Binary relations lattice

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joint work with Joel Gay, Vincent Pilaud et Viviane Pons

Université Paris-Est Marne-la-Vallée

SLC 76 05/04/2016

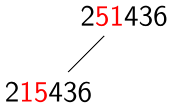
Transpositions

251436

Transpositions

251436

Transpositions



Transpositions

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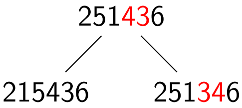
Transpositions



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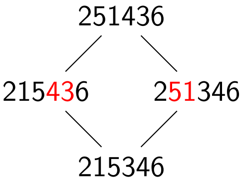
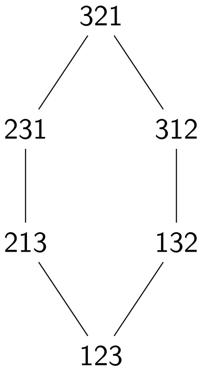


Weak order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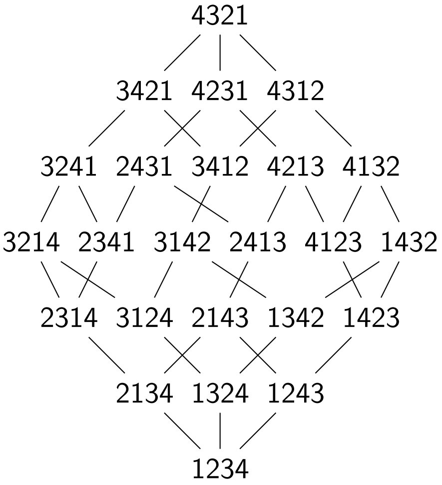
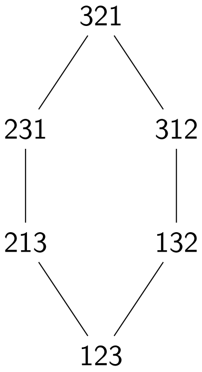
231
312
213
132
123


4321
3421 4231 4312
3241 2431 3412 4213 4132
3214 2341 3142 2413 4123 1432
2314 3124 2143 1342 1423
2134 1324 1243
1234

Weak order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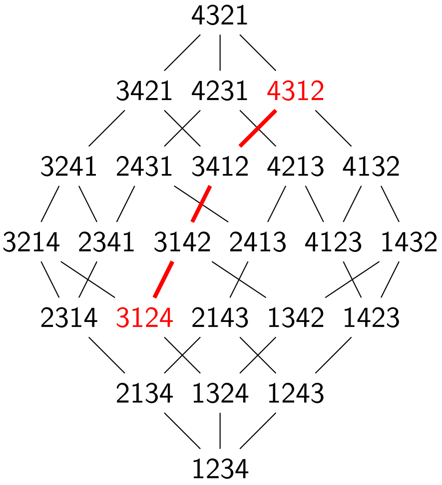
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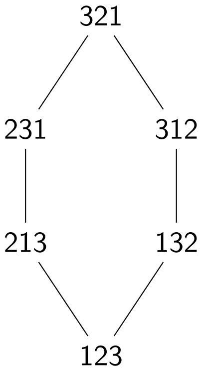
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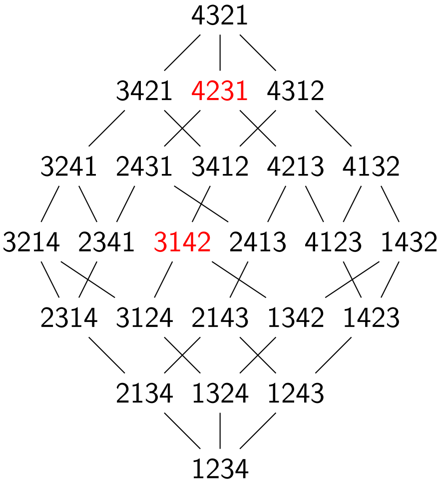
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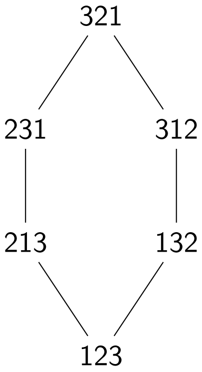
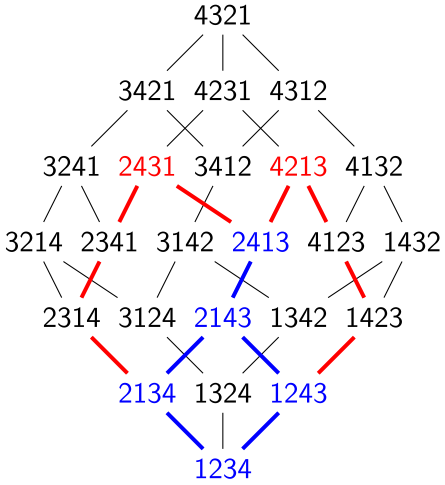
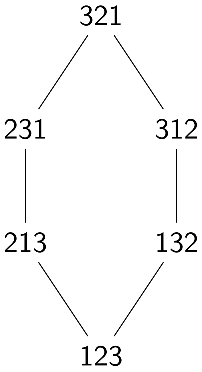
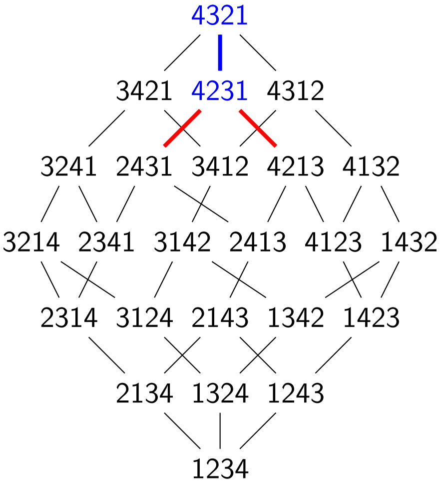




$$2413 \wedge 4213 = 2413$$

Weak order



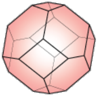


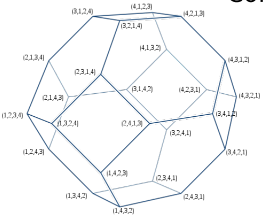
$2413 \wedge 4213 = 2413$

$2413 \vee 4213 = 4231$

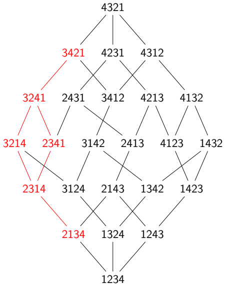
Triple interpretation

Geometry





Combinatorial interpretation



Algebra

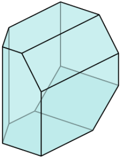
$$\mathbb{F}_{21} \,.\, \mathbb{F}_{12} = \mathbb{F}_{21 \sqcup\!\sqcup 12}$$

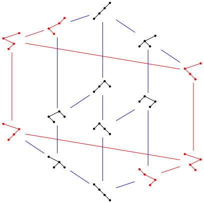
$$= \mathbb{F}_{2134} + \mathbb{F}_{2314} + \mathbb{F}_{2341} + \mathbb{F}_{3214} + \mathbb{F}_{3241} + \mathbb{F}_{3421}$$

Triple interpretation

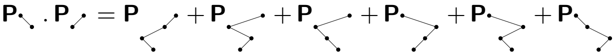
Combinatorial interpretation

Geometry





Algebra



Permutation graph

4312

Permutation graph

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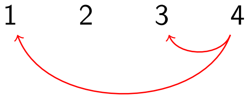

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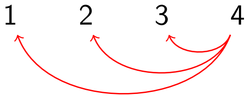

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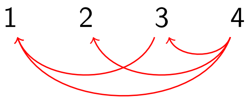

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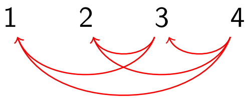
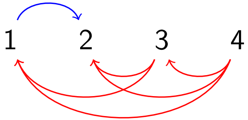


Permutation graph

4312



Permutation graph

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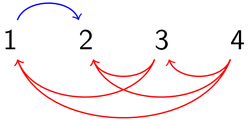

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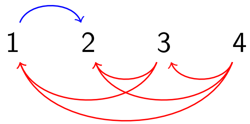
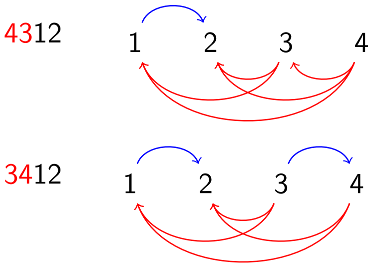


3412

Permutation graph



Permutation graph

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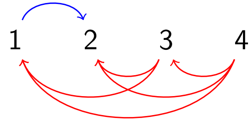

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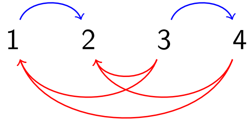

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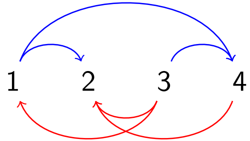

Binary relations on integer

Let R be a relation of size n.

$$1 \quad 2 \quad \cdots \quad i \quad \cdots \quad j \quad \cdots \quad k \quad \cdots \quad n$$

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$i \,\mathrm{R}\, j$

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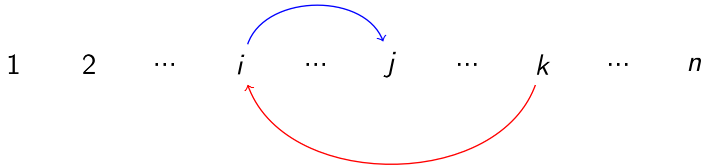

$i \mathrel{\mathrm{R}} j$ $\qquad$ $k \mathrel{\mathrm{R}} i$

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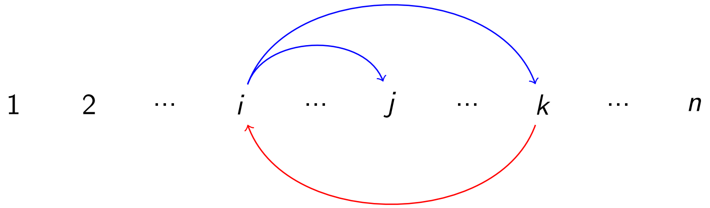

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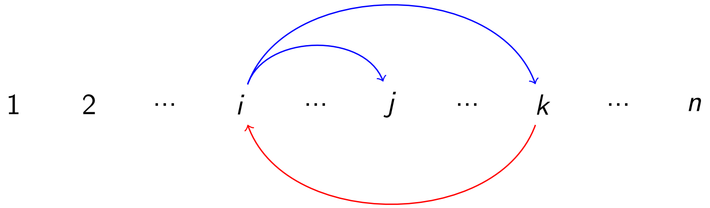


$i \mathrel{\mathrm{R}} j$ $\quad$ $k \mathrel{\mathrm{R}} i$ $\quad$ $i \mathrel{\mathrm{R}} k$

There is $2^{n(n-1)}$ binary relations.

Partial order on relations

Let R be a binary relation

$$\mathrm{R}^{\mathsf{Inc}} = \{i \, \mathrm{R} \, j, i < j\}$$
$$\mathrm{R}^{\mathsf{Dec}} = \{j \, \mathrm{R} \, i, i < j\}$$

Partial order on relations

Let R be a binary relation

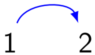
$$\mathrm{R}^{\mathsf{Inc}} = \{i \,\mathrm{R}\, j, i < j\}$$
$$\mathrm{R}^{\mathsf{Dec}} = \{j \,\mathrm{R}\, i, i < j\}$$

Let R and S be two binary relations,

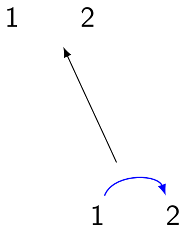
$$\mathrm{R} \preccurlyeq \mathrm{S} \Leftrightarrow \qquad \mathrm{R}^{\mathsf{Inc}} \supseteq \mathrm{S}^{\mathsf{Inc}} \text{ and } \mathrm{R}^{\mathsf{Dec}} \subseteq \mathrm{S}^{\mathsf{Dec}}$$

Relations of size 2

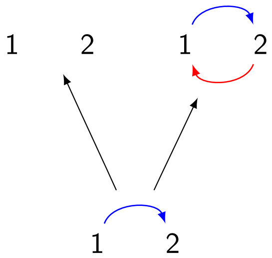
Relations of size 2



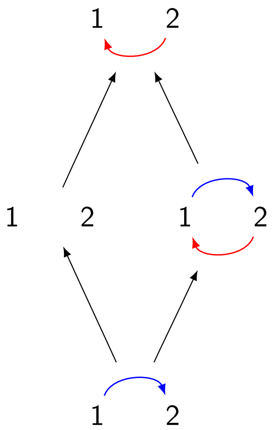
Relations of size 2

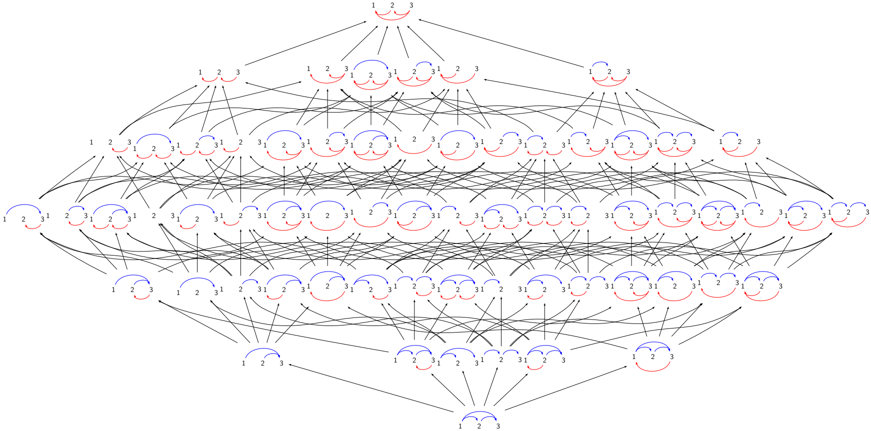


Relations of size 2

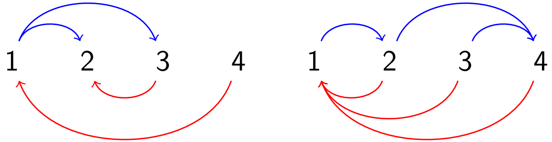


Relations of size 2





Meet and Join



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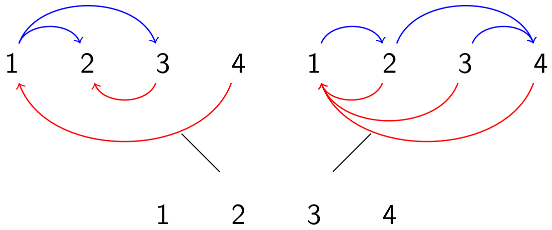

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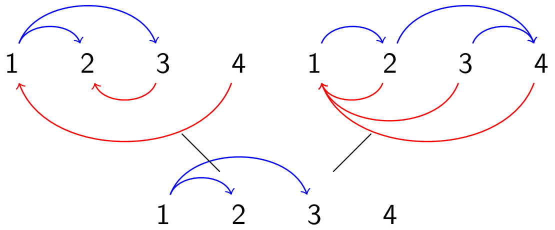

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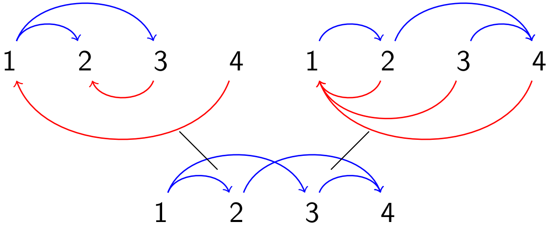

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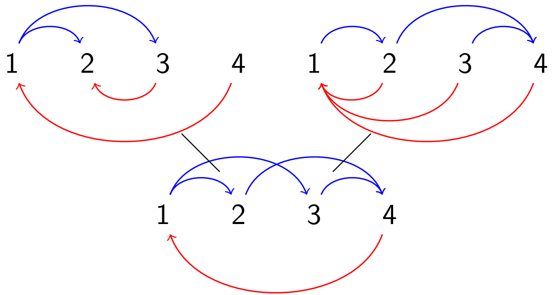

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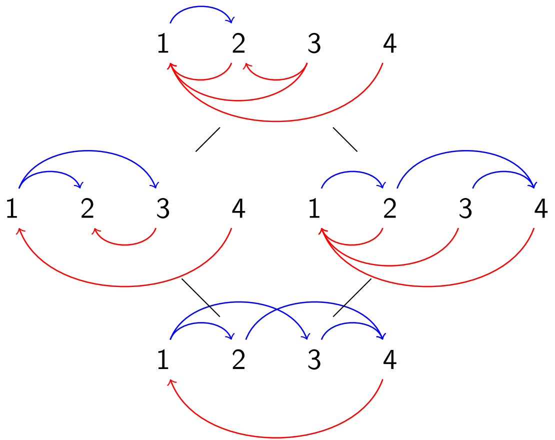
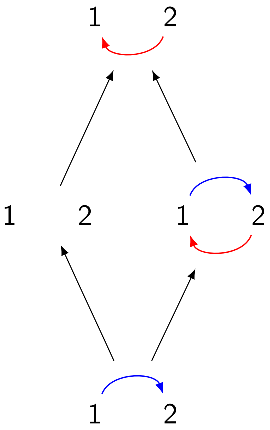


We want to keep the relations that are:

- antisymmetric
- transitives

(posets)

Antisymmetry



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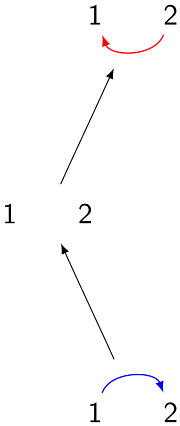
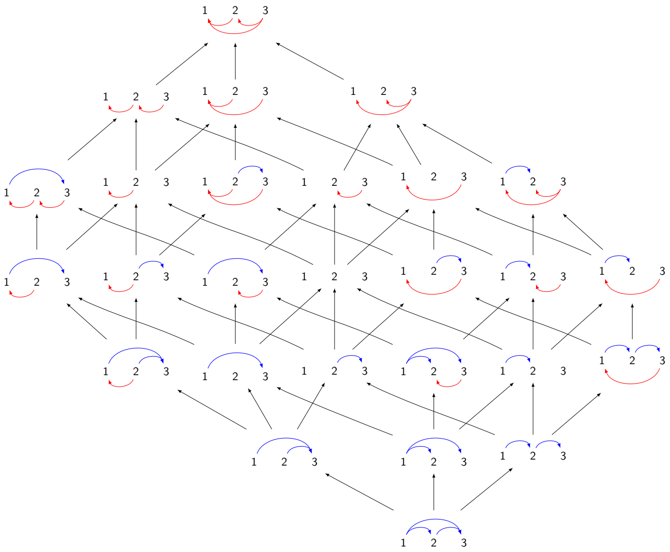
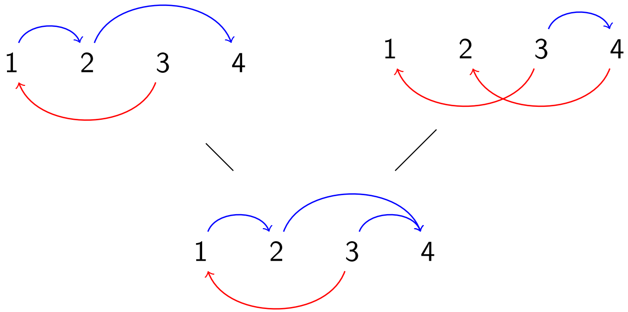




Sublattice?

If R and S are antisymmetric, is $\mathrm{R} \wedge \mathrm{S}$ also antisymmetric?



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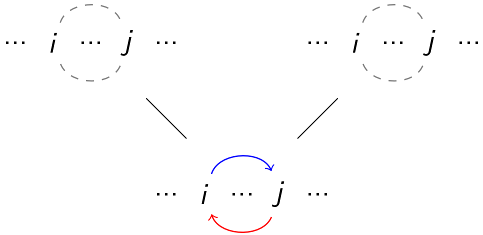

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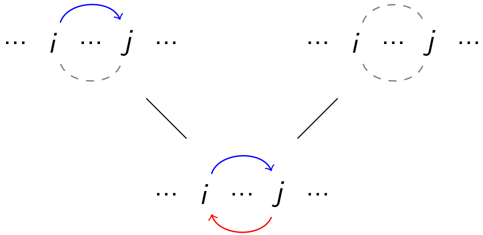

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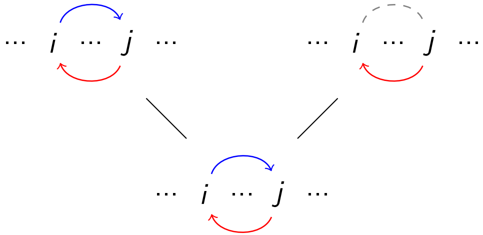
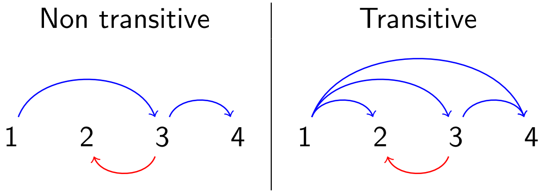
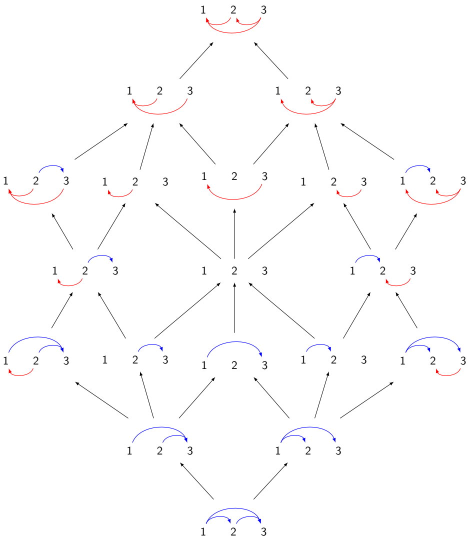


Transitivity





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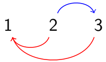

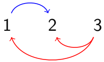

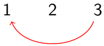

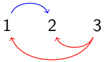

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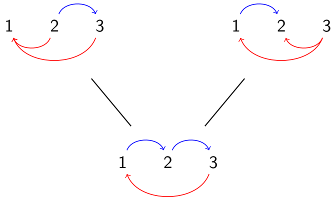

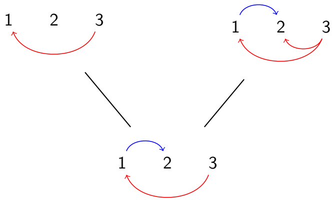

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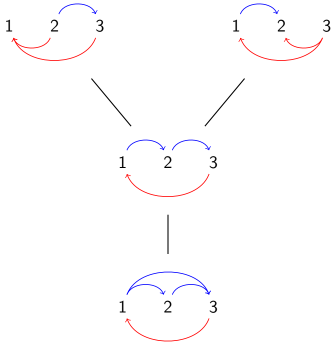

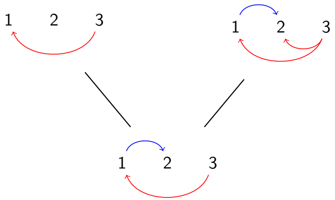

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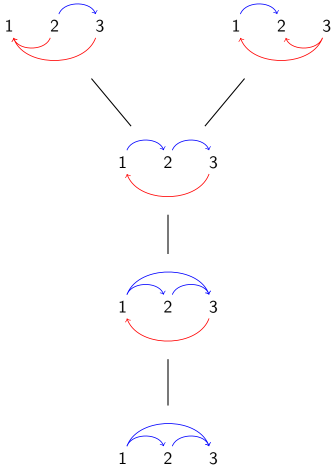
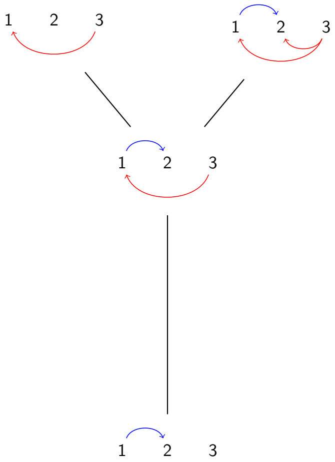
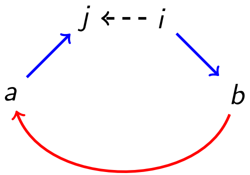




Transitive decreasing deletion



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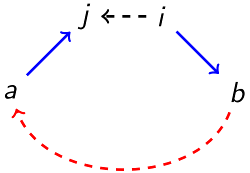

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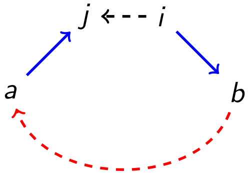

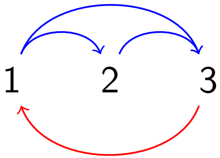

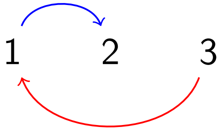

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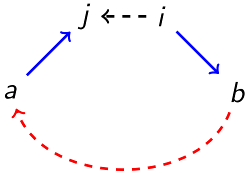
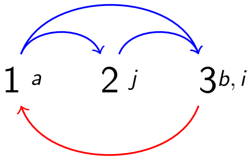
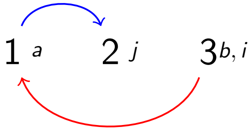
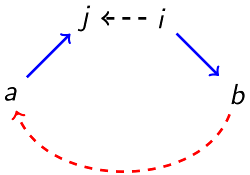
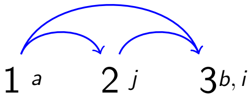


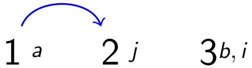




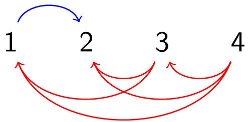
Transitive decreasing deletion







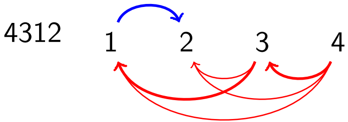
Back to the permutations

4312 

We have $i \mathrm{R} j$ iff the number i is on the left of j in the permutation. The relation is

- antisymmetric
- transitive
- **and total**

Back to the permutations



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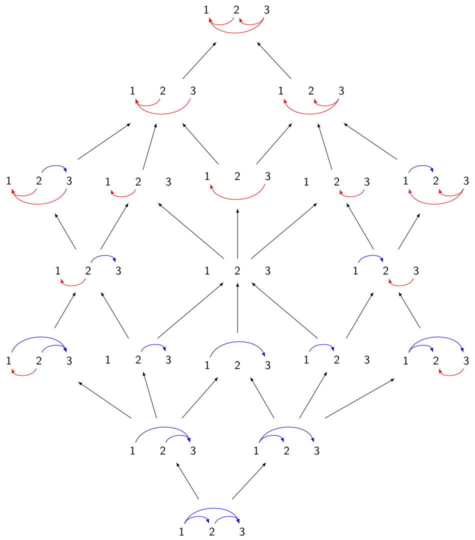
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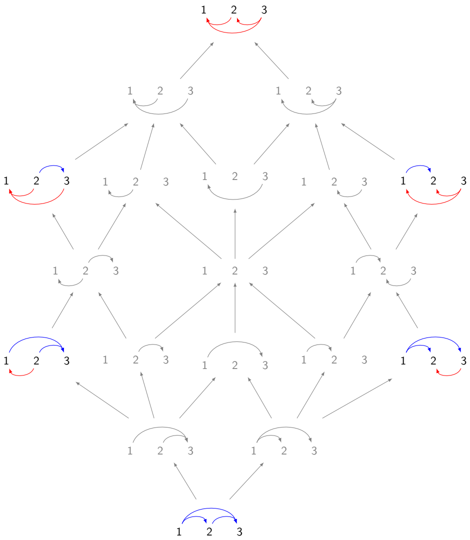

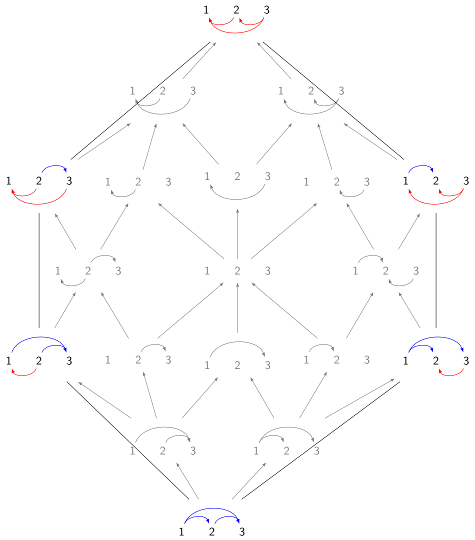

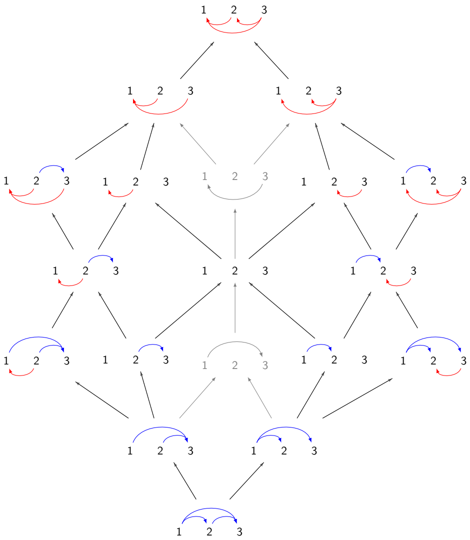
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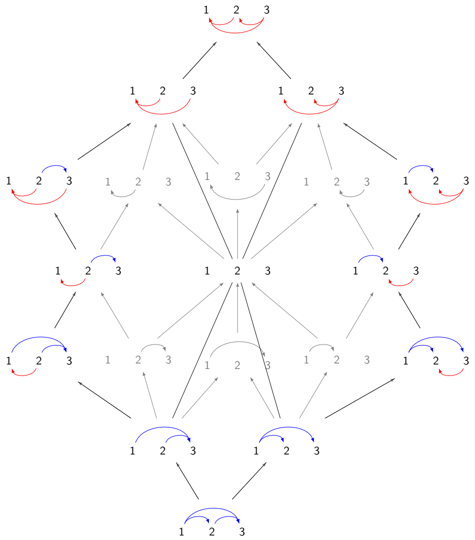

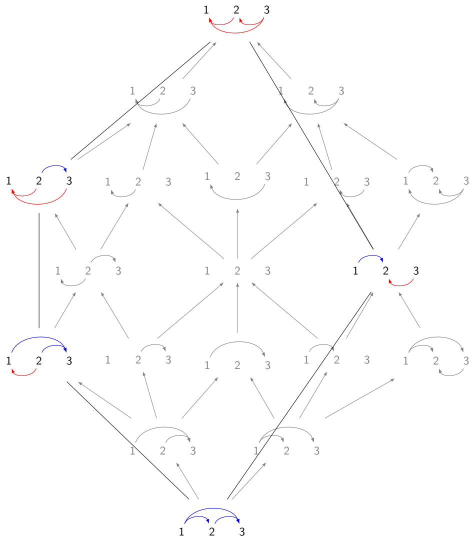

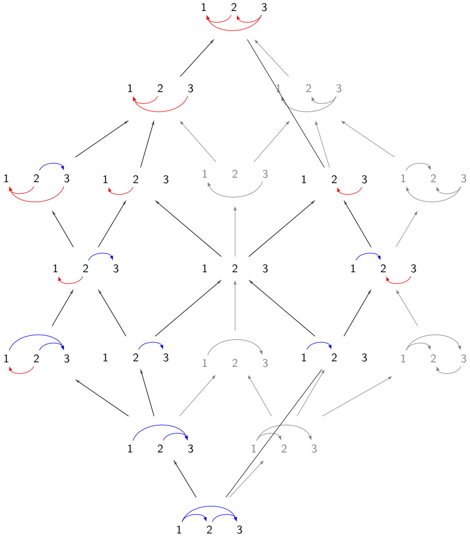

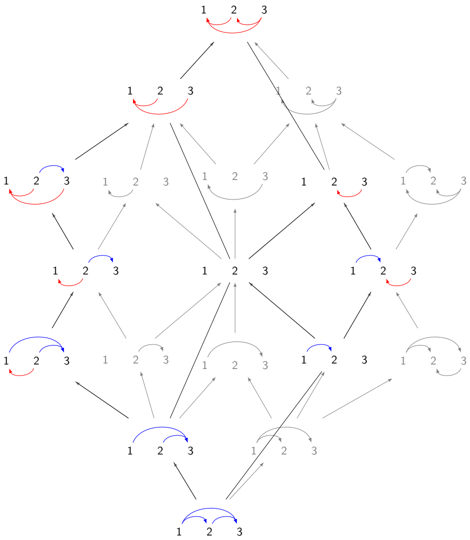

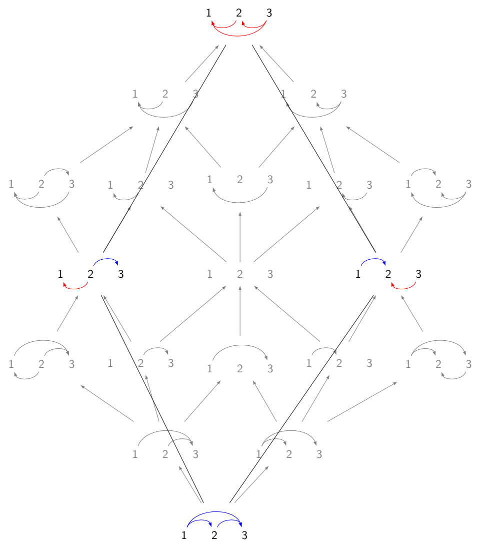

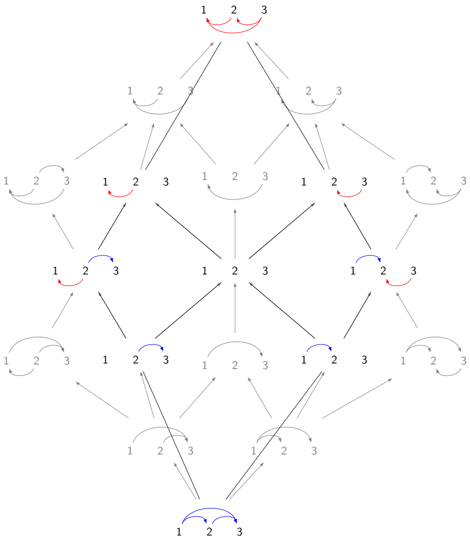
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Results

- A set of rules on posets that automatically yields sublattices.

Work in progress

- Study associated Hopf algebras.
- Study associated polytopes.
- Study the same problems in the Coxeter group framework.