

# Counting smaller trees in the Tamari order

Grégory Chatel, Viviane Pons

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- 1 Introduction
  - Basic definitions
  - Tamari lattice
  - Goal
  
- 2 Our work
  - Main result
  - Example
  - Sketch of proof

# Permutation and weak order

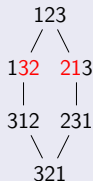
## Permutations

A permutation  $\sigma$  is a word of size  $n$  where every letter of  $\{1, \dots, n\}$  appear only once.

Ex : 1234, 15234, 4231.

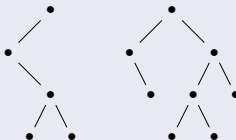
## Weak order : partial order on permutations

At each step, we inverse two increasing consecutive values.

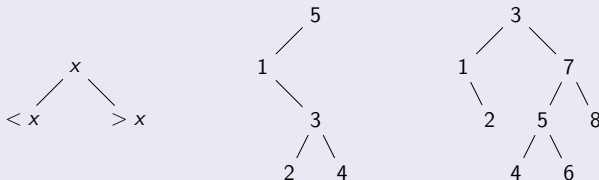


# Binary trees

## Binary trees



## Canonical labelling



# Permutation to binary tree

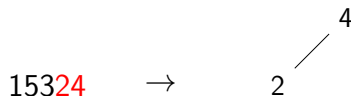
## Binary search tree insertion

15324  $\rightarrow$

4

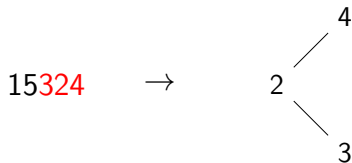
# Permutation to binary tree

Binary search tree insertion



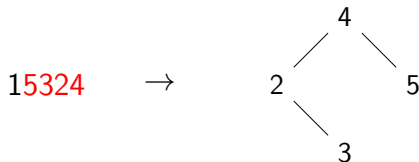
# Permutation to binary tree

Binary search tree insertion



# Permutation to binary tree

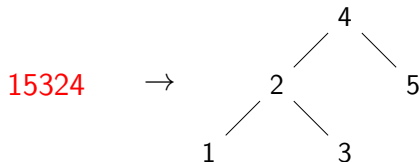
Binary search tree insertion





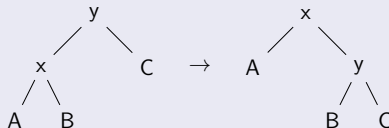
# Permutation to binary tree

Binary search tree insertion



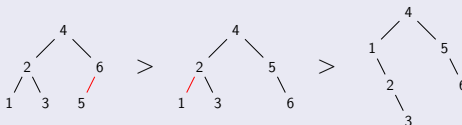
# Order relation

## Right rotation



This gives a partial order on binary trees.

## Example



# Tamari lattice

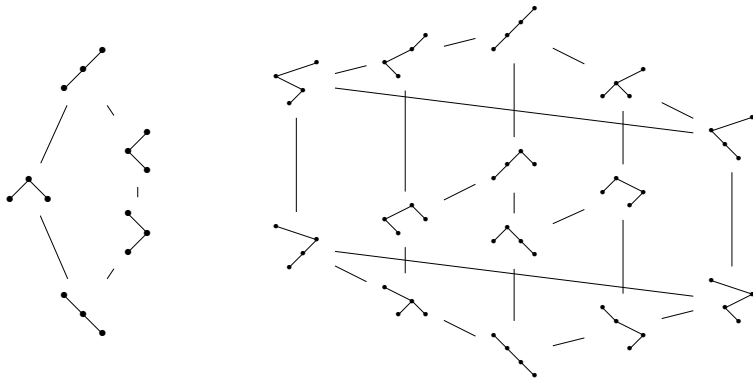
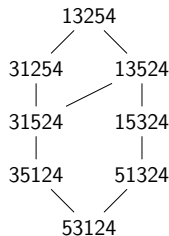
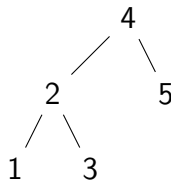
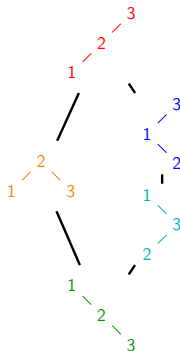
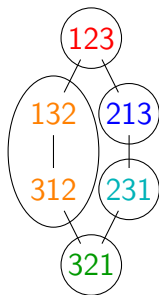


Figure : Tamari lattices of size 3 and 4.

# Tamari lattice as a quotient of the weak order

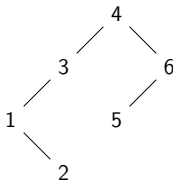


# What do we want to do ?

## Goal

We want a formula that computes, for any given tree  $T$  the number of trees smaller than  $T$  in the Tamari order.

Example : how many trees are smaller than this one ?



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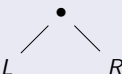
# Tamari polynomials

## Tamari polynomials

Given a binary tree  $T$ , we define:

$$\mathcal{B}_{\emptyset} := 1 \quad (1)$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x) \frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1} \quad (2)$$

with  $T =$  

# Main theorem

## Theorem

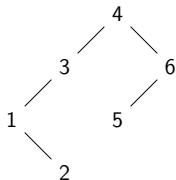
Let  $T$  be a binary tree. Its Tamari polynomial  $\mathcal{B}_T(x)$  counts the trees smaller than  $T$  in the Tamari order according to the number of nodes on their left border. In particular,  $\mathcal{B}_T(1)$  is the number of trees smaller than  $T$ .



# Example

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x) \frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

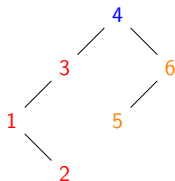


# Example

$$\mathcal{B}_\emptyset := 1$$

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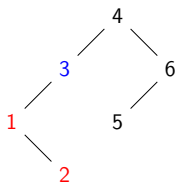
$$\bullet \mathcal{B}_T(x) = \mathcal{B}_4(x) = x\mathcal{B}_3(x) \frac{\mathcal{B}_6(x) - \mathcal{B}_6(1)}{x-1}$$



# Example

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x) \frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

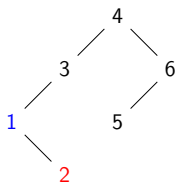


- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x\mathcal{B}_3(x) \frac{\mathcal{B}_6(x) - \mathcal{B}_6(1)}{x - 1}$
- $\mathcal{B}_3(x) = x\mathcal{B}_1(x)$

# Example

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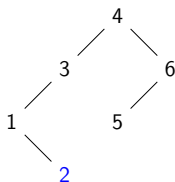


- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x\mathcal{B}_3(x) \frac{\mathcal{B}_6(x) - \mathcal{B}_6(1)}{x-1}$
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# Example

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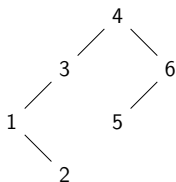


- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x\mathcal{B}_3(x) \frac{\mathcal{B}_6(x) - \mathcal{B}_6(1)}{x-1}$
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- $\mathcal{B}_1(x) = x \frac{\mathcal{B}_2(x) - \mathcal{B}_2(1)}{x-1}$
- $\mathcal{B}_2(x) = x$

# Example

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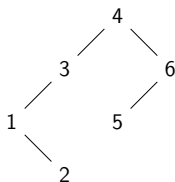


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- $\mathcal{B}_2(x) = x$
- $\mathcal{B}_1(x) = x(1+x) = x + x^2$

# Example

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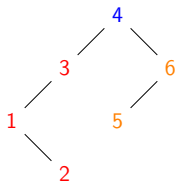


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- $\mathcal{B}_2(x) = x$
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- $\mathcal{B}_3(x) = x(x+x^2) = x^2 + x^3$

# Example

$$\mathcal{B}_{\emptyset} := 1$$

$$\mathcal{B}_T(x) := x \mathcal{B}_L(x) \frac{x \mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$



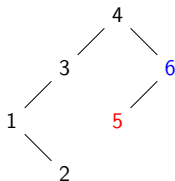
$$\bullet \mathcal{B}_T(x) = \mathcal{B}_4(x) = x(x^2 + x^3) \frac{\mathcal{B}_6(x) - \mathcal{B}_6(1)}{x - 1}$$



# Example

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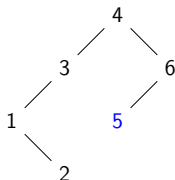


- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x(x^2 + x^3) \frac{\mathcal{B}_6(x) - \mathcal{B}_6(1)}{x-1}$
- $\mathcal{B}_6(x) = x\mathcal{B}_5(x)$

# Example

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$$\mathcal{B}_T(x) := x\mathcal{B}_L(x) \frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

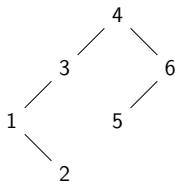


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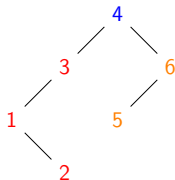


- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x(x^2 + x^3) \frac{\mathcal{B}_6(x) - \mathcal{B}_6(1)}{x-1}$
- $\mathcal{B}_6(x) = x\mathcal{B}_5(x)$
- $\mathcal{B}_5(x) = x$
- $\mathcal{B}_6(x) = xx = x^2$

# Example

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x) \frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

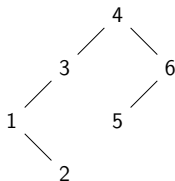


$$\bullet \mathcal{B}_4(x) = x(x^2 + x^3)(1 + x + x^2)$$

# Example

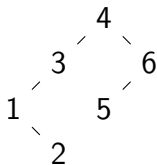
$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x) \frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$



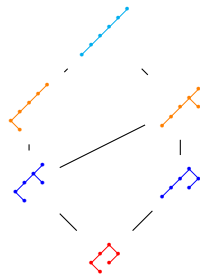
- $\mathcal{B}_4(x) = x(x^2 + x^3)(1 + x + x^2)$
- $\mathcal{B}_4(x) = x^6 + 2x^5 + 2x^4 + x^3$

# Example



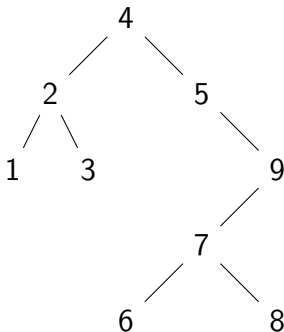
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$

$$\mathcal{B}_T(1) = 6$$



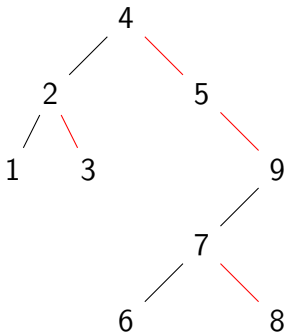
# Sketch of proof

Increasing and decreasing forests



# Sketch of proof

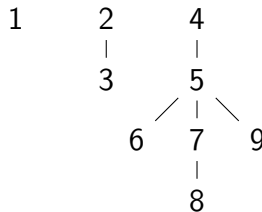
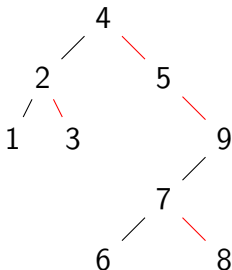
Increasing and decreasing forests





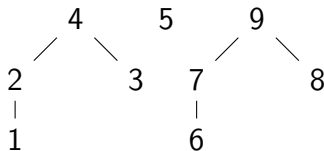
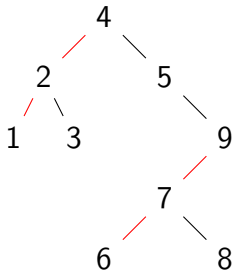
# Sketch of proof

## Increasing and decreasing forests

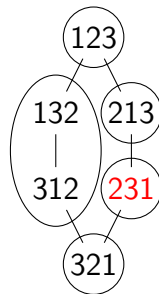
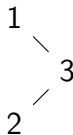
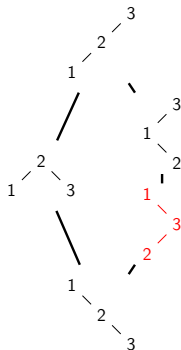


# Sketch of proof

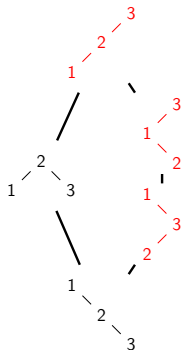
Increasing and decreasing forests



# Initial and final intervals



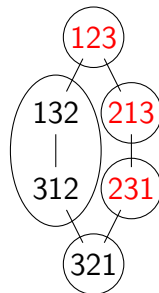
# Initial and final intervals



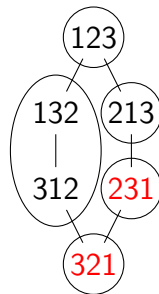
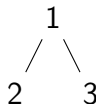
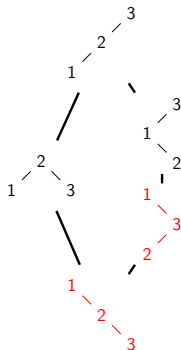
1

3

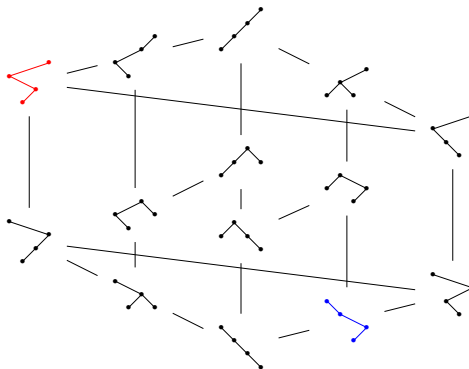
2



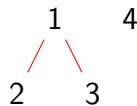
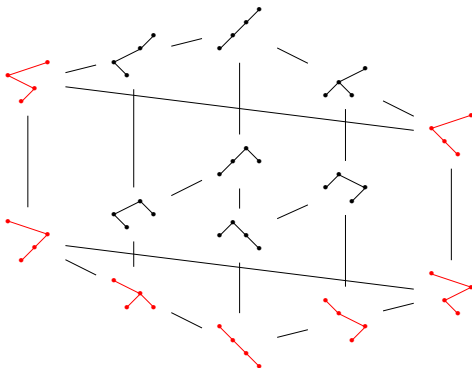
# Initial and final intervals



# Linear extensions of any interval



# Linear extensions of any interval



1

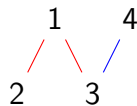
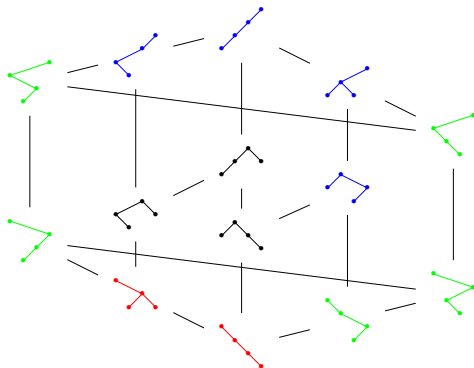
2

4

3



# Linear extensions of any interval



# Sketch of proof

## Bilinear form

$$\mathcal{B}_T(x) = x\mathcal{B}_L(x) \frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

$$\mathcal{B}(f, g) = xf(x) \frac{xg(x) - g(1)}{x - 1}$$

# Sketch of proof

## Combinatorial interpretation

$$\sum_{T' < T} [T', T] = \mathbb{B} \left( \sum_{T1' < T1} [T1', T1], \sum_{T2' < T2} [T2', T2] \right)$$

with  $T =$  

$$\mathbb{B}\left( \begin{array}{c} 1 \\ | \\ 3 \end{array}, \begin{array}{c} 2 \\ / \quad \backslash \\ 1 \quad 3 \end{array} \right) = \begin{array}{c} 1 \\ | \\ 3 \end{array} \begin{array}{c} 2 \\ / \quad \backslash \\ 1 \quad 3 \end{array}$$

$$\mathbb{B}\left( \begin{array}{c} 1 \\ | \\ 3 \end{array}, \begin{array}{c} 2 \\ / \quad \backslash \\ 1 \quad 3 \end{array} \right) = \begin{array}{c} 4 \\ | \\ 3 \end{array} \begin{array}{c} 2 \\ / \quad \backslash \\ 5 \quad 7 \end{array}$$

$$\mathbb{B}\left( \begin{array}{c} 1 \\ | \\ 3 \end{array}, \begin{array}{c} 2 \\ / \quad \backslash \\ 1 \quad 3 \end{array} \right) = \begin{array}{c} 4 \\ / \quad | \\ 1 \quad 2 \\ | \\ 3 \end{array} \begin{array}{c} 6 \\ / \quad \backslash \\ 5 \quad 7 \end{array}$$

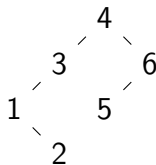
$$\mathbb{B}\left( \begin{array}{c} 1 \\ | \\ 3 \end{array}, \begin{array}{c} 2 \\ / \quad \backslash \\ 1 \quad 3 \end{array} \right) = \begin{array}{c} 4 \\ / \quad | \\ 1 \quad 2 \\ | \\ 3 \end{array} + \begin{array}{c} 6 \\ / \quad \backslash \\ 5 \quad 7 \end{array} + \begin{array}{c} 4 \\ / \quad | \quad \backslash \\ 1 \quad 2 \quad 6 \\ | \quad \quad / \quad \backslash \\ 3 \quad 5 \quad 7 \end{array}$$

The diagram shows the decomposition of a binary tree into three components. The first component is a tree with root 1 and child 3. The second component is a tree with root 2, left child 1, and right child 3. The third component is a tree with root 4, left child 1, and right child 2, which in turn has a left child 3. The fourth component is a tree with root 6, left child 5, and right child 7. The fifth component is a tree with root 4, left child 1, and right child 2, which in turn has a left child 3. The right child 2 of the fifth tree has a right child 6, which in turn has a left child 5 and a right child 7. The edge connecting 2 and 6 in the fifth tree is highlighted in red.

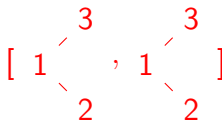
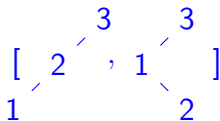
$$\mathbb{B}\left( \begin{array}{c} 1 \\ | \\ 3 \end{array}, \begin{array}{c} 2 \\ / \backslash \\ 1 \quad 3 \end{array} \right) = \begin{array}{c} 4 \\ / \quad | \\ 1 \quad 2 \\ | \\ 3 \end{array} \begin{array}{c} 6 \\ / \backslash \\ 5 \quad 7 \end{array} + \begin{array}{c} 4 \\ / \quad | \quad \backslash \\ 1 \quad 2 \quad 6 \\ | \quad \backslash \\ 3 \quad 5 \quad 7 \end{array}$$

$$+ \begin{array}{c} 4 \\ / \quad | \quad \backslash \\ 1 \quad 2 \quad 6 \\ | \quad / \backslash \\ 3 \quad 5 \quad 7 \end{array}$$





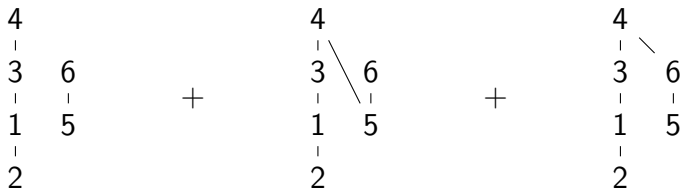
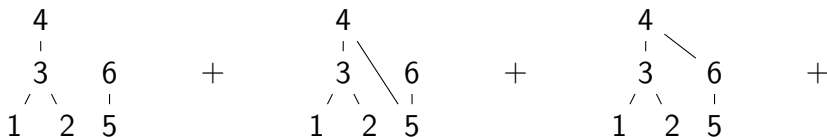
$$\mathbb{B} \left( \begin{array}{c} 3 \\ 1 \quad 2 \end{array} + \begin{array}{c} 3 \\ 1 \quad 2 \end{array}, \begin{array}{c} 2 \\ | \\ 1 \end{array} \right)$$



$$\mathbb{B}\left( \begin{array}{c} 3 \\ / \quad \backslash \\ 1 \quad 2 \end{array} + \begin{array}{c} 3 \\ / \quad \backslash \\ 1 \quad 2 \end{array}, \begin{array}{c} 2 \\ | \\ 1 \end{array} \right) =$$

$$\begin{array}{c} 4 \\ | \\ 3 \quad 6 \\ / \quad \backslash \quad | \\ 1 \quad 2 \quad 5 \end{array} + \begin{array}{c} 4 \\ | \\ 3 \quad 6 \\ / \quad \backslash \quad | \\ 1 \quad 2 \quad 5 \end{array} + \begin{array}{c} 4 \\ | \\ 3 \quad 6 \\ / \quad \backslash \quad | \\ 1 \quad 2 \quad 5 \end{array} +$$

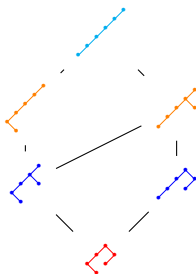
$$\begin{array}{c} 4 \\ | \\ 3 \quad 6 \\ | \quad | \\ 1 \quad 5 \\ | \\ 2 \end{array} + \begin{array}{c} 4 \\ | \\ 3 \quad 6 \\ | \quad | \\ 1 \quad 5 \\ | \\ 2 \end{array} + \begin{array}{c} 4 \\ | \\ 3 \quad 6 \\ | \quad | \\ 1 \quad 5 \\ | \\ 2 \end{array}$$



$$\begin{aligned}
 & \left[ \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagup \\ \bullet \\ \diagup \\ \bullet \\ \diagup \\ \bullet \end{array}, \begin{array}{c} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{array} \right] + \left[ \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array}, \begin{array}{c} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{array} \right] + \left[ \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array}, \begin{array}{c} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{array} \right] + \\
 & \qquad \qquad \qquad x^6 \qquad \qquad \qquad + \qquad \qquad \qquad x^5 \qquad \qquad \qquad + \qquad \qquad \qquad x^4 \qquad \qquad \qquad +
 \end{aligned}$$

$$\begin{aligned}
 & \left[ \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagup \\ \bullet \\ \diagup \\ \bullet \end{array}, \begin{array}{c} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{array} \right] + \left[ \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array}, \begin{array}{c} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{array} \right] + \left[ \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array}, \begin{array}{c} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{array} \right] \\
 & \qquad \qquad \qquad x^5 \qquad \qquad \qquad + \qquad \qquad \qquad x^4 \qquad \qquad \qquad + \qquad \qquad \qquad x^3
 \end{aligned}$$

# Example



# Questions ?