $\mathbb{K}\langle D\rangle$ with the structure of a $\mathbb{K}\langle C\rangle$-algebra, for any $x \in C(n)$, $\sigma \in \mathfrak{S}(n)$, and $(a_1, \dots, a_n) \in \mathbf{List}_{\{n\}}(D)$,

$$\bullet_n (\circledcirc_\sigma(x), (a_1, \dots, a_n)) = \bullet_n \left(x, \left(a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)}\right)\right). \tag{2.3.13}$$

As for operads, one can regard each homogeneous element of arity $n \in \mathbb{N}_{\geqslant 1}$ of the symmetric operad $\mathbb{K}\langle C\rangle$ as a complete product of arity n on $\mathbb{K}\langle D\rangle$.

For instance, let us consider algebras on the symmetric operad **NAP**. We denote by $\mathfrak{t}$ the single generator of **NAP** appearing in (2.3.10). Since this generator is subjected to Relation (2.3.11), for any $f_1, f_2, f_3 \in \mathbb{K}\langle D\rangle$,

$$\begin{aligned} 0 &= (\mathfrak{t} \circ_1 \mathfrak{t} - \circledcirc_{132} (\mathfrak{t} \circ_1 \mathfrak{t}))(f_1, f_2, f_3) \\ &= (\mathfrak{t} \circ_1 \mathfrak{t})(f_1, f_2, f_3) - (\circledcirc_{132} (\mathfrak{t} \circ_1 \mathfrak{t}))(f_1, f_2, f_3) \\ &= (\mathfrak{t} \circ_1 \mathfrak{t})(f_1, f_2, f_3) - (\mathfrak{t} \circ_1 \mathfrak{t})(f_1, f_3, f_2) \\ &= \mathfrak{t}(\mathfrak{t}(f_1, f_2), f_3) - \mathfrak{t}(\mathfrak{t}(f_1, f_3), f_2). \end{aligned} \tag{2.3.14}$$

This is equivalent to the relation

$$(f_1 \,\mathfrak{t}\, f_2) \,\mathfrak{t}\, f_3 - (f_1 \,\mathfrak{t}\, f_3) \,\mathfrak{t}\, f_2 = 0 \tag{2.3.15}$$

written in infix way.

3. Product categories

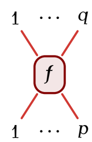
A very intuitive generalization of operads arises when one thinks of considering elements with several outputs, instead of only one as in the case of operads. This leads to a sort of extension of operads, called pros. We present here these algebraic structures.

3.1. Abstract bioperators. While operads are defined over augmented graded polynomial spaces, pros require spaces on 2-graded collections. Let us provide elementary definitions about these.

3.1.1. *Bigraded polynomial spaces.* Given a 2-graded collection C (see Section 1.1.3 of Chapter 1), the polynomial space $\mathbb{K}\langle C\rangle$ is said *bigraded*. The *arity* (resp. *coarity*) of a nonzero (p,q)-homogeneous element f of $\mathbb{K}\langle C\rangle$ is p (resp. q). Moreover, to not overload the notation, we denote by $\mathbb{K}\langle C\rangle(p,q)$ the homogeneous component $\mathbb{K}\langle C\rangle((p,q))$ of $\mathbb{K}\langle C\rangle$ for any $(p,q)\in\mathbb{N}^2$. When $\mathbb{K}\langle C\rangle$ is combinatorial, its *Hilbert series* is the $\mathbb{N}^2$-series

$$\mathbb{H}_{\mathbb{K}\langle C\rangle} := \mathbf{I}(C) = \sum_{(p,q)\in\mathbb{N}^2} (\dim \mathbb{K}\langle C\rangle(p,q))\,(p,q). \tag{3.1.1}$$

3.1.2. *Abstract bioperators.* We regard any homogeneous element f of a bigraded polynomial space $\mathbb{K}\langle C\rangle$ as an *abstract bioperator*, that is, an abstract operator having zero or more inputs and zero or more outputs. These abstract bioperators are depicted by following the drawing conventions of biproducts exposed in Section 2.1.1 of Chapter 3. Therefore, if f is of arity p and of coarity q, f is depicted by

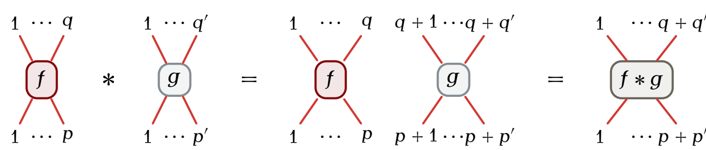


. (3.1.2)

3.1.3. *Composing abstract bioperators.* Let $\mathbb{K}\langle C\rangle$ be bigraded polynomial space. A *horizontal composition map* on $\mathbb{K}\langle C\rangle$ is a complete binary ω-concentrated product on $\mathbb{K}\langle C\rangle$ of the form

$$* : \mathbb{K}\langle [C, C]_\times\rangle \to \mathbb{K}\langle C\rangle \tag{3.1.3}$$

for the map $\omega : \mathbb{N}^2\times\mathbb{N}^2 \to \mathbb{N}^2$ satisfying $\omega((p_1,q_1),(p_2,q_2)) := (p_1+p_2, q_1+q_2)$. On abstract bioperators, for any $f\in\mathbb{K}\langle C\rangle(p,q)$ and $g\in\mathbb{K}\langle C\rangle(p',q')$, $f*g$ is the abstract bioperator

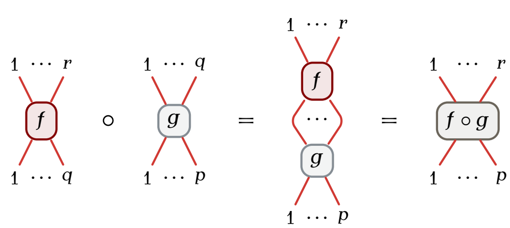


. (3.1.4)

A *vertical composition map* on $\mathbb{K}\langle C\rangle$ is a binary product on $\mathbb{K}\langle C\rangle$ of the form, for any $p,q,r\in\mathbb{N}$,

$$\circ^{(p,q,r)} : \mathbb{K}\langle [C(q,r), C(p,q)]_\times\rangle \to \mathbb{K}\langle C\rangle(p,r). \tag{3.1.5}$$

By a slight abuse of notation, we shall sometimes omit the (p,q,r) in the notation of $\circ^{(p,q,r)}$ in order to denote it in a more concise way by $\circ$. On abstract bioperators, for any $f\in\mathbb{K}\langle C\rangle(q,r)$ and $g\in\mathbb{K}\langle C\rangle(p,q)$, $f\circ g$ is the abstract bioperator

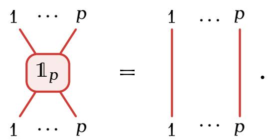


. (3.1.6)

Finally, a *unit map* on $\mathbb{K}\langle C\rangle$ is a product of arity zero on $\mathbb{K}\langle C\rangle$ of the form, for any $p \in \mathbb{N}$,

$$\mathbb{1}_p : \mathbb{K}\langle [\,]_\times \rangle \to \mathbb{K}\langle C\rangle(p,p). \tag{3.1.7}$$

For any $p \in \mathbb{N}$, $\mathbb{1}_p$ is the abstract bioperator



(3.1.8)

3.2. Pros. Pros are algebraic structures furnishing a formalization of the notion of abstract bioperators and their compositions. We provide here definitions about these structures and about bialgebras over pros.

3.2.1. *Elementary definitions.* The bigraded space $\mathbb{K}\langle C\rangle$ is a *product category* (or, for short, a *pro*) if it is endowed with a horizontal composition map $*$, vertical composition maps $\circ$, and unit maps $\mathbb{1}_p$, $p \in \mathbb{N}$, satisfying the following six relations (3.2.1), (3.2.2), (3.2.3), (3.2.4), (3.2.5) and (3.2.6).

First, $*$ is associative, that is, for any $x, y, z \in C$,

$$(x * y) * z = x * (y * z). \tag{3.2.1}$$

Second, the products $\circ$ are associative, that is, for any $x \in C(r,s)$, $y \in C(q,r)$, $z \in C(p,q)$,

$$(x \circ y) \circ z = x \circ (y \circ z). \tag{3.2.2}$$

Moreover, the products $*$ and $\circ$ satisfy the *square compatibility relation*, that is for any $x \in C(q,r)$, $y \in C(p,q)$, $x' \in C(q',r')$, $y' \in C(p',q')$,

$$(x \circ y) * (x' \circ y') = (x * x') \circ (y * y'). \tag{3.2.3}$$

Finally, the unit maps satisfy the following three sorts of relations. For any $p, q \in \mathbb{N}$,

$$\mathbb{1}_p * \mathbb{1}_q = \mathbb{1}_{p+q}, \tag{3.2.4}$$

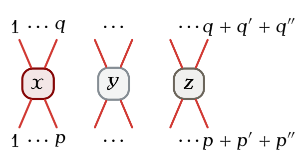
for any $x \in C(p,q)$,

$$x * \mathbb{1}_0 = x = \mathbb{1}_0 * x, \tag{3.2.5}$$

and for any $x \in C(p,q)$,

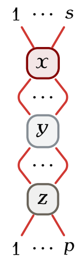
$$x \circ \mathbb{1}_p = x = \mathbb{1}_q \circ x. \tag{3.2.6}$$

Let us understand these relations with the help of abstract bioperators and the behavior of their compositions. First, the left and right members of (3.2.1) are both equal to



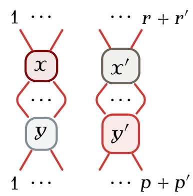
, (3.2.7)

where $x \in C(p, q)$, $y \in C(p', q')$, and $z \in C(p'', q'')$. Second, the left and right members of (3.2.2) are both equal to



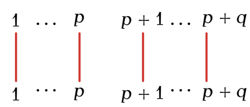
. (3.2.8)

Moreover, the left and right members of (3.2.3) are both equal to



. (3.2.9)

Finally, concerning the relations involving the unit maps, the left and right members of (3.2.4) are both equal to



. (3.2.10)

Relation (3.2.5) expresses the fact that $\mathbb{1}_0$ is the unit for the product $*$, and Relation (3.2.6) says that $\mathbb{1}_p$, $p \in \mathbb{N}$, is the unit of the products $\circ^{(p,p,q)}$ and $\circ^{(q,p,p)}$ for any $q \in \mathbb{N}$.

Since a pro is a particular polynomial algebra, all the properties and definitions about polynomial algebras exposed in Section 2.3 of Chapter 3 remain valid for pros (like pros morphisms, sub-pros, generating sets, pro ideals and quotients, *etc.*).

3.2.2. *Categorical definition.* In the same way as monoids, operads, and colored operads can be defined precisely and concisely by using the language of category theory (see Section 2.1.4), pros admit a similar definition using this language. Indeed, a pro $\mathbb{K}\langle C\rangle$ can be seen as a category where the objects are the elements of $\mathbb{N}$ and which is equipped with a bifunctor $*$ defined by $p * p' := p + p'$. The elements of $\mathbb{K}\langle C\rangle$ are interpreted as maps $\phi : p \to q$. The horizontal composition of $\mathbb{K}\langle C\rangle$ translates as the bifunctor of the category, the vertical composition translates as the composition of morphisms, and the unit map translates as the identity maps $\mathbb{1}_p : p \to p$ of the category.

3.2.3. *Bialgebras over pros.* Let $\mathbb{K}\langle C\rangle$ be a pro. A *bialgebra over* $\mathbb{K}\langle C\rangle$ (or, for short, a $\mathbb{K}\langle C\rangle$*-bialgebra*) is a polynomial space $\mathbb{K}\langle D\rangle$, where D is a (not necessarily 2-graded) collection, which is endowed for all $p, q \in \mathbb{N}$ with linear maps

$$\bullet_{p,q} : \mathbb{K}\left\langle \left[C(p,q), \mathbf{List}_{\{p\}}(D) \right]_{\times} \right\rangle \to \mathbb{K}\left\langle \mathbf{List}_{\{q\}}(D) \right\rangle \tag{3.2.11}$$

satisfying the relations imposed by the pro structure of $\mathbb{K}\langle C\rangle$, that are, for all $x \in C(p', q')$, $y \in C(p'', q'')$, and $(a_1, \dots, a_{p'+p''}) \in \mathbf{List}_{\{p'+p''\}}(D)$, by denoting by $\cdot$ the concatenation of tuples extended by linearity,

$$\bullet_{p'+p'',q'+q''}\left(x * y, \left(a_1, \dots, a_{p'}, a_{p'+1}, \dots, a_{p'+p''}\right)\right) = \bullet_{p',q'}\left(x, \left(a_1, \dots, a_{p'}\right)\right) \cdot \bullet_{p'',q''}\left(y, \left(a_{p'+1}, \dots, a_{p'+p''}\right)\right), \quad (3.2.12a)$$

for all $x \in C(q, r)$, $y \in C(p, q)$, and $(a_1, \dots, a_p) \in \mathbf{List}_{\{p\}}(D)$,

$$\bullet_{p,r}\left(x \circ y, \left(a_1, \dots, a_p\right)\right) = \bullet_{q,r}\left(x, \bullet_{p,q}\left(y, \left(a_1, \dots, a_p\right)\right)\right), \quad (3.2.12b)$$

and for all $(a_1, \dots, a_p) \in \mathbf{List}_{\{p\}}(D)$,

$$\bullet_{p,p}\left(\mathbb{1}_p, \left(a_1, \dots, a_p\right)\right) = \left(a_1, \dots, a_p\right). \quad (3.2.12c)$$

In other words, any object x of C of arity p and coarity q plays the role of a complete biproduct (in the sense of Section 2.1.1 of Chapter 3) of the form

$$x : \mathbb{K}\left\langle \mathbf{List}_{\{p\}}(D)\right\rangle \to \mathbb{K}\left\langle \mathbf{List}_{\{q\}}(D)\right\rangle, \quad (3.2.13)$$

defined, for any $(a_1, \dots, a_p) \in \mathbf{List}_{\{p\}}(D)$ by

$$x\left(a_1, \dots, a_p\right) := \bullet_{p,q}\left(x, \left(a_1, \dots, a_p\right)\right). \quad (3.2.14)$$

3.3. Main pros. We provide here classical examples of pros involving for most of these combinatorial objects (see Table 5.1). The first one is the associative pro, a sort of generalization of the associative operad. The next one is a pro of matrices, where the usual matrix operations (direct sum and multiplication) are revisited in the context of pros. This structure contains a lot of interesting sub-pros, like a pro of binary relations, a pro of maps, and a pro of permutations.

Pro	Objects	Arity	Coarity
PAs	Pairs (p, q) of nonneg. int.	p	q
$\mathbf{Mat}_{\mathbb{S}}$	Matrices on $\mathbb{S}$	num. of rows	num. of columns
BRel	Binary relations	card. of domain	card. of codomain
$\mathbb{K}\langle\mathfrak{Map}\rangle$	Maps	card. of domain	card. of codomain
$\mathbb{K}\langle\mathfrak{NDMap}\rangle$	Nondecreasing maps	card. of domain	card. of codomain
$\mathbb{K}\langle\mathfrak{IMap}\rangle$	Injective maps	card. of domain	card. of codomain
$\mathbb{K}\langle\mathfrak{SMap}\rangle$	Surjective maps	card. of domain	card. of codomain
$\mathbb{K}\langle\mathfrak{BMap}\rangle$	Bijective maps	card. of domain	card. of domain

TABLE 5.1. Overview of some pros. Here, $\mathbb{S}$ is a semiring.

3.3.1. *Associative pro.* Let $A := \{\mathfrak{a}_{p,q} : p, q \in \mathbb{N}\}$ be the 2-graded collection where the index of each $\mathfrak{a}_{p,q}$, $p, q \in \mathbb{N}$, is (p, q). The *associative pro* **PAs** is the space $\mathbb{K}\langle A\rangle$ endowed with the horizontal composition map $*$ defined linearly, for any $\mathfrak{a}_{p,q} \in A(p, q)$ and $\mathfrak{a}_{p',q'} \in A(p', q')$ by $\mathfrak{a}_{p,q} * \mathfrak{a}_{p',q'} := \mathfrak{a}_{p+p',q+q'}$, with the vertical composition maps $\circ$ defined linearly, for any $\mathfrak{a}_{q,r} \in A(q, r)$, $\mathfrak{a}_{p,q} \in A(p, q)$ by $\mathfrak{a}_{q,r} \circ \mathfrak{a}_{p,q} := \mathfrak{a}_{p,r}$, and with the unit maps $\mathbb{1}_p$, $p \in \mathbb{N}$, defined by $\mathbb{1}_p := \mathfrak{a}_{p,p}$.

3.3.2. *Pro of matrices.* Let $\mathbb{S}$ be a semiring (that is, a ring such without the condition that each element have an additive inverse). Let $\mathfrak{Mat}_{\mathbb{S}}$ be the 2-graded collection of all matrices on $\mathbb{S}$ where $\mathfrak{Mat}_{\mathbb{S}}(p, q)$ is the set of such matrices of dimension $p \times q$, $p, q \in \mathbb{N}$. The *pro of matrices* $\mathbf{Mat}_{\mathbb{S}}$ is the space $\mathbb{K}\langle\mathfrak{Mat}_{\mathbb{S}}\rangle$ endowed with the horizontal composition map $*$ defined linearly, for any $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathbf{Mat}_{\mathbb{S}}$ by

$$\mathfrak{m}_1 * \mathfrak{m}_2 := \mathfrak{m}_1 \oplus \mathfrak{m}_2, \tag{3.3.1}$$

where $\oplus$ is the matrix direct sum. Moreover, $\mathbf{Mat}_{\mathbb{S}}$ is endowed with the vertical composition maps $\circ$ defined linearly, for any $\mathfrak{m}_1 \in \mathfrak{Mat}_{\mathbb{S}}(q, r)$ and $\mathfrak{m}_2 \in \mathfrak{Mat}_{\mathbb{S}}(p, q)$, $p, q, r \in \mathbb{N}$, by

$$\mathfrak{m}_1 \circ \mathfrak{m}_2 := \mathfrak{m}_2 \cdot \mathfrak{m}_1, \tag{3.3.2}$$

where $\cdot$ is the matrix multiplication. Finally, we define the unit maps $\mathbb{1}_p$, $p \in \mathbb{N}$, of $\mathbf{Mat}_{\mathbb{S}}$ as the identity matrix of order p. For instance, by setting $\mathbb{S}$ as the semiring $(\mathbb{N}, +, \cdot)$,

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix} \tag{3.3.3}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \tag{3.3.4}$$

are, respectively, horizontal and vertical compositions in $\mathbf{Mat}_{\mathbb{S}}$.

3.3.3. *Pro of binary relations.* Let $\mathbb{B}$ be the Boolean semiring, that is the set $\{0, 1\}$ equipped with the addition $+$ satisfying $0 + 0 = 0$ and $0 + 1 = 1 + 0 = 1 + 1 = 1$, and the multiplication $\cdot$ satisfying $1 \cdot 1 = 1$ and $0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$. The *pro of binary relations* **BRel** is the pro $\mathbf{Mat}_{\mathbb{B}}$. By definition, the collection of all matrices on $\mathbb{B}$, called *Boolean matrices* forms a basis of **BRel**. For instance,

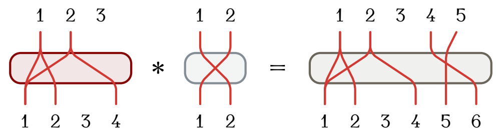
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.3.5}$$

and

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{3.3.6}$$

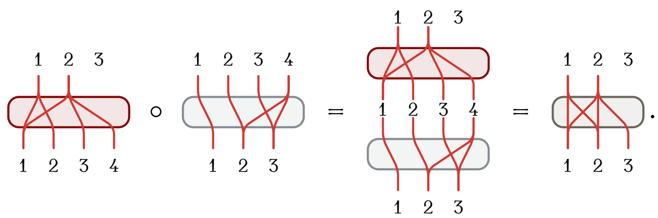
are, respectively, horizontal and vertical compositions in **BRel**. There is, for any $p, q \in \mathbb{N}$, a one-to-one correspondence between the set $\mathbf{Mat}_{\mathbb{B}}(p, q)$ and the set of all binary relations between $[p]$ and $[q]$. Indeed, a matrix $\mathfrak{m} \in \mathbf{Mat}_{\mathbb{B}}(p, q)$, $p, q \in \mathbb{N}$, and a binary relation $\mathfrak{R}$ between $[p]$ and $[q]$ are in correspondence if, for any $x \in [p]$ and $y \in [q]$, one has $\mathfrak{m}_{x,y} = 1$ if and only if $x \,\mathfrak{R}\, y$. By using this correspondence, and by drawing a binary relation $\mathfrak{R}$

through a graph connecting x to y if $x \,\mathfrak{R}\, y$, where the elements of the domain are depicted below and the elements of the codomain are depicted above, (3.3.5) and (3.3.6) translate, respectively, as



(3.3.7)

and



(3.3.8)

Therefore, $*$ is the concatenation of binary relations and $\circ$ is their composition.

3.3.4. *Pros of maps and variations.* The pro **BRel** admits many sub-pros by considering subspaces of binary relations satisfying particular conditions. For instance, by setting $\mathfrak{Map}$ as the subcollection of $\mathfrak{Mat}_{\mathbb{B}}$ restrained on binary relations that are maps (that is, $\mathfrak{m} \in \mathfrak{Map}(p,q)$ if for any $x \in [p]$, there is exactly one $y \in [q]$ such that $\mathfrak{m}_{x,y} = 1$), $\mathbb{K}\langle \mathfrak{Map} \rangle$ is a sub-pro of **BRel**. Moreover, by setting $\mathfrak{NDMap}$ as the subcollection of $\mathfrak{Map}$ restrained on maps that are nondecreasing (that is, $\mathfrak{m} \in \mathfrak{NDMap}(p,q)$ if for any $x, x' \in [p]$ such that $x \leqslant x'$, $\mathfrak{m}_{x,y} = \mathfrak{m}_{x',y'} = 1$ implies $y \leqslant y'$), $\mathbb{K}\langle \mathfrak{NDMap} \rangle$ is a sub-pro of $\mathbb{K}\langle \mathfrak{Map} \rangle$. Besides, by setting $\mathfrak{IMap}$ (resp. $\mathfrak{SMap}$) as the subcollection of $\mathfrak{Map}$ restrained on maps that are injections (resp. surjections), $\mathbb{K}\langle \mathfrak{IMap} \rangle$ (resp. $\mathbb{K}\langle \mathfrak{SMap} \rangle$) is a sub-pro of $\mathbb{K}\langle \mathfrak{Map} \rangle$. Finally, by setting $\mathfrak{BMap}$ as the subcollection of $\mathfrak{Map}$ restrained on maps that are bijections, $\mathbb{K}\langle \mathfrak{BMap} \rangle$ is a sub-pro of both $\mathbb{K}\langle \mathfrak{IMap} \rangle$ and $\mathbb{K}\langle \mathfrak{SMap} \rangle$.

Bibliographic notes

About series on operads. Our approach concerning formal power series through the framework of collections encompasses the classical case of power series, as explained in Section 1. Since the introduction of formal power series, a lot of generalizations were proposed in order to extend the range of enumerative problems they can help to solve. The most obvious ones are multivariate series allowing to count objects not only with respect to their sizes but additionally with respect to various other statistics (see Section 1.1.3 of Chapter 1). Such series are elements of $\mathbb{K}\langle\langle \mathbf{MSet}(\{t_1, \dots, t_k\}) \rangle\rangle$ where all the t_i, $i \in [k]$, are atomic objects. Another one consists in considering noncommutative series on words [**Eil74**,**SS78**,**BR10**] (and thus, elements of $\mathbb{K}\langle\langle A^* \rangle\rangle$, where A is an alphabet), or even, pushing the generalization one step further, on elements of a monoid [**Sak09**] (and thus, elements of $\mathbb{K}\langle\langle \mathcal{M} \rangle\rangle$, where $\mathcal{M}$ is a monoid). Besides, as another generalization, series on trees have been considered [**BR82**, **Boz01**]. Series on operads increase the list of these generalizations. Chapoton was the first to have considered such series on operads [**Cha02**, **Cha08**, **Cha09**]. Several authors have contributed to this field by considering slight variations in the definitions of these series. Among these, one can cite van der Laan [**vdL04**], Frabetti [**Fra08**], and Loday and Nikolov [**LN13**]. In this text, we have presented series on set-operads as powerful tools to provide descriptions of the generating series of some combinatorial graded collections C. All this relies on a set-operad structure on C having the property to be a Koszul operad and admitting a Poincaré-Birkhoff-Witt basis. The obtained descriptions of the generating series of C are in fact the generating series of the syntax trees on the generators of C (as an operad) that avoid some patterns (that are the left members of an orientation of the space of relations of C, see Proposition 2.3.1 of Chapter 4). Similar ideas were brought by Khoroshkin and Piontkovski [**KP15**], focused on the theory of Gröbner bases for symmetric operads. In [**Gir19**], the emphasis was put on the combinatorial and enumerative consequences of set-operads admitting Poincaré-Birkhoff-Witt bases, leading to refinements of their Hilbert series.

About colored operads. Classical references about colored operads are [**BV73**] and [**Yau16**]. By looking at this theory in the shoes of a computer scientist, one can think the colors as data types in computer programming. In the same way as two functions can be composed only if the output type of the one is equal to the type of an input of the second, the (partial and full) composition maps of a colored operad require a property on the colors of the operands. Moreover, to pursue the analogy, the elements of arity one of a colored operad having different input and output colors can be interpreted as casting operators in computer programming (that are, operators taking as input one object and changing its type). Besides, colored operads are interesting devices for enumerative prospects when combined with series on operads [**Gir16a**]. In this cited work, a generalization of both context-free grammars (see [**Har78**,**HMU06**]) and regular tree grammars (see [**CDG⁺07**]) using colored operads was proposed. The colors play here the role of terminal and nonterminal symbols of the grammars. The bud operad construction presented in Section 2.1.6 appears in this context.

About cyclic operads. In a cyclic operad, the distinction between inputs and outputs of the elements is diminished due to the fact that the cycle maps change outputs into inputs and conversely. These structures appeared first in [**GK95**]. Alternative descriptions of cyclic operads have been provided. Among them, in [**CO17**], the authors proposed an axiomatization wherein composition maps are parametrized by two vertices (corresponding to inputs or outputs of the elements involved in the composition). Intuitively, this amounts to create a link between the two vertices of the abstract operators.

About symmetric operads. In the literature, symmetric operads are simply called "operads" and are the main considered variants among all the algebraic structures of this sort. There are several alternative ways to define symmetric operads, and an interesting one relies on the theory of species of structures [**Mén15**] (see the bibliographic notes of Chapter 1). Besides, some of the tools and notions presented in Chapter 4 admit generalizations for symmetric operads. Among these, free structures can be described by syntax trees with labeled leaves, Poincaré-Birkhoff-Witt bases can be described by pattern avoidance in these trees, and Koszul duality remains a well-defined notion. General references about symmetric operads are [**Mar08**, **LV12**, **Mén15**]. As a matter of fact, Koszul duality is a very important topic in the theory of symmetric operads. One of the most beautiful properties of this duality is the fact that the symmetric operad of commutative associative algebras is the Koszul dual of the symmetric operad of Lie algebras. An important object here is the operadic butterfly [**Lod01**, **Lod06**], a diagram of symmetric operads related by symmetric operad morphisms and links established by Koszul duality. This diagram contains, for instance, the operads of associative algebras, commutative associative algebras, Lie algebras, and dendriform algebras.

About pros. Surprisingly, even if pros are in some sense generalizations of operads, they appeared earlier than these lasts in the work of Mac Lane [**ML65**]. Equally surprising is the fact that the axiomatization of pros is in some way simpler than the one of operads (there is no need to do arithmetic on the indexes to describe the relations the horizontal and vertical composition maps have to satisfy, contrarily to partial composition maps of operads). Basic and modern references about pros are [**Lei04**] and [**Mar08**]. Besides, pros are related to the Hopf bialgebra theory since in [**BG16**] a construction from set-pros to Hopf bialgebras was proposed. Moreover, in [**Laf03**, **Laf11**], several examples of pros were provided, and links with rewrite systems on elements of free pros were presented. As a last remark, free pros are difficult objects to describe explicitly due mainly to the fact that they can contain elements of arity or coarity zero. The works [**Cor18**, **LLMN18**] collected results in this direction.

Bibliography

[Agu00] M. Aguiar. Pre-Poisson algebras. *Lett. Math. Phys.*, 54(4):263–277, 2000. 78

[AL04] M. Aguiar and J.-L Loday. Quadri-algebras. *J. Pure Appl. Algebra*, 191(3):205–221, 2004. 78

[AL07] M. Aguiar and M. Livernet. The associative operad and the weak order on the symmetric groups. *J. Homotopy Relat. Str.*, 2(1):57–84, 2007. 114

[AP17] J. Alm and D. Petersen. Brown's dihedral moduli space and freedom of the gravity operad. *Ann. Sci. Éc. Norm. Supér.*, 50(5):1081–1122, 2017. 114

[AVL62] G.M. Adelson-Velsky and E. M. Landis. An algorithm for the organization of information. *Soviet Mathematics Doklady*, 3:1259–1263, 1962. 30

[BD16] M. R. Bremner and V Dotsenko. *Algebraic Operads: An Algorithmic Companion*. Chapman and Hall/CRC, 2016. Pages xvii+365. 51

[BDO18] E. Burgunder and B. Delcroix-Oger. Confluence laws and Hopf-Borel type theorem for operads. arXiv:1701.01323v2, 2018. 78

[BF03] C. Brouder and A. Frabetti. QED Hopf algebras on planar binary trees. *J. Algebra*, no. 1:298–322, 2003. 79

[BG16] J.-P. Bultel and S. Giraudo. Combinatorial Hopf algebras from PROs. *J. Algebr. Comb.*, 44(2):455–493, 2016. 138

[BGS82] A. Björner, A. M. Garsia, and R. P. Stanley. *An introduction to Cohen-Macaulay partially ordered sets*, volume 83 of *NATO Adv. Study Inst. Ser. C: Math. Phys. Sci.* Reidel, Dordrecht-Boston, Mass., 1982. 113

[Bjö84] A Björner. Orderings of Coxeter groups. In *Combinatorics and algebra*, volume 34 of *Contemp. Math.*, pages 175–195. Amer. Math. Soc., Providence, RI, 1984. 31

[BL10] N. Bergeron and M. Livernet. The non-symmetric operad pre-Lie is free. *J. Pure Appl. Algebra*, 214(7):1165–1172, 2010. 114

[BLL98] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial species and tree-like structures*, volume 67 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998. 30

[BLL13] F. Bergeron, G. Labelle, and P. Leroux. *Introduction to the Theory of Species of Structures*. Université du Québec à Montréal, 2013. 30

[BMFPR11] M. Bousquet-Mélou, É. Fusy, and L.-F. Préville-Ratelle. The number of intervals in the m-Tamari lattices. *Electron. J. Comb.*, 18(2), 2011. Paper 31. 30

[BN98] F. Baader and T. Nipkow. *Term rewriting and all that*. Cambridge University Press, Cambridge, 1998. Pages xii+301. 31

[Boz01] S. Bozapalidis. Context-free series on trees. *Inform. Comput.*, 169(2):186–229, 2001. 137

[BPR12] F. Bergeron and L.-F. Préville-Ratelle. Higher trivariate diagonal harmonics via generalized Tamari posets. *J. Comb.*, 3(3):317–341, 2012. 30

[BR82] J. Berstel and C. Reutenauer. Recognizable formal power series on trees. *Theor. Comput. Sci.*, 18(2):115–148, 1982. 137

[BR10] J. Berstel and C. Reutenauer. *Noncommutative rational series with applications*, volume 137. Cambridge University Press, 2010. Pages xiv+248. 136

[Buc76] B. Buchberger. A theoretical basis for the reduction of polynomials to canonical forms. *ACM SIGSAM Bull.*, 10(3), 1976. 31

[BV73] J. M. Boardman and R. M. Vogt. Homotopy invariant algebraic structures on topological spaces. *Lect. Notes Math.*, 347, 1973. 112, 137

[Cay57] A. Cayley. On the Theory of the Analytical Forms called Trees. *Phil. Mag.*, 13:172–176, 1857. 51

[CCG18] C. Chenavier, C. Cordero, and S. Giraudo. Generalizations of the associative operad and convergent rewrite systems. *HDRA*, 2018. 51, 113

[CCM11] P. Caron, J.-C. Champarnaud, and L. Mignot. Multi-Bar and Multi-Tilde Regular Operators. *J. Autom. Lang. Comb.*, 16(1):11–26, 2011. 114

[CDG$^+$07] H. Comon, M. Dauchet, R. Gilleron, F. Jacquemard, C. Löding, D. Lugiez, S. Tison, and M. Tommasi. *Tree Automata Techniques and Applications.* `http://www.grappa.univ-lille3.fr/tata`, 2007. 137

[CG14] F. Chapoton and S. Giraudo. Enveloping operads and bicoloured noncrossing configurations. *Exp. Math.*, 23(3):332–349, 2014. 113, 114

[Cha02] F. Chapoton. Un théorème de Cartier-Milnor-Moore-Quillen pour les bigèbres dendriformes et les algèbres braces. *J. Pure Appl. Algebra*, 168(1):1–18, 2002. 78, 137

[Cha05] F. Chapoton. On some anticyclic operads. *Algebr. Geom. Topol.*, 5:53–69, 2005. 114

[Cha06] F. Chapoton. Sur le nombre d'intervalles dans les treillis de Tamari. *Sém. Lothar. Combin.*, B55f:18 pp., 2006. 30, 113

[Cha07] F. Chapoton. The anticyclic operad of moulds. *Int. Math. Res. Notices*, 20:1–36, 2007. Art. ID rnm078. 114

[Cha08] F. Chapoton. Operads and algebraic combinatorics of trees. *Sém. Lothar. Combin.*, B58c:27 pp., 2008. 51, 78, 113, 137

[Cha09] F. Chapoton. A rooted-trees q-series lifting a one-parameter family of Lie idempotents. *Algebra & Number Theory*, 3(6):611–636, 2009. 137

[Cha14] F. Chapoton. Flows on rooted trees and the Menous-Novelli-Thibon idempotents. *Math. Scand.*, 115(1), 2014. 113

[CL01] F. Chapoton and M. Livernet. Pre-Lie algebras and the rooted trees operad. *Int. Math. Res. Notices*, 8:395–408, 2001. 78, 113, 114

[CL07] F. Chapoton and M. Livernet. Relating two Hopf algebras built from an operad. *Int. Math. Res. Notices*, 2007(24):1–27, 2007. Art. ID rnm131. 113

[CLRS09] T.H. Cormen, C. E. Leiserson, R.L. Rivest, and C. Stein. *Introduction to algorithms*. The MIT Press, third edition, 2009. 51

[CO17] P.-L. Curien and J. Obradović. A formal language for cyclic operads. *Higher Structures*, 1(1):2255, 2017. 137

[Cor18] C. Cordero. Enumerative Combinatorics of Prographs. *Formal Power Series and Algebraic Combinatorics*, 2018. 138

[Cox34] H. S. M. Coxeter. Discrete groups generated by reflections. *Ann. Math.*, 35(3):588–621, 1934. 31

[CP17] G. Châtel and V. Pilaud. Cambrian Hopf algebras. *Adv. Math.*, 311:598–633, 2017. 79

[DHT02] G. Duchamp, F. Hivert, and J.-Y. Thibon. Noncommutative Symmetric Functions VI: Free Quasi-Symmetric Functions and Related Algebras. *Int. J. of Algebr. Comput.*, 12:671–717, 2002. 79

[DK10] V. Dotsenko and A. Khoroshkin. Gröbner bases for operads. *Duke Math. J.*, 153(2):363–396, 2010. 51, 113

[DM47] A. Dvoretzky and Th. Motzkin. A problem of arrangements. *Duke Math. J.*, 14(2):305–313, 1947. 51

[EFM09] K. Ebrahimi-Fard and D. Manchon. Dendriform equations. *J. Algebra*, 322(11):4053–4079, 2009. 78

[EFM14] K. Ebrahimi-Fard and D. Manchon. The Magnus expansion, trees and Knuth's rotation correspondence. *Found. Comput. Math.*, 14(1):1–25, 2014. 51

[EFMP08] K. Ebrahimi-Fard, D. Manchon, and F. Patras. New identities in dendriform algebras. *J. Algebra*, 320(2):708–727, 2008. 78

[Eil74] S. Eilenberg. *Automata, languages, and machines. Vol. A*, volume 58 of *Pure and Applied Mathematics*. Academic Press, New York, 1974. Pages xvi+451. 136

[Foi07] L. Foissy. Bidendriform bialgebras, trees, and free quasi-symmetric functions. *J. Pure Appl. Algebra*, 209(2):439–459, 2007. 78

[Foi12] L. Foissy. Ordered forests and parking functions. *Int. Math. Res. Notices*, 2012(7):1603–1633, 2012. 79

[Foi15] L. Foissy. Examples of Com-PreLie Hopf algebras. `arXiv:1501.06375v1`, 2015. 79

[Fra08] A. Frabetti. Groups of tree-expanded series. *J. Algebra*, 319(1):377–413, 2008. 137

[FS09] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009. 30, 51

[Ger63] M. Gerstenhaber. The cohomology structure of an associative ring. *Ann. Math.*, 78:267–288, 1963. 78, 112

[Get94] E. Getzler. Two-dimensional topological gravity and equivariant cohomology. *Commun. Math. Phys.*, 163(3):473–489, 1994. 114

[Gir12] S. Giraudo. Algebraic and combinatorial structures on pairs of twin binary trees. *J. Algebra*, 360:115–157, 2012. 79

[Gir15] S. Giraudo. Combinatorial operads from monoids. *J. Algebr. Comb.*, 41(2):493–538, 2015. 113, 114

[Gir16a] S. Giraudo. Colored operads, series on colored operads, and combinatorial generating systems. `arXiv:1605.04697v1`, 2016. 137

[Gir16b] S. Giraudo. Operads from posets and Koszul duality. *Eur. J. Combin.*, 56C:1–32, 2016. 51, 113

[Gir16c] S. Giraudo. Pluriassociative algebras I: The pluriassociative operad. *Adv. Appli. Math.*, 77:1–42, 2016. 78, 113

[Gir16d] S. Giraudo. Pluriassociative algebras II: The polydendriform operad and related operads. *Adv. Appli. Math.*, 77:43–85, 2016. 78

[Gir17] S. Giraudo. Combalgebraic structures on decorated cliques. *Formal Power Series and Algebraic Combinatorics*, 2017. 113, 114

[Gir19] S. Giraudo. Tree series and pattern avoidance in syntax trees. `arXiv:1903.00677v1`, 2019. 137

[GK94] V. Ginzburg and M. M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994. 113

[GK95] E. Getzler and M. M. Kapranov. Cyclic operads and cyclic homology. In *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 167–201. Int. Press, Cambridge, MA, 1995. 137

[GKL+95] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, and J.-Y. Thibon. Noncommutative symmetric functions I. *Adv. Math.*, 112:218–348, 1995. 79

[GLMN16] S. Giraudo, J.-G. Luque, L. Mignot, and F. Nicart. Operads, quasiorders, and regular languages. *Adv. Appl. Math.*, 75:56–93, 2016. 114

[GR63] G. Th. Guilbaud and P. Rosenstiehl. Analyse algébrique d'un scrutin. *Mathématiques et sciences humaines*, tome 4:9–33, 1963. 30

[Har78] M. A. Harrison. *Introduction to formal language theory*. Addison-Wesley Publishing Co., Reading, Mass., 1978. Pages xiv+594. 137

[Hiv99] F. Hivert. *Combinatoire des fonctions quasi-symétriques*. PhD thesis, Université de Marne-la-Vallée, 1999. 79

[Hiv03] F. Hivert. An introduction to Combinatorial Hopf Algebras. *IOS Press*, 2003. 79

[HMU06] J. E. Hopcroft, R. Matwani, and J. D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Pearson, third edition, 2006. 137

[HNT05] F. Hivert, J.-C. Novelli, and J.-Y. Thibon. The Algebra of Binary Search Trees. *Theor. Comput. Sci.*, 339(1):129–165, 2005. 31, 51, 79

[Hof10] E. Hoffbeck. A Poincaré-Birkhoff-Witt criterion for Koszul operads. *Manuscripta Math.*, 131(1-2):87–110, 2010. 51, 113

[HT72] S. Huang and D. Tamari. Problems of associativity: A simple proof for the lattice property of systems ordered by a semi-associative law. *J. Comb. Theory A*, 13:7–13, 1972. 30

[Joy81] A. Joyal. Une théorie combinatoire des séries formelles. *Adv. Math.*, 42(1):1–82, 1981. 30

[KB70] D. Knuth and P. Bendix. Simple word problems in universal algebras. In *Computational Problems in Abstract Algebra*, pages 263–297. Pergamon, Oxford, 1970. 31

[Knu97] D. Knuth. *The Art of Computer Programming, volume 1: Fundamental Algorithms*. Addison Wesley Longman, third edition, 1997. Pages xx+650. 51

[Knu98] D. Knuth. *The art of computer programming, volume 3: Sorting and searching*. Addison Wesley Longman, 1998. Pages xiv+780. 30, 31, 51

[Koc09] J. Kock. *Notes on Polynomial Functors*. Unpublished, 2009. URL `http://mat.uab.es/~kock/cat/notes-on-polynomial-functors.html`. 30

[KP15] A. Khoroshkin and D. Piontkovski. On generating series of finitely presented operads. *J. Algebra*, 426:377–429, 2015. 137

[Lab81] G. Labelle. Une nouvelle démonstration combinatoire des formules d'inversion de Lagrange. *Adv. Math.*, 42(3):217–247, 1981. 51

[Laf03] Y. Lafont. Towards an algebraic theory of Boolean circuits. *J. Pure Appl. Algebra*, 184(2-3):257–310, 2003. 138

[Laf11] Y. Lafont. Diagram rewriting and operads. *Sémin. Congr.*, 26:163–179, 2011. 138

[Lei04] T. Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004. Pages xiv+433. 138

[Ler04] P. Leroux. Ennea-algebras. *J. Algebra*, 281(1):287–302, 2004. 78

[Ler07] P. Leroux. A simple symmetry generating operads related to rooted planar m-ary trees and polygonal numbers. *J. Integer Seq.*, 10(4):1–23, 2007. Article 07.4.7. 78

[Ler11] P. Leroux. L-algebras, triplicial-algebras, within an equivalence of categories motivated by graphs. *Comm. Algebra*, 39(8):2661–2689, 2011. 114

[Liv06] M. Livernet. A rigidity theorem for pre-Lie algebras. *J. Pure Appl. Algebra*, 207(1):1–18, 2006. 79, 113, 114

[LLMN18] É. Laugerotte, J.-G. Luque, L. Mignot, and F. Nicart. Multilinear representations of Free PROs. `arXiv:1803.00228v1`, 2018. 138

[LMN13] J.-G. Luque, L. Mignot, and F. Nicart. Some Combinatorial Operators in Language Theory. *J. Autom. Lang. Comb.*, 18(1):27–52, 2013. 114

[LN13] J.-L. Loday and N. M. Nikolov. Operadic construction of the renormalization group. In *Lie theory and its applications in physics*, volume 36 of *Springer Proc. Math. Stat.*, pages 191–211. Springer, Tokyo, 2013. 137

[Lod96] J.-L. Loday. La renaissance des opérades. *Séminaire Bourbaki*, 37(792):47–74, 1996. 113

[Lod01] J.-L. Loday. Dialgebras. In *Dialgebras and related operads*, volume 1763 of *Lecture Notes in Math.*, pages 7–66. Springer, Berlin, 2001. 78, 113, 114, 138

[Lod02] J.-L. Loday. Arithmetree. *J. Algebra*, 258(1):275–309, 2002. Special issue in celebration of Claudio Procesi's 60th birthday. 78

[Lod05] J.-L. Loday. Inversion of integral series enumerating planar trees. *Sém. Lothar. Combin.*, 53, 2005. Art. B53d, 16. 113

[Lod06] J.-L. Loday. Completing the operadic butterfly. *Georgian Math. J.*, 13(4):741–749, 2006. 138

[Lod08] J.-L. Loday. Generalized Bialgebras and Triples of Operads. *Astérisque*, 320:1–114, 2008. 78, 79, 113, 114

[Lod10] J.-L. Loday. On the operad of associative algebras with derivation. *Georgian Math. J.*, 17(2):347–372, 2010. 114

[LR98] J.-L. Loday and M. Ronco. Hopf Algebra of the Planar Binary Trees. *Adv. Math.*, 139:293–309, 1998. 31, 51, 78, 79

[LR02] J.-L. Loday and M. Ronco. Order Structure on the Algebra of Permutations and of Planar Binary Trees. *J. Algebr. Comb.*, 15(3):253–270, 2002. 31

[LR03] J.-L. Loday and M. Ronco. Algèbres de Hopf colibres. *C. R. Math.*, 337(3):153–158, 2003. 114

[LR04] J.-L. Loday and M. Ronco. Trialgebras and families of polytopes. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 369–398. Amer. Math. Soc., Providence, RI, 2004. 78, 114

[LR06] J.-L. Loday and M. Ronco. On the structure of cofree Hopf algebras. *J. Reine Angew. Math.*, 592:123–155, 2006. 114

[LR12] S. Law and N. Reading. The Hopf algebra of diagonal rectangulations. *J. Comb. Theory A*, 119(3):788–824, 2012. 79

[LV12] J.-L. Loday and B. Vallette. *Algebraic Operads*, volume 346 of *Grundlehren der mathematischen Wissenschaften*. Springer, Heidelberg, 2012. Pages xxiv+634. 51, 78, 113, 138

[Mac15] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Classic Texts in the Physical Sciences. Oxford University Press, second edition, 2015. 79

[Man11] D. Manchon. A short survey on pre-Lie algebras. In *Noncommutative geometry and physics: renormalisation, motives, index theory*, ESI Lect. Math. Phys., pages 89–102. Eur. Math. Soc., Zürich, 2011. 78

[Mar08] M. Markl. Operads and PROPs. In *Handbook of Algebra*, volume 5, pages 87–140. Elsevier/North-Holland, Amsterdam, 2008. 113, 138

[May72] J. P. May. *The geometry of iterated loop spaces*, volume 271 of *Lectures Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1972. Pages viii+175. 112

[Mén15] M. A. Méndez. *Set operads in combinatorics and computer science*. SpringerBriefs in Mathematics. Springer, Cham, 2015. Pages xvi+129. 30, 113, 138

[ML65] S. Mac Lane. Categorical algebra. *Bull. Amer. Math. Soc.*, 71:40–106, 1965. 138

[MR95] C. Malvenuto and C. Reutenauer. Duality between quasi-symmetric functions and the Solomon descent algebra. *J. Algebra*, 177(3):967–982, 1995. 79

[MY91] M. Méndez and J. Yang. Möbius Species. *Adv. Math.*, 85(1):83–128, 1991. 113

[Nar55] T.V. Narayana. Sur les treillis formés par les partitions d'un entier et leurs applications à la théorie des probabilités. *C. R. Acad. Sci. Paris*, 240:1188–1189, 1955. 51

[New42] M. H. A. Newman. On theories with a combinatorial definition of "equivalence.". *Ann. Math.*, 43(2):223–243, 1942. 31

[Nov14] J.-C. Novelli. m-dendriform algebras. `arXiv:1406.1616v1`, 2014. 78

[NT07] J.-C. Novelli and J.-Y. Thibon. Hopf algebras and dendriform structures arising from parking functions. *Fund. Math.*, 193:189–241, 2007. 79

[NT13] J.-C. Novelli and J.-Y. Thibon. Duplicial algebras and Lagrange inversion. `arXiv:1209.5959v2`, 2013. 51

[PR95] S. Poirier and C. Reutenauer. Algèbres de Hopf de tableaux. *Ann. Sci. Math. Québec*, 19:79–90, 1995. 79

[Pri70] S. B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.*, 152:39–60, 1970. 113

[Ree58] R. Ree. Lie elements and an algebra associated with shuffles. *Ann. Math.*, 68(2):210–220, 1958. 78

[Rey07] M. Rey. Algebraic constructions on set partitions. *Formal Power Series and Algebraic Combinatorics*, 2007. 79

[Rot64] G.-C. Rota. On the foundations of combinatorial theory. I. Theory of Möbius functions. *Z. Wahrscheinlichkeit*, 2:340–368, 1964. 77

[Saï14] A. Saïdi. The pre-Lie operad as a deformation of NAP. *J. Algebra Appl.*, 13(1):23, 2014. 114

[Sak09] J. Sakarovitch. *Elements of Automata Theory*. Cambridge University Press, 2009. 136

[Slo] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. URL `https://oeis.org/`. 20, 21, 22, 23, 38, 39, 49, 102, 105, 107, 109, 110, 111

[SS78] A. Salomaa and M. Soittola. *Automata-theoretic aspects of formal power series*. Texts and Monographs in Computer Science. Springer-Verlag, New York-Heidelberg, 1978. Pages x+171. 136

[Sta11] R. P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, second edition, 2011. 77

[Tam62] D. Tamari. The algebra of bracketings and their enumeration. *Nieuw Arch. Wisk.*, 10(3):131–146, 1962. 30

[Val07] B. Vallette. Homology of generalized partition posets. *J. Pure Appl. Algebra*, 208(2):699–725, 2007. 113

[vdL04] P. van der Laan. *Operads. Hopf algebras and coloured Koszul duality*. PhD thesis, Universiteit Utrecht, 2004. 137

[Vin63] È. B. Vinberg. The theory of homogeneous convex cones. *Trudy Moskov. Mat. Obšč.*, 12:303–358, 1963. 78

[Yau16] D. Yau. *Colored Operads*. Graduate Studies in Mathematics. American Mathematical Society, 2016. 113, 137

[YO69] T. Yanagimoto and M. Okamoto. Partial orderings of permutations and monotonicity of a rank correlation statistic. *Ann. I. Stat. Math.*, 21:489–506, 1969. 30

[Zin12] G. W. Zinbiel. Encyclopedia of types of algebras 2010. In *Operads and universal algebra*, volume 9 of *Nankai Ser. Pure Appl. Math. Theoret. Phys.*, pages 217–297. World Sci. Publ., Hackensack, NJ, 2012. 114