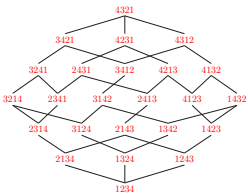
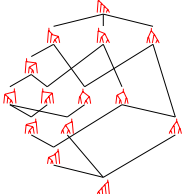
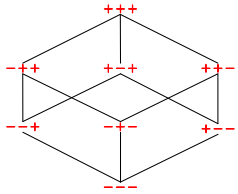
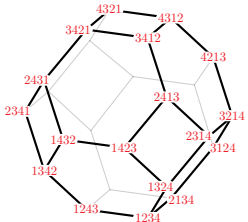
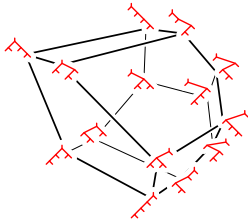
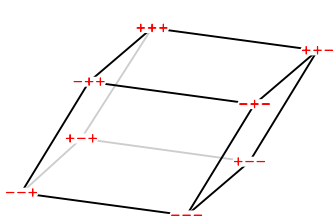


The Cambrian Hopf Algebra

G. Châtel

September 21st, 2015

Joint work with V. Pilaud

	permutations	binary trees	binary sequences
Combinatorics			
Algebra	Malvenuto-Reutenauer algebra $\text{FQSym} = \text{vect} \langle \mathbb{F}_\tau \mid \tau \in \mathfrak{S} \rangle$	Loday-Ronco algebra $\text{PBT} = \text{vect} \langle \mathbb{P}_T \mid T \in \mathcal{BT} \rangle$	Solomon algebra $\text{Rec} = \text{vect} \langle \mathbb{X}_\eta \mid \eta \in \pm^* \rangle$
Geometry			

1 Combinatorics

- Binary trees
- Cambrian trees
- Cambrian lattices

2 Algebra

- FQSym
- The Cambrian algebra

1 Combinatorics

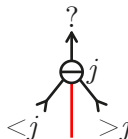
- Binary trees
- Cambrian trees
- Cambrian lattices

2 Algebra

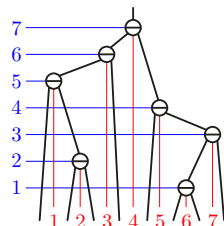
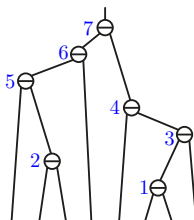
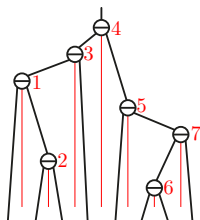
- FQSym
- The Cambrian algebra

Binary trees

Binary search tree = directed and labeled tree such that



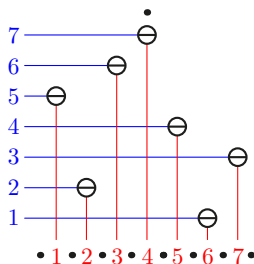
increasing tree = directed and labeled tree such that labels increase along arcs
leveled binary tree = directed tree with a binary search tree labeling and an increasing labeling



Permutations to leveled binary trees

The **sylvester correspondence** = permutations $\mapsto$ leveled binary trees.

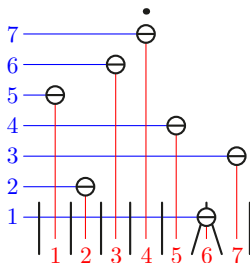
Exm: permutation 6275134



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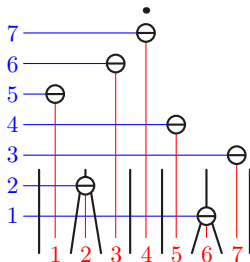
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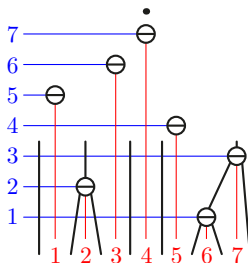
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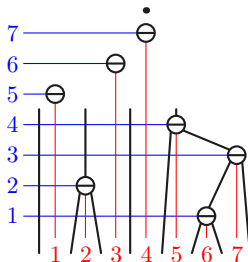
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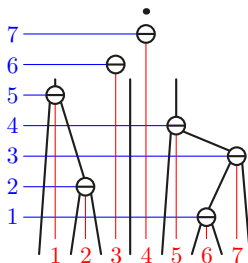
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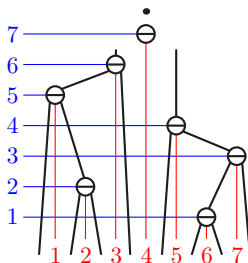
Exm: permutation 6275134



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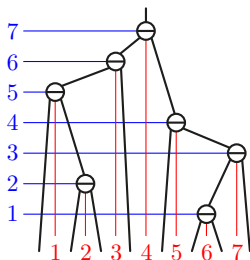
Exm: permutation 6275134



Permutations to leveled binary trees

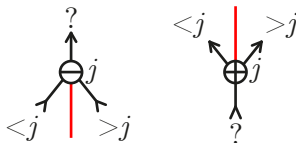
The **sylvester correspondence** = permutations $\mapsto$ leveled binary trees.

Exm: permutation 6275134



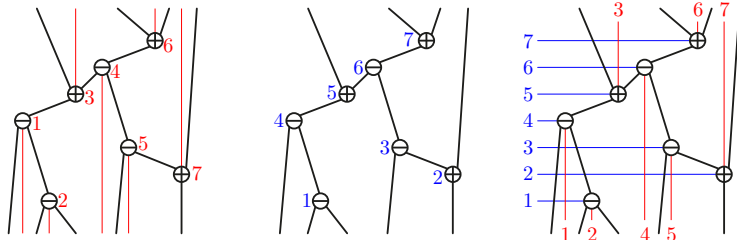
Cambrian trees

Cambrian tree = directed and labeled tree such that



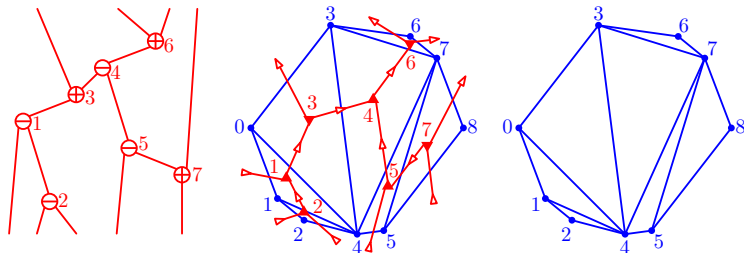
increasing tree = directed and labeled tree such that labels increase along arcs

leveled Cambrian tree = directed tree with a Cambrian labeling and an increasing labeling



Cambrian trees and triangulations

Cambrian trees are dual to triangulations of polygons



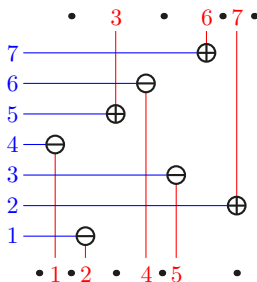
signature $\longleftrightarrow$ vertices above or below $[0, 8]$
 node j $\longleftrightarrow$ triangle $i < j < k$

For any signature ε , there are $C_n = \frac{1}{n+1} \binom{2n}{n}$ ε -Cambrian trees.

Signed permutations to Cambrian trees

Cambrian correspondence = signed permutation $\mapsto$ leveled Cambrian tree.

Exm: signed permutation $\underline{2}\bar{7}\underline{5}\underline{1}\bar{3}\underline{4}\bar{6}$

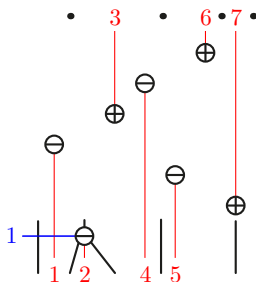


Reading. Cambrian lattices. 2006
Lange-Pilaud. Associahedra via spines. 2015

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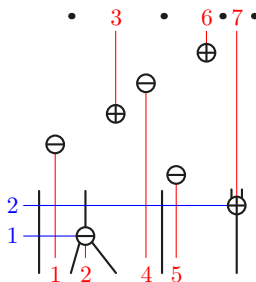


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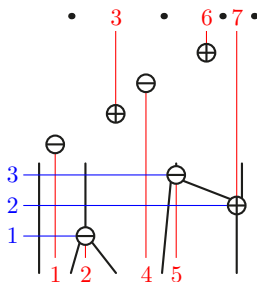


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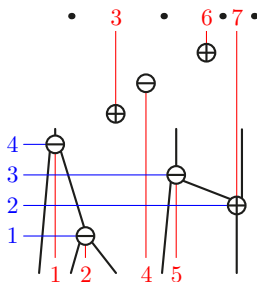


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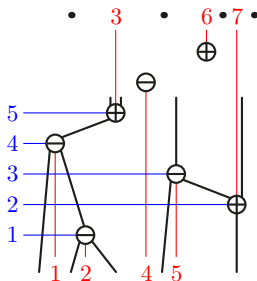


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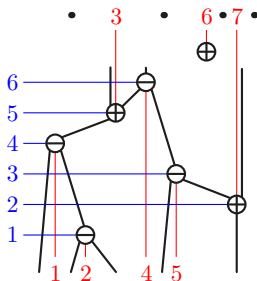


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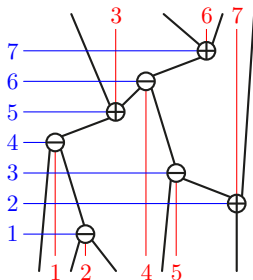


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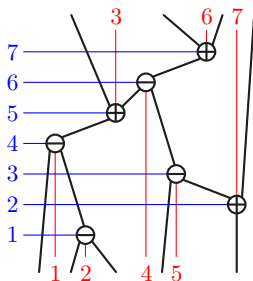


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Signed permutations to Cambrian trees

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Exm: signed permutation $\underline{2}\overline{7}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}$



$P(\tau)$ = **P**-symbol of τ = Cambrian tree produced by the Cambrian corresp.

$Q(\tau)$ = **Q**-symbol of τ = increasing tree produced by the Cambrian corresp.

(analogous to the Robinson-Schensted algorithm)

The Cambrian congruence

ε -Cambrian congruence = transitive closure of the rewriting rules

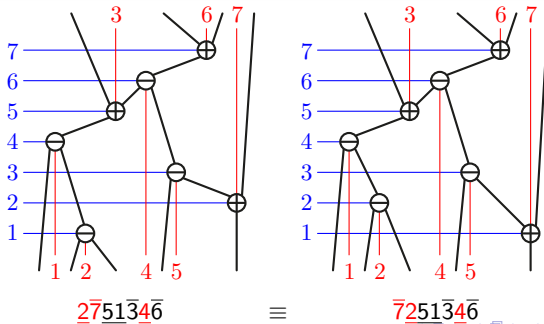
$$\dots ac \dots \underline{b} \dots \equiv_{\varepsilon} \dots ca \dots \underline{b} \dots \quad \text{if } a < b < c \text{ and } \varepsilon_b = -$$

$$\dots \bar{b} \dots ac \dots \equiv_{\varepsilon} \dots \bar{b} \dots ca \dots \quad \text{if } a < b < c \text{ and } \varepsilon_b = +$$

where a, b, c are elements of $[n]$.

Proposition [reformulating Reading 2006]

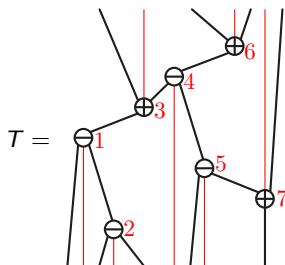
$$\tau \equiv_{\varepsilon} \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$$



Cambrian trees to signed permutations

Proposition [reformulating Reading 2006]

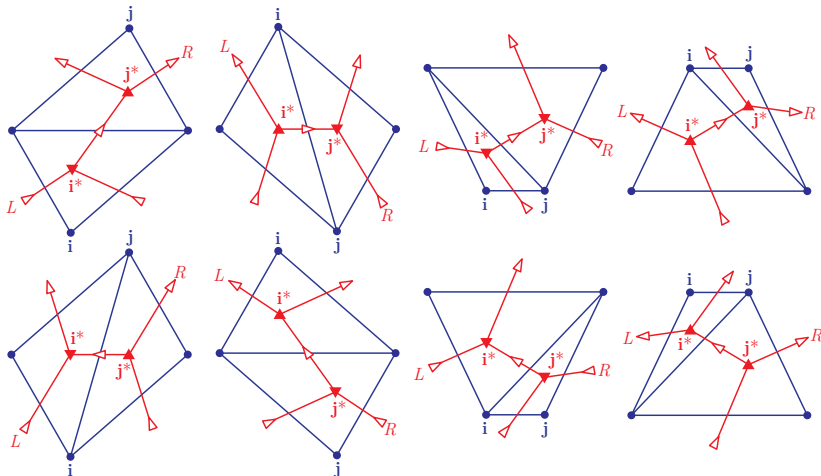
$$\mathbf{P}^{-1}(T) = \mathcal{L}(T)$$



$$\mathbf{P}^{-1}(T) = \mathcal{L}(T) = \{ \underline{2137546}, \underline{2173546}, \underline{2175346}, \underline{2713546}, \underline{2715346}, \underline{2751346}, \underline{7213546}, \underline{7215346}, \underline{7251346}, \underline{7521346} \}$$

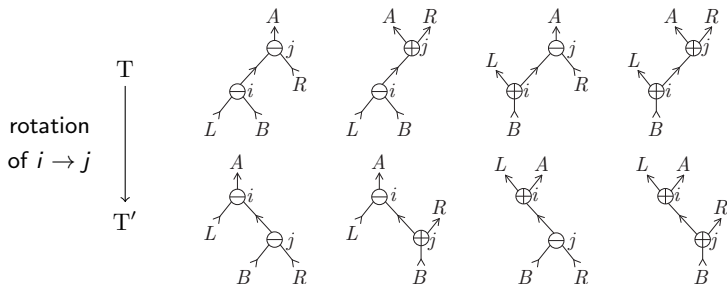
Rotations and flips

Rotation on Cambrian trees $\longleftrightarrow$ flips on triangulations.



Rotation and Cambrian lattices

Rotation operation preserves Cambrian trees:

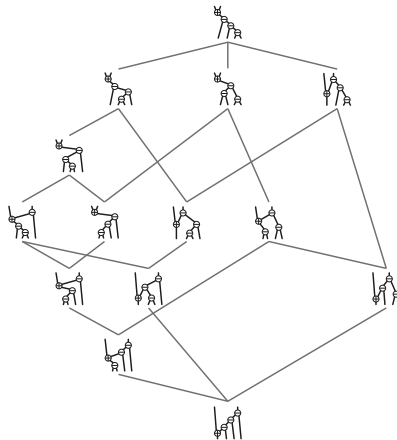
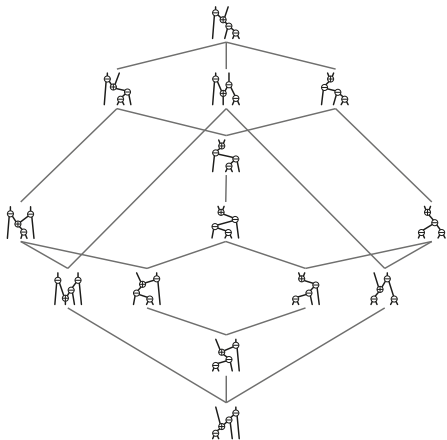


increasing rotation = rotation of edge $i \rightarrow j$ where $i < j$

Proposition [reformulating Reading 2006]

The transitive closure of the increasing rotation graph is the **Cambrian lattice**.
P defines a lattice homomorphism from the weak order to the Cambrian lattice.

Rotations and Cambrian lattices



1 Combinatorics

- Binary trees
- Cambrian trees
- Cambrian lattices

2 Algebra

- FQSym
- The Cambrian algebra

Two products on permutations

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$ with $a \in [n]$, $b \in [n']$, define

shifted concatenation $\tau \bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1) + n, \dots, \tau'(n') + n] \in \mathfrak{S}_{n+n'}$

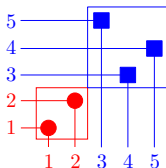
shifted shuffle product $\tau \sqcup \tau' = au \sqcup bv = a(u \sqcup bv) + (b + |au|)(au \sqcup v)$

convolution product $\tau \star \tau' = (\tau^{-1} \sqcup \tau'^{-1})^{-1}$

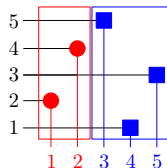
When we compute products of permutations, there is no multiplicities so we can consider that the output of the shuffle is a set of permutations.

$$12 \sqcup 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$$

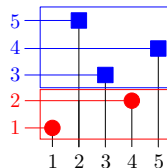
$$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$$



concatenation



shuffle



convolution

The Malvenuto-Reutenauer algebra

The Malvenuto-Reutenauer algebra = Hopf algebra FQSym with basis $(F_\tau)_{\tau \in \mathfrak{S}}$ and where

$$F_\tau \cdot F_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} F_\sigma \quad \text{and} \quad \Delta F_\sigma = \sum_{\sigma \in \tau \star \tau'} F_\tau \otimes F_{\tau'}$$

Malvenuto-Reutenauer. Duality between Quasi-Symmetric functions and the Solomon Descent Algebra. 1995

The Malvenuto-Reutenauer algebra

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Malvenuto-Reutenauer. Duality between Quasi-Symmetric functions and the Solomon Descent Algebra. 1995

Definition: Combinatorial Hopf Algebras

A *Combinatorial Hopf Algebra* = combinatorial vector space $\mathcal{B}$ endowed with

$$\begin{aligned} \text{product } \cdot : \mathcal{B} \otimes \mathcal{B} &\rightarrow \mathcal{B} \\ \text{coproduct } \Delta : \mathcal{B} &\rightarrow \mathcal{B} \otimes \mathcal{B} \end{aligned}$$

which are “compatible”, i.e.,

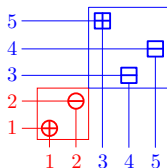
$$\Delta(f \cdot g) = \Delta(f) \cdot \Delta(g)$$

Two products on signed permutations

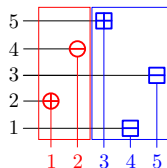
For signed permutations:

$$\underline{12} \sqcup \underline{231} = \{\underline{12453}, \underline{14253}, \underline{14523}, \underline{14532}, \underline{41253}, \underline{41523}, \underline{41532}, \underline{45123}, \underline{45132}, \underline{45312}\},$$

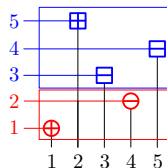
$$\underline{12} \star \underline{231} = \{\underline{12453}, \underline{13452}, \underline{14352}, \underline{15342}, \underline{23451}, \underline{24351}, \underline{25341}, \underline{34251}, \underline{35241}, \underline{45231}\}.$$



concatenation



shuffle



convolution

Signed analog of Malvenuto-Reutenauer [Novelli, Thibon 2010]

$\text{FQSym}_{\pm}$ = Hopf algebra with basis $(F_{\tau})_{\tau \in \mathfrak{S}_{\pm}}$ and where

$$F_{\tau} \cdot F_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} F_{\sigma} \quad \text{and} \quad \Delta F_{\sigma} = \sum_{\sigma \in \tau \star \tau'} F_{\tau} \otimes F_{\tau'}$$

The Cambrian algebra as a subalgebra of $\text{FQSym}_{\pm}$

The Cambrian algebra = subspace Camb of $\text{FQSym}_{\pm}$ generated by

$$\mathbb{P}_T := \sum_{\substack{\tau \in \mathfrak{S}_{\pm} \\ \mathbf{P}(\tau) = T}} \mathbb{F}_{\tau} = \sum_{\tau \in \mathcal{L}(T)} \mathbb{F}_{\tau},$$

for all Cambrian trees T .

$$\mathbb{P} = \mathbb{F}_{\underline{2137546}} + \mathbb{F}_{\underline{2173546}} + \mathbb{F}_{\underline{2175346}} + \mathbb{F}_{\underline{2713546}} + \mathbb{F}_{\underline{2715346}} + \mathbb{F}_{\underline{2751346}} + \mathbb{F}_{\underline{7213546}} + \mathbb{F}_{\underline{7215346}} + \mathbb{F}_{\underline{7251346}} + \mathbb{F}_{\underline{7251346}} + \mathbb{F}_{\underline{7521346}}$$


Theorem [C.-Pilaud]

Camb is a Hopf subalgebra of $\text{FQSym}_{\pm}$.

(i.e., the Cambrian congruence is “compatible” with the product and coproduct in $\text{FQSym}_{\pm}$)

GAME: Explain the product and coproduct directly on the Cambrian trees...

Product in the Cambrian algebra

$$\begin{aligned}
 \mathbb{P} \cdot \mathbb{P} &= \mathbb{F}_{\underline{12}} \cdot (\mathbb{F}_{\underline{213}} + \mathbb{F}_{\underline{231}}) \\
 &= \left(\begin{array}{l} \mathbb{F}_{\underline{12435}} + \mathbb{F}_{\underline{12453}} + \mathbb{F}_{\underline{14235}} \\ + \mathbb{F}_{\underline{14253}} + \mathbb{F}_{\underline{14523}} + \mathbb{F}_{\underline{41235}} \\ + \mathbb{F}_{\underline{41253}} + \mathbb{F}_{\underline{41523}} + \mathbb{F}_{\underline{45123}} \end{array} \right) + \left(\begin{array}{l} \mathbb{F}_{\underline{14325}} + \mathbb{F}_{\underline{14352}} \\ + \mathbb{F}_{\underline{14532}} + \mathbb{F}_{\underline{41325}} \\ + \mathbb{F}_{\underline{41352}} + \mathbb{F}_{\underline{41532}} \\ + \mathbb{F}_{\underline{45132}} \end{array} \right) + \left(\begin{array}{l} \mathbb{F}_{\underline{43125}} + \mathbb{F}_{\underline{43152}} \\ + \mathbb{F}_{\underline{43512}} + \mathbb{F}_{\underline{45312}} \end{array} \right) \\
 &= \mathbb{P} + \mathbb{P} + \mathbb{P}
 \end{aligned}$$

Proposition [C.-Pilaud]

For any Cambrian trees T and T' ,

$$\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_{\substack{T \nearrow \overline{T'} \\ \leq_{\text{Camb}} S \leq_{\text{Camb}} T \\ \nwarrow \overline{T}}} \mathbb{P}_S$$

Product in the Cambrian algebra

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- For every Cambrian tree T , $\mathcal{L}(T)$ is an interval of the weak order.

Product in the Cambrian algebra

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- $[\sigma, \tau] \sqcup [\sigma', \tau'] = [\sigma\overline{\sigma'}, \overline{\tau'}\tau]$.

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- For every Cambrian tree T , $\mathcal{L}(T)$ is an interval of the weak order.
- $[\sigma, \tau] \sqcup [\sigma', \tau'] = [\sigma\overline{\sigma'}, \overline{\tau'}\tau]$.
- $\mathcal{L}(T) = [\sigma, \tau], \mathcal{L}(T') = [\sigma', \tau']$

$$\mathbf{P}(\sigma\overline{\sigma'}) = \begin{array}{c} \overline{T'} \\ \nearrow \\ T \end{array} \quad \mathbf{P}(\overline{\tau'}\tau) = \begin{array}{c} T \\ \nwarrow \\ \overline{T'} \end{array}$$

Product in the Cambrian algebra

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- For every Cambrian tree T , $\mathcal{L}(T)$ is an interval of the weak order.
- $[\sigma, \tau] \sqcup [\sigma', \tau'] = [\sigma\overline{\sigma'}, \overline{\tau'}\tau]$.
- $\mathcal{L}(T) = [\sigma, \tau], \mathcal{L}(T') = [\sigma', \tau']$

$$\mathbf{P}(\sigma\overline{\sigma'}) = \begin{array}{c} \nearrow \overline{T'} \\ T \end{array} \qquad \mathbf{P}(\overline{\tau'}\tau) = \begin{array}{c} T \nwarrow \\ \overline{T'} \end{array}$$

- Permutations in $[\sigma\overline{\sigma'}, \overline{\tau'}\tau]$ form a union of Cambrian classes because Camb is a subalgebra of $\text{FQSym}_{\pm}$.

Product in the Cambrian algebra

Proposition [C.-Pilaud]

For any Cambrian trees T and T' ,

$$\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_{\begin{array}{c} T \nearrow \overline{T'} \\ \leq_{\text{Camb}} S \leq_{\text{Camb}} T \nwarrow \overline{T'} \end{array}} \mathbb{P}_S$$

- For every Cambrian tree T , $\mathcal{L}(T)$ is an interval of the weak order.
- $[\sigma, \tau] \sqcup [\sigma', \tau'] = [\sigma\overline{\sigma'}, \overline{\tau'}\tau]$.
- $\mathcal{L}(T) = [\sigma, \tau], \mathcal{L}(T') = [\sigma', \tau']$

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- Permutations in $[\sigma\overline{\sigma'}, \overline{\tau'}\tau]$ form a union of Cambrian classes because Camb is a subalgebra of $\text{FQSym}_{\pm}$.
- These trees form an interval because $\mathbf{P}$ is a lattice homomorphism.

Coproduct in the Cambrian algebra

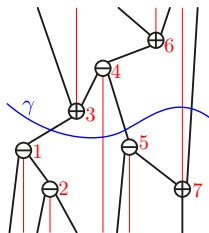
$$\begin{aligned}
\Delta \mathbb{P} &= \Delta (\mathbb{F}_{213} + \mathbb{F}_{231}) \\
&= 1 \otimes (\mathbb{F}_{213} + \mathbb{F}_{231}) + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{1\bar{2}} + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{2\bar{1}} + \mathbb{F}_{2\bar{1}} \otimes \mathbb{F}_{\bar{1}} + \mathbb{F}_{1\bar{2}} \otimes \mathbb{F}_{\bar{1}} + (\mathbb{F}_{213} + \mathbb{F}_{231}) \otimes 1 \\
&= 1 \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1 \\
&= 1 \otimes \mathbb{P} + \mathbb{P} \otimes (\mathbb{P} \cdot \mathbb{P}) + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1.
\end{aligned}$$

Proposition [C.-Pilaud]

For any Cambrian tree S ,

$$\Delta \mathbb{P}_S = \sum_{\gamma} \left(\prod_{T \in B(S, \gamma)} \mathbb{P}_T \right) \otimes \left(\prod_{T' \in A(S, \gamma)} \mathbb{P}_{T'} \right)$$

where γ runs over all cuts of S , and $A(S, \gamma)$ and $B(S, \gamma)$ denote the Cambrian forests above and below γ respectively.



Coproduct in the Cambrian algebra

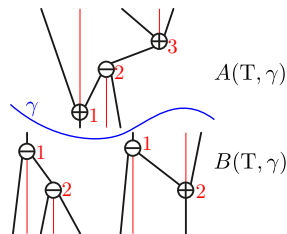
$$\begin{aligned}
\Delta \mathbb{P} &= \Delta (\mathbb{F}_{213} + \mathbb{F}_{231}) \\
&= 1 \otimes (\mathbb{F}_{213} + \mathbb{F}_{231}) + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{1\bar{2}} + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{2\bar{1}} + \mathbb{F}_{2\bar{1}} \otimes \mathbb{F}_{\bar{1}} + \mathbb{F}_{1\bar{2}} \otimes \mathbb{F}_{\bar{1}} + (\mathbb{F}_{213} + \mathbb{F}_{231}) \otimes 1 \\
&= 1 \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1 \\
&= 1 \otimes \mathbb{P} + \mathbb{P} \otimes (\mathbb{P} \cdot \mathbb{P}) + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1.
\end{aligned}$$

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Extensions

- Study of multiplicative basis.
- Hopf algebra on twin Cambrian trees.
- Hopf algebra on Schröder-Cambrian trees.

