Counting smaller trees in the Tamari order

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1. Introduction
   - Basic definitions
   - Tamari lattice
   - Goal

2. Our work
   - Main result
   - Example
   - Sketch of proof

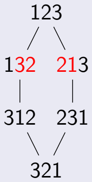
Permutation and weak order

Permutations

A permutation σ is a word of size n where every letter of $\{1, \dots, n\}$ appear only once.
Ex : 1234, 15234, 4231.

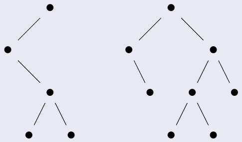
Weak order : partial order on permutations

At each step, we inverse two increasing consecutives values.

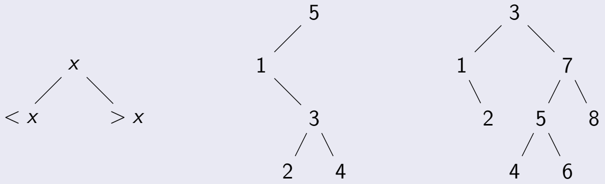


Binary trees

Binary trees



Canonical labelling



Permutation to binary tree

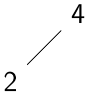
Binary search tree insertion

1532**4** $\longrightarrow$ 4

Permutation to binary tree

Binary search tree insertion

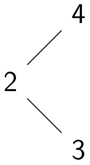
15324 →



Permutation to binary tree

Binary search tree insertion

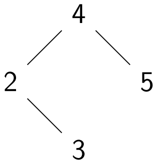
15324 →



Permutation to binary tree

Binary search tree insertion

1**5324** $\rightarrow$



Permutation to binary tree
Binary search tree insertion

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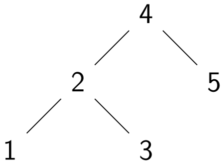
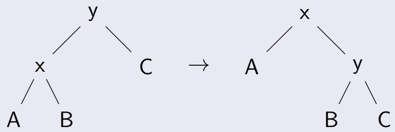


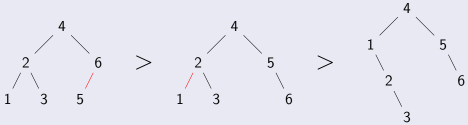
Order relation

Right rotation



This gives a partial order on binary trees.

Example



Tamari lattice

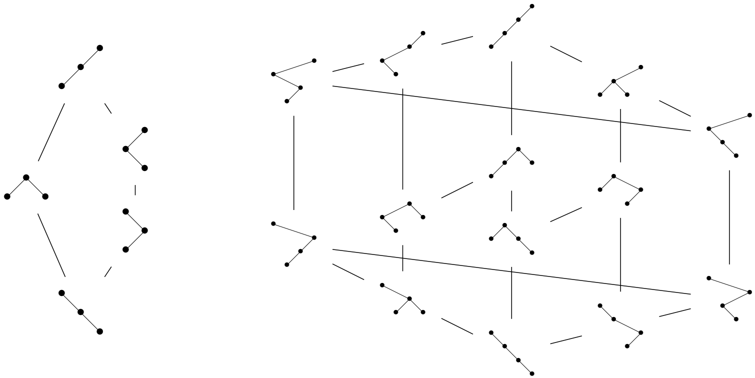
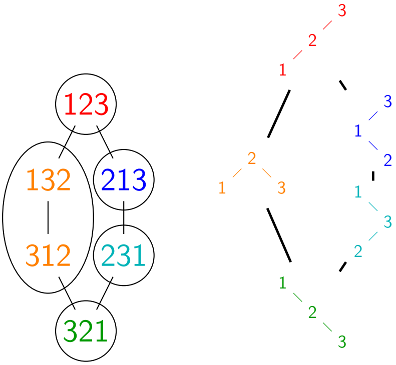
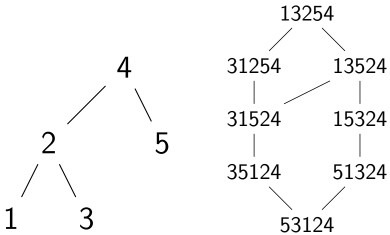


Figure : Tamari lattices of size 3 and 4.

Tamari lattice as a quotient of the weak order



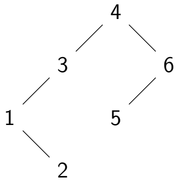


What do we want to do ?

Goal

We want a formula that computes, for any given tree T the number of trees smaller than T in the Tamari order.

Example : how many trees are smaller than this one ?



Tamari polynomials

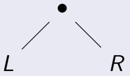
Tamari polynomials

Given a binary tree T, we define:

$$\mathcal{B}_\emptyset := 1 \tag{1}$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1} \tag{2}$$

with T =



Main theorem

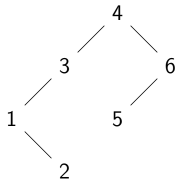
Theorem

Let T be a binary tree. Its Tamari polynomial $\mathcal{B}_T(x)$ counts the trees smaller than T in the Tamari order according to the number of nodes on their left border. In particular, $\mathcal{B}_T(1)$ is the number of trees smaller than T.

Example

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$



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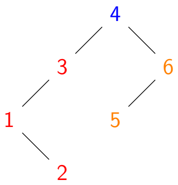

- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = xB_3(x)\frac{B_6(x)-B_6(1)}{x-1}$

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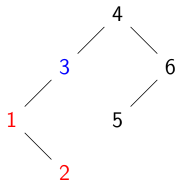

- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = xB_3(x)\frac{B_6(x)-B_6(1)}{x-1}$
- $\mathcal{B}_3(x) = xB_1(x)$

Example

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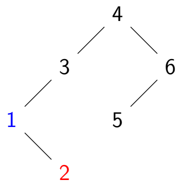

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Example

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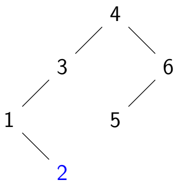

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- $\mathcal{B}_2(x) = x$

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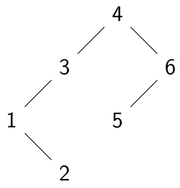

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- $\mathcal{B}_2(x) = x$
- $\mathcal{B}_1(x) = x(1+x) = x + x^2$

Example

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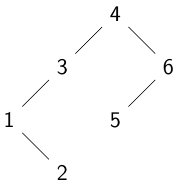

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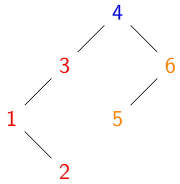

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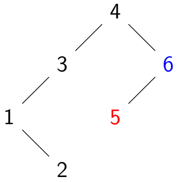

- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x(x^2 + x^3)\frac{B_6(x)-B_6(1)}{x-1}$
- $\mathcal{B}_6(x) = x\mathcal{B}_5(x)$

Example

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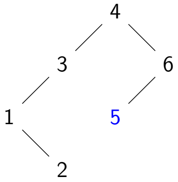

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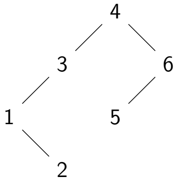

- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x(x^2 + x^3)\frac{\mathcal{B}_6(x) - \mathcal{B}_6(1)}{x-1}$
- $\mathcal{B}_6(x) = x\mathcal{B}_5(x)$
- $\mathcal{B}_5(x) = x$
- $\mathcal{B}_6(x) = xx = x^2$

Example

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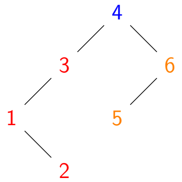
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$



- $\mathcal{B}_4(x) = x(x^2 + x^3)(1 + x + x^2)$

Example

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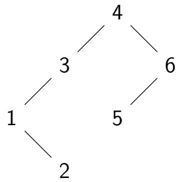
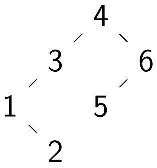


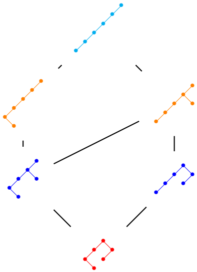
- $\mathcal{B}_4(x) = x(x^2 + x^3)(1 + x + x^2)$
- $\mathcal{B}_4(x) = x^6 + 2x^5 + 2x^4 + x^3$

Example

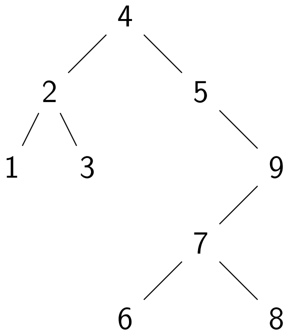


$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$

$$\mathcal{B}_T(1) = 6$$



Sketch of proof
Increasing and decreasing forests



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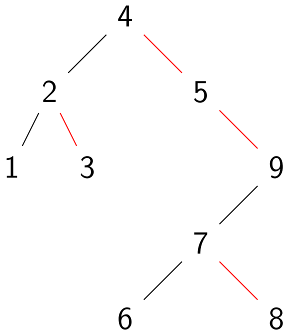

Sketch of proof

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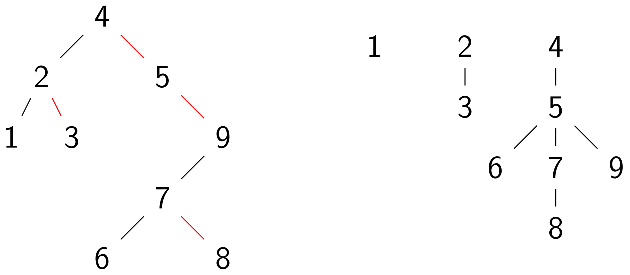
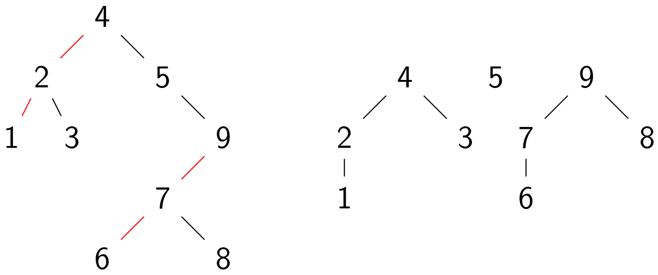
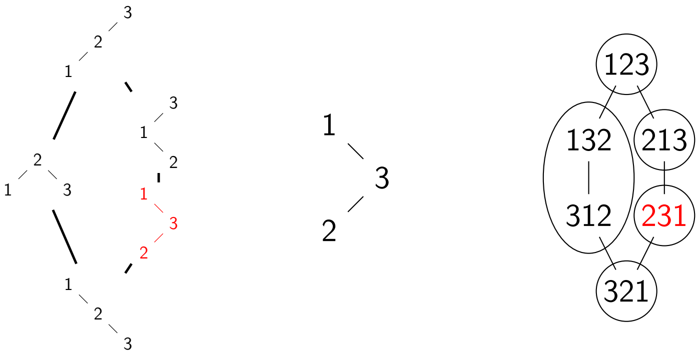


Sketch of proof

Increasing and decreasing forests



Initial and final intervals



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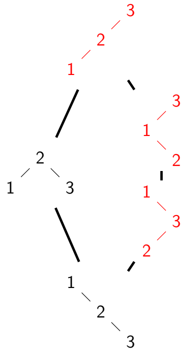

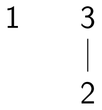

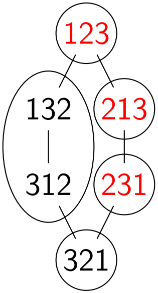

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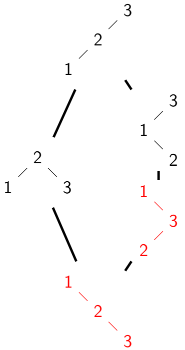
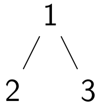
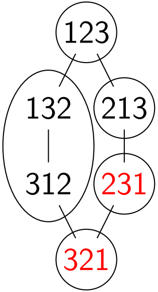
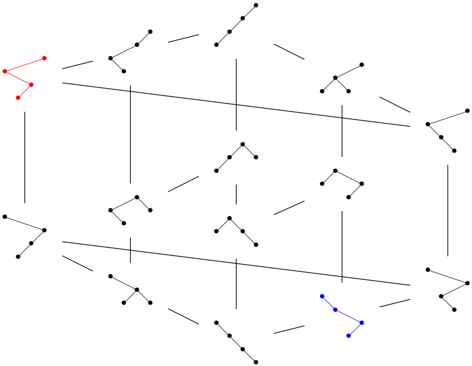






Linear extensions of any interval



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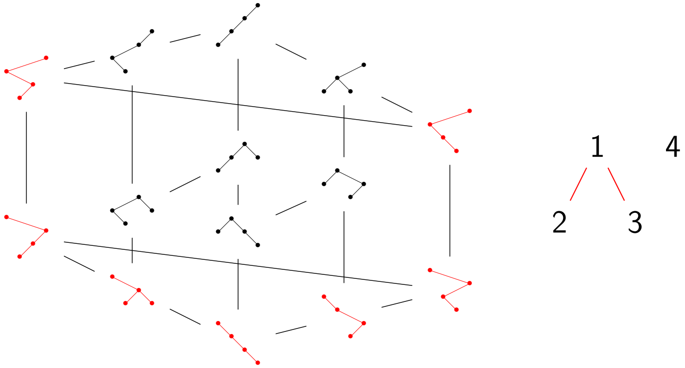

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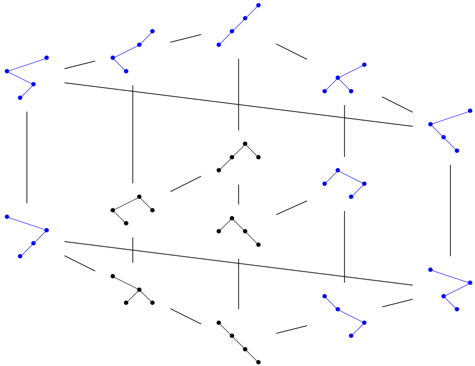

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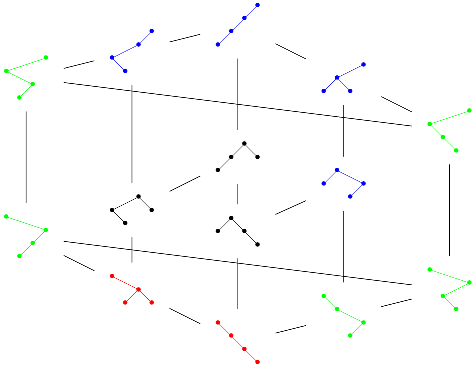
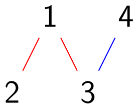




Sketch of proof
Bilinear form

$$\mathcal{B}_T(x) = x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

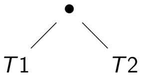
$$\mathcal{B}(f,g) = xf(x)\frac{xg(x) - g(1)}{x-1}$$

Sketch of proof

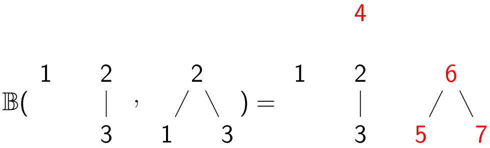
Combinatorial interpretation

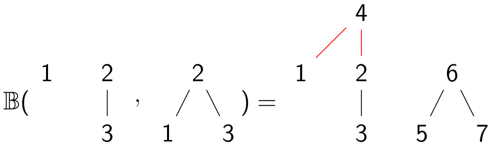
$$\sum_{T' < T} [T', T] = \mathbb{B}\Big(\sum_{T1' < T1} [T1', T1], \sum_{T2' < T2} [T2', T2] \Big)$$

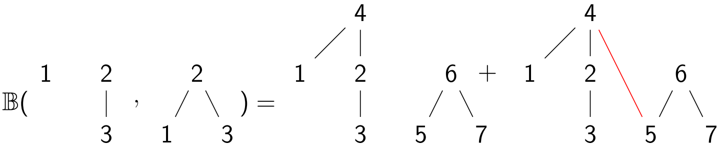
with T =

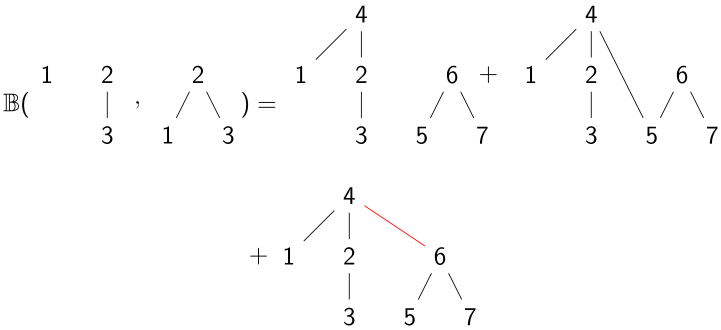


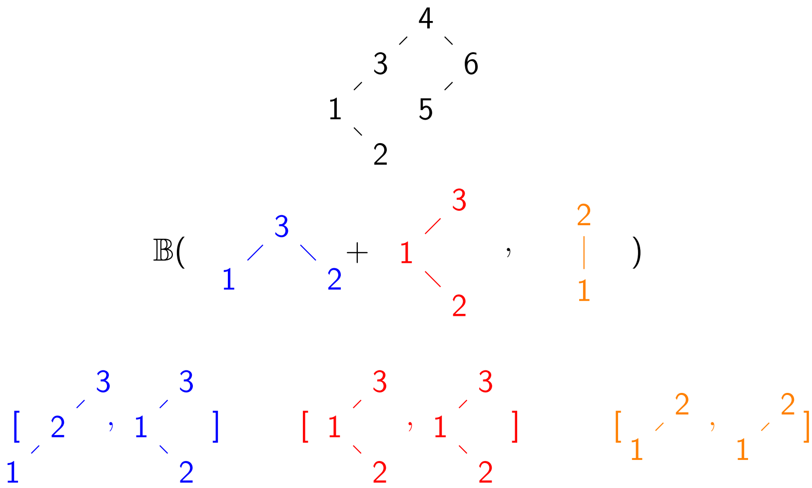
$$\mathbb{B}\left(\begin{array}{ccc} 1 & & 2 \\ & & | \\ & & 3 \end{array} \; , \; \begin{array}{ccc} & 2 & \\ / & & \backslash \\ 1 & & 3 \end{array} \right) = \begin{array}{ccc} 1 & & 2 \\ & & | \\ & & 3 \end{array} \quad \begin{array}{ccc} & 2 & \\ / & & \backslash \\ 1 & & 3 \end{array}$$

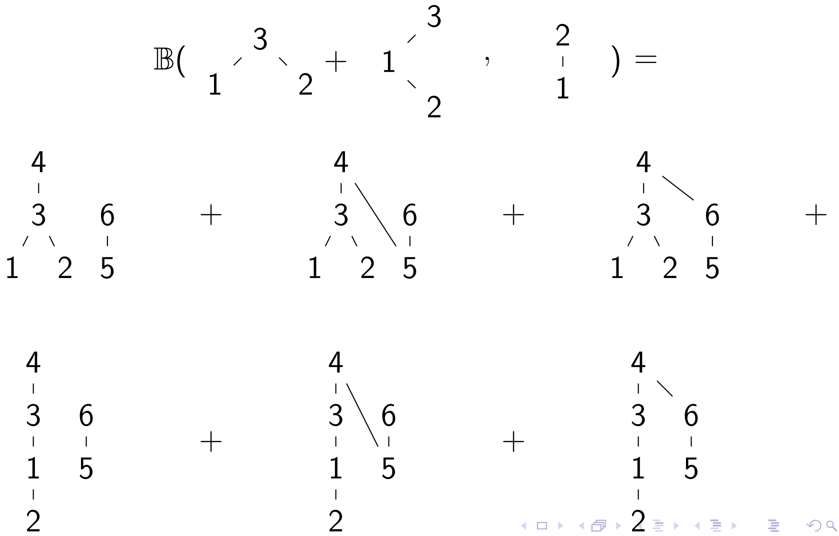

𝔹(
1 2 3
,
2 1 3
) =
4
1 2 3
6 5 7

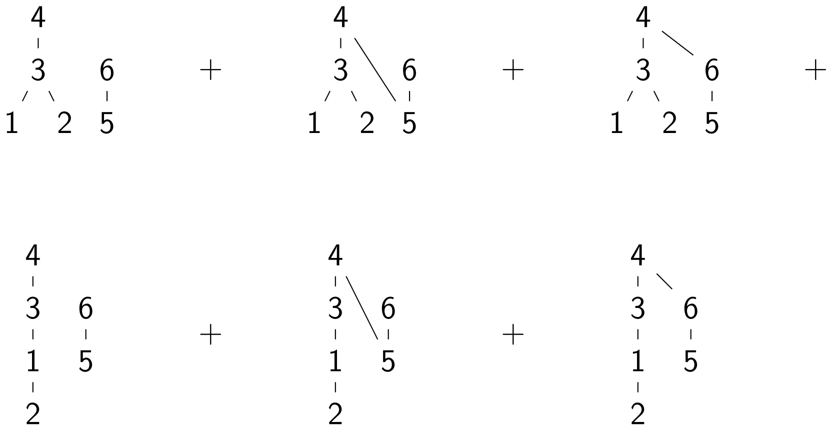

𝔹(
1
2
3
,
2
1
3
) =
4
1
2
3
6
5
7

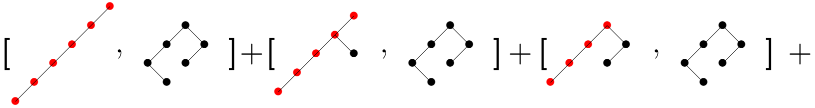

𝔹(
1 2 3 , 2 1 3
) =
4 1 2 3 6 5 7
+
4 1 2 3 6 5 7



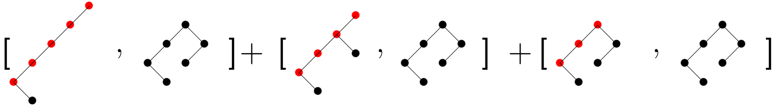






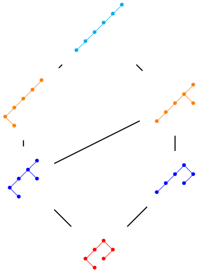


$x^6 + x^5 + x^4 +$



$x^5 + x^4 + x^3$

Example



Questions ?